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Weakly convex sets and modulus of nonconvexity

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ABSTRACT

We consider a definition of a weakly convex set which is a generalization of the notion of a weakly convex set in the sense of Vial and a proximally smooth set in the sense of Clarke, from the case of the Hilbert space to a class of Banach spaces with the modulus of convexity of the second order. Using the new definition of the weakly convex set with the given modulus of nonconvexity we prove a new retraction theorem and we obtain new results about continuity of the intersection of two continuous set-valued mappings (one of which has nonconvex images) and new affirmative solutions of the splitting problem for selections. We also investigate relationship between the new definition and the definition of a proximally smooth set and a smooth set.

1. Introduction

We begin by some definitions for a Banach space $(E, \|\cdot\|)$ over $\mathbb{R}$. Let $B_r(a) = \{x \in E \mid \|x-a\| \leqslant r\}$. Let $\operatorname{cl} A$ denote the closure and $\operatorname{int} A$ the interior of the subset $A \subset E$. The *diameter* of the subset $A \subset E$ is defined as $\operatorname{diam} A = \sup_{x,y\in A}\|x-y\|$. The *distance* from the point $x \in E$ to the subset $A \subset E$ is defined as $\varrho(x, A) = \inf_{a\in A}\|x-a\|$. For a subset $A \subset E$, let $U_d(A)$ be the *open d-neighborhood* of A, i.e.

$$U_d(A) = \bigl\{x \in E \bigm| \varrho(x, A) < d\bigr\}.$$

The *Hausdorff distance* between two subsets $A, B \subset E$ is defined as follows

$$h(A, B) = \max\Bigl\{\sup_{a\in A}\varrho(a, B), \sup_{b\in B}\varrho(b, A)\Bigr\}.$$

We denote the convex hull of the set A by $\operatorname{co} A$.

Definition 1.1. (See [2,10].) Let $x_0, x_1 \in E$, $\|x_1 - x_0\| \leqslant 2d$. The set

$$D_d(x_0, x_1) = \bigcap_{a\in E:\ \{x_0,x_1\}\subset B_d(a)} B_d(a)$$

is called a *strongly convex segment of radius d*, and the set

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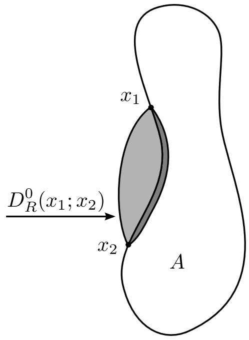

Fig. 1.

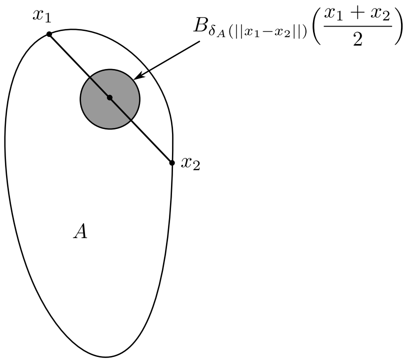

Fig. 2.

$$D_d^0(x_0,x_1) = D_d(x_0,x_1) \setminus \{x_0,x_1\}$$

is called a *strongly convex segment of radius d without extreme points*.

Definition 1.2. (See Vial [19] and Fig. 1.) A subset A of a normed space is called *weakly convex* (*in the sense of Vial*) with constant $R > 0$, if for any pair of points $x_0, x_1 \in A$ such that $0 < \|x_1 - x_0\| < 2R$ the set $A \cap D_R^0(x_0,x_1)$ is nonempty.

Definition 1.3. (See Clarke et al. [7,8].) A subset A of a normed space E is called *proximally smooth* with constant $d > 0$, if the distance function $x \to \varrho(x, A)$ is Frechet differentiable on the tube $U_d(A)\setminus A$.

Definition 1.4. (See Polyak [17].) Let E be a Banach space and let a subset $A \subset E$ be convex and closed. *The modulus of convexity* $\delta_A : [0, \operatorname{diam} A) \to [0,+\infty)$ is the function defined by

$$\delta_A(\varepsilon) = \sup\left\{\delta \geqslant 0 \;\middle|\; B_\delta\left(\frac{x_1+x_2}{2}\right) \subset A,\ \forall x_1, x_2 \in A\colon\ \|x_1 - x_2\| = \varepsilon\right\}.$$

It is obvious that $\delta_A(0) = 0$.

Definition 1.5. (See Polyak [17] and Fig. 2.) Let E be a Banach space and let a subset $A \subset E$ be convex and closed. If the modulus of convexity $\delta_A(\varepsilon)$ is strictly positive for all $\varepsilon \in (0, \operatorname{diam} A)$, then we call the set A *uniformly convex* (*with modulus* $\delta_A(\cdot)$).

We proved in [3] that every uniformly convex set $A \neq E$ is bounded and if the Banach space E contains a nonsingleton uniformly convex set $A \neq E$ then it admits a uniformly convex equivalent norm. We also proved that the function $\varepsilon \to \delta_A(\varepsilon)/\varepsilon$ is increasing (see also [14, Lemma 1.e.8]), and for any uniformly convex set $A \neq E$ there exists a constant $C > 0$ such that $\delta_A(\varepsilon) \leqslant C\varepsilon^2$ [3].

Let δ_E be the modulus of convexity for the Banach space E, i.e. the modulus of convexity for the closed unit ball in E.

Definition 1.6. Let E be a Banach space. Let a subset $A \subset E$ be closed and $d \in (0, \operatorname{diam} A)$. The *modulus of nonconvexity* $\gamma_A : [0, d) \to [0,+\infty)$ is defined as

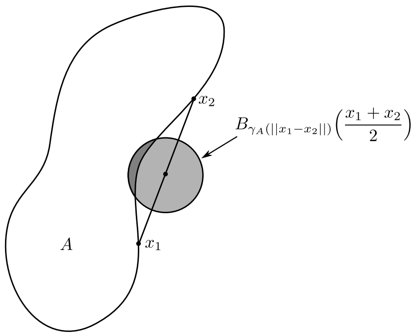

Fig. 3.

$$\gamma_A(\varepsilon) = \inf\left\{\gamma > 0 \;\middle|\; B_\gamma\left(\frac{x_1+x_2}{2}\right) \cap A \neq \emptyset,\ \forall x_1, x_2 \in A\colon\ \|x_1 - x_2\| \leqslant \varepsilon\right\}$$

and $\gamma_A(0) = 0$.

It is easy to see that the modulus of nonconvexity is a nondecreasing function. Besides, we shall further suppose that the modulus of nonconvexity is continuous from the right. Otherwise we shall redefine the modulus by continuity from the right.

Definition 1.7. (See Fig. 3.) Let E be a Banach space, and let a subset $A \subset E$ be closed. We shall call the set A *weakly convex with modulus of nonconvexity* $\gamma_A(\varepsilon)$, $\varepsilon \in [0, d)$ ($d \leqslant \operatorname{diam} A$), if the modulus of nonconvexity γ_A satisfies the inequality

$$0 \leqslant \gamma_A(\varepsilon) < \frac{\varepsilon}{2}, \quad \forall \varepsilon \in [0, d).$$

It is obvious that the equality $\gamma_A(\varepsilon) = 0$ for all $\varepsilon \in [0, \operatorname{diam} A)$ means (for the closed set A) convexity of the set A.

Hereafter the text "weakly convex" means weakly convex in the sense of Definition 1.7.

Example 1.1. Let $E = \mathcal{H}$ be the Hilbert space and $\delta_{\mathcal{H}}(\varepsilon) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4}}$ be the modulus of convexity of $\mathcal{H}$. A weakly convex subset $A \subset \mathcal{H}$ with modulus $\gamma_A(\varepsilon) = d\delta_{\mathcal{H}}(\varepsilon/d)$, $\varepsilon \in [0, d)$, $d > 0$, is weakly convex in the sense of Vial with constant d and proximally smooth with constant d (see [6–8,10,19], in particular [2]). These three properties are equivalent in the Hilbert space.

The relationship between weak convexity in the sense of Vial and proximal smoothness of a set in a Banach space is much more complicated (see [2] for details).

The next lemma is a simple consequence of similarity.

Lemma 1.1. *(See [16, Lemma 2.7.1].) Let a space E be uniformly convex with modulus δ_E. Then for all $x, y \in B_1(0)$, such that $\|x - y\| = \varepsilon > 0$, and for any $\beta \in (0, \frac{1}{2}]$ the following inequality holds*

$$B_{2\beta\delta_E(\varepsilon)}\big((1-\beta)x + \beta y\big) \subset B_1(0).$$

Lemma 1.2. *Let a space E be uniformly convex with modulus δ_E. Then for any ε, η such that $0 < \varepsilon/2 < \eta < \varepsilon < 2$ the following inequality holds*

$$\frac{\delta_E(\eta)}{\eta} \leqslant \frac{\delta_E(\varepsilon)}{\varepsilon} - 2\frac{\varepsilon - \eta}{\varepsilon \cdot \eta}\delta_E\big(r(\varepsilon)\big),$$

where $r(\varepsilon) = \frac{1}{4}(\frac{\varepsilon}{2} - \delta_E(\varepsilon))$.

By the Day–Nordlander theorem [9], $\delta_E(\varepsilon) \leqslant \frac{\varepsilon^2}{4} < \varepsilon/2$ for all $\varepsilon \in (0, 2)$. Hence $r(\varepsilon) > 0$ for all $\varepsilon \in (0, 2)$.

Proof. Let's fix $\varepsilon \in (0, 2)$, $\alpha \in (0, \frac{\varepsilon}{4} - \frac{1}{2}\delta_E(\varepsilon))$ and $\lambda \in (0, 1)$. Choose points $x_1, x_2 \in \partial B_1(0)$, such that $\|x_1 - x_2\| = \varepsilon$ and $\delta_E(\varepsilon) + \alpha > \delta$, where $\delta = \sup\{r \geqslant 0 \mid B_r(z) \subset B_1(0)\}$ and $z = \frac{1}{2}(x_1 + x_2)$.

For any natural number k we define the point $a_k \in \partial B_1(0)$ with the property $\|a_k - z\| \leqslant \delta + \frac{1}{k}$. Let y_i^k be the homothetic image of the point x_i under the homothety with center a_k and coefficient λ, $i = 1, 2$; let z_k be the homothetic image of the point z under the homothety with center a_k and coefficient λ.

By construction, $\|y_1^k - y_2^k\| = \lambda\varepsilon$ and $\|z_k - a_k\| \leqslant \lambda\delta + \lambda\frac{1}{k}$. By the triangle inequality and by the property of chosen points x_i, a_k we have $\|x_i - a_k\| \geqslant \|\frac{x_1-x_2}{2}\| - \|a_k - z\| \geqslant \frac{1}{4}(\frac{\varepsilon}{2} - \delta_A(\varepsilon)) = r(\varepsilon) > 0$ for $i = 1, 2$ and sufficiently large k. Let $\beta = \min\{\lambda, 1-\lambda\} \in (0, \frac{1}{2}]$. By Lemma 1.1 we have $B_{2\beta\delta_E(r(\varepsilon))}(y_i^k) \subset B_1(0)$, $i = 1, 2$. Hence

$$\delta_E(\lambda\varepsilon) \leqslant \left\| \frac{y_1^k + y_2^k}{2} - a_k \right\| - 2\beta\delta_E\big(r(\varepsilon)\big) = \|z_k - a_k\| - 2\beta\delta_E\big(r(\varepsilon)\big) \leqslant \lambda\delta_E(\varepsilon) + \lambda\alpha + \lambda\frac{1}{k} - 2\beta\delta_E\big(r(\varepsilon)\big).$$

Letting $\alpha \to +0$, $k \to \infty$, we obtain

$$\delta_E(\lambda\varepsilon) \leqslant \lambda\delta_E(\varepsilon) - 2\beta\delta_E\big(r(\varepsilon)\big).$$

The desired estimate appears if we put $\lambda = \eta/\varepsilon$. $\square$

One of the important motivations for consideration of weakly convex sets in the sense of Definition 1.7 is given by the next theorem.

Theorem 1.1. *Let a space E be uniformly convex with modulus δ_E, $d > 0$. Let $A \subset E$ be a weakly convex set with modulus of nonconvexity γ_A, and suppose that function $d\delta_E(\varepsilon/d) - \gamma_A(\varepsilon)$ is positive for all $\varepsilon \in (0, \min\{2d, \operatorname{diam} A\})$. Then for any point $x \in U_d(A)$ the set*

$$P_A x = \big\{a \in A \;\big|\; \|x - a\| = \varrho(x, A)\big\}$$

is a singleton.

Proof. (1) *Nonemptiness of $P_A x$.* Let's fix $x \in U_d(A)\backslash A$. Let points $a_k \in A$ be such that $\|x - a_k\| \to \varrho(x, A)$. Define nonnegative numbers $\varepsilon_k = \|x - a_k\| - \varrho(x, A)$.

Suppose that the sequence $\{a_k\}_{k=1}^{\infty}$ has no converging subsequence. Without loss of generality we may assume that there exists a number $\varepsilon_0 > 0$ such that for any natural k, m the following inequality holds: $\|a_k - a_m\| \geqslant \varepsilon_0$. By the definition of ε_k, ε_m we have

$$\max\big\{\|x - a_k\|, \|x - a_m\|\big\} \leqslant \varepsilon_k + \varepsilon_m + \varrho(x, A).$$

Let $\varrho = \varrho(x, A)$. Then

$$\left\| x - \frac{a_k + a_m}{2} \right\| \leqslant \varrho + \varepsilon_k + \varepsilon_m - (\varrho + \varepsilon_k + \varepsilon_m)\delta_E\big(\|a_k - a_m\|/(\varrho + \varepsilon_k + \varepsilon_m)\big).$$

Due to the weak convexity of the set A for any $\alpha > 0$ there exists

$$a_{km} \in B_{\gamma_A(\|a_k - a_m\|)+\alpha}\left(\frac{a_k + a_m}{2}\right) \cap A.$$

Hence

$$\begin{aligned}\|x - a_{km}\| &\leqslant \left\| x - \frac{a_k + a_m}{2} \right\| + \gamma_A\big(\|a_k - a_m\|\big) + \alpha \\ &\leqslant \varrho + \varepsilon_k + \varepsilon_m + \alpha - (\varrho + \varepsilon_k + \varepsilon_m)\delta_E\big(\|a_k - a_m\|/(\varrho + \varepsilon_k + \varepsilon_m)\big) + \gamma_A\big(\|a_k - a_m\|\big).\end{aligned}$$

Let's choose $d_1 \in (\frac{1}{2}d, d)$ and a sequence $\alpha_k > 0$, $\alpha_k \to 0$, such that for all sufficiently large k, m the inequality $\varrho + \varepsilon_k + \varepsilon_m + \alpha_k < d_1 < d$ holds. Then by Lemma 1.2

$$(\varrho + \varepsilon_k + \varepsilon_m)\delta_E\big(\|a_k - a_m\|/(\varrho + \varepsilon_k + \varepsilon_m)\big) \geqslant d_1\delta_E\left(\frac{\|a_k - a_m\|}{d_1}\right)$$

and we have the estimate

$$\begin{aligned}\|x - a_{km}\| &\leqslant \varrho + \varepsilon_k + \varepsilon_m + \alpha_k - d\delta_E\left(\frac{\|a_k - a_m\|}{d}\right) + \gamma_A\big(\|a_k - a_m\|\big) - \left(d_1\delta_E\left(\frac{\|a_k - a_m\|}{d_1}\right) - d\delta_E\left(\frac{\|a_k - a_m\|}{d}\right)\right) \\ &\leqslant \varrho + \varepsilon_k + \varepsilon_m + \alpha_k - \left(d_1\delta_E\left(\frac{\|a_k - a_m\|}{d_1}\right) - d\delta_E\left(\frac{\|a_k - a_m\|}{d}\right)\right).\end{aligned}$$

By Lemma 1.2 it follows that

$$d_1\delta_E\left(\frac{\|a_k-a_m\|}{d_1}\right)-d\delta_E\left(\frac{\|a_k-a_m\|}{d}\right)\geqslant 2(d-d_1)\delta_E\big(r\big(\|a_k-a_m\|/d_1\big)\big).$$

From the inequalities $\varepsilon_0\leqslant\|a_k-a_m\|<2d_1$ and $r(\varepsilon)=\frac{1}{4}(\frac{\varepsilon}{2}-\delta_E(\varepsilon))\geqslant\frac{1}{4}(\frac{\varepsilon}{2}-\frac{\varepsilon^2}{4})>0$, it follows that for all k,m the value $\delta_E(r(\|a_k-a_m\|/d_1))$ is bounded from below by a positive constant $c>0$. Hence for sufficiently large k,m (when $\varepsilon_k+\varepsilon_m+\alpha_k<2(d-d_1)c$), $\|x-a_{km}\|<\varrho$. Contradiction. Therefore, the sequence $\{a_k\}_{k=1}^{\infty}$ has a converging subsequence and $P_Ax\neq\emptyset$.

(2) *The set P_Ax is a singleton*. The proof is similar to the step (1). If $\varrho(x,A)=\|x-a_i\|$, $i=1,2$, $a_1,a_2\in A$, then we have

$$\left\|x-\frac{a_1+a_2}{2}\right\|\leqslant\varrho(x,A)-\varrho(x,A)\delta_E\big(\|a_1-a_2\|/\varrho(x,A)\big),$$

and for all $\alpha>0$,

$$\exists a\in B_{\gamma_A(\|a_1-a_2\|)+\alpha}\left(\frac{a_1+a_2}{2}\right)\cap A.$$

Now by choosing $0<\alpha<\varrho(x,A)\delta_E(\|a_1-a_2\|/\varrho(x,A))-\gamma_A(\|a_1-a_2\|)$, we obtain that

$$\begin{aligned}\|x-a\|&\leqslant\left\|x-\frac{a_1+a_2}{2}\right\|+\gamma_A\big(\|a_1-a_2\|\big)+\alpha\leqslant\varrho(x,A)-\varrho(x,A)\delta_E\big(\|a_1-a_2\|/\varrho(x,A)\big)\\&\quad+\gamma_A\big(\|a_1-a_2\|\big)+\alpha<\varrho(x,A).\end{aligned}$$

Contradiction. □

By Theorem 1.1 and the results from [2] it follows that if the space E is additionally uniformly smooth then each weakly convex set with the modulus γ_A (for which $d\delta_E(\varepsilon/d)-\gamma_A(\varepsilon)>0$) is proximally smooth with constant $d>0$. We note that d is not the largest possible constant for the proximal smoothness of the set A.

It's easy to see that the proximal smoothness with constant $d>0$ implies the weak convexity. Suppose that (for simplicity) the subset $A\subset E$ is compact in the strong topology of the Banach space E and proximally smooth with constant $d>0$. Then the set A is weakly convex with some modulus of nonconvexity $\gamma_A(\varepsilon)$, $\varepsilon\in(0,\min\{2d,\operatorname{diam}A\})$. Indeed, the compactness of the set A implies that the values of modulus from Definition 1.6 are achieved for every $\varepsilon\in(0,\min\{2d,\operatorname{diam}A\})$ at some points $a_1,a_2\in A$, $\|a_1-a_2\|\leqslant\varepsilon$. This means that for the point $x=\frac{1}{2}(a_1+a_2)$ we have $\varrho(x,A)=\gamma_A(\varepsilon)\geqslant 0$. Using inequality $\varrho(x,A)\leqslant\frac{1}{2}\varepsilon$, we obtain from the estimate $\frac{1}{2}\varepsilon<d$ and from proximal smoothness of the set A (see [2, Theorem 2.4]) that the set P_Ax is a singleton and $\varrho(x,A)=\gamma_A(\varepsilon)<\frac{1}{2}\varepsilon$.

2. The order of function γ_A

Before further considerations we shall make some remarks. Consider for simplicity a set A on the Euclidean plane. Let the boundary ∂A be a smooth closed curve $x=x(s)$, $y=y(s)$, where s is the natural parameter. Suppose that the curve ∂A contains no straight segments. In this case the radius of curvature of ∂A at the point $(x(s),y(s))$ equals $R(s)=(x'^2(s)+y'^2(s))^{3/2}/|x''(s)y'(s)-y''(s)x'(s)|$. If the radius $R(s)$ is finite and positive at the point $(x(s),y(s))$ (and this takes place for a.e. values of parameter s), then the curve at the neighborhood of the point $(x(s),y(s))$ is similar to the circle of radius $R(s)$.

If additionally, the set A is not locally convex at the point $(x(s),y(s))$ (i.e. for any $r>0$ the set $A\cap B_r((x(s),y(s)))$ is nonconvex), then for a small $\varepsilon>0$ the function $\gamma_A(\varepsilon)$ has the order no smaller than ε^2 (more precisely, $\gamma_A(\varepsilon)\geqslant R^2(s)-\sqrt{R^2(s)-\frac{\varepsilon^2}{4}}$).

We shall show that the situation above is typical: if the set A is nonconvex, then the modulus of nonconvexity for A satisfies the estimate $\gamma_A(\varepsilon)\geqslant\mathrm{Const}\cdot\varepsilon^2$. As we have mentioned above the modulus of nonconvexity for convex set A equals zero.

If the subset A of the Banach space E is a symmetric *cavern* (i.e. $A=\mathrm{cl}(E\backslash B)$, where B is a closed convex bounded symmetric body), then $\gamma_A(\varepsilon)\geqslant\mathrm{Const}\cdot\varepsilon^2$. The proof follows by the fact that in this case the function $\gamma_A(\varepsilon)$ has the same order as the function

$$\sigma_{E,B}(\varepsilon)=\sup\left\{1-\frac{\|x+y\|_B}{2}\;\middle|\;x,y\in\partial B,\ \|x-y\|_B\leqslant\varepsilon\right\},$$

introduced in [4]. Here $\|\cdot\|_B$ is the norm in the space E with the unit ball B. In [4] and [5] the inequality $1-\sqrt{1-\frac{\varepsilon^2}{4}}\leqslant\sigma_{E,B}(\varepsilon)$ was proved. In fact, it is the "dual" of the Day–Nordlander theorem. It can be proved similarly as the Day–Nordlander theorem (see [9, §3, pp. 60–62] for details). The proof is the same except that instead of function $\delta_X(\varepsilon)=\inf_\varphi\Delta(\varepsilon,\varphi)$ one should consider on page 62 the function $\sigma_X(\varepsilon)=\sup_\varphi\Delta(\varepsilon,\varphi)$.

Let the subset $A \subset E$ from a Banach space E be a *cavern*, i.e. $A = \mathrm{cl}(E \backslash B)$ where the set $B \subset E$ is a closed convex and bounded body, $0 \in \operatorname{int} B$. We shall estimate the value of $\gamma_A(\varepsilon)$.

For any closed convex and bounded set $B \subset E$, $0 \in \operatorname{int} B$, we define the *Minkowski* function

$$\mu_B(x) = \inf\{t > 0 \mid x \in tB\}, \quad \forall x \in E.$$

For any bounded set $C \subset E$ we define *B-diameter* of the set C as follows

$$\operatorname{diam}_B C = \sup_{x, y \in C} \mu_B(x - y).$$

For any closed convex bounded sets $A, B, C \subset E$ we define the modulus

$$\sigma_C^{A,B}(\varepsilon) = \inf\left\{\sigma \geqslant 0 \,\middle|\, \left(\sigma A + \frac{x+y}{2}\right) \cap (E \setminus \operatorname{int} C) \neq \emptyset,\ \forall x, y \in A\colon\ \mu_B(x - y) \leqslant \varepsilon\right\}$$

and the modulus

$$\sigma_C(\varepsilon) = \sigma_C^{B_1(0), B_1(0)}(\varepsilon). \tag{2.1}$$

Moduli $\sigma_C^{A,B}$ and σ_C generalize the definition from [4] to arbitrary convex sets.

It is obvious from the definition of σ_C that if C is a convex body then we have for all admissible $\varepsilon > 0$ for the set $A = \mathrm{cl}(E \backslash C) = \mathrm{cl}(E \setminus \operatorname{int} C)$,

$$\gamma_A(\varepsilon) = \sigma_C(\varepsilon). \tag{2.2}$$

The next lemmas are direct consequences of the definition of $\sigma_C^{A,B}$.

Lemma 2.1. *For any bounded closed convex bodies $A, B, C \subset E$ and $t > 0$, the following holds*:

(1) $$\sigma_C^{tA,B}(\varepsilon) = \frac{1}{t}\sigma_C^{A,B}(\varepsilon), \quad \forall \varepsilon \in (0, \operatorname{diam}_B C);$$

(2) $$\sigma_C^{A,tB}(\varepsilon) = \sigma_C^{A,B}(t\varepsilon), \quad \forall \varepsilon \in \left(0, \frac{1}{t}\operatorname{diam}_B C\right); \quad \textit{and}$$

(3) $$\sigma_{tC}^{A,B}(\varepsilon) = t\sigma_C^{A,B}\left(\frac{\varepsilon}{t}\right), \quad \forall \varepsilon \in (0, t \operatorname{diam}_B C).$$

Lemma 2.2. *For any bounded closed convex bodies A', B', $A, B, C \subset E$ and $\varepsilon \in (0, \operatorname{diam}_B C)$,*

(1) *if $A' \subset A$ then $\sigma_C^{A',B}(\varepsilon) \geqslant \sigma_C^{A,B}(\varepsilon)$; and*
(2) *if $B' \subset B$ then $\sigma_C^{A,B'}(\varepsilon) \leqslant \sigma_C^{A,B}(\varepsilon)$.*

Theorem 2.1. *Suppose that the subset $A \subset E$ of a Banach space E is a cavern. Let $B_r(0) \subset \mathrm{cl}(E \backslash A) \subset B_R(0)$. Then for all $\varepsilon \in (0, 2r)$ we have*

$$\gamma_A(\varepsilon) \geqslant \frac{\varepsilon^2}{8R^2} r.$$

Proof. Let $B = \mathrm{cl}(E \backslash A)$ be a closed convex body. Using Lemmas 2.1 and 2.2 we get

$$\sigma_B(\varepsilon) = \sigma_B^{B_1(0), B_1(0)}(\varepsilon) = r\sigma_B^{B_r(0), B_R(0)}\left(\frac{\varepsilon}{R}\right) \geqslant r\sigma_B^{B,B}\left(\frac{\varepsilon}{R}\right).$$

Using the result of Banaś [4] we obtain

$$\sigma_B^{B,B}\left(\frac{\varepsilon}{R}\right) \geqslant \sigma_{\mathcal{H}}\left(\frac{\varepsilon}{R}\right) = 1 - \sqrt{1 - \frac{\varepsilon^2}{4R^2}} \geqslant \frac{\varepsilon^2}{8R^2}.$$

By invoking formula (2.2) we complete the proof. □

Of course, a weakly convex set is not necessarily a cavern. But if such set A is connected and nonconvex, then it has "cavern-like" parts, and hence $\gamma_A(\varepsilon) \geqslant \mathrm{Const} \cdot \varepsilon^2$.

Hereafter all Banach spaces will have the modulus of convexity of the second order at zero and will contain weakly convex sets with modulus of nonconvexity of the second order, too. There are many such spaces besides Hilbert spaces, for example $l_p,\ p \in (1,2)$ (see [2,6,9] for details).

We shall define a special condition.

Definition 2.1. Let δ be the modulus of convexity for some closed convex set A and γ the modulus of nonconvexity for some closed weakly convex set B. We shall say that *condition* (i) is valid for the moduli δ and γ if:

(1) for all $s \in [0, s_0]$ there exists a solution $t = t_s > s$ of the equation $\delta(t-s) - \gamma(t) = 0$,
(2) the function $\delta(t-s) - \gamma(t)$ is positive and increasing for all admissible $t > t_s$ and
(3) for all $s \in [0, s_0]$ there exists a solution $t(s)$ of the equation $\delta(t-s) - \gamma(t) = \frac{s}{2}$.

In the case when $\delta(t) = d\delta_E(t/d)$ is the modulus of convexity for the ball $B_d(0)$ in the Banach space E we shall define the solution $t(s)$ as $t_E(s)$.

Remark 2.1. Definition 2.1 has a technical character (it is useful for further proofs) and it is not so exotic. Suppose that for sufficiently small $t > 0$ our moduli have the second order at zero and are defined by formulae $\delta(t) = c_1 t^2 + o(t^2)$, $t \to +0$, $\gamma(t) = c_2 t^2 + o(t^2)$, $t \to +0$, and $c_1 > c_2 > 0$. Then for sufficiently small numbers $s > 0$ the function $t \to \delta(t-s) - \gamma(t)$ is positive and increasing, and $t(s) \asymp \sqrt{s}$, $s \to +0$.

Remark 2.2. Suppose that a Banach space E has modulus of convexity δ_E of the second order and a closed subset $A \subset E$ is weakly convex with the modulus of nonconvexity γ_A of the second order, too. Taking into account that for all $0 < d < d_1$, $d\delta_E(\varepsilon/d) \geqslant d_1\delta_E(\varepsilon/d_1)$, $\forall \varepsilon \in (0, 2d)$ (see [3, Lemma 2.1]), and $d\delta_E(\varepsilon/d) \asymp \frac{\varepsilon^2}{d}$, $\varepsilon \to +0$, we can conclude that there exists a number $d > 0$ such that $d\delta_E(\varepsilon/d) > \gamma_A(\varepsilon)$, $\forall \varepsilon \in (0, 2d)$. If additionally the space E is smooth then by Theorem 1.1 and [2, Theorem 2.4] we obtain that the set A is proximally smooth with constant d.

3. Properties of weakly convex sets

Theorem 3.1. *Let $d > 0$. Let a subset A of a Banach space E be weakly convex with modulus $\gamma_A(\varepsilon)$, $\varepsilon \in [0, d)$ and the subset $B \subset E$ be uniformly convex with modulus $\delta_B(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} B)$ and $\operatorname{diam} B < d$. Let $\delta_B(\varepsilon) > \gamma_A(\varepsilon)$ for all $\varepsilon \in [0, \operatorname{diam} B)$. Then the set $A \cap B$, if nonempty, is weakly convex with modulus $\gamma_{A\cap B}(\varepsilon) \leqslant \gamma_A(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} A \cap B)$ and connected.*

Proof. The weak convexity of the intersection and the estimate for the modulus follows by definitions.

Suppose that the set $A \cap B$ is not connected. This means that there exist two nonempty closed disjoint sets $A_1 \subset A \cap B$ and $A_2 = (A \cap B)\backslash A_1$. Choose $k = 1$ and points $a_1 \in A_1$, $b_1 \in A_2$.

Due to weak convexity of the set $A \cap B$ there exists a point $w \in \frac{1}{2}(a_k + b_k) + (\gamma_A(\|a_k - b_k\|) + \alpha_k)B_1(0)$, $w \in A$. The numbers α_k are chosen by the conditions $\alpha_k \to 0$ and $0 < \alpha_k < \frac{1}{2}(\frac{1}{2}\|a_k - b_k\| - \gamma_A(\|a_k - b_k\|))$.

One of the inclusions $w \in A_1$ or $w \in A_2$ is true. If $w \in A_1$, then denote $a_{k+1} = w$, $b_{k+1} = b_k$. If $w \in A_2$, then denote $a_{k+1} = a_k$, $b_{k+1} = w$. In this way we build the sequences $\{a_k\}_{k=1}^{\infty} \subset A_1$, $\{b_k\}_{k=1}^{\infty} \subset A_2$.

Let $l_k = \|a_k - b_k\|$. Then $0 \leqslant l_{k+1} \leqslant \frac{1}{2}l_k + \gamma_A(l_k) + \alpha_k < l_k$. Hence $l_k \to l \geqslant 0$. Taking the limit $k \to \infty$ and using continuity of the function γ_A from the right we get $\frac{1}{2}l \leqslant \gamma_A(l)$. It follows from Definitions 1.6 and 1.7 that $l = 0$. Therefore, $\|a_k - b_k\| \to 0$.

We proved in [3] that for any uniformly convex set B there exists a number $c > 0$ such that the modulus of convexity for the set B can be estimated as follows $\delta_B(\varepsilon) \leqslant c\varepsilon^2$, $\varepsilon \in (0, \operatorname{diam} B)$.

It follows from the construction of points a_{k+1} that $a_{k+1} = a_k$, or

$$\|a_{k+1} - a_k\| \leqslant \frac{1}{2}\|a_k - b_k\| + \gamma_A\big(\|a_k - b_k\|\big) + \alpha_k \leqslant \frac{3}{4}l_k + \frac{c}{2}l_k^2.$$

In the latter case $b_{k+1} = b_k$ and

$$l_{k+1} = \|a_{k+1} - b_k\| \leqslant \frac{3}{4}l_k + \frac{c}{2}l_k^2 = \left(\frac{3}{4} + \frac{c}{2}l_k\right)l_k \leqslant \frac{4}{5}l_k,$$

for all $k > k_0$. Thus there exists a number $d > 0$, such that $l_k \leqslant d(\frac{4}{5})^k$. It follows from the estimate

$$\|a_{k+1} - a_k\| \leqslant \frac{3d}{4}\left(\frac{4}{5}\right)^k + \frac{c}{2}d\left(\frac{4}{5}\right)^{2k} \leqslant K\left(\frac{4}{5}\right)^k,$$

which is valid for sufficiently large k, that for such k and $m > k$,

$$\|a_m - a_k\| = \sum_{n=k}^{m-1}\|a_{n+1} - a_n\| \leqslant \sum_{n=k}^{m-1} K\left(\frac{4}{5}\right)^n \leqslant 5K\left(\frac{4}{5}\right)^k,$$

the latter means that the sequence $\{a_k\}$ is fundamental. Since $\|a_k - b_k\| \to 0$ thus the sequence $\{b_k\}$ is also fundamental. By the closedness of the sets A_1 and A_2 and from the condition $\|a_k - b_k\| \to 0$ we conclude that $a_k \to x \in A_1$, $b_k \to x \in A_2$. Hence $A_1 \cap A_2 \neq \emptyset$. □

For any closed subset $A \subset E$, a point $x \in U_d(A)$ and a number $s > 0$ we define the *set-valued projection*

$$P_A(x, s) = \{a \in A \mid \|x - a\| \leqslant \varrho(x, A) + s\}.$$

It follows by definition that $P_A(x, s) \neq \emptyset$ for all $s > 0$. Apart from this, under conditions of Theorem 1.1, $P_A(x, 0) = P_A x$ is a singleton.

Theorem 3.2. *Let a Banach space E be uniformly convex with modulus δ_E. Let a subset $A \subset E$ be weakly convex with modulus $\gamma_A(\varepsilon)$, $\varepsilon \in [0, 2d)$. Let $d\delta_E(\varepsilon/d) > \gamma_A(\varepsilon)$ for all $\varepsilon \in (0, 2d)$. Suppose that the condition* (i) *from Definition* 2.1 *is satisfied. Then*

$$P_A(x, s) \subset B_{t_E(s)}(P_A x).$$

Proof. Let $a = P_A x$ and $b \in P_A(x, s)$, $s \leqslant s_0$. Let's define the point $y \in [b, x]$ by the condition $\|x - y\| = \varrho(x, A) = \|x - a\|$. Let $w = \frac{a+b}{2}$, $z = \frac{a+y}{2}$.

It follows from the inequality $\|b - y\| \leqslant s$ that $\|w - z\| \leqslant s/2$. In the triangle bya we see that $\|y - a\| \geqslant \|a - b\| - s$.

Let $\varrho = \varrho(x, A)$. Note that $\|a - b\| - s \leqslant \|y - a\| < 2\varrho < 2d$. If the inequality $\varrho\delta_E((\|a - b\| - s)/\varrho) - \gamma_A(\|a - b\|) > \frac{s}{2}$ holds, then for some $\alpha > 0$ we have $\varrho\delta_E((\|a - b\| - s)/\varrho) - \gamma_A(\|a - b\|) > \frac{s}{2} + \alpha$. Using the inequality $\varrho\delta_E((\|a - b\| - s)/\varrho) < \varrho\delta_E((\|a - y\|)/\varrho)$, we obtain

$$a_0 \in B_{\gamma_A(\|a-b\|)+\alpha}(w) \cap A, \qquad B_{\gamma_A(\|a-b\|)+\alpha}(w) \subset \operatorname{int} B_{\varrho\delta_E((\|a-b\|-s)/\varrho)}(z).$$

Hence

$$\begin{aligned}\|a_0 - x\| &\leqslant \|a_0 - w\| + \|w - z\| + \|z - x\| \leqslant \gamma_A(\|a - b\|) + \alpha + \frac{s}{2} + \varrho(x, A) \\ &\quad - \varrho(x, A)\delta_E\big((\|a - b\| - s)/\varrho(x, A)\big) < \varrho(x, A).\end{aligned}$$

This contradiction shows that $d\delta_E((\|a - b\| - s)/d) - \gamma_A(\|a - b\|) < \varrho\delta_E((\|a - b\| - s)/\varrho) - \gamma_A(\|a - b\|) \leqslant \frac{s}{2}$ and by the conditions of the theorem, $\|a - b\| \leqslant t_E(s)$. The point $b \in P_A(x, s)$ was arbitrary and the theorem is thus proved. □

Corollary 3.1. *Under the assumptions of Theorem* 3.2*, the projection $P_A x$ uniformly continuously depends on x. More precisely, if $\|x_1 - x_2\| < s_0$ and $x_1, x_2 \in U_d(A)$, then $\|P_A x_1 - P_A x_2\| \leqslant t_E(\|x_1 - x_2\|)$. Moreover, $t_E(s) \asymp \sqrt{s}$, $s \to +0$.*

Theorem 3.3. *Suppose that the assumptions of Theorem* 3.2 *hold and $d_1 \in (0, d)$. Then for any point $x \in E$ the set $A \cap B_{d_1}(x)$, if nonempty, is weakly convex with modulus $\gamma_{A\cap B_{d_1}(x)}(\varepsilon) \leqslant \gamma_A(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} A \cap B_{d_1}(x))$, and path connected.*

Proof. Weak convexity of the intersection follows from the definitions.

Fix any pair of points $x, y \in A$ such that $0 < \|x - y\| < 2d_1$. For any number $t \in [0; 1]$ we denote $z_t = (1 - t)x + ty$. The map $z \mapsto P_A z$ is single-valued and continuous (Corollary 3.1) on the set $U_{d_1}(A)$, hence it is single-valued and continuous on the set $U' = U_{d_1}(A) \cup A$. Since $z_t \in U'$ for all $t \in [0; 1]$ there is a unique point $a(t)$ with $\{a(t)\} = P_A z_t$. The function $a : [0; 1] \to A$ is continuous and defines the desired curve $\Gamma = \{a(t):\ t \in [0; 1]\}$ which connects points x and y. □

Theorem 3.4. *Let E be a uniformly convex space with modulus δ_E. Let $A \subset E$ be a weakly convex set with modulus of nonconvexity $\gamma_A(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} A)$. Let $A \subset B_r(a)$, $2r < d$ and $\gamma_A(\varepsilon) < d\delta_E(\varepsilon/d)$ for all $\varepsilon \in [0, \operatorname{diam} A)$. Suppose that the assumptions of Theorem* 3.2 *hold. Then the set A is a continuous retract of E.*

Proof. Let $x \in E \backslash A$. Let $B = \operatorname{cl}\operatorname{co} A \subset B_r(a)$, $y = P_B x$. Due to the uniform convexity of the space E the metric projection on the set B is continuous. We observe that this projection is uniformly continuous (see [13] and [3, Example 3.2]) on the balls.

Since $y = y(x) \in B \subset B_r(a)$ we have $\varrho(y, A) \leqslant 2r < d$. By Theorem 1.1 and Corollary 3.1 there exists a unique metric projection $z = P_A y$ which uniformly continuously depends on y. Therefore, $z(x) = P_A(P_B x)$ is the desired retraction, see Fig. 4. □

Remark 3.1. We remark that function $z(x)$ from Theorem 3.4 is uniformly continuous on the balls.

Let us also mention that Theorem 3.4 remains valid in any uniformly convex and smooth Banach space for any proximally smooth set A with constant d and $A \subset B_r(a)$, $d < 2r$. Instead of Theorem 1.1 and Corollary 3.1 one must use the results from [2, Theorem 2.4].

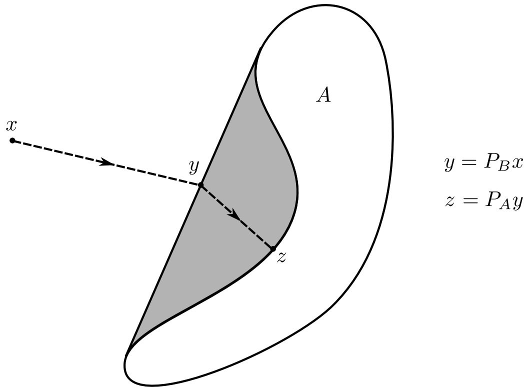

Fig. 4. Scheme of retraction.

Theorem 3.5. *Let (T, ρ) be a metric space. Let $F_1, F_2 : (T, \rho) \to 2^{\mathbb{R}^n}$ be set-valued mappings, continuous in the Hausdorff metric. Suppose that for a point $t_0 \in T$ the set $F_1(t_0)$ is uniformly convex with modulus $\delta(\varepsilon)$, and the set $F_2(t_0)$ is weakly convex with modulus $\gamma(\varepsilon)$. Let $\gamma(\varepsilon) < \delta(\varepsilon)$ for all $\varepsilon < \min\{\operatorname{diam} F_1(t_0), \operatorname{diam} F_2(t_0)\}$. Let $H(t) = F_1(t) \cap F_2(t) \neq \emptyset$ for all $t \in T$. Then the mapping $H(t)$ is continuous at the point $t = t_0$ in the Hausdorff metric.*

Proof. It follows from the uniform convexity of the set $F_1(t_0)$ that it is bounded (see [12], [3, Theorem 2.1]). Due to the continuity in the Hausdorff metric we conclude that there exists a number $\delta > 0$ such that the set $\operatorname{cl} \bigcup_{\rho(t,t_0)<\delta} F(t)$ is compact. By the Closed Graph Theorem [1] the set-valued mapping $H(t)$ is upper semicontinuous at the point $t = t_0$, i.e.

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall t\colon \quad \rho(t, t_0) < \delta, \qquad H(t) \subset H(t_0) + B_\varepsilon(0),$$

or

$$\limsup_{t \to t_0} H(t) \subset H(t_0). \tag{3.1}$$

If the set $H(t_0)$ is a singleton then the continuity of the set-valued mapping H at the point t_0 follows by its upper semicontinuity. Next we shall assume that the set $H(t_0)$ consists of more than one point.

Suppose that lower semicontinuity fails, i.e. that

$$H(t_0) \not\subset \liminf_{t \to t_0} H(t).$$

Thus there exist a number $\varepsilon_0 > 0$ and points $t_k \in T$, $t_k \to t_0$, such that

$$H(t_0) \not\subset H(t_k) + B_{\varepsilon_0}(0), \quad \text{for any natural } k.$$

For any k there exists a point $h_k \in H(t_0)$ with

$$h_k \notin H(t_k) + B_{\varepsilon_0}(0).$$

Since the set $H(t_0)$ is compact, thus without loss of generality we may assume that $h_k \to h_0 \in H(t_0)$ and

$$h_0 \notin H(t_k) + B_{\varepsilon_0/2}(0), \quad \text{for any natural } k. \tag{3.2}$$

Let us define the set $H_0 = \limsup_{k\to\infty} H(t_k) \subset H(t_0)$. By construction $h_0 \in H(t_0) \backslash H_0$, hence $H_0 \neq H(t_0)$.

Let $x_0 \in P_{H_0} h_0$. For $h_0 \in F_1(t_0) \cap F_2(t_0)$ and $x_0 \in H_0$ put $l = \|h_0 - x_0\| > 0$.

Using uniform convexity of $F_1(t_0)$ we get

$$B_{\delta(l)}\left(\frac{x_0 + h_0}{2}\right) \subset F_1(t_0).$$

Due to the weak convexity of the set $F_2(t_0)$ and finite dimension of images of the mapping F_2 we can find a point

$$f \in B_{\gamma(l)}\left(\frac{x_0 + h_0}{2}\right) \cap F_2(t_0).$$

By continuity of the map F_2 there exist points $f_k \in F_2(t_k)$ with $f_k \to f$. Besides, for $\varepsilon = (\delta(l) - \gamma(l))/3$ we can find a natural number k_0, such that for all $k > k_0$ the following holds:

$$f_k \in B_{\delta(l)-\varepsilon}\left(\frac{x_0+h_0}{2}\right) \subset F_1(t_k), \tag{3.3}$$

and

$$f_k \in B_{\gamma(l)+\varepsilon}\left(\frac{x_0+h_0}{2}\right) \cap F_2(t_k). \tag{3.4}$$

By the formulae (3.3) and (3.4) it follows that $f_k \in H(t_k)$ for all $k > k_0$. Hence $f = \lim_{k\to\infty} f_k \in H_0$.

At the same time

$$\|h_0 - f\| \leqslant \left\|h_0 - \frac{x_0+h_0}{2}\right\| + \left\|f - \frac{x_0+h_0}{2}\right\| \leqslant \frac{1}{2}\|x_0-h_0\| + \gamma\big(\|x_0-h_0\|\big) < \|x_0-h_0\|.$$

This contradicts with the inclusion $x_0 \in P_{H_0}h_0$. Thus, $H(t)$ is lower semicontinuous at the point $t = t_0$. □

Let $F : (T,\rho) \to 2^{(E,\|\cdot\|)}$ be a set-valued mapping. If for any $t \in T$ the set $F(t)$ is uniformly convex with modulus $\delta_{F(t)}(\varepsilon) \geqslant \delta(\varepsilon) > 0$, $\varepsilon \in (0, \operatorname{diam} F(t))$, and δ is an increasing function, then we shall say that the set-valued mapping F is uniformly convex with modulus δ.

If for any $t \in T$ the set $F(t)$ is weakly convex with modulus of nonconvexity $\gamma_{F(t)}(\varepsilon) \leqslant \gamma(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} F(t))$, $\gamma(0) = 0$, $\gamma(\varepsilon) < \frac{\varepsilon}{2}$ for admissible $\varepsilon > 0$ and function γ is continuous from the right and nondecreasing then we shall say that the set-valued mapping F is uniformly weakly convex with modulus γ.

Definition 3.1. Let a set-valued mapping $F_1 : (T,\rho) \to 2^{(E,\|\cdot\|)}$ be uniformly convex with modulus δ and a set-valued mapping $F_2 : (T,\rho) \to 2^{(E,\|\cdot\|)}$ uniformly weakly convex with modulus γ. We shall say that *condition* (ii) is valid for the moduli δ and γ if

(1) for all $s \in [0, s_0]$ there exists a solution $t = t_s > s$ of the equation $\delta(t-s) - \gamma(t) = 0$,
(2) the function $\delta(t-s) - \gamma(t)$ is positive and increasing for all admissible $t > t_s$,
(3) for all $s \in [0, s_0]$ there exists a solution $t(s)$ of the equation $\delta(t-s) - \gamma(t) = \frac{s}{2}$.

It follows from the results of the second paragraph that condition (ii) is possible only if moduli δ and γ are of the second order at zero.

We say that set-valued mapping F is uniformly continuous with modulus of continuity $\omega \geqslant 0$ if for any $t_1, t_2 \in T$ the inequality $h(F(t_1), F(t_2)) \leqslant \omega(\rho(t_1,t_2))$ holds.

Theorem 3.6. *Let $F_1, F_2 : (T,\rho) \to 2^{(E,\|\cdot\|)}$. Let the values of F_2 be uniformly convex with modulus $\delta(\varepsilon)$. Let the values of F_1 be uniformly weakly convex with modulus $\gamma(\varepsilon)$. Suppose that set-valued mapping F_i is uniformly continuous with modulus ω_i, $i = 1,2$. Let the condition* (ii) *holds.*

Let $H(t) = F_1(t) \cap F_2(t) \neq \emptyset$ for all $t \in T$ and suppose that for some $M > 0$ the inclusion $\bigcup_{t\in T} H(t) \subset B_M(0)$ holds. Then

$$h\big(H(t_1), H(t_2)\big) \leqslant \begin{cases} 2\omega_1 + 3\omega_2 + t(\frac{\omega_1+\omega_2}{2}), & \frac{\omega_1+\omega_2}{2} < s_0, \\ \frac{\omega_1+\omega_2}{s_0}M, & \frac{\omega_1+\omega_2}{2} \geqslant s_0. \end{cases} \tag{3.5}$$

Here $\omega_i = \omega_i(\rho(t_1,t_2))$, $i = 1,2$.

Proof. Let $c_1 \in H(t_1)$. We shall show that for any number λ, which is strictly larger than the right side of the formula (3.5), there exists a point $c_2 \in H(t_2)$ with $\|c_1 - c_2\| \leqslant \lambda$. This will prove the theorem.

Fix $d \in H(t_2)$. If $\omega_1 + \omega_2 \geqslant 2s_0$, then, taking $c_2 = d$, we obtain that

$$h\big(H(t_1), H(t_2)\big) \leqslant \|c_1 - c_2\| \leqslant 2M \leqslant \frac{\omega_1+\omega_2}{s_0}M.$$

Suppose that $\omega_1 + \omega_2 < 2s_0$. Fix $k > 1$, such that inequality $k\omega_1 + k^2\omega_2 < 2s_0$ holds. For the point $c_1 \in H(t_1) = F_1(t_1) \cap F_2(t_1)$ we can find the point $b \in F_2(t_2)$ such that $\|b - c_1\| \leqslant k\omega_2$.

Fix the point $b_\pi \in F_1(t_2)$ which satisfies the condition $\|b - b_\pi\| \leqslant k \cdot \varrho(b, F_1(t_2))$. Invoking the inequality $\varrho(b, F_1(t_1)) \leqslant \|b - c_1\| \leqslant k\omega_2$ we get the following estimate

$$\|b - b_\pi\| \leqslant k\varrho\big(b, F_1(t_2)\big) \leqslant kh\big(F_1(t_1), F_1(t_2)\big) + k\varrho\big(b, F_1(t_1)\big) \leqslant k\omega_1 + k\|b - c_1\| \leqslant k\omega_1 + k^2\omega_2.$$

Define the point $a \in [d,b] \cap H(t_2)$ as the one which is nearest to the point b. The set $[d,b] \cap H(t_2)$ is nonempty because it contains the point d. Put $n = 1$, $a_1 = a$.

Consider the following cases:

(1) $\delta(\|a_n - b\|) > \gamma(\|a_n - b_\pi\|) + \frac{1}{2}\|b - b_\pi\|$ or
(2) $\delta(\|a_n - b\|) \leqslant \gamma(\|a_n - b_\pi\|) + \frac{1}{2}\|b - b_\pi\|$.

If the case (1) takes place then we choose

$$\alpha_n = \min\left\{\frac{1}{n}, \frac{1}{2}\left(\delta\big(\|a_n - b\|\big) - \gamma\big(\|a_n - b_\pi\|\big) - \frac{1}{2}\|b - b_\pi\|\right), \frac{1}{2}\left(\frac{\|a_n - b_\pi\|}{2} - \gamma\big(\|a_n - b_\pi\|\big)\right)\right\} > 0.$$

By the uniform weak convexity of F_1 with the modulus γ there exists a point

$$w \in B_{\gamma(\|a_n - b_\pi\|)+\alpha_n}\left(\frac{a_n + b_\pi}{2}\right) \cap F_1(t_2) \subset B_{\delta(\|a_n - b\|)}\left(\frac{a_n + b}{2}\right) \subset F_2(t_2), \tag{3.6}$$

and

$$\|b_\pi - w\| \leqslant \left\|b_\pi - \frac{a_n + b_\pi}{2}\right\| + \left\|\frac{a_n + b_\pi}{2} - w\right\| \leqslant \frac{1}{2}\|a_n - b_\pi\| + \gamma\big(\|a_n - b_\pi\|\big) + \alpha_n < \|a_n - b_\pi\|.$$

Now we put $n = n + 1$, $a_n = w$ and again consider cases (1) or (2).

If the case (2) does not take place for all natural n, then we obtain from the construction of the points $\{a_n\}$ that the sequence $l_n = \|a_n - b_\pi\|$ satisfies the condition $0 \leqslant l_{n+1} \leqslant \frac{l_n}{2} + \gamma(l_n) + \alpha_n < l_n$. It follows by the Weierstrass theorem that the sequence l_n converges to some number $l \geqslant 0$ from the right. Using the continuity of the function γ from the right and taking the limit we deduce that $\frac{l}{2} \leqslant \gamma(l)$. The latter is possible only in the case $l = 0$ (see the definition of γ).

Thus, if for all $n \in \mathbb{N}$ the case (2) does not take place then $H(t_2) \ni a_n \to b_\pi$, i.e. $b_\pi \in H(t_2)$. Taking $c_2 = b_\pi$ we have

$$\|c_1 - c_2\| = \|c_1 - b_\pi\| \leqslant \|c_1 - b\| + \|b - b_\pi\| \leqslant k\omega_2 + k\omega_1 + k^2\omega_2.$$

The number $k > 1$ was arbitrary, hence

$$h\big(H(t_1), H(t_2)\big) \leqslant \omega_1 + 2\omega_2.$$

Suppose that for some $n \in \mathbb{N}$ the case (2) occurs and $\|a_n - b_\pi\| > \|b - b_\pi\|$. Taking into account that $\|a_n - b\| > \|a_n - b_\pi\| - \|b - b_\pi\|$, we conclude from the inequality of the case (2), that

$$\delta\big(\|a_n - b_\pi\| - \|b - b_\pi\|\big) - \gamma\big(\|a_n - b_\pi\|\big) \leqslant \frac{1}{2}\|b - b_\pi\|.$$

From the condition (ii) of the theorem we get

$$\|a_n - b_\pi\| \leqslant t\left(\frac{1}{2}\|b - b_\pi\|\right) \leqslant t\left(\frac{k\omega_1 + k^2\omega_2}{2}\right).$$

By choosing $c_2 = a_n$ we obtain

$$h\big(H(t_1), H(t_2)\big) \leqslant \|c_1 - c_2\| \leqslant \|c_1 - b\| + \|b - b_\pi\| + \|a_n - b_\pi\| \leqslant k\omega_2 + k\omega_1 + k^2\omega_2 + t\left(\frac{k\omega_1 + k^2\omega_2}{2}\right).$$

By taking the limit $k \to 1 + 0$, we finally prove the theorem. The case $\|a_n - b_\pi\| \leqslant \|b - b_\pi\|$, which follows from the last estimate and from the inequality $\|b - b_\pi\| \leqslant k\omega_1 + k^2\omega_2$, also gives formula (3.5). □

Remark 3.2. For convex valued mapping F_2 a similar result was proved in [3, Theorem 3.1].

Remark 3.3. If additionally the conditions of Theorem 3.1 hold for sets $F_1(t)$ and $F_2(t)$, then the values of the map H in Theorem 3.6 are connected.

Remark 3.4. In our case the moduli δ and γ are of the second order and we have that $t(s)$ is of the order $\sqrt{s}$ when $s \to 0$. For the Hilbert space this result was proved by Ivanov [10].

4. Application to selection problems

Theorem 4.1. *Let E be a uniformly convex Banach space with modulus δ_E. Let $\Phi \subset E$ be a collection of weakly convex sets with modulus of nonconvexity $\gamma(\varepsilon)$ (in the sense of Section 3), and suppose that all sets from Φ are contained in some ball. Let $d > 0$. Let the condition (ii) holds for moduli $d\delta_E(t/d)$ and $\gamma(t)$. Let $d\delta_E(t/d) > \gamma(t)$ for all admissible $t > 0$. Suppose that any set $H \in \Phi$ is contained in (each in its own) ball of radius $r > 0$ and $2r < d$.*

Then the collection Φ has a uniformly continuous selection, i.e. there exists a uniformly continuous in the Hausdorff metric function $s : \Phi \to E$ such that for all $H \in \Phi$ we have $s(H) \in H$.

Proof. Without loss of generality we shall assume that the sets from the family Φ are contained in the ball $B_R(0)$ and for any $H \in \Phi$ we have $\varrho(0, \mathrm{cl\,co}\, H) \geqslant r_1 > 0$. Consider $\Psi = \{\mathrm{cl\,co}\, H \mid H \in \Phi\}$. The metric projection of zero $y(H) = P_{\mathrm{cl\,co}\, H} 0$ on the sets from Ψ is a uniformly continuous selection defined of Ψ. We have proved in [3, Lemma 3.1] that for any $H_1, H_2 \in \Phi$ (taking into account the condition $h(\mathrm{cl\,co}\, H_1, \mathrm{cl\,co}\, H_1) \leqslant h(H_1, H_1)$)

$$\big\| y(H_1) - y(H_2) \big\| \leqslant 2h(H_1, H_2) + f_E\big(h(H_1, H_2)\big),$$

where

$$f_E(t) = \begin{cases} \delta^{-1}(t/2), & t < 2\Delta_E, \\ \frac{Rt}{\Delta_E}, & t \geqslant 2\Delta_E. \end{cases}$$

Here $\delta(\varepsilon) = R\delta_E(\varepsilon/R)$, $\Delta_E = \delta(2r_1)$.

Let $y = y(H)$. From $\varrho(y, H) \leqslant 2r < d$ using Theorem 1.1 we conclude that there exists a unique metric projection $z(H) = P_H y$.

If $2h(H_1, H_2) + f_E(h(H_1, H_2)) < (d - 2r)/2$, then, by defining $y_i = y(H_i)$, $z_i = z(H_i)$, $i = 1, 2$, we get $\|y_1 - y_2\| < (d - 2r)/2$.

Consider a metric subspace T of the metric space $((E, \Phi), (\|\cdot,\cdot\| + h(\cdot,\cdot)))$. Elements of T are pairs $(x, H) \in (E, \Phi)$ such that $\varrho(x, H) < d$. Consider the set-valued mappings $F_1(x, H) = H$, $F_2(x, H) = B_{\varrho(x,H)}(x)$ from T into E. The set-valued mapping F_1 is uniformly weakly convex with modulus γ and uniformly continuous with modulus $\omega_1(t) = t$. The set-valued mapping F_2 is uniformly convex with modulus $d\delta_E(\varepsilon/d)$.

For points (y_i, H_i), $i = 1, 2$, we have

$$\begin{aligned} h\big(F_2(y_1, H_1), F_2(y_2, H_2)\big) &= \|y_1 - y_2\| + \big|\varrho(y_1, H_1) - \varrho(y_2, H_2)\big| \\ &\leqslant \|y_1 - y_2\| + \big|\varrho(y_1, H_1) - \varrho(y_1, H_2)\big| + \big|\varrho(y_1, H_2) - \varrho(y_2, H_2)\big|, \end{aligned}$$

$|\varrho(y_1, H_1) - \varrho(y_1, H_2)| \leqslant h(H_1, H_2)$, and from the condition $\|y_1 - y_2\| \leqslant (d - 2r)/2$ we obtain that

$$\varrho(y_1, H_2) \leqslant \|y_1 - y_2\| + \varrho(y_2, H_2) \leqslant (d - 2r)/2 + 2r = (d + 2r)/2 < d.$$

Put $z_{12} \in H_2$: $\|y_1 - z_{12}\| = \varrho(y_1, H_2)$. Using Corollary 3.1 we get $|\varrho(y_1, H_2) - \varrho(y_2, H_2)| \leqslant \|y_1 - y_2\| + \|z_2 - z_{12}\| \leqslant \|y_1 - y_2\| + t_E(\|y_1 - y_2\|)$.

Thus in the case $2h(H_1, H_2) + f_E(h(H_1, H_2)) < (d - 2r)/2$ projections $z_i = F_1(y_i, H_i) \cap F_2(y_i, H_i)$ uniformly continuously depend on sets H_i, $i = 1, 2$, by Theorem 3.6, i.e.

$$\|z_1 - z_2\| \leqslant \omega\big(h(H_1, H_2)\big),$$

where $\omega(h(H_1, H_2))$ is superposition of the function $2h(H_1, H_2) + f_E(h(H_1, H_2))$ and the function from the right side of formula (3.5).

If $2h(H_1, H_2) + f_E(h(H_1, H_2)) \geqslant (d - 2r)/2$, then by the strict monotonicity (increasing) of the function f_E, there exists a number $C > 0$, such that $h(H_1, H_2) > C$. In this case

$$\|z_1 - z_2\| \leqslant 2R \leqslant \frac{2R}{C} h(H_1, H_2).$$

Therefore, $s(H) = z(H)$ is a uniformly continuous selection. □

Example 4.1. One can apply these results to certain questions about continuous selections of set-valued mappings [15,18]. Let a space E be uniformly convex, a sets $A, B \subset E$ be such that B is uniformly convex with modulus $\delta(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} B)$, and A is weakly convex with modulus $\gamma(\varepsilon)$, $\varepsilon \in [0, \operatorname{diam} A)$. Let for some $d > 0$ the inequalities $2 \operatorname{diam} B < d$ and $\gamma(\varepsilon) < d\delta_E(\varepsilon/d)$ for all $\varepsilon \in [0, \min\{2d, \operatorname{diam} A\})$ hold. Suppose that the condition (i) is valid for pairs $\delta(\varepsilon)$, $\gamma(\varepsilon)$ and $d\delta_E(\varepsilon/d)$, $\gamma(\varepsilon)$. Then there exist uniformly continuous functions $a : A + B \to A$ and $b : A + B \to B$ such that for any $c \in A + B$ we have $a(c) + b(c) = c$.

Proof. By Theorem 3.6 the set-valued mapping $A + B \ni c \to H(c) = B \cap (c - A)$ is uniformly continuous. By definition the set $H(c)$ is weakly convex with modulus of nonconvexity γ for all $c \in A + B$. By the boundedness of the set B, all sets $H(c)$, $c \in A + B$, are contained in some ball. Furthermore for any point $c \in A + B$ each set $H(c)$ is contained in the ball of radius no larger than $2 \operatorname{diam} B < d$. By Theorem 4.1 there exists a uniformly continuous selection $b(c) = s(H(c)) \in B$, where the function $s(\cdot)$ is from Theorem 4.1; $a(c) = c - b(c) \in A$. $\square$

5. A class of weakly convex sets

We shall show that simple smooth closed surfaces of codimension 1 are weakly convex sets. In this section the space E will be an arbitrary reflexive Banach space.

We introduce the *normal cone* $N(A, x)$ to the set A at the point $x \in A$ as follows

$$N(A,x) = \{p \in E^* \mid (p, x - a) \geqslant -\alpha_x(\|x - a\|) \cdot \|x - a\| \cdot \|p\|,\ \forall a \in A\},$$

where the function $\alpha_x : [0, \operatorname{diam} A) \to [0, +\infty)$ and $\lim_{t\to+0} \alpha_x(t) = 0$.

Let $A \subset E$ be any closed set with the property $\operatorname{cl}\operatorname{int} A = A$ and $x \in A$. Suppose that the set ∂A has the following properties: ∂A is path connected, $\forall x \in \partial A$,

$$N(\partial A, x) \cap \partial B_1^*(0) = \big(N(A, x) \cap \partial B_1^*(0)\big) \cup \big(N\big(\operatorname{cl}(E\backslash A), x\big) \cap \partial B_1^*(0)\big),$$

where

$$N(A, x) \cap \partial B_1^*(0) = \{p\}, \qquad N\big(\operatorname{cl}(E\backslash A), x\big) \cap \partial B_1^*(0) = \{-p\},$$

and there exists infinitely small at zero function $\alpha : [0, \operatorname{diam} A) \to [0, +\infty)$ with the property

$$N(A,x) = \{p \in E^* \mid (p, x - a) \geqslant -\alpha(\|x - a\|) \cdot \|x - a\| \cdot \|p\|,\ \forall a \in A\}, \quad \forall x \in \partial A,$$
$$N\big(\operatorname{cl}(E\backslash A), x\big) = \{p \in E^* \mid (p, x - a) \geqslant -\alpha(\|x - a\|) \cdot \|x - a\| \cdot \|p\|,\ \forall a \in \operatorname{cl}(E\backslash A)\}, \quad \forall x \in \partial A.$$

Then we say that the set ∂A is a *smooth closed surface of codimension* 1 *with a function of smoothness* α. Roughly speaking, smooth closed surface of codimension 1 is the smooth path connected boundary between some set A and its complementary set $\operatorname{cl}(E\backslash A)$.

Let $r > 0$. Define for any point $x \in \partial A$ and for unit vector $p \in N(\partial A, x)$ the vector $y \in E$ with $\|y\| = (p, y) = 1$. We say that a smooth closed surface ∂A is *simple*, if for any 2-dimensional affine plane L, such that $\{x, x + y\} \subset L$, the intersection $L \cap \partial A \cap B_r(x)$ is a path connected planar curve. We call $r > 0$ the *parameter of simplicity*.

Theorem 5.1. *Let E be a reflexive Banach space. Suppose that $A \subset E$ is a closed set,* $\operatorname{cl}\operatorname{int} A = A$, *and ∂A is a simple smooth closed surface of codimension* 1 *with the function of smoothness α and the parameter of simplicity $r > 0$. Then the set ∂A is weakly convex with the modulus of nonconvexity $\gamma_A(\varepsilon) \leqslant \varepsilon(\alpha(\varepsilon) + \alpha(\varepsilon/2))$ for all $\varepsilon \in [0, \min\{r, \varepsilon_0\})$; where $\varepsilon_0 = \sup\{t > 0 \mid \alpha(\tau) + \alpha(\tau/2) < \frac{1}{2}$, $\forall \tau \in (0, t)\}$.*

Proof. Let $\varepsilon \in (0, \min\{r, \varepsilon_0\})$, $x_1, x_2 \in \partial A$ such that $\|x_1 - x_2\| = \varepsilon$. Let $p_1 \in N(A, x_1) \cap \partial B_1^*(0)$ or $p_1 \in N(\operatorname{cl}(E\backslash A), x_1) \cap \partial B_1^*(0)$. Let $H_1 = \{x \in E \mid (p_1, x_1 - x) = 0\} = x_1 + \ker p_1$.

Consider Fig. 5. Define $M = \{x \in E \mid |(p_1, x_1 - x)| \leqslant \alpha(\|x_1 - x\|) \cdot \|x_1 - x\|\}$. Using the reflexivity of the space E let $y \in E$, $\|y\| = 1$, $(p_1, y) = 1$. Let $L = \operatorname{aff}\{x_1, x_1 + y, x_2\}$. Let $\tilde{x} = \frac{1}{2}(x_1 + x_2)$, from the definition of L and from the definition of smooth closed surface $\varrho(x_2, H_1 \cap L) = \varrho(x_2, H_1) \leqslant \alpha(\varepsilon)\varepsilon$, hence $\varrho(\tilde{x}, H_1 \cap L) \leqslant \frac{\varepsilon}{2}\alpha(\varepsilon)$. Let $\tilde{x}_1 \in H_1 \cap L$ be a point such that $\|\tilde{x} - \tilde{x}_1\| = \varrho(\tilde{x}, H_1 \cap L)$.

Let γ be the connected part of the planar curve $L \cap \partial B_{\varepsilon/2}(x_1) \cap M$, which intersects the line $H_1 \cap L$, and lies in the same hyperplane with the point $\tilde{x}$ with respect to the line $\operatorname{aff}\{x_1, x_1 + y\}$. By the simplicity of the surface ∂A and by the inequality $\varepsilon < r$ there exists $z \in \gamma \cap \partial A$. From the inclusion $z \in M$ we have $\varrho(z, H_1 \cap L) \leqslant \alpha(\varepsilon/2)\frac{\varepsilon}{2}$. Let $z_1 \in H_1 \cap L$ be a point such that $\|z - z_1\| = \varrho(z, H_1 \cap L)$.

Choose the right direction of the line $H_1 \cap L$ from the point x_1 to the point $w = H_1 \cap L \cap \gamma$.

By the triangle inequality we have from the triangle $x_1\tilde{x}\tilde{x}_1$,

$$\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\alpha(\varepsilon) \leqslant \|x_1 - \tilde{x}_1\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\alpha(\varepsilon)$$

and from the triangle $x_1 z z_1$,

$$\frac{\varepsilon}{2} - \frac{\varepsilon}{2}\alpha\left(\frac{\varepsilon}{2}\right) \leqslant \|x_1 - z_1\| \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\alpha\left(\frac{\varepsilon}{2}\right).$$

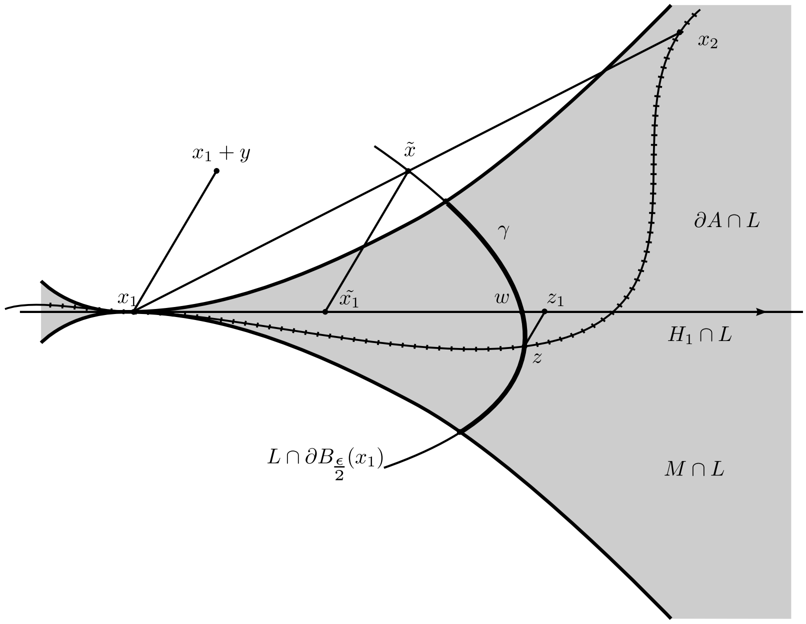

Fig. 5. Proof of Theorem 5.1.

If the point $\tilde{x}_1$ lies to the right of the point w then

$$\|\tilde{x}_1 - w\| = \|x_1 - \tilde{x}_1\| - \|x_1 - w\| \leqslant \frac{\varepsilon}{2}\alpha(\varepsilon).$$

If the point $\tilde{x}_1$ lies to the left of the point w then

$$\|\tilde{x}_1 - w\| = \|x_1 - w\| - \|x_1 - \tilde{x}_1\| \leqslant \frac{\varepsilon}{2}\alpha(\varepsilon).$$

In both cases $\|\tilde{x}_1 - w\| \leqslant \frac{\varepsilon}{2}\alpha(\varepsilon)$.

In the same way we obtain that $\|z_1 - w\| \leqslant \frac{\varepsilon}{2}\alpha(\frac{\varepsilon}{2})$.

From this we deduce that

$$\|z_1 - \tilde{x}_1\| \leqslant \|\tilde{x}_1 - w\| + \|z_1 - w\| \leqslant \frac{\varepsilon}{2}\alpha(\varepsilon) + \frac{\varepsilon}{2}\alpha\left(\frac{\varepsilon}{2}\right).$$

Finally, for any $x_1, x_2 \in \partial A$, $\|x_1 - x_2\| = \varepsilon$, $\tilde{x} = \frac{1}{2}(x_1 + x_2)$ there exists $z \in \partial A$ such that

$$\|\tilde{x} - z\| \leqslant \|\tilde{x} - \tilde{x}_1\| + \|z - z_1\| + \|\tilde{x}_1 - z_1\| \leqslant \varepsilon\left(\alpha(\varepsilon) + \alpha\left(\frac{\varepsilon}{2}\right)\right) < \frac{\varepsilon}{2}. \qquad \square$$

We observe that under the conditions of Theorem 5.1 both sets A and $\operatorname{cl}(E\backslash A)$ are weakly convex. The proof easily follows from Theorem 5.1.

Let E be a Banach space and a subset $A \subset E$ be closed. We shall say that unit vector $n \in E$ is a *proximall normal* to the set A at the point $x \in \partial A$ if there exists $r > 0$ such that

$$A \cap \operatorname{int} B_r(x + rn) = \emptyset.$$

Theorem 5.2. *Let space E be uniformly convex with modulus of convexity of the second order and uniformly smooth, and let subsets $A \subset E$ and $\operatorname{cl}(E\backslash A)$ be weakly convex with modulus $\gamma(\varepsilon)$ of the second order and* $\operatorname{cl}\operatorname{int} A = A$. *Then $N(A, x) \cap \partial B_1^*(0) = \{p(x)\}$, $N(\operatorname{cl}(E\backslash A), x) \cap \partial B_1^*(0) = \{-p(x)\}$, at any point $x \in \partial A$ and $p(x)$ uniformly continuously depends on the point $x \in \partial A$.*

Proof. By Remark 2.2 the sets A and $\operatorname{cl}(E\backslash A)$ are proximally smooth with some parameter $d > 0$. Using the results of Ivanov [11, Theorem 2] we have that for any point $x \in \partial A$ there exists a proximally normal vector $n(x)$ to the set A at the point $x \in \partial A$ (and proximally normal vector $-n(x)$ to the set $\operatorname{cl}(E\backslash A)$ at the point $x \in \partial A$) and $n(x)$ uniformly continuously depends on x. By uniform smoothness of the space E for any vector $n \in E$, $\|n\| = 1$, there exists unit vector $p(n) \in E^*$ with $(p(n), n) = 1$ and $p(n)$ uniformly continuously depends on n (see [9,14]). Again by the smoothness of the space E we have that $p(x) = p(n(x)) \in N(A, x)$ is uniformly continuous on $x \in \partial A$. $\square$

6. Epilogue

1. We see from the results above that in the spaces with modulus of convexity of the second order the notion of weakly convex set is very effective.

2. Some of the results can be proved in a more general setting of uniformly convex Banach spaces: Theorem 3.4, or Theorems 3.1, 3.2, and 3.3 (see [2] for details). However, the proofs in [2] are much more complicated.

We wish to point out that the results above are interesting and nontrivial even in the finite-dimensional case.

3. We agree with Banaś that the modulus σ from [4] and [5] is sometimes much more convenient in applications than the standard modulus of smoothness [9,14]. In fact, the modulus of nonconvexity is a modification of modulus σ from [4] for the nonconvex case.

Also, Theorem 5.1 shows a deep relationship between weakly convex sets and smooth sets. In conclusion, we formulate the following:

Conjecture 6.1. *Let the space E be uniformly convex (and smooth?), the subsets $A \subset E$ and $\mathrm{cl}(E \backslash A)$ closed and weakly convex with modulus $\gamma(\varepsilon)$ and $\mathrm{cl}\,\mathrm{int}\,A = A$. If $\lim_{\varepsilon \to +0} \gamma(\varepsilon)/\varepsilon = 0$, then the unit normal vector to the set A at the point $x \in \partial A$ uniformly continuously depends on the point $x \in \partial A$.*

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