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Algebraic geometry of the eigenvector mapping for a free rigid body

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ABSTRACT

The present paper deals with the algebro-geometric aspects of the eigenvector mapping for a free rigid body. The eigenvector mapping is regarded as a rational mapping to the complex projective plane from the product of the elliptic curves, one of which is the integral curve and the other the spectral curve. This is the space of the necessary data to determine the eigenvectors. The eigenvector mapping admits a factorisation through a Kummer surface, which is a double covering of the projective plane branched along a sextic curve associated with the dynamics. The key of the argument is the Cremona transformation of the projective plane and some elliptic fibrations of the Kummer surface.
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1. Introduction

The dynamical system for a free rigid body on $SO(3)$, which is nothing but the dynamical system of the geodesic flow on $SO(3)$ with respect to a left-invariant Riemannian metric from the viewpoint of differential geometry, is a typical example of completely integrable systems. The motion of a free rigid body can be described by the Euler equation posed on the angular momentum, after using the symplectic reduction procedure. By means of the two first integrals, the kinetic energy and the norm of the angular momentum, the integral curve of the Euler equation coincides with one of the connected components of the intersection of the quadric level surfaces of the first integrals, which is generally a (real) smooth elliptic curve. From the viewpoint of the theory of integrable systems, often considered is the Manakov equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M+\lambda \mathrm{J}^2\right)=\left[M+\lambda \mathrm{J}^2, \Omega+\lambda \mathrm{J}\right],$$

which is a Lax equation with a complex parameter λ, equivalent to the Euler equation. By the preservation of the eigenvalues of the matrix $M+\lambda \mathrm{J}^2$ appearing in the left-hand side of the Lax equation, it is natural to consider the spectral curve associated with the Manakov equation, which is an affine cubic curve in $\mathbb{C}^2$ and whose completion is, in general, a smooth

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elliptic curve. It is known that the above two elliptic curves are isogenous. (This was shown by L. Haine in [7]. See also [4] or [12].) On the other hand, the eigenvectors of the matrix $M + \lambda J^2$ give rise to holomorphic line bundles over the spectral curve parameterized by the integral curve. In other words, these eigenvectors describe a family of holomorphic mappings of the spectral curve to the complex projective plane $P_2(\mathbb{C})$ with the parameters in the intersection of the quadric level surfaces.

As to the eigenvector mapping associated with a general Lax equation, P. Griffiths makes a cohomological interpretation of the linearization of the flows on the Jacobian variety of the spectral curve for a large number of integrable systems including the free rigid bodies. See, e.g., [6,2,4] for more detail. For the three-dimensional free rigid body, this eigenvector mapping can be utilized to give the isogeny between the quadrics intersection and the spectral curve. See [7] or [4] for these.

In comparison with these works, the present paper gives further study on the eigenvector mapping for the three-dimensional free rigid bodies from the viewpoint of the theory of complex algebraic surfaces. In fact, it is quite natural to regard the eigenvector mapping associated with the Manakov equation as a rational mapping to the complex projective plane from the product of two elliptic curves, one of which is the intersection of two quadric level surfaces and the other of which is the spectral curve. However, the structure of this rational mapping itself does not seem to be clear. The aim of the present paper is to study this rational mapping from the view point of the theory of complex algebraic surfaces. As the main result, the eigenvector mapping can be understood as the double covering of the Kummer surface associated with an Abelian surface of product type, which is the double covering of $P_2(\mathbb{C})$ branched over the sextic curve canonically defined by the dynamical system.

It is very interesting that the eigenvector mapping for the $SO(3)$ free rigid body is related to the geometry of the Cremona transformation of $P_2(\mathbb{C})$ and a certain (generalized) del Pezzo surface of degree two, which is obtained from $P_2(\mathbb{C})$ through blowing-ups with special seven points as its centres. Furthermore, it is shown that there are some elliptic fibrations of the Kummer surface onto $P_1(\mathbb{C})$, which are related to the free rigid body dynamics.

The structure of the present paper is as follows: In Section 2 of the present paper, a brief review is given on the free rigid body dynamics. The formulation of the problem is given in Section 3. The main theorem is stated at the end of this section. The proof of the main theorem is given in Section 4, and included are the detailed description of the structure of the eigenvector mapping as a rational mapping, several important elliptic fibrations of the Kummer surface in relation to the dynamical system, the intrinsic characterization of the Cremona transformation of $P_2(\mathbb{C})$, and the associated del Pezzo surface of degree two.

2. Free rigid body dynamics

In this section, presented is a brief review on the free rigid body dynamics. See, e.g., [1,3,4,11] for more detail. It is well known that the motion of a free rigid body can be described by the Euler equation

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p \times \left(\mathsf{A}^{-1} p\right). \tag{2.1}$$

Here, $p \in \mathbb{R}^3$ is the angular momentum, $\times$ is the exterior product of $\mathbb{R}^3$ with respect to the ordinary Euclidean metric, and A stands for the inertia tensor of the rigid body, which is in fact a positive-definite 3×3 symmetric matrix. The most important property of the system (2.1) is that there are two first integrals: the energy $H(p) = \frac{1}{2} p^{\mathsf{T}} \mathsf{A}^{-1} p$ and the half of the squared norm of the angular momentum $L(p) = \frac{1}{2} p^{\mathsf{T}} p$, $p \in \mathbb{R}^3$. Here, the superscript $^{\mathsf{T}}$ denotes the transposition of matrices. The intersection of (2.1) coincide with a connected component of the intersection of the quadric level surfaces of these first integrals, which can be described by

$$\begin{cases} \dfrac{1}{I_1} p_1^2 + \dfrac{1}{I_2} p_2^2 + \dfrac{1}{I_3} p_3^2 = 2h, \\ p_1^2 + p_2^2 + p_3^2 = 2l, \end{cases} \tag{2.2}$$

with a suitable coordinates which diagonalize the matrix A into $\operatorname{diag}(I_1, I_2, I_3)$. Here, h and l are the values of H and L, respectively, determined by the initial conditions. With the further transformation $p_1 = \sqrt{-2l}\frac{x_0}{x_3}$, $p_2 = \sqrt{-2l}\frac{x_1}{x_3}$, $p_3 = \sqrt{-2l}\frac{x_2}{x_3}$, $I_1 = \frac{1}{a_0}$, $I_2 = \frac{1}{a_1}$, $I_3 = \frac{1}{a_2}$, and $\frac{h}{l} = a_3$, the equation of the quadrics intersection can be written as

$$\begin{cases} a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0, \\ x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0. \end{cases} \tag{2.3}$$

As will be seen in the next section, this equation defines an elliptic curve.

Through the Lie algebra isomorphism $R : \mathfrak{so}(3, \mathbb{R}) \xrightarrow{\sim} (\mathbb{R}^3, \times)$, given by

$$\begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix},$$

the Euler equations transformed into

$$\frac{\mathrm{d}M}{\mathrm{d}t} = [M, \Omega], \tag{2.4}$$

where $M = R^{-1}(p)$, $\Omega = R^{-1}(\mathsf{A}^{-1}p) \in \mathfrak{so}(3)$ are related by $M = \mathsf{J}\Omega + \Omega\mathsf{J}$, where J is the symmetric matrix determined by A. If we assume that $\mathsf{A} = \operatorname{diag}(I_1, I_2, I_3)$, we can set $\mathsf{J} = \operatorname{diag}(J_1, J_2, J_3)$, such that $I_1 = J_2 + J_3$, $I_2 = J_3 + J_1$, and $I_3 = J_1 + J_2$. It is easily checked that the Euler equation (2.4) is equivalent to the following Lax equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M + \lambda \mathsf{J}^2\right) = \left[M + \lambda \mathsf{J}^2, \Omega + \lambda \mathsf{J}\right]$$

with a parameter $\lambda \in \mathbb{C}$, which is called the Manakov equation [9]. Since the eigenvalues of the matrix $M + \lambda \mathsf{J}^2$ is invariant along the integral curve, one is led to consider the characteristic equation

$$\det\left(M + \lambda \mathsf{J}^2 - \mu \mathsf{E}\right) = 0, \tag{2.5}$$

where E stands for the unit matrix. If we regard (2.5) as the equation posed on $(\lambda, \mu) \in \mathbb{C}^2$, it defines an affine cubic curve. The curve, as well as its completion in $P_2(\mathbb{C})$, is called the spectral curve associated with the Manakov equation. It is to be noted that the spectral curve is, in general, an elliptic curve which is defined independently of the variables p_1, p_2, p_3, since (2.5) is written as

$$\left(J_1^2\lambda - \mu\right)\left(J_2^2\lambda - \mu\right)\left(J_3^2\lambda - \mu\right) + 2h'\lambda - 2l\mu = 0,$$

where the coefficients J_1^2, J_2^2, J_3^2, and $h' = \frac{1}{2}(J_1^2p_1^2 + J_2^2p_2^2 + J_3^2p_3^2)$ are the invariants of the dynamical system. In fact, we have

$$h' = I_1I_2I_3h + \frac{(I_1 + I_2 + I_3)^2 - 4(I_1I_2 + I_2I_3 + I_3I_1)}{4}l.$$

Further, we can consider the eigenvector of the matrix $M + \lambda \mathsf{J}^2$. We choose the parameters I_1, I_2, I_3, h, and l to be generic. Then, for any point $(p_1, p_2, p_3)^{\mathrm{T}}$ in the integral curve (2.2) and for any point $(\lambda, \mu) \in \mathbb{C}^2$ in the spectral curve (2.5), the eigenvector $(x, y, z)^{\mathrm{T}} \in \mathbb{C}^3$ which satisfies

$$\begin{pmatrix} \lambda J_1^2 - \mu & -p_3 & p_2 \\ p_3 & \lambda J_2^2 - \mu & -p_1 \\ -p_2 & p_1 & \lambda J_3^2 - \mu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0, \tag{2.6}$$

is uniquely determined up to the scalar multiplication. Now, we have a correspondence of the point in the product of the integral curve and the spectral curve to the eigenvector. The aim of the present paper is to study this correspondence from the complex algebro-geometric point of view.

3. Eigenvector mapping as a rational mapping

Following the review of free rigid body dynamics given in the previous section, we will formulate the problem in order to study the eigenvectors of the matrix $M + \lambda \mathsf{J}^2$ from the viewpoint of the theory of complex algebraic surfaces. For this purpose, all variables and parameters are assumed to be complex numbers.

Keeping the integral curve of the Euler equation (2.1) in mind, we consider the algebraic curve C in $P_3(\mathbb{C})$ defined by (2.3), where we regard not only $(x_0 : x_1 : x_2 : x_3)$ but also $(a_0 : a_1 : a_2 : a_3)$ as the homogeneous coordinate systems of two complex projective spaces. Recall that the parameter $(a_0 : a_1 : a_2 : a_3)$ were originally coming from the constants I_1, I_2, I_3, h, and l of motion. As to the structure of the curve C, we have the following well-known proposition.

Proposition 3.1. *If a_0, a_1, a_2, and a_3 are distinct, then the space curve C defined by (2.3) is a smooth elliptic curve, which is isomorphic to the double covering of $P_1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$ branched at a_0, a_1, a_2, and a_3.*

For the proof, see, e.g., [4] or [12].

On the other hand, the spectral curve associated with the Manakov equation defines an affine cubic curve C':

$$\left(J_1^2\lambda - \mu\right)\left(J_2^2\lambda - \mu\right)\left(J_3^2\lambda - \mu\right) + 2h'\lambda - 2l\mu = 0. \tag{3.1}$$

The completion in $P_2(\mathbb{C})$ of this algebraic curve is denoted by the same symbol. We easily have the following proposition. For brevity, we set

$$b_0 = \frac{h'}{l}, \qquad b_1 = J_1^2, \qquad b_2 = J_2^2, \qquad b_3 = J_3^2.$$

Proposition 3.2. *If b_0, b_1, b_2, and b_3 are distinct, the completion C' of the plane algebraic curve (3.1) is a smooth elliptic curve, which has the structure of the double covering of $P_1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$ branched at b_0, b_1, b_2, and b_3.*

In fact, the two elliptic curves C and C' are isomorphic. For the proof, see [4] or [12].

Now, let us assume the generic condition that b_0, b_1, b_2, and b_3 are distinct. If we choose a point (p_1, p_2, p_3) in the quadrics intersection C and another (λ, μ) in the spectral curve C', we have the (non-zero) eigenvector $(x, y, z)^{\mathrm{T}} \in \mathbb{C}^3$ satisfying

$$\begin{pmatrix} b_1\lambda - \mu & -p_3 & p_2 \\ p_3 & b_2\lambda - \mu & -p_1 \\ -p_2 & p_1 & b_3\lambda - \mu \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0. \tag{3.2}$$

Since the eigenvector $(x, y, z)^{\mathrm{T}}$ is unique up to scalar multiplication of non-zero complex numbers, it is thought to define a point in $P_2(\mathbb{C})$. We regard (3.2) as a rational mapping from the product $C \times C'$ of the elliptic curves to $P_2(\mathbb{C})$, which is denoted by

$$f : C \times C' - \cdots \to P_2(\mathbb{C}).$$

The following theorem is the main results of the present paper.

Theorem 3.3. *There is a Kummer surface F between $C \times C'$ and $P_2(\mathbb{C})$, such that the eigenvector mapping f factors as the composition of two* 2 : 1 *rational mappings as follows*:

$$\begin{array}{ccc} C \times C' & \overset{f}{- \cdots \to} & P_2(\mathbb{C}) \\ & \searrow \quad \nearrow & \\ & F & \end{array}$$

The mapping $C \times C' - \cdots \to F$ is the one given by the quotient by the canonical involution of the Abelian surface $C' \times C'$ followed by the blowing-ups and composed with an isomorphism between $C \times C'$ and $C' \times C'$.

This theorem will be shown in the next section. The structures of the two Abelian surfaces will be clarified in Section 4.6.

4. Algebraic geometry of the eigenvector mapping

4.1. Projective geometry associated with the eigenvector mapping

In order to characterize the rational mapping $f : C \times C' - \cdots \to P_2(\mathbb{C})$ from the viewpoint of the theory of complex algebraic surfaces, we determine the variables (p_1, p_2, p_3) and (λ, μ) for any given point $(x : y : z) \in P_2(\mathbb{C})$ through (3.2). Eq. (3.2) can be written as

$$b_1\lambda x = \mu x + p_3 y - p_2 z, \tag{4.1}$$

$$b_2\lambda y = \mu y + p_1 z - p_3 x, \tag{4.2}$$

$$b_3\lambda z = \mu z + p_2 x - p_1 y. \tag{4.3}$$

From $x \times$ (4.1) $+ y \times$ (4.2) $+ z \times$ (4.3) and the formula $(x, y, z)^{\mathrm{T}}\{(x, y, z) \times (p_1, p_2, p_3)\} = 0$, we have

$$\frac{\mu}{\lambda} = \frac{b_1 x^2 + b_2 y^2 + b_3 z^2}{x^2 + y^2 + z^2}. \tag{4.4}$$

Further, Eqs. (4.1), (4.2), and (4.3) are rewritten as

$$y p_3 - z p_2 = (b_1\lambda - \mu)x, \tag{4.5}$$

$$z p_1 - x p_3 = (b_2\lambda - \mu)y, \tag{4.6}$$

$$x p_2 - y p_1 = (b_3\lambda - \mu)z. \tag{4.7}$$

Adding up the squares of (4.5), (4.6), (4.7) and by using the formula $|(x, y, z) \times (p_1, p_2, p_3)|^2 = |(x, y, z)|^2 |(p_1, p_2, p_3)|^2 - |(x, y, z)(p_1, p_2, p_3)^{\mathrm{T}}|^2$, we deduce

$$(x p_1 + y p_2 + z p_3)^2 = 2l\left(x^2 + y^2 + z^2\right) - \lambda^2 \frac{(b_1 - b_2)^2 x^2 y^2 + (b_2 - b_3)^2 y^2 z^2 + (b_3 - b_1)^2 z^2 x^2}{x^2 + y^2 + z^2}. \tag{4.8}$$

From (3.1) and (4.4), we have

$$\lambda^2 = \frac{-2h' + 2l(\mu/\lambda)}{(J_1^2 - \mu/\lambda)(J_2^2 - \mu/\lambda)(J_3^2 - \mu/\lambda)} = \frac{2l(x^2+y^2+z^2)^2\{(b_1-b_0)x^2 + (b_2-b_0)y^2 + (b_3-b_0)z^2\}}{\{(b_1-b_2)y^2 + (b_1-b_3)z^2\}\{(b_2-b_3)z^2 + (b_2-b_1)x^2\}\{(b_3-b_1)x^2 + (b_3-b_2)y^2\}}. \tag{4.9}$$

Through (4.4), (4.8), and (4.9), we now observe that it is necessary to take a bi-double covering of $P_2(\mathbb{C})$, in order to determine the variables (p_1, p_2, p_3) and (λ, μ) from $(x : y : z)$. For this, we can utilize the following homogeneous variables:

$$\sigma = \frac{\sqrt{-2l}(x^2+y^2+z^2)}{\lambda},$$
$$\tau = \frac{xp_1 + yp_2 + zp_3}{\lambda}.$$

In fact, we have, from (4.8) and (4.9),

$$\sigma^2 = -\frac{\{(b_1-b_2)y^2 + (b_1-b_3)z^2\}\{(b_2-b_3)z^2 + (b_2-b_1)x^2\}\{(b_3-b_1)x^2 + (b_3-b_2)y^2\}}{(b_1-b_0)x^2 + (b_2-b_0)y^2 + (b_3-b_0)z^2}, \tag{4.10}$$

$$\tau^2 = \frac{(b_1-b_2)^2(b_0-b_3)x^2y^2 + (b_2-b_3)^2(b_0-b_1)y^2z^2 + (b_3-b_1)^2(b_0-b_2)z^2x^2}{(b_1-b_0)x^2 + (b_2-b_0)y^2 + (b_3-b_0)z^2}, \tag{4.11}$$

and Eqs. (4.5), (4.6), and (4.7), together with $xp_1 + yp_2 + zp_3 = \sqrt{-2l}\frac{\tau}{\sigma}(x^2+y^2+z^2)$, yield

$$p_1 = \sqrt{-2l}\frac{x\tau + (b_2-b_3)yz}{\sigma},$$
$$p_2 = \sqrt{-2l}\frac{y\tau + (b_3-b_1)zx}{\sigma},$$
$$p_3 = \sqrt{-2l}\frac{z\tau + (b_1-b_2)xy}{\sigma}.$$

Furthermore, we can deduce from the definition of σ and from (4.4)

$$\lambda = \frac{\sqrt{-2l}(x^2+y^2+z^2)}{\sigma},$$
$$\mu = \frac{\sqrt{-2l}(b_1x^2 + b_2y^2 + b_3z^2)}{\sigma}.$$

Hence, we can determine the points $(p_1, p_2, p_3) \in C$ and $(\lambda, \mu) \in C'$ for any given point $(x : y : z) \in P_2(\mathbb{C})$ by means of σ and τ. In other words, an isomorphism is given between the product $C \times C'$ of the elliptic curves and the bi-double covering A of $P_2(\mathbb{C})$ described as (4.10) and (4.11).

For the later convenience, we put $x = \sqrt{b_2-b_3}X$, $y = \sqrt{b_3-b_1}Y$, $z = \sqrt{b_1-b_2}Z$, $\sigma = (b_1-b_2)(b_2-b_3)(b_3-b_1)S$, $\tau = \sqrt{-(b_1-b_2)(b_2-b_3)(b_3-b_1)}T$, and $c_1 = (b_0-b_1)(b_2-b_3)$, $c_2 = (b_0-b_2)(b_3-b_1)$, $c_3 = (b_0-b_3)(b_1-b_2)$. Then, (4.10) and (4.11) are written, respectively, as

$$S^2 = \frac{(X^2-Y^2)(Y^2-Z^2)(Z^2-X^2)}{c_1X^2 + c_2Y^2 + c_3Z^2}, \tag{4.12}$$

and

$$T^2 = \frac{c_1Y^2Z^2 + c_2Z^2X^2 + c_3X^2Y^2}{c_1X^2 + c_2Y^2 + c_3Z^2}. \tag{4.13}$$

Note that $c_1 + c_2 + c_3 = 0$ is satisfied. In order to prove Theorem 3.3, we describe the bi-double covering of $P_2(\mathbb{C})$ given by (4.12) and (4.13) step by step. Denote the double covering (4.13) by d_1 and the one (4.12) by d_2. The branch locus of the bi-double covering is the sum of the following divisors:

$$Q_0 : c_1Y^2Z^2 + c_2Z^2X^2 + c_3X^2Y^2 = 0, \qquad C_0 : c_1X^2 + c_2Y^2 + c_3Z^2 = 0,$$
$$C_1 : Y^2 - Z^2 = 0, \qquad C_2 : Z^2 - X^2 = 0, \qquad C_3 : X^2 - Y^2 = 0.$$

In fact, the branch locus of the double covering d_1 consists of Q_0 and C_0, while that of d_2 consists of C_0, C_1, C_2, and C_3. The conics C_1, C_2, C_3 are pairs of lines. Let us denote the irreducible components of C_1, C_2, and C_3 by

$$C_1^{\pm}: Y = \pm Z, \qquad C_2^{\pm}: Z = \pm X, \qquad C_3^{\pm}: X = \pm Y,$$

and the four points where three of $C_1^{\pm}$, $C_2^{\pm}$, and $C_3^{\pm}$ meet together by

$$P_0 = C_1^+ \cap C_2^+ \cap C_3^+ : (X:Y:Z) = (1:1:1), \qquad P_1 = C_1^+ \cap C_2^- \cap C_3^- : (X:Y:Z) = (-1:1:1),$$
$$P_2 = C_1^- \cap C_2^+ \cap C_3^- : (X:Y:Z) = (1:-1:1), \qquad P_3 = C_1^- \cap C_2^- \cap C_3^+ : (X:Y:Z) = (1:1:-1).$$

It is to be noted that all the conics C_0, C_1, C_2, and C_3 are included in the pencil $\mathcal{L}$ of conics passing through the four points P_0, P_1, P_2, P_3. In particular, the conics C_1, C_2, and C_3 are the singular conics in $\mathcal{L}$. Any conic in $\mathcal{L}$ is described as

$$\alpha_1 X^2 + \alpha_2 Y^2 + \alpha_3 Z^2 = 0, \tag{4.14}$$

which we denote by $C_{(\alpha_1:\alpha_2:\alpha_3)}$, and where, for the parameter $(\alpha_1 : \alpha_2 : \alpha_3) \in P_2(\mathbb{C})$, $\alpha_1 + \alpha_2 + \alpha_3 = 0$ is satisfied. The smooth conic C_0 trivially corresponds to the parameter $(\alpha_1 : \alpha_2 : \alpha_3) = (c_1 : c_2 : c_3)$, while the singular conics C_1, C_2, and C_3 correspond to the parameters $(\alpha_1 : \alpha_2 : \alpha_3) = (0 : 1 : -1)$, $(-1 : 0 : 1)$, and $(1 : -1 : 0)$, respectively. Further, the quartic Q_0 is tangent to the conic C_0 at each point P_0, P_1, P_2, and P_3, and has double points at

$$P_4 : (X:Y:Z) = (1:0:0), \qquad P_5 : (X:Y:Z) = (0:1:0), \qquad P_6 : (X:Y:Z) = (0:0:1),$$

where the pairs $C_1^{\pm}$, $C_2^{\pm}$, and $C_3^{\pm}$ of the two lines intersect, respectively. For the later convenience, we denote the diagonal lines $P_5P_6 : X = 0$, $P_6P_4 : Y = 0$, and $P_4P_5 : Z = 0$ by G_4, G_5, and G_6, respectively. Note that each of these diagonal lines are tangent to one of the branches of the quartic Q_0 at its singular points P_4, P_5, and P_6.

Here, we mention two families of curves in $P_2(\mathbb{C})$, which play important roles in the study on the elliptic fibrations of the Kummer surface F below. These families are also closely related to the Cremona transformation as mentioned in Section 4.3. First, we take the net of cubic curves which pass through the seven points P_i $(i = 0, 1, \ldots, 6)$. A general member of the net is given in the form

$$\beta_1 X\big(Y^2 - Z^2\big) + \beta_2 Y\big(Z^2 - X^2\big) + \beta_3 Z\big(X^2 - Y^2\big) = 0, \tag{4.15}$$

which is denoted by $D_{(\beta_1:\beta_2:\beta_3)}$, so that the net is as a space identified with the projective plane $P_2(\mathbb{C}) : (\beta_1 : \beta_2 : \beta_3)$. We consider the conic $\mathcal{M}$ in the net defined by $c_1\beta_1^2 + c_2\beta_2^2 + c_3\beta_3^2 = 0$, and we see that $\mathcal{M}$ consists of the members tangent to both C_0 and Q_0. The singular member of $\mathcal{M}$ are $D_{(1:\pm1:\pm1)}$, each of which are a triple of three lines as follows:

$$D_{(1:1:1)} = C_1^+ + C_2^+ + C_3^+ : (X - Y)(Y - Z)(Z - X) = 0,$$
$$D_{(-1:1:1)} = C_1^+ + C_2^- + C_3^- : (X + Y)(Y - Z)(Z + X) = 0,$$
$$D_{(1:-1:1)} = C_1^- + C_2^+ + C_3^- : (X + Y)(Y + Z)(Z - X) = 0,$$
$$D_{(1:1:-1)} = C_1^- + C_2^- + C_3^+ : (X - Y)(Y + Z)(Z + X) = 0.$$

The tangent point of $D_{(\beta_1:\beta_2:\beta_3)}$ for $(\beta_1 : \beta_2 : \beta_3) \neq (1 : \pm1 : \pm1)$ and C_0 is $(X : Y : Z) = (\beta_1 : \beta_2 : \beta_3)$, while the tangent point of $D_{(\beta_1:\beta_2:\beta_3)}$ and Q_0 is $(X : Y : Z) = (\beta_2\beta_3 : \beta_3\beta_1 : \beta_1\beta_2)$.

We can also consider the pencil $\mathcal{N}$ of quartic curves having double points at P_0, P_1, P_2, P_3. Any member of the pencil $\mathcal{N}$ of quartic curves is given as

$$\gamma_1 Y^2 Z^2 + \gamma_2 Z^2 X^2 + \gamma_3 X^2 Y^2 = 0, \tag{4.16}$$

where $(\gamma_1 : \gamma_2 : \gamma_3) \in P_2(\mathbb{C})$ satisfies $\gamma_1 + \gamma_2 + \gamma_3 = 0$. We write this quartic curve (4.16) as $Q_{(\gamma_1:\gamma_2:\gamma_3)}$. The singular member of the pencil $\mathcal{N}$ are

$$Q_{(0:1:-1)} = C_1^+ + C_1^- + 2G_4 : X^2(Y - Z)(Y + Z) = 0,$$
$$Q_{(-1:0:1)} = C_2^+ + C_2^- + 2G_5 : Y^2(Z - X)(Z + X) = 0,$$
$$Q_{(1:-1:0)} = C_3^+ + C_3^- + 2G_6 : Z^2(X - Y)(X + Y) = 0.$$

From the viewpoint of the original setting for the eigenvector mapping, it should be mentioned that the pencil $\mathcal{L}$ of conics corresponds to the images of the members in the family of the holomorphic mappings of the integral curves into the projective plane parameterized by the point in the spectral curve,

$$f'_{(\lambda,\mu)} : C \to P_2(\mathbb{C}), \quad (\lambda, \mu) \in C',$$

which is given by fixing the point $(\lambda, \mu) \in C'$ for the eigenvector mapping $f : C \times C' \to P_2(\mathbb{C})$. In fact, the images $f'_{(\lambda,\mu)}(C) \subset P_2(\mathbb{C})$ are described by (4.4), or by $C_{(\alpha_1:\alpha_2:\alpha_3)}$ with $(\alpha_1 : \alpha_2 : \alpha_3) = ((\mu - b_1\lambda)(b_2 - b_3) : (\mu - b_2)(b_3 - b_1) : (\mu - b_3)(b_1 - b_2))$. Regarding the eigenvector mapping as such a family of the holomorphic mappings of the spectral curve, L. Haine proved that the integral curve and the spectral curve are isogenous. Note that this kind of viewpoint is standard in the theory

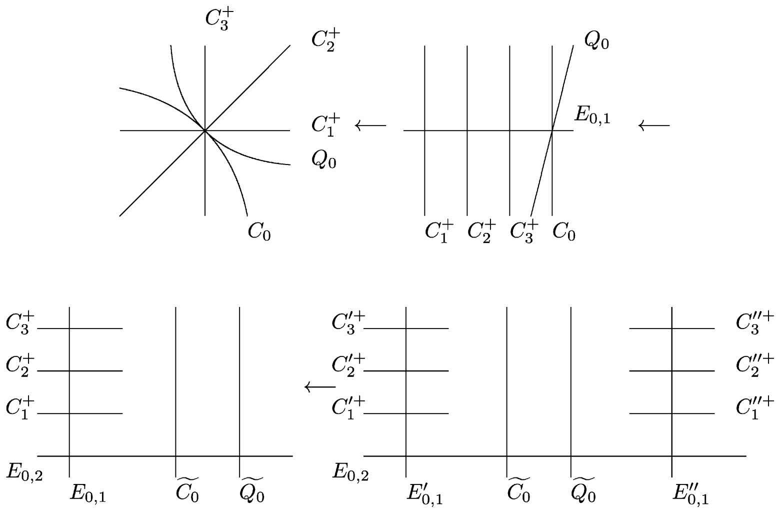

Fig. 1. Blowing-up with the centre at P_0 and the double covering d_1.

of integrable systems. See [4,6,7] for these. On the other hand, the family $\mathcal{M}$ of cubics is the images of the members in the family of the holomorphic mappings of the spectral curve into the projective plane parameterized by the point in the integral curve,

$$f''_M : C' \to P_2(\mathbb{C}), \quad M \in C,$$

which is given by fixing the point $M \in C$ for the eigenvector mapping f. Using (4.4), (4.5), and (4.6), the images $f''_M(C')$ are shown to be given by

$$p_1 x\{(b_1 - b_2)y^2 + (b_1 - b_3)z^2\} + p_2 y\{(b_2 - b_3)z^2 + (b_2 - b_1)x^2\} + p_3 z\{(b_3 - b_1)x^2 + (b_3 - b_2)y^2\} = 0,$$

or by $D_{(\beta_1:\beta_2:\beta_3)}$ with $(\beta_1 : \beta_2 : \beta_3) = (\frac{p_1}{\sqrt{b_2-b_3}} : \frac{p_2}{\sqrt{b_3-b_1}} : \frac{p_3}{\sqrt{b_1-b_2}})$.

4.2. *Three rational surfaces and Kummer surface F*

We describe the double covering by (4.13). Since the three points P_4, P_5, and P_6, are the double points of the quartic curve Q_0, which is an irreducible component of the branch locus of the double covering d_1, we blow up the original projective plane $P_2(\mathbb{C})$ with these three points as the centres, separately. The resulting surface is denoted by R_3. Further, since the points P_0, P_1, P_2, and P_3 are the tangent points of the irreducible components Q_0 and C_0 of the branch locus, we blow up $P_2(\mathbb{C}) : (X : Y : Z)$ with these four points as the centre, and subsequently with the four intersection points of the proper transforms of Q_0 and C_0 through the first blowing-ups. The exceptional divisors over P_i $(i = 0, 1, 2, 3)$ are denoted by $E_{i,1}$ and $E_{i,2}$, according to the order of the blowing up. The exceptional divisors over the three points P_4, P_5, and P_6 are denoted by F_4, F_5, and F_6, respectively. The proper transforms of Q_0 and C_0 are written as $\widetilde{Q}_0$ and $\widetilde{C}_0$, respectively. We denote the intermediate surface obtained through the seven-point blowing-up in P_i $(i = 0, 1, 2, 3, 4, 5, 6)$ by R_7. Note that R_7 is a so-called generalized del Pezzo surface of degree two. (See [10] for more details about del Pezzo surfaces.) The finally obtained surface from R_7 through the four-point blowing-up is denoted by R_{11}. Note that $\widetilde{Q}_0$ and $\widetilde{C}_0$ are disjoint from each other on R_{11}. At this stage, the branch locus $\widetilde{Q}_0 + \widetilde{C}_0$ is non-singular and the double covering by (4.13) provides two copies $C_k'^{\pm}$ and $C_k''^{\pm}$ of the line $C_k^{\pm}$, $k = 1, 2, 3$. Further, this double covering d_1 also supplies two copies $E'_{i,1}$ and $E''_{i,1}$ of the exceptional divisor $E_{i,1}$, $i = 0, 1, 2, 3$. The procedure of the blowing-ups followed by the double covering is described around P_0 and around P_4 in Figs. 1 and 2, respectively. All the irreducible components in the last of these figures has self-intersection number -2.

Since the singular points of the sextic $Q_0 + C_0$ are simple, the covering surface F by d_1 over the 11-point blowing-up of $P_2(\mathbb{C})$ is shown to have the trivial canonical bundle from III. Theorem 7.2 or V. Section 22 in [5], so that we can conclude that F is a K3 surface. (The four singular points P_0, P_1, P_2, and P_3 are of type A_3 and the three P_4, P_5, and P_6 are of type A_1. See, e.g., II. Section 8 of [5] for these.) In fact, F is a Kummer surface. This can be shown as follows: Pulling back the rational function (4.12) through $d_1 : F \to P_2(\mathbb{C})$, it is naturally seen that the double covering d_2 induces a double covering of F, which we denote by $\tilde{d}_2$. The branch locus of $\tilde{d}_2$ is the sum of the 16 disjoint (-2)-curves $C_i'^{\pm}$, $C_i''^{\pm}$, $i = 1, 2, 3$, and $E_{j,2}$, $j = 0, 1, 2, 3$. Note that the pulling back of the quadric curve C_0 through the double covering d_1 is out of the branch locus of $\tilde{d}_2$. Since every K3 surface which has a double covering branched along disjoint 16 (-2)-curves is a Kummer surface (cf. VIII. Proposition 6.1 in [5]), F is a Kummer surface. However, in order to show that F is the Kummer surface obtained from the Abelian surface $C' \times C'$ by means of its canonical involution, we have to look into the details of the eigenvector mapping

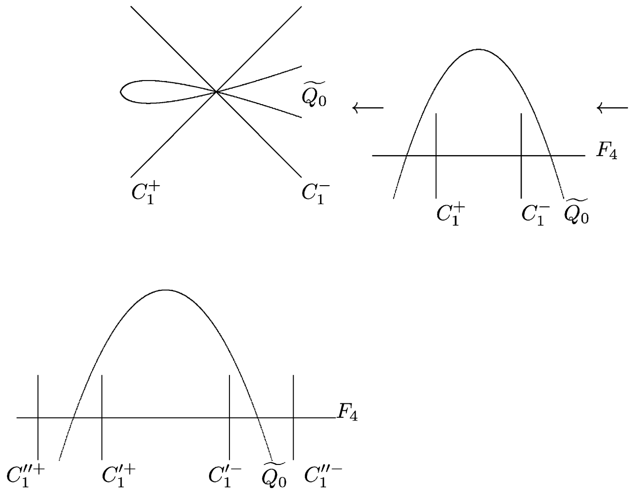

Fig. 2. Blowing-up with the centre at P_4 and the double covering d_1.

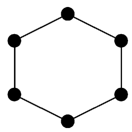

Fig. 3. Hexagonal dual graph G.

and the surface F. As before, the double covering of the Kummer surface F through $\tilde{d}_2$ is denoted by A. Note that there is an isomorphism between A and $C \times C'$, as was mentioned in Section 4.1.

4.3. Cremona transformation

Through the argument in the previous subsections, we observe that the roles of the two curves C_0 and Q_0 are symmetric. This can be clarified by the action of the Cremona transformation $\iota : X \mapsto \frac{1}{X}$, $Y \mapsto \frac{1}{Y}$, $Z \mapsto \frac{1}{Z}$, which gives rise to a holomorphic involution of the surface R_3. It is easily seen that ι maps C_0 onto Q_0 and vice versa. This involution of R_3, which we denote by the same symbol ι, is naturally lifted to the generalized del Pezzo surface R_7, for the fixed points of ι are P_0, P_1, P_2, and P_3, which are the centre of the blowing-up $R_7 \to R_3$. The exceptional curves $E_{0,1}$, $E_{1,1}$, $E_{2,1}$, $E_{3,1}$ exactly form the fixed-point set of ι. The proper transforms of the six lines $C_1^{\pm}$, $C_2^{\pm}$, $C_3^{\pm}$ are (-2)-curves and each of them is mapped onto itself by ι, so that their quotient are six (-1)-curves of the quotient surface $R_3/\langle\iota\rangle$, which are disjoint from each other. By blowing them down separately, we obtain again a projective plane on which we have the six special points as the images of $C_j^{\pm}$ $(j = 1, 2, 3)$ and the four special lines as the images of the quotient of $E_{i,1}$ $(i = 0, 1, 2, 3)$. The six points are exactly the intersection of the four lines. This suggests us another construction of R_7, to which we return later in Section 4.5. Instead, we will explain the process from R_3 to R_7, emphasizing the role of the involution ι. In fact, the centre of the further blowing up $R_7 \to R_3$ was the fixed point set of ι.

Now, forgetting the explicit construction of R_3 and R_7, we denote by V_3 the blowing up of $P_2(\mathbb{C})$ with arbitrary three non-collinear points as its centre. As is well known, there are exactly six (-1)-curves on V_3 whose intersection behavior is described as the hexagonal dual graph G (see Fig. 3).

In order to obtain $P_2(\mathbb{C})$, it suffices to blow down three disjoint (-1)-curves. We have essentially two possibilities for the choice of the three. There is a canonical homomorphism of the holomorphic automorphism group $\mathrm{Aut}(V_3)$ onto the automorphism group $\mathrm{Aut}(G)$ of the graph G, which is isomorphic to the dihedral group of order 12. The only non-trivial central element of $\mathrm{Aut}(G)$ is the antipodal mapping, which is induced by the original involution ι. On the other hand, we can prove that any element of $\mathrm{Aut}(V_3)$ which induces the antipodal mapping is an involution. We call such a holomorphic automorphism of V_3 a Cremona involution of V_3. We can even prove that all Cremona involution are conjugate to each other, *i.e.* they are essentially unique. Now, if we pick up a Cremona involution χ, then it has exactly four fixed points and we obtain the four-point blowing-up V_7 of V_3 with these four points as the centre. The involution χ is naturally lifted to an involution of V_7.

We mention the relation between the Cremona transformation ι and the families of curves $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$. By (4.14) and (4.16), a member $C_{(\alpha_1:\alpha_2:\alpha_)}$ of $\mathcal{L}$ is mapped through the Cremona transformation ι onto the quartic curve $Q_{(\alpha_1:\alpha_2:\alpha_3)}$ in $\mathcal{N}$, and visa versa. On the other hand, each member $D_{(\beta_1:\beta_2:\beta_3)}$ of the family $\mathcal{M}$ of cubics is invariant through ι, which is obvious from (4.15).

4.4. Elliptic fibrations of the Kummer surface F

It is widely known that the structure of elliptic fibrations play a crucial role in the study of K3 surfaces. See, e.g., [13] for the detail. Here, we quote a general theorem on elliptic fibrations of K3 surfaces.

Theorem 4.1. *(Cf. [13, Theorem 1].) Let V be a K3 surface. If there is an effective divisor D consisting of (-2)-curves and if D has self-intersection number 0, then there is an elliptic fibration $\pi : V \to P_1(\mathbb{C})$ which has D as one of its singular fibres.*

Using this theorem, we can find several elliptic fibrations of the Kummer surface F.

- One can easily observe that the following disjoint four divisors consisting of (-1)-curves have self-intersection number 0:

$$2\widetilde{C}_0 + E_{0,2} + E_{1,2} + E_{2,2} + E_{3,2}, \qquad 2F_4 + C_1'^{+} + C_1'^{-} + C_1''^{+} + C_1''^{-},$$
$$2F_5 + C_2'^{+} + C_2'^{-} + C_2''^{+} + C_2''^{-}, \qquad 2F_6 + C_3'^{+} + C_3'^{-} + C_3''^{+} + C_3''^{-}. \tag{4.17}$$

  By Theorem 4.1, we have an elliptic fibration $\pi_0 : F \to P_1(\mathbb{C})$ whose singular fibres are of type I_0^* in Kodaira's notation [8] given as (4.17). Note that the divisors $E'_{0,1}$, $E'_{1,1}$, $E'_{2,1}$, $E'_{3,1}$, $E''_{0,1}$, $E''_{1,1}$, $E''_{2,1}$, $E''_{3,1}$ give rise to sections of the fibration π_0. It is obvious that these four divisors are the pulling-backs of the singular conics C_1, C_2, C_3 and C_0 through the covering $d_1 : F \to P_2(\mathbb{C})$. Thus, the elliptic fibration π_0 is coming from the pencil $\mathcal{L}$ of conics.

Remark 1. Each member $C_{(\alpha_1:\alpha_2:\alpha_3)}$ of the pencil $\mathcal{L}$ has the intersection with $C_0 + Q_0$ at $(X : Y : Z) = (\sqrt{\frac{c_1}{\alpha_1}} : \pm\sqrt{\frac{c_2}{\alpha_2}} : \pm\sqrt{\frac{c_3}{\alpha_3}})$ except P_0, P_1, P_2, and P_3. In fact, these four points are intersection points of $C_{(\alpha_1:\alpha_2:\alpha_3)}$ and Q_0. If we take the biholomorphic mapping $C_{(\alpha_1:\alpha_2:\alpha_3)} \ni (X : Y : Z) \mapsto (\sqrt{\alpha_2}Y + \sqrt{-\alpha_3}Z : \sqrt{\alpha_1}X) \in P_1(\mathbb{C})$, it is easy to check the cross ratio of the four points $(X : Y : Z) = (\sqrt{\frac{c_1}{\alpha_1}} : \sqrt{\frac{c_2}{\alpha_2}} : \sqrt{\frac{c_3}{\alpha_3}})$, $(-\sqrt{\frac{c_1}{\alpha_1}} : \sqrt{\frac{c_2}{\alpha_2}} : \sqrt{\frac{c_3}{\alpha_3}})$, $(\sqrt{\frac{c_1}{\alpha_1}} : -\sqrt{\frac{c_2}{\alpha_2}} : \sqrt{\frac{c_3}{\alpha_3}})$, $(\sqrt{\frac{c_1}{\alpha_1}} : \sqrt{\frac{c_2}{\alpha_2}} : -\sqrt{\frac{c_3}{\alpha_3}})$ is $-\frac{c_1}{c_3}$, which is equal to the one for the integral curve or the spectral curve, independently of $(\alpha_1 : \alpha_2 : \alpha_3)$. Thus, the members of the pencil $\mathcal{L}$ give rise to elliptic curves whose j-invariants are constant with respect to the parameter $(\alpha_1 : \alpha_2 : \alpha_3)$.

- Similar to the case of the previous elliptic fibration π_0, we can see that the following disjoint four divisors consisting of (-1)-curves have self-intersection number 0:

$$2\widetilde{Q}_0 + E_{0,2} + E_{1,2} + E_{2,2} + E_{3,2}, \qquad 2G_4 + C_1'^{+} + C_1'^{-} + C_1''^{+} + C_1''^{-},$$
$$2G_5 + C_2'^{+} + C_2'^{-} + C_2''^{+} + C_2''^{-}, \qquad 2G_6 + C_3'^{+} + C_3'^{-} + C_3''^{+} + C_3''^{-}. \tag{4.18}$$

  By Theorem 4.1, there is an elliptic fibration $\pi_1 : F \to P_1(\mathbb{C})$ whose singular fibres are of type I_0^* given as in (4.18). The divisors $E'_{0,1}$, $E'_{1,1}$, $E'_{2,1}$, $E'_{3,1}$, $E''_{0,1}$, $E''_{1,1}$, $E''_{2,1}$, $E''_{3,1}$ also gives rise to sections of π_1. These four divisors in (4.18) are pulling-backs of the singular elements in the pencil $\mathcal{N}$ of quartics through the covering $d_1 : F \to P_2(\mathbb{C})$. The elliptic fibration π_1 comes from the pencil $\mathcal{N}$ of quartics. In view of the Cremona transformation in the previous subsection, it is easy to show that each member $Q_{(\gamma_1:\gamma_2:\gamma_3)}$ has intersection points with $C_0 + Q_0$ at $(X : Y : Z) = (\sqrt{\frac{\gamma_1}{c_1}} : \pm\sqrt{\frac{\gamma_2}{c_2}} : \pm\sqrt{\frac{\gamma_3}{c_3}})$ other than P_0, P_1, P_2, and P_3. Further, the Cremona transformation ι obviously induces an involution of the Kummer surface F which maps the elliptic fibrations π_0 and π_1 biholomorphically to each other. Note that this involution, which is denoted also by ι, is different from the covering automorphism associated with the double covering d_1.
- Further, one can observe that there is a network of (-2)-curves on F as the dual diagram in Fig. 4.
  Using this diagram, we can see that the following four divisors are disjoint and have self-intersection number 0:

$$2E'_{0,1} + E_{0,2} + C_1'^{+} + C_2'^{+} + C_3'^{+}, \qquad 2E'_{1,1} + E_{1,2} + C_1''^{+} + C_2'^{-} + C_3'^{-},$$
$$2E''_{2,1} + E_{2,2} + C_1''^{-} + C_2''^{+} + C_3''^{-}, \qquad 2E''_{3,1} + E_{3,2} + C_1'^{-} + C_2''^{-} + C_3''^{+}. \tag{4.19}$$

  Again by Theorem 4.1, we have an elliptic fibration $\pi_2 : F \to P_1(\mathbb{C})$ whose singular fibres are of type I_0^* as in (4.19). Since the singular members of the family $\mathcal{M}$ of cubics coincide with the images of the four divisors in (4.19) through the double covering $d_1 : F \to P_1(\mathbb{C})$, this elliptic fibration π_2 comes from the family $\mathcal{M}$ of cubics. In fact, the proper transform of each member $D_{(\beta_1:\beta_2:\beta_3)}$ of the family $\mathcal{M}$ of cubics through the canonical mapping $R_{11} \to P_2(\mathbb{C})$, which is a smooth rational curve, is tangent to $\widetilde{C}_0$ and to $\widetilde{Q}_0$ respectively at one point, and has no other intersection point with $\widetilde{C}_0$ and $\widetilde{Q}_0$. Since $\widetilde{C}_0 + \widetilde{Q}_0$ is the branch locus of d_1, the pulling-back of $D_{(\beta_1:\beta_2:\beta_3)}$ has two irreducible curves each of which is an elliptic curve. Let $D'_{(\beta_1:\beta_2:\beta_3)}$ and $D''_{(\beta_1:\beta_2:\beta_3)}$ denote these two irreducible components. Then, $D'_{(\beta_1:\beta_2:\beta_3)} \cdot D''_{(\beta_1:\beta_2:\beta_3)} = 2$, where $\cdot$ denotes the intersection form. It is obvious that, if $(\beta_1 : \beta_2 : \beta_3) \neq (\beta'_1 : \beta'_2 : \beta'_3)$, the proper transforms of $D_{(\beta_1:\beta_2:\beta_3)}$ and $D_{(\beta'_1:\beta'_2:\beta'_3)}$ on R_7, as well as those on R_{11}, have intersection number 2 for they are cubics. We can easily have $(D'_{(\beta_1:\beta_2:\beta_3)} + D''_{(\beta_1:\beta_2:\beta_3)}) \cdot (D'_{(\beta'_1:\beta'_2:\beta'_3)} + D''_{(\beta_1:\beta_2:\beta_3)}) = 4$ on the Kummer surface F. Choosing the

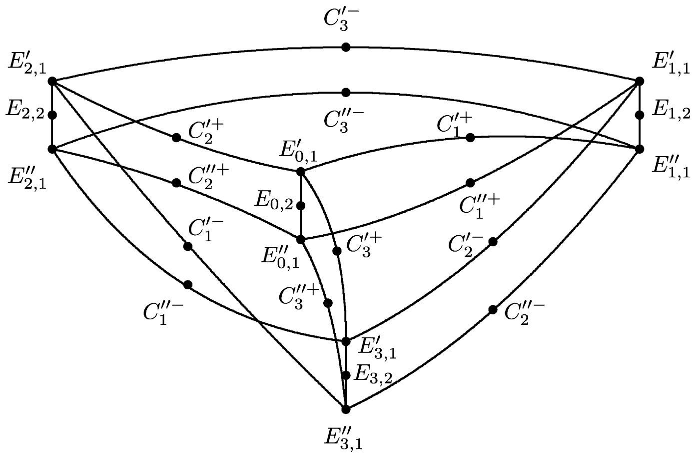

Fig. 4. Network of some (-2)-curves on F.

labeling of the irreducible components suitably, we can show that $D'_{(\beta_1:\beta_2:\beta_3)} \cdot D'_{(\beta'_1:\beta'_2:\beta'_3)} = 0$ and $D''_{(\beta_1:\beta_2:\beta_3)} \cdot D''_{(\beta'_1:\beta'_2:\beta'_3)} = 0$, if $(\beta_1 : \beta_2 : \beta_3) \neq (\beta'_1 : \beta'_2 : \beta'_3)$. Thus, these two kinds of irreducible components give rise to two elliptic fibrations of F. One of these two fibrations is π_2 and the other π_3 below.

- The above diagram (Fig. 4) also tells us that there are disjoint four divisors with self-intersection number 0 as follows:

$$2E''_{0,1} + E_{0,2} + C_1''^+ + C_2''^+ + C_3''^+, \qquad 2E''_{1,1} + E_{1,2} + C_1'^+ + C_2''^- + C_3''^-,$$
$$2E'_{2,1} + E_{2,2} + C_1'^- + C_2'^+ + C_3'^-, \qquad 2E'_{3,1} + E_{3,2} + C_1''^- + C_2'^- + C_3'^+. \tag{4.20}$$

By Theorem 4.1, there is an elliptic fibration $\pi_3 : F \to P_1(\mathbb{C})$ whose singular fibres are of type I_0^* as in (4.20). These four divisors are mapped onto the singular members of the family $\mathcal{M}$ of cubics. Thus, the elliptic fibration π_3 comes from the family $\mathcal{M}$ of cubics. The elliptic fibrations π_2 and π_3 are biholomorphically mapped to each other through the covering automorphism of the double covering d_1.

4.5. Quotient plane and its double coverings

We consider the quotient of the del Pezzo surface R_7 with respect to the Cremona involution ι. First, we take the holomorphic mapping to $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$

$$(X : Y : Z) \mapsto \left(\frac{Y}{Z}, \frac{Z}{X}, \frac{X}{Y}\right), \tag{4.21}$$

which is defined on the open set $P_2(\mathbb{C}) \setminus (\{X = 0\} \cup \{Y = 0\} \cup \{Z = 0\})$. Attaching the six punctured lines $\mathbb{C}^* \times \{\infty\} \times \{0\}$, $\mathbb{C}^* \times \{0\} \times \{\infty\}$, $\{0\} \times \mathbb{C}^* \times \{\infty\}$, $\{\infty\} \times \mathbb{C}^* \times \{0\}$, $\{\infty\} \times \{0\} \times \mathbb{C}^*$, and $\{0\} \times \{\infty\} \times \mathbb{C}^*$ to the image of the mapping (4.21), we obtain the embedding of R_3 into $P_1(\mathbb{C}) \times P_1(\mathbb{C}) \times P_1(\mathbb{C})$, which is the completion of $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$. Since ι acts on $P_2(\mathbb{C})$ as $(X : Y : Z) \mapsto (\frac{1}{X} : \frac{1}{Y} : \frac{1}{Z})$, the meromorphic mapping between projective planes

$$(X : Y : Z) \mapsto (\bar{X} : \bar{Y} : \bar{Z}) := \left(\frac{Y}{Z} - \frac{Z}{Y} : \frac{Z}{X} - \frac{X}{Z} : \frac{X}{Y} - \frac{Y}{X}\right) \tag{4.22}$$

gives rise to the quotient mapping $d_3 : R_7 \to B \cong P_2(\mathbb{C})$, where B is obtained from the quotient surface $R_7/\langle\iota\rangle$ through the blowing down of the (-1)-curves which are the images of the six lines $C_1^{\pm} : Y = \pm Z$, $C_2^{\pm} : Z = \pm X$, and $C_3^{\pm} : X = \pm Y$. The mapping (4.22) can be understood as that of R_3, embedded in $P_1(\mathbb{C}) \times P_1(\mathbb{C}) \times P_1(\mathbb{C})$ by (4.21), onto another projective plane. Note that the images of the six lines $Y = \pm Z$, $Z = \pm X$, and $X = \pm Y$ are the points $(\bar{X} : \bar{Y} : \bar{Z}) = (0 : 1 : \mp 1)$, $(\mp 1 : 0 : 1)$, $(1 : \mp 1 : 0)$, respectively. Since the four exceptional divisors $E_{0,1}$, $E_{1,1}$, $E_{2,1}$, and $E_{3,1}$ are stabilized by the action of the Cremona involution ι, the quotient mapping d_3 is the double covering of $P_2(\mathbb{C})$ branched along the four lines, $\bar{X} + \bar{Y} + \bar{Z} = 0$, $-\bar{X} + \bar{Y} + \bar{Z} = 0$, $\bar{X} - \bar{Y} + \bar{Z} = 0$, and $\bar{X} + \bar{Y} - \bar{Z} = 0$ which are the images of $E_{0,1}$, $E_{1,1}$, $E_{2,1}$, and $E_{3,1}$. The double covering d_3 is described by the equation

$$\bar{W}^2 = (\bar{X} + \bar{Y} + \bar{Z})(-\bar{X} + \bar{Y} + \bar{Z})(\bar{X} - \bar{Y} + \bar{Z})(\bar{X} + \bar{Y} - \bar{Z}). \tag{4.23}$$

Using (4.22), we can put

$$\bar{X} = X\big(Y^2 - Z^2\big), \qquad \bar{Y} = Y\big(Z^2 - X^2\big), \qquad \bar{Z} = Z\big(X^2 - Y^2\big). \tag{4.24}$$

Then, setting $\bar{W} := \sqrt{-1}(X^2 - Y^2)(Y^2 - Z^2)(Z^2 - X^2)$, (4.23) is shown to be satisfied. It is easy to see that the image of the quadric curve C_0 and the quartic curve Q_0 is the quadric curve

$$c_2c_3\bar{X}^2 + c_3c_1\bar{Y}^2 + c_1c_2\bar{Z}^2 = 0$$

on B, which we write as $\bar{C}_0$. The image on B of a conic $C_{(\alpha_1:\alpha_2:\alpha_3)}$ in the pencil $\mathcal{L}$ is the quadric curve

$$\alpha_2\alpha_3\bar{X}^2 + \alpha_3\alpha_1\bar{Y}^2 + \alpha_1\alpha_2\bar{Z}^2 = 0,$$

which will be denoted by $\bar{C}_{(\alpha_1:\alpha_2:\alpha_3)}$, as well as that of the quartic $Q_{(\alpha_1:\alpha_2:\alpha_3)}$ in the pencil $\mathcal{N}$. It should be pointed out that the family of quadric curves $\bar{C}_{(\alpha_1:\alpha_2:\alpha_3)}$, for $(\alpha_1 : \alpha_2 : \alpha_3) \in P_2(\mathbb{C})$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$, is the pencil $\bar{\mathcal{L}}$ of conics which are tangent to the fixed four lines $\bar{X} \pm \bar{Y} \pm \bar{Z} = 0$. The tangent points are $(X : Y : Z) = (\sqrt{\frac{\alpha_1}{\alpha_2\alpha_3}} : \pm\sqrt{\frac{\alpha_2}{\alpha_3\alpha_1}} : \pm\sqrt{\frac{\alpha_3}{\alpha_1\alpha_2}})$, respectively. On the other hand, the member $D_{(\beta_1:\beta_2:\beta_3)}$ of the family $\mathcal{M}$ of cubics is mapped onto the line

$$\beta_1\bar{X} + \beta_2\bar{Y} + \beta_3\bar{Z} = 0,$$

which we denote by $\bar{D}_{(\beta_1:\beta_2:\beta_3)}$. It is also to be pointed out that the line $\bar{D}_{(\beta_1:\beta_2:\beta_3)}$ is, if $(\beta_1 : \beta_2 : \beta_3) \neq (1 : \pm1 : \pm1)$, tangent to the fixed conic $\bar{C}_0$ at the point $(\bar{X} : \bar{Y} : \bar{Z}) = (c_1\beta_1 : c_2\beta_2 : c_3\beta_3)$. These lines give rise to a family of lines in $B \cong P_2(\mathbb{C})$, which we denote by $\bar{\mathcal{M}}$.

The double covering d_1 of the original projective plane gives rise to the one of the quotient projective plane, which we denote by $\bar{d}_1$. Since the image of the branch locus $C_0 + Q_0$ is the conic $\bar{C}_0$, the induced double covering $\bar{d}_1$ has the branch locus along the conic $\bar{C}_0$. The resulting surface from $\bar{d}_1$ is $P_1(\mathbb{C}) \times P_1(\mathbb{C})$. We explain this explicitly below, although it is immediate from the general results on the double coverings of $P_2(\mathbb{C})$ (cf. V. 22 of [5]).

We set

$$U := \{((\beta_1 : \beta_2 : \beta_3), (\beta_1' : \beta_2' : \beta_3')) \in P_2(\mathbb{C}) \times P_2(\mathbb{C}) \mid c_1\beta_1^2 + c_2\beta_2^2 + c_3\beta_3^2 = 0,\ c_1\beta_1'^2 + c_2\beta_2'^2 + c_3\beta_3'^2 = 0\},$$

which is biholomorphic to $P_1(\mathbb{C}) \times P_1(\mathbb{C})$. The double covering $\bar{d}_1 : U \to B$ is given by putting $\bar{d}_1((\beta_1 : \beta_2 : \beta_3), (\beta_1' : \beta_2' : \beta_3')) = (\beta_2\beta_3' - \beta_3\beta_2' : \beta_3\beta_1' - \beta_1\beta_3' : \beta_1\beta_2' - \beta_2\beta_1')$, which is nothing but the intersection $\bar{D}_{(\beta_1:\beta_2:\beta_3)} \cap \bar{D}_{(\beta_1':\beta_2':\beta_3')}$ of the two lines on B. We can take the coordinate system

$$s := -c_3\frac{\beta_3 + \beta_1}{\beta_1 - \beta_2} = -c_2\frac{\beta_1 + \beta_2}{\beta_3 - \beta_1},$$
$$t := -c_3\frac{\beta_3' + \beta_1'}{\beta_1' - \beta_2'} = -c_2\frac{\beta_1 + \beta_2}{\beta_3' - \beta_1'}$$

defined on the open neighbourhood $\beta_1 - \beta_2 \neq 0$, $\beta_1' - \beta_2' \neq 0$ in U. Using this coordinate system, the double covering is realized as

$$\begin{aligned}&\bar{d}_1((\beta_1 : \beta_2 : \beta_3), (\beta_1' : \beta_2' : \beta_3'))\\ &\quad = (c_1(st + c_2c_3) : c_2((s + c_3)(t + c_3) + c_3c_1) : c_3((s - c_2)(t - c_2) + c_1c_2)).\end{aligned} \tag{4.25}$$

The four branch lines $\bar{X} + \bar{Y} + \bar{Z} = 0$, $-\bar{X} + \bar{Y} + \bar{Z} = 0$, $\bar{X} - \bar{Y} + \bar{Z} = 0$, $\bar{X} + \bar{Y} - \bar{Z} = 0$ of the double covering d_3 are pulled back to the pairs of the two rational curves $1/st = 0$, $st = 0$, $(s + c_3)(t + c_3) = 0$, $(s - c_2)(t - c_2) = 0$, respectively.

The pulling-back of the generic member $\bar{C}_{(\alpha_1:\alpha_2:\alpha_3)}$ of the pencil $\bar{\mathcal{L}}$ through $\bar{d}_1$ is the $(2,2)$-curve

$$\{(c_2\alpha_3 - c_3\alpha_2)st - c_2c_3\alpha_1(s + t) - (c_2\alpha_3 - c_3\alpha_2)c_2c_3\}^2 - 4c_1^2c_2c_3\alpha_2\alpha_3 st = 0, \tag{4.26}$$

which is tangent to the eight rational curves $s = \infty$, $t = \infty$, $s = 0$, $t = 0$, $(s - c_2) = 0$, $(t - c_2) = 0$, $(s + c_3) = 0$, $(t + c_3) = 0$. The tangent points are $(s,t) = (\infty, \frac{c_2c_3\alpha_1}{c_2\alpha_3 - c_3\alpha_2})$, $(\frac{c_2c_3\alpha_1}{c_2\alpha_3 - c_3\alpha_2}, \infty)$, $(0, -\frac{c_2\alpha_3 - c_3\alpha_2}{\alpha_1})$, $(-\frac{c_2\alpha_3 - c_3\alpha_2}{\alpha_1}, 0)$, $(c_2, \frac{c_3\alpha_2}{\alpha_3})$, $(\frac{c_3\alpha_2}{\alpha_3}, c_2)$, $(-c_3, -\frac{c_2\alpha_3}{\alpha_2})$, $(-\frac{c_2\alpha_3}{\alpha_2}, -c_3)$, respectively. We denote the $(2,2)$-curve described as (4.26) by $\widetilde{C}_{(\alpha_1:\alpha_2:\alpha_3)}$. This curve is, in fact, an elliptic curve, since it is the double covering of the quadric curve $\bar{C}_{(\alpha_1:\alpha_2:\alpha_3)}$ branched at the four points $(X : Y : Z) = (\sqrt{c_1\alpha_1(c_2\alpha_3 - c_3\alpha_2)} : \pm\sqrt{c_2\alpha_2(c_3\alpha_1 - c_1\alpha_3)} : \pm\sqrt{c_3\alpha_3(c_1\alpha_2 - c_2\alpha_1)})$. These four points are the intersection points of $\bar{C}_{(\alpha_1:\alpha_2:\alpha_3)}$ and $\bar{C}_{(c_1:c_2:c_3)}$ and their cross ratio is calculated to be $-\frac{c_1}{c_3}$ as before, so that $\widetilde{C}_{(\alpha_1:\alpha_2:\alpha_3)}$, for $(\alpha_1 : \alpha_2 : \alpha_3) \in P_2(\mathbb{C})$ with $\alpha_1 + \alpha_2 + \alpha_3 = 0$, gives rise to a pencil of elliptic curves, whose j-invariants are constant, on $U \cong P_1(\mathbb{C}) \times P_1(\mathbb{C})$. We denote this pencil by $\tilde{\mathcal{L}}$.

The member $\bar{D}_{(\beta_1:\beta_2:\beta_3)}$ of $\bar{\mathcal{M}}$, where $c_1\beta_1^2 + c_2\beta_2^2 + c_3\beta_3^2 = 0$, is pulled back through $\bar{d}_1$ to the pairs of the two lines

$$\{(\beta_2 - \beta_3)s + (c_1\beta_1 - c_2\beta_2 - c_3\beta_3)\}\{(\beta_2 - \beta_3)t + (c_1\beta_1 - c_2\beta_2 - c_3\beta_3)\} = 0.$$

These pairs of lines give rise to the families of rational curves on U with the parameter $(\beta_1 : \beta_2 : \beta_3) \in P_2(\mathbb{C})$ satisfying $c_1\beta_1^2 + c_2\beta_2^2 + c_3\beta_3^2 = 0$, which we denote by $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}'$.

4.6. Double covering of U

We consider the double covering $\tilde{d}_3$ of U induced by $d_3 : R_7 \to B$. The branch locus of $\tilde{d}_3$ is the sum of the eight rational curves $s = \infty$, $t = \infty$, $s = 0$, $t = 0$, $(s - c_2) = 0$, $(t - c_2) = 0$, $(s + c_3) = 0$, $(t + c_3) = 0$. By (4.24) and (4.25), we can put

$$\frac{Y}{Z} - \frac{Z}{Y} = \rho c_1 \xi, \qquad \frac{Z}{X} - \frac{X}{Z} = \rho c_2 \eta, \qquad \frac{X}{Y} - \frac{Y}{X} = \rho c_2 \zeta, \tag{4.27}$$

where we set

$$\xi = (st + c_2 c_3), \qquad \eta = (s + c_3)(t + c_3) + c_3 c_1, \qquad \zeta = (s - c_2)(t - c_2) + c_1 c_2.$$

Note that $c_1\xi + c_2\eta + c_3\zeta + c_1c_2c_3 = 0$ holds. For the later convenience, we mention the following formulae:

$$\begin{aligned}
\frac{\bar{X} + \bar{Y} + \bar{Z}}{XYZ} &= \frac{(X - Y)(Y - Z)(Z - X)}{XYZ} = 2c_1c_2c_3\rho, \\
\frac{-\bar{X} + \bar{Y} + \bar{Z}}{XYZ} &= -\frac{(X + Y)(Y - Z)(Z + X)}{XYZ} = -2c_1\rho st, \\
\frac{\bar{X} - \bar{Y} + \bar{Z}}{XYZ} &= -\frac{(X + Y)(Y + Z)(Z - X)}{XYZ} = -2c_2\rho(s + c_3)(t + c_3), \\
\frac{\bar{X} + \bar{Y} - \bar{Z}}{XYZ} &= -\frac{(X - Y)(Y + Z)(Z + X)}{XYZ} = -2c_3\rho(s - c_2)(t - c_2).
\end{aligned} \tag{4.28}$$

Putting $w := \frac{\bar{W}}{(XYZ)^2} / (\frac{\bar{X}+\bar{Y}+\bar{Z}}{XYZ})^2$, we have from (4.23) the expression of the double covering $\tilde{d}_3$ as

$$w^2 = st(s - c_2)(t - c_2)(s + c_3)(t + c_3), \tag{4.29}$$

on the open set $\beta_1 - \beta_2 \neq 0$, $\beta_1' - \beta_2' \neq 0$ of U. We can determine ρ in (4.27) as follows: Using (4.24), we have

$$\frac{\sqrt{-1}\bar{W}}{(\bar{X} + \bar{Y} + \bar{Z})(-\bar{X} + \bar{Y} + \bar{Z})} = \frac{Y/Z + 1}{Y/Z - 1}. \tag{4.30}$$

The left-hand side of (4.30) is equal to $\frac{w}{c_1 st}$, so that $\frac{Y}{Z} = \frac{w + c_1 st}{w - c_1 st}$, and consequently we have $\rho = \frac{4w}{\xi\eta\zeta}$. Using this, we can describe w in terms of X, Y, Z as

$$w = c_1c_2c_3 \frac{(X + Y)(Y + Z)(Z + X)}{(X - Y)(Y - Z)(Z - X)}. \tag{4.31}$$

In fact, the above equation $\frac{Y}{Z} = \frac{w + c_1 st}{w - c_1 st}$ tells us that

$$\frac{Y}{Z} + \frac{Z}{Y} = \frac{2(w^2 + c_1^2 s^2 t^2)}{w^2 - c_1^2 s^2 t^2} = \frac{2\{(s - c_2)(t - c_2)(s + c_3)(t + c_3) + c_1^2 st\}}{(s - c_2)(t - c_2)(s + c_3)(t + c_3) - c_1^2 st},$$

which infers, together with $\frac{Y^2 - Z^2}{YZ} = \frac{4c_1 w}{\eta\zeta}$, that $\frac{c_1 w}{(s-c_2)(t-c_2)(s+c_3)(t+c_3)} = \frac{Y-Z}{Y+Z}$. Similarly, we have $\frac{c_2 w}{(s-c_2)(t-c_2)st} = \frac{Z-X}{Z+X}$ and $\frac{c_3 w}{st(s+c_3)(t+c_3)} = \frac{Y-Z}{Y+Z}$. Then, the desired equation (4.31) is obvious from (4.29).

Now, we consider the surface F' obtained from U through the double covering $\tilde{d}_3$. It will be shown that F' is isomorphic to F, after blowing down the (-1)-curves in order to get the minimal model of the Kummer surface. We have to show that the meromorphic function field of F and that of F' are isomorphic. For this, it suffices to prove that the meromorphic functions T, X/Z, and Y/Z on the Kummer surface F are included in the meromorphic function field of F' and that w, s, and t are included in the function field of F. By the above argument, it is obvious that X/Z and Y/Z are included in the function field of F'. To show that the function T is included in the meromorphic function field over F', we observe the equation

$$\begin{aligned}
\left(\frac{(c_1X^2 + c_2Y^2 + c_3Z^2)T}{XYZ}\right)^2 &= \frac{(c_1X^2 + c_2Y^2 + c_3Z^2)(c_1Y^2Z^2 + c_2Z^2X^2 + c_3X^2Y^2)}{X^2Y^2Z^2} \\
&= c_2c_3\left(\frac{Y}{Z} - \frac{Z}{Y}\right)^2 + c_3c_1\left(\frac{Z}{X} - \frac{X}{Z}\right)^2 + c_1c_2\left(\frac{X}{Y} - \frac{Y}{X}\right)^2 \\
&= -c_1^2c_2^2c_3^2\rho^2(s - t)^2,
\end{aligned} \tag{4.32}$$

where we use (4.27). Since X/Z and Y/Z are in the function field of F', (4.32) tells us that T is also included in this function field. On the other hand, (4.28) gives rise to the expression of any symmetric functions in s and t in terms of X,

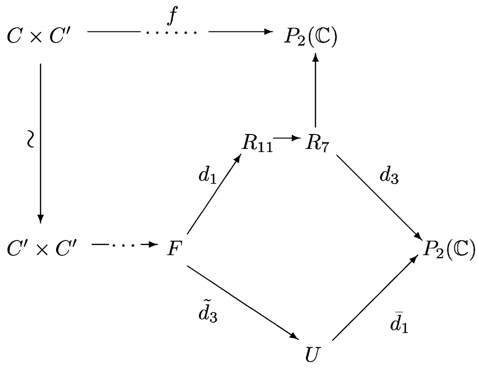

Fig. 5. Diagram for the proof of Theorem 3.3.

Y, and Z and by (4.32) we can determine $s-t$ in terms of X, Y, Z, and $\widetilde{W}$, so that the functions s and t are shown to be included in the function field of F. By (4.31), it is obvious that w is in the function field of F. This ends the proof.

As to the elliptic fibrations on the Kummer surface F, the elliptic fibrations π_0 and π_1 are corresponding to the families $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}'$ of rational curves, while π_2 and π_3 are corresponding to the pencil $\widetilde{\mathcal{L}}$ of elliptic $(2,2)$-curves on U.

It is obvious that the Kummer surface F' is that obtained from the Abelian surface $C' \times C'$ by the quotient by its canonical involution, since the double covering of F' is given by

$$w'^2 = s(s-c_2)(s+c_3),$$
$$w''^2 = t(t-c_2)(t+c_3),$$

which are related to the covering $\tilde{d}_3$ by $w = w'w''$. The branch locus of the double covering is the sum of the 16 (-1)-curves which are the exceptional divisors through the desingularization of the 16 A_1-singularities of the branch divisor $\{s=0\}+\{t=0\}+\{(s-c_2)=0\}+\{(t-c_2)=0\}+\{(s+c_3)=0\}+\{(t+c_3)=0\}+\{s=\infty\}+\{t=\infty\}$ of the double covering $\tilde{d}_3 : F' \to U$. Thus, we have $A \cong C' \times C'$.

Finally, we add the diagram (see Fig. 5) which describes the objects appearing in our study. The isomorphism $C \times C' \xrightarrow{\sim} C' \times C'$ is the composition of the above one $A \cong C' \times C'$ and the one $C \times C' \cong A$ which was mentioned in Section 4.1. The rational mapping $C' \times C' - \cdots \to F$ denotes the quotient mapping by the canonical involution with the blowing down of disjoint 16 (-1)-curves.

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