



Texture zeros and majorana phases of the neutrino mass matrix

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Abstract

We present the generic formulas to calculate the ratios of neutrino masses and the Majorana phases of CP violation from the neutrino mass matrix with two independent vanishing entries in the flavor basis where the charged lepton mass matrix is diagonal. An order-of-magnitude illustration is given for seven experimentally acceptable textures of the neutrino mass matrix, and some analytical approximations are made for their phenomenological consequences at low energy scales.

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1. The atmospheric and solar neutrino oscillations observed in the Super-Kamiokande experiment [1] have provided robust evidence that neutrinos are massive and lepton flavors are mixed. A full description of the mass spectrum and flavor mixing in the framework of three lepton families requires twelve real parameters: three charged lepton masses (m_e, m_μ, m_τ), three neutrino masses (m_1, m_2, m_3), three flavor mixing angles ($\theta_x, \theta_y, \theta_z$), one Dirac-type CP-violating phase (δ) and two Majorana-type CP-violating phases (ρ and σ). So far only the masses of charged leptons have been accurately measured [2]. Although we have achieved some preliminary knowledge on two neutrino mass-squared differences and three flavor mixing angles from current neutrino oscillation experiments, much more effort is needed to determine these parameters precisely. The more challenging task is to pin down the absolute neutrino mass scale and the CP-violating phases. Towards reaching these goals, a number of new neutrino experiments have been proposed [3].

After sufficient information on neutrino masses and lepton flavor mixing parameters is experimentally accumulated, a determination of the textures of lepton mass matrices should become possible. On the other hand, the textures of charged lepton and neutrino mass matrices may finally be derived from a fundamental theory of lepton mass generation, which is unfortunately unknown for the time being. It is therefore important in phenomenology to investigate how the textures of lepton mass matrices can link up with the observables of lepton flavor mixing.

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Recently Frampton, Glashow and Marfatia [4] have examined the possibility that a restricted class of lepton mass matrices may suffice to describe current experimental data. They find seven acceptable textures of the neutrino mass matrix with two independent vanishing entries in the flavor basis where the charged lepton mass matrix is diagonal.

In this Letter we carry out a further study of two-zero textures of the neutrino mass matrix. Our work is different from Ref. [4] in several aspects: (a) we write out the generic constraint equations for the neutrino mass matrix with two independent vanishing entries, from which the analytically exact expressions of neutrino mass ratios can be derived; (b) the formulas to calculate the Majorana-type CP-violating phases are presented; (c) the relative magnitudes of neutrino masses, the Majorana phases, the ratio of two neutrino mass-squared differences, and the effective mass term of the neutrinoless double beta decay are estimated by taking typical inputs of the flavor mixing angles and the Dirac-type CP-violating phase; and (d) an order-of-magnitude illustration is given for seven two-zero textures of the neutrino mass matrix.

2. In the flavor basis where the charged lepton mass matrix is diagonal, the neutrino mass matrix can be written as

$$M = V \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} V^T, \quad (1)$$

where m_i (for $i = 1, 2, 3$) denote the real and positive neutrino masses, and V is the lepton flavor mixing matrix linking the neutrino mass eigenstates (ν_1, ν_2, ν_3) to the neutrino flavor eigenstates (ν_e, ν_μ, ν_τ) in the chosen basis. A full description of V needs six real parameters: three mixing angles and three CP-violating phases. Note that V can always be expressed as a product of the Dirac-type flavor mixing matrix U (consisting of three mixing angles and one CP-violating phase) and a diagonal phase matrix P (consisting of two nontrivial Majorana phases): $V = UP$. Then we may rewrite M in Eq. (1) as

$$M = U \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} U^T, \quad (2)$$

where two Majorana-type CP-violating phases are included into the complex neutrino mass eigenvalues λ_i , and the relation $|\lambda_i| = m_i$ holds. Without loss of generality, we take

$$\lambda_1 = m_1 e^{2i\rho}, \quad \lambda_2 = m_2 e^{2i\sigma}, \quad \lambda_3 = m_3. \quad (3)$$

In the following we shall show that both the neutrino mass ratios (m_1/m_3 and m_2/m_3) and the Majorana phases (ρ and σ) can be determined, if two independent entries of M vanish.

As M is symmetric, it totally has six independent complex entries. If two of them vanish, i.e., $M_{ab} = M_{\alpha\beta} = 0$, we obtain the following constraint relations:

$$\sum_{i=1}^3 (U_{ai} U_{bi} \lambda_i) = 0, \quad \sum_{i=1}^3 (U_{\alpha i} U_{\beta i} \lambda_i) = 0, \quad (4)$$

where each of the four subscripts runs over e, μ and τ , but $(\alpha, \beta) \neq (a, b)$. Solving Eq. (4), we find

$$\frac{\lambda_1}{\lambda_3} = \frac{U_{a3} U_{b3} U_{\alpha 2} U_{\beta 2} - U_{a2} U_{b2} U_{\alpha 3} U_{\beta 3}}{U_{a2} U_{b2} U_{\alpha 1} U_{\beta 1} - U_{a1} U_{b1} U_{\alpha 2} U_{\beta 2}}, \quad (5)$$

and

$$\frac{\lambda_2}{\lambda_3} = \frac{U_{a1} U_{b1} U_{\alpha 3} U_{\beta 3} - U_{a3} U_{b3} U_{\alpha 1} U_{\beta 1}}{U_{a2} U_{b2} U_{\alpha 1} U_{\beta 1} - U_{a1} U_{b1} U_{\alpha 2} U_{\beta 2}}. \quad (6)$$

One can observe that the left-hand sides of Eqs. (5) and (6) are associated with the Majorana-type CP-violating phases, while the right-hand sides of Eqs. (5) and (6) are associated with the Dirac-type CP-violating phase hidden in the elements of U . Therefore, two Majorana phases must depend upon the Dirac-type CP-violating phase. This dependence results simply from the texture zeros of M that we have taken.

Comparing Eq. (5) or Eq. (6) with Eq. (3), we arrive at the expressions of two neutrino mass ratios as follows:

$$\frac{m_1}{m_3} = \left| \frac{U_{a3}U_{b3}U_{\alpha 2}U_{\beta 2} - U_{a2}U_{b2}U_{\alpha 3}U_{\beta 3}}{U_{a2}U_{b2}U_{\alpha 1}U_{\beta 1} - U_{a1}U_{b1}U_{\alpha 2}U_{\beta 2}} \right|, \quad \frac{m_2}{m_3} = \left| \frac{U_{a1}U_{b1}U_{\alpha 3}U_{\beta 3} - U_{a3}U_{b3}U_{\alpha 1}U_{\beta 1}}{U_{a2}U_{b2}U_{\alpha 1}U_{\beta 1} - U_{a1}U_{b1}U_{\alpha 2}U_{\beta 2}} \right|. \quad (7)$$

Furthermore, the expressions of two Majorana phases are found to be

$$\rho = \frac{1}{2} \arg \left[\frac{U_{a3}U_{b3}U_{\alpha 2}U_{\beta 2} - U_{a2}U_{b2}U_{\alpha 3}U_{\beta 3}}{U_{a2}U_{b2}U_{\alpha 1}U_{\beta 1} - U_{a1}U_{b1}U_{\alpha 2}U_{\beta 2}} \right],$$

$$\sigma = \frac{1}{2} \arg \left[\frac{U_{a1}U_{b1}U_{\alpha 3}U_{\beta 3} - U_{a3}U_{b3}U_{\alpha 1}U_{\beta 1}}{U_{a2}U_{b2}U_{\alpha 1}U_{\beta 1} - U_{a1}U_{b1}U_{\alpha 2}U_{\beta 2}} \right]. \quad (8)$$

With the inputs of three flavor mixing angles and the Dirac-type CP-violating phase, we are able to predict the relative magnitudes of three neutrino masses and the values of two Majorana phases. This predictability allows us to examine whether the chosen texture of M with two independent vanishing entries is empirically acceptable or not.

Indeed, the prediction for m_1/m_2 and m_2/m_3 in a given pattern of M is required to be compatible with the hierarchy of solar and atmospheric neutrino mass-squared differences:

$$R_\nu \equiv \left| \frac{m_2^2 - m_1^2}{m_3^2 - m_2^2} \right| \approx \frac{\Delta m_{\text{sun}}^2}{\Delta m_{\text{atm}}^2} \ll 1. \quad (9)$$

The magnitude of R_ν depends upon the explicit solution to the solar neutrino problem. For the large-angle Mikheyev–Smirnov–Wolfenstein (MSW) oscillation of solar neutrinos [5], which is most favored by the present Super-Kamiokande [1] and SNO [6] data, we have $R_\nu \sim \mathcal{O}(10^{-2})$. Because of $|V_{e3}|^2 = |U_{e3}|^2 \ll 1$ [7], the atmospheric neutrino oscillation is approximately decoupled from the solar neutrino oscillation.

With the help of Eqs. (7) and (8), one can calculate the effective mass term of the neutrinoless double beta decay, whose magnitude amounts to $|M_{ee}|$. The explicit expression of $|M_{ee}|$ reads as follows:

$$|M_{ee}| = m_3 \left| \frac{m_1}{m_3} U_{e1}^2 e^{2i\rho} + \frac{m_2}{m_3} U_{e2}^2 e^{2i\sigma} + U_{e3}^2 \right|. \quad (10)$$

The Heidelberg–Moscow Collaboration has reported $|M_{ee}| < 0.34$ eV at the 90% confidence level [8]. Useful information on the absolute mass scale of neutrinos could in principle be extracted from a more accurate measurement of $|M_{ee}|$ in the future.

3. As already pointed out in Ref. [4], there are totally fifteen logical possibilities for the texture of M with two independent vanishing entries, but only seven of them are in accord with current experimental data and empirical hypotheses. The seven acceptable patterns of M are listed in Table 1, where all the non-vanishing entries are symbolized by \times 's. To work out the explicit expressions of λ_1/λ_3 and λ_2/λ_3 in each case, we adopt the following parametrization for the Dirac-type flavor mixing matrix U :

$$U = \begin{pmatrix} c_x c_z & s_x c_z & s_z \\ -c_x s_y s_z - s_x c_y e^{-i\delta} & -s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\ -c_x c_y s_z + s_x s_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z \end{pmatrix}, \quad (11)$$

where $s_x \equiv \sin \theta_x$, $c_x \equiv \cos \theta_x$, and so on. The advantage of this phase choice is that the Dirac-type CP-violating phase δ does not appear in the effective mass term of the neutrinoless double beta decay [9]. In other words, the

Table 1

Seven patterns of the neutrino mass matrix M with two independent vanishing entries, which are in accord with current experimental data and empirical hypotheses. An order-of-magnitude illustration of M is given by using typical inputs of $\theta_x, \theta_y, \theta_z$ and δ , as explained in the text

Pattern	Texture of M	Order of magnitude
A ₁	$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \times \\ \mathbf{0} & \times & \times \\ \times & \times & \times \end{pmatrix}$	$\sim m_3 \begin{pmatrix} \mathbf{0} & \mathbf{0} & 0.1 \\ \mathbf{0} & 0.4 & 0.5 \\ 0.1 & 0.5 & 0.6 \end{pmatrix}$
A ₂	$\begin{pmatrix} \mathbf{0} & \times & \mathbf{0} \\ \times & \times & \times \\ \mathbf{0} & \times & \times \end{pmatrix}$	$\sim m_3 \begin{pmatrix} \mathbf{0} & 0.1 & \mathbf{0} \\ 0.1 & 0.4 & 0.5 \\ \mathbf{0} & 0.5 & 0.6 \end{pmatrix}$
B ₁	$\begin{pmatrix} \times & \times & \mathbf{0} \\ \times & \mathbf{0} & \times \\ \mathbf{0} & \times & \times \end{pmatrix}$	$\sim m_3 \begin{pmatrix} 0.7 & 0.06 & \mathbf{0} \\ 0.06 & \mathbf{0} & 0.8 \\ \mathbf{0} & 0.8 & 0.3 \end{pmatrix}$
B ₂	$\begin{pmatrix} \times & \mathbf{0} & \times \\ \mathbf{0} & \times & \times \\ \times & \times & \mathbf{0} \end{pmatrix}$	$\sim m_1 \begin{pmatrix} 1.0 & \mathbf{0} & 0.05 \\ \mathbf{0} & 0.3 & 0.8 \\ 0.05 & 0.8 & \mathbf{0} \end{pmatrix}$
B ₃	$\begin{pmatrix} \times & \mathbf{0} & \times \\ \mathbf{0} & \mathbf{0} & \times \\ \times & \times & \times \end{pmatrix}$	$\sim m_3 \begin{pmatrix} 0.7 & \mathbf{0} & 0.07 \\ \mathbf{0} & \mathbf{0} & 0.8 \\ 0.07 & 0.8 & 0.3 \end{pmatrix}$
B ₄	$\begin{pmatrix} \times & \times & \mathbf{0} \\ \times & \times & \times \\ \mathbf{0} & \times & \mathbf{0} \end{pmatrix}$	$\sim m_1 \begin{pmatrix} 1.0 & 0.04 & \mathbf{0} \\ 0.04 & 0.3 & 0.8 \\ \mathbf{0} & 0.8 & \mathbf{0} \end{pmatrix}$
C	$\begin{pmatrix} \times & \times & \times \\ \times & \mathbf{0} & \times \\ \times & \times & \mathbf{0} \end{pmatrix}$	$\sim m_3 \begin{pmatrix} 1.0 & 0.06 & 0.2 \\ 0.06 & \mathbf{0} & 1.0 \\ 0.2 & 1.0 & \mathbf{0} \end{pmatrix}$

latter depends only upon the Majorana phases ρ and σ in our phase convention. Without loss of generality, three mixing angles $(\theta_x, \theta_y, \theta_z)$ can all be arranged to lie in the first quadrant. Three CP-violating phases (δ, ρ, σ) may take arbitrary values from $-\pi$ to $+\pi$ (or from 0 to 2π).

Now let us calculate λ_1/λ_3 and λ_2/λ_3 for each pattern of M with the help of Eqs. (5), (6) and (11). The instructive results for $m_1/m_3, m_2/m_3, \rho, \sigma, R_\nu$ and $|M_{ee}|$ may then be obtained.

Pattern A₁. $M_{ee} = M_{e\mu} = 0$ (i.e., $a = b = e; \alpha = e$ and $\beta = \mu$). We obtain

$$\frac{\lambda_1}{\lambda_3} = +\frac{s_z}{c_z^2} \left(\frac{s_x s_y}{c_x c_y} e^{i\delta} - s_z \right), \quad \frac{\lambda_2}{\lambda_3} = -\frac{s_z}{c_z^2} \left(\frac{c_x s_y}{s_x c_y} e^{i\delta} + s_z \right). \tag{12}$$

As current experimental data favor $\sin^2 2\theta_x \sim \mathcal{O}(1)$, $\sin^2 2\theta_y \approx 1$ and $\sin^2 2\theta_z \leq 0.1$ [1,6,7], one may make an analytical approximation for the exact result obtained above. By use of Eqs. (7)–(10), we arrive explicitly at

$$\begin{aligned} \frac{m_1}{m_3} &\approx t_x t_y s_z, & \frac{m_2}{m_3} &\approx \frac{t_y}{t_x} s_z, & \rho &\approx \frac{\delta}{2}, & \sigma &\approx \frac{\delta}{2} \pm \frac{\pi}{2}, \\ R_\nu &\approx \frac{t_y^2}{t_x^2} |1 - t_x^4| s_z^2, & |M_{ee}| &= 0 \end{aligned} \tag{13}$$

to lowest order, where $t_x \equiv \tan \theta_x$ and so on. Taking the typical inputs $\theta_x = 30^\circ, \theta_y = 40^\circ, \theta_z = 5^\circ$ and $\delta = 90^\circ$, we obtain $m_1/m_3 \approx 0.04, m_2/m_3 \approx 0.13, \rho \approx 45^\circ$, and $\sigma \approx 135^\circ$ (or -45°). In addition, we get $R_\nu \approx 0.014$, consistent with our empirical hypothesis that the solar neutrino deficit is attributed to the large-angle MSW oscillation. The vanishing of $|M_{ee}|$ implies that it is in practice impossible to detect the neutrinoless double beta decay.

Pattern A₂. $M_{ee} = M_{e\tau} = 0$ (i.e., $a = b = e$; $\alpha = e$ and $\beta = \tau$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{s_z}{c_z^2} \left(\frac{s_x c_y}{c_x s_y} e^{i\delta} + s_z \right), \quad \frac{\lambda_2}{\lambda_3} = +\frac{s_z}{c_z^2} \left(\frac{c_x c_y}{s_x s_y} e^{i\delta} - s_z \right). \quad (14)$$

In the lowest-order approximation, we explicitly obtain

$$\begin{aligned} \frac{m_1}{m_3} &\approx \frac{t_x}{t_y} s_z, & \frac{m_2}{m_3} &\approx \frac{1}{t_x t_y} s_z, & \rho &\approx \frac{\delta}{2} \pm \frac{\pi}{2}, & \sigma &\approx \frac{\delta}{2}, \\ R_\nu &\approx \frac{1}{t_x^2 t_y^2} |1 - t_x^4| s_z^2, & |M_{ee}| &= 0. \end{aligned} \quad (15)$$

Using the same inputs as above, we get $m_1/m_3 \approx 0.06$, $m_2/m_3 \approx 0.18$, $\rho \approx 135^\circ$ (or -45°), $\sigma \approx 45^\circ$, and $R_\nu \approx 0.03$. We see that the phenomenological consequences of Patterns A₁ and A₂ are nearly the same [4]. However, Pattern A₂ seems to be more interesting for model building [10], in particular when the spirit of lepton-quark similarity is taken into account.

Pattern B₁. $M_{\mu\mu} = M_{e\tau} = 0$ (i.e., $a = b = \mu$; $\alpha = e$ and $\beta = \tau$). We obtain

$$\begin{aligned} \frac{\lambda_1}{\lambda_3} &= \frac{s_x c_x s_y (2c_y^2 s_z^2 - s_y^2 c_z^2) - c_y s_z (s_x^2 s_y^2 e^{+i\delta} + c_x^2 c_y^2 e^{-i\delta})}{s_x c_x s_y c_y^2 + (s_x^2 - c_x^2) c_y^3 s_z e^{i\delta} + s_x c_x s_y s_z^2 (1 + c_y^2)} e^{2i\delta}, \\ \frac{\lambda_2}{\lambda_3} &= \frac{s_x c_x s_y (2c_y^2 s_z^2 - s_y^2 c_z^2) + c_y s_z (c_x^2 s_y^2 e^{+i\delta} + s_x^2 c_y^2 e^{-i\delta})}{s_x c_x s_y c_y^2 + (s_x^2 - c_x^2) c_y^3 s_z e^{i\delta} + s_x c_x s_y s_z^2 (1 + c_y^2)} e^{2i\delta}. \end{aligned} \quad (16)$$

The smallness of s_z^2 allows us to make a similar analytical approximation as before. To lowest order, we find

$$\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx t_y^2, \quad \rho \approx \sigma \approx \delta \pm \frac{\pi}{2}, \quad R_\nu \approx \frac{1 + t_x^2}{t_x} |t_{2y} c_\delta| s_z, \quad |M_{ee}| \approx m_3 t_y^2, \quad (17)$$

where $t_{2y} \equiv \tan 2\theta_y$ and $c_\delta \equiv \cos \delta$. Note that

$$\frac{m_1}{m_3} - \frac{m_2}{m_3} \approx \frac{4s_z c_\delta}{s_{2x} s_{2y}}, \quad \sigma - \rho \approx \frac{2s_z s_\delta}{t_y^2 s_{2x} t_{2y}} \quad (18)$$

in the next-to-leading order approximation, where $s_\delta \equiv \sin \delta$ and $s_{2x} \equiv \sin 2\theta_x$. Typically taking $\theta_x = 30^\circ$, $\theta_y = 40^\circ$, $\theta_z = 5^\circ$ and $\delta = 89^\circ$, we arrive at $m_1/m_3 \approx m_2/m_3 \approx 0.7$ with a difference of about 0.007, $\sigma \approx \rho \approx 179^\circ$ (or -1°) with a difference of about 3° , $R_\nu \approx 0.02$, and $|M_{ee}|/m_3 \approx 0.7$. One can see that $|\delta| \approx 90^\circ$ is required in this texture of M for plausible inputs of θ_y and θ_z , such that R_ν gets suppressed sufficiently. If Pattern B₁ is realistically correct, large CP-violating effects may be observable in long-baseline neutrino oscillations. It is also worth mentioning that a typical upper bound on three nearly degenerate neutrino masses can be extracted from the Heidelberg–Moscow experiment [8]: $m_1 \approx m_2 \approx 0.7m_3 \approx |M_{ee}| < 0.34$ eV. This bound is certainly compatible with the present direct-mass-search experiments [2], in particular, for the electron neutrino.

Pattern B₂. $M_{\tau\tau} = M_{e\mu} = 0$ (i.e., $a = b = \tau$; $\alpha = e$ and $\beta = \mu$). We obtain

$$\begin{aligned} \frac{\lambda_1}{\lambda_3} &= \frac{s_x c_x c_y (2s_y^2 s_z^2 - c_y^2 c_z^2) + s_y s_z (s_x^2 c_y^2 e^{+i\delta} + c_x^2 s_y^2 e^{-i\delta})}{s_x c_x s_y^2 c_y - (s_x^2 - c_x^2) s_y^3 s_z e^{i\delta} + s_x c_x c_y s_z^2 (1 + s_y^2)} e^{2i\delta}, \\ \frac{\lambda_2}{\lambda_3} &= \frac{s_x c_x c_y (2s_y^2 s_z^2 - c_y^2 c_z^2) - s_y s_z (c_x^2 c_y^2 e^{+i\delta} + s_x^2 s_y^2 e^{-i\delta})}{s_x c_x s_y^2 c_y - (s_x^2 - c_x^2) s_y^3 s_z e^{i\delta} + s_x c_x c_y s_z^2 (1 + s_y^2)} e^{2i\delta}. \end{aligned} \quad (19)$$

In the lowest-order approximation, we explicitly obtain

$$\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx \frac{1}{t_y^2}, \quad \rho \approx \sigma \approx \delta \pm \frac{\pi}{2}, \quad R_\nu \approx \frac{1+t_x^2}{t_x} |t_{2y} c_\delta| s_z, \quad |M_{ee}| \approx \frac{m_3}{t_y^2}, \quad (20)$$

together with

$$\frac{m_2}{m_3} - \frac{m_1}{m_3} \approx \frac{4s_z c_\delta}{s_{2x} s_{2y}}, \quad \sigma - \rho \approx \frac{2t_y^2 s_z s_\delta}{s_{2x} t_{2y}}. \quad (21)$$

Using the same inputs as in Pattern B₁, we get $m_2/m_3 \approx m_1/m_3 \approx 1.4$ with a difference of about 0.007, $\sigma \approx \rho \approx 179^\circ$ (or -1°) with a difference of about 1.4° , $R_\nu \approx 0.02$, and $|M_{ee}|/m_3 \approx 1.4$. Because of $t_y \sim \mathcal{O}(1)$, the phenomenological consequences of Patterns B₁ and B₂ are almost the same.

Pattern B₃. $M_{\mu\mu} = M_{e\mu} = 0$ (i.e., $a = b = \mu$; $\alpha = e$ and $\beta = \mu$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{s_y}{c_y} \cdot \frac{s_x s_y - c_x c_y s_z e^{-i\delta}}{s_x c_y + c_x s_y s_z e^{+i\delta}} e^{2i\delta}, \quad \frac{\lambda_2}{\lambda_3} = -\frac{s_y}{c_y} \cdot \frac{c_x s_y + s_x c_y s_z e^{-i\delta}}{c_x c_y - s_x s_y s_z e^{+i\delta}} e^{2i\delta}. \quad (22)$$

The approximate expressions for the neutrino mass ratios, the Majorana phases and the observables R_ν and $|M_{ee}|$ turn out to be

$$\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx t_y^2, \quad \rho \approx \sigma \approx \delta \pm \frac{\pi}{2}, \quad R_\nu \approx \frac{1+t_x^2}{t_x} t_y^2 |t_{2y} c_\delta| s_z, \quad |M_{ee}| \approx m_3 t_y^2. \quad (23)$$

In addition,

$$\frac{m_2}{m_3} - \frac{m_1}{m_3} \approx \frac{4t_y^2 s_z c_\delta}{s_{2x} s_{2y}}, \quad \rho - \sigma \approx \frac{2s_z s_\delta}{s_{2x} t_{2y}} \quad (24)$$

in the next-to-leading order approximation. Taking the same inputs as in Pattern B₁, we find $m_2/m_3 \approx m_1/m_3 \approx 0.7$ with a difference of about 0.005, $\rho \approx \sigma \approx 179^\circ$ (or -1°) with a difference of about 2° , $R_\nu \approx 0.014$, and $|M_{ee}|/m_3 \approx 0.7$. One can see that the phenomenological consequences of Pattern B₃ are essentially the same as those of Pattern B₁. This point has been observed in Ref. [4].

Pattern B₄. $M_{\tau\tau} = M_{e\tau} = 0$ (i.e., $a = b = \tau$; $\alpha = e$ and $\beta = \tau$). We obtain

$$\frac{\lambda_1}{\lambda_3} = -\frac{c_y}{s_y} \cdot \frac{s_x c_y + c_x s_y s_z e^{-i\delta}}{s_x s_y - c_x c_y s_z e^{+i\delta}} e^{2i\delta}, \quad \frac{\lambda_2}{\lambda_3} = -\frac{c_y}{s_y} \cdot \frac{c_x c_y - s_x s_y s_z e^{-i\delta}}{c_x s_y + s_x c_y s_z e^{+i\delta}} e^{2i\delta}. \quad (25)$$

To lowest order, we get the following approximate results:

$$\frac{m_1}{m_3} \approx \frac{m_2}{m_3} \approx \frac{1}{t_y^2}, \quad \rho \approx \sigma \approx \delta \pm \frac{\pi}{2}, \quad R_\nu \approx \frac{1+t_x^2}{t_x t_y^2} |t_{2y} c_\delta| s_z, \quad |M_{ee}| \approx \frac{m_3}{t_y^2}, \quad (26)$$

together with

$$\frac{m_1}{m_3} - \frac{m_2}{m_3} \approx \frac{4s_z c_\delta}{s_{2x} s_{2y} t_y^2}, \quad \rho - \sigma \approx \frac{2s_z s_\delta}{s_{2x} t_{2y}}. \quad (27)$$

Using the same inputs as in Pattern B₁, we obtain $m_1/m_3 \approx m_2/m_3 \approx 1.4$ with a difference of about 0.01, $\rho \approx \sigma \approx 179^\circ$ (or -1°) with a difference of about 2° , $R_\nu \approx 0.03$, and $|M_{ee}|/m_3 \approx 1.4$. One can see that the phenomenological consequences of Patterns B₁, B₂, B₃ and B₄ are nearly the same. Therefore, it is very difficult, even impossible, to distinguish one of them from the others in practical experiments. Nevertheless, one of the four textures might be more favored than the others in model building, when underlying flavor symmetries responsible for those texture zeros are taken into account.

Pattern C. $M_{\mu\mu} = M_{\tau\tau} = 0$ (i.e., $a = b = \mu$ and $\alpha = \beta = \tau$). We obtain

$$\begin{aligned}\frac{\lambda_1}{\lambda_3} &= -\frac{c_x c_z^2}{s_z} \cdot \frac{c_x (s_y^2 - c_y^2) + 2s_x s_y c_y s_z e^{i\delta}}{2s_x c_x s_y c_y - (s_x^2 - c_x^2)(s_y^2 - c_y^2)s_z e^{i\delta} + 2s_x c_x s_y c_y s_z^2 e^{2i\delta}} e^{i\delta}, \\ \frac{\lambda_2}{\lambda_3} &= +\frac{s_x c_z^2}{s_z} \cdot \frac{s_x (s_y^2 - c_y^2) - 2c_x s_y c_y s_z e^{i\delta}}{2s_x c_x s_y c_y - (s_x^2 - c_x^2)(s_y^2 - c_y^2)s_z e^{i\delta} + 2s_x c_x s_y c_y s_z^2 e^{2i\delta}} e^{i\delta}.\end{aligned}\quad (28)$$

Assuming $s_z^2 \ll 1$ and $t_x \sim t_y \sim \mathcal{O}(1)$, we may make an analytical approximation for the exact result in Eq. (28). To lowest order, we get

$$\begin{aligned}\frac{m_1}{m_3} &\approx \sqrt{1 - \frac{2c_\delta}{t_x t_y s_z} + \frac{1}{t_x^2 t_y^2 s_z^2}}, & \frac{m_2}{m_3} &\approx \sqrt{1 + \frac{2t_x c_\delta}{t_y s_z} + \frac{t_x^2}{t_y^2 s_z^2}}, \\ R_\nu &\approx \frac{1 + t_x t_y}{t_x t_y} \left| \frac{2}{t_x} \frac{1 - t_x t_y s_z c_\delta}{t_x + 2t_y s_z c_\delta} \right|, & |M_{ee}| &\approx m_3 \sqrt{1 - \frac{4c_\delta}{t_x t_y s_z} + \frac{4}{t_x t_y s_z^2}}\end{aligned}\quad (29)$$

as well as

$$\rho \approx \delta + \epsilon \pm \frac{\pi}{2} \quad \text{with} \quad t_\epsilon = \frac{s_\delta}{t_x t_y s_z - c_\delta}, \quad \sigma \approx \delta - \epsilon \pm \frac{\pi}{2} \quad \text{with} \quad t_\epsilon = \frac{t_x s_\delta}{t_y s_z + t_x c_\delta}.\quad (30)$$

One can observe that a small value of R_ν is possible if and only if the condition $t_x t_y s_z c_\delta \approx 1$ is satisfied [4]. Some fine tuning of the inputs seems unavoidable in this case. Taking $\theta_x = \theta_y = 44.8^\circ$, $\theta_z = 5^\circ$ and $\delta = 90^\circ$, for example, we find $R_\nu \approx 0.03$, $\rho \approx +5^\circ$ (or 185°), $\sigma \approx -5^\circ$ (or 175°), and $m_1 \approx m_2 \approx m_3 \approx |M_{ee}|$. If this texture of M is realistically correct, large CP violation may manifest itself in neutrino oscillations.

4. As shown above, the seven patterns of M can be classified into three distinct categories [4]: A (with A_1 and A_2), B (with B_1 , B_2 , B_3 and B_4), and C. It is experimentally difficult or impossible to distinguish the textures of M within each category. However, category A is experimentally distinguishable from category B or C. To be specific, let us summarize the main phenomenological consequences of each category:

- (1) The neutrino mass spectrum: $m_1 \sim m_2 \ll m_3$ in category A; $m_1 \sim m_2 \sim m_3$ in category B; and $m_1 \sim m_2 \sim m_3$ in category C.
- (2) The Dirac phase of CP violation: δ is not constrained in category A; $|\delta| \approx \pi/2$ in category B (for plausible inputs of θ_y and θ_z); and δ is sensitive to the values of three mixing angles in category C.
- (3) The Majorana phases of CP violation: $|\sigma - \rho| \approx \pi/2$ in category A; $\sigma \approx \rho$ in category B; and $\sigma \sim \rho$ in category C.
- (4) The neutrinoless double beta decay: $|M_{ee}| \approx 0$ in category A; $|M_{ee}| \sim m_3$ in category B; and $|M_{ee}| \sim m_3$ in category C.

We see that it is not easy to distinguish between category B and category C, unless the values of flavor mixing angles (θ_x , θ_y , θ_z) and the ratio of solar and atmospheric neutrino mass-squared differences (R_ν) can be accurately determined.

It is worth remarking that the “inverse” neutrino mass hierarchy $m_1 \gg m_2 \gg m_3$ cannot be incorporated with three categories of M discussed above. The reason is simply that such a hierarchy conflicts with our empirical hypotheses [4], i.e., $\Delta m_{\text{sun}}^2 = |m_2^2 - m_1^2|$ and $\Delta m_{\text{atm}}^2 = |m_3^2 - m_2^2|$. If $m_1 \gg m_2 \gg m_3$ were assumed, one would inevitably be led to $R_\nu \equiv |m_2^2 - m_1^2|/|m_3^2 - m_2^2| \approx m_1^2/m_2^2 \gg 1$, contrary to the prerequisite $R_\nu \approx \Delta m_{\text{sun}}^2/\Delta m_{\text{atm}}^2 \ll 1$ set in Eq. (9). Therefore, we conclude that only the normal hierarchy or near degeneracy of neutrino masses is allowed for seven two-zero patterns of the neutrino mass matrix under consideration.

To give an order-of-magnitude illustration of the neutrino mass matrix, we calculate the elements of M for each pattern by using the formula

$$M_{ab} = \sum_{i=1}^3 (V_{ai} V_{bi} m_i) = \sum_{i=1}^3 (U_{ai} U_{bi} \lambda_i) \quad (31)$$

and the typical inputs taken before. The rough results are listed in Table 1. We see that there is no clear hierarchy among the non-vanishing elements of M , unlike the familiar case of quark mass matrices [11].

Of course, the specific textures of lepton mass matrices cannot be preserved to all orders or at any energy scales in the unspecified interactions which generate lepton masses [4]. Nevertheless, those phenomenologically favored textures at low energy scales may shed light on the underlying flavor symmetries responsible for the generation of lepton masses at high energy scales. It is expected that more precise data of neutrino oscillations in the future could help select the *most* favorable pattern of lepton mass matrices.

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