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1-Torsion of finite modules over semiperfect rings

Alex Martsinkovsky

Mathematics Department, Northeastern University, Boston, MA 02115, USA

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ABSTRACT

We initiate the study of 1-torsion of finite modules over two-sided noetherian semiperfect rings. In particular, we give a criterion for determining when the 1-torsion submodule contains minimal generators of the module. We also provide an explicit construction for a projective cover of the submodule generated by the torsion elements in the top of the module. Some of the obtained results hold without the noetherian assumption. We also give several applications to local algebra.

1. Introduction

The goal of this paper is to study the kernel of the canonical map from a finite module over a semiperfect ring to its double dual. Such kernels will be referred to as 1-torsion. In particular, we want to understand under what conditions the 1-torsion contains minimal generators of the ambient module. The original motivation for this problem came from a question raised by Reiffen and Vetter [11] in their work on Pfaffian forms on complex spaces. An algebraic reformulation of it, due to G. Scheja, is discussed in detail in E. Platte's paper [10]. We quickly recall the basic facts. Let k be a valued field of characteristic zero and A a reduced equidimensional local analytic k-algebra with (universally finite) module of Kähler differentials $D_k(A)$. The torsion problem can be stated as follows: (if $k = \mathbb{C}$) is it possible for $D_k(A)$ to have common minimal generators with its torsion submodule?

After mentioning several cases with a negative answer in [10], Platte constructs a class of examples showing that indeed the module of differentials can have torsion elements among its minimal generators. At the end of the paper, he mentions another question, raised by Scheja, whether the torsion submodule of the module of differentials can be a direct summand. He then quotes a result of Scheja that for hypersurface rings the new problem is equivalent to the original problem, elevates the question to a conjecture (i.e., the torsion submodule is never a direct summand) and remarks that, if

E-mail address: alexmart@neu.edu.

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true, it would provide a quick proof of Grothendieck's version of the purity of the branch locus for complete intersections [6]. Platte concludes his paper with a remark that "[u]nfortunately, a proof of the weakened torsion conjecture seems to be [methodologically] remote".

In the present paper, we shall show how methods of stable module theory can be used to provide new insights and perspectives on the problems of Reiffen–Vetter and Platte–Scheja. One may begin, for example, by asking a natural question: given the module of differentials of an algebra, how does one determine whether or not this *specific* module has torsion elements among its minimal generators? In fact, properties of the torsion submodule of *any* finitely presented module is a topic of interest in its own right and the same question can be posed for any finitely generated module over a commutative noetherian local ring. Moreover, there is no reason not to pose this question in the utmost generality, for any finitely generated module over a two-sided noetherian semiperfect ring. In that setup, the torsion submodule should be replaced by the more general concept of 1-torsion.

The main result of this paper (Theorem 5) provides a verifiable module-theoretic criterion for an *arbitrary* finitely generated module over an *arbitrary* noetherian semiperfect ring to have 1-torsion elements among its minimal generators. More precisely, this happens exactly when the first syzygy module of the Auslander transpose of the module has a projective summand. This has immediate applications in commutative algebra. First, we have an interesting consequence for finite modules over artinian commutative rings: the 1-torsion submodule can never contain minimal generators of the module. Secondly, the non-existence of projective summands in the syzygy modules can be deduced from the vanishing of the ξ-invariants of the module (see below for details). Roughly speaking, those invariants measure the difference between the cohomology and the Tate–Vogel cohomology of the module. The latter is an example of an abstract stable homotopy theory, based on the Eckmann–Hilton homotopy groups of modules.

We also remark that, in equationally defined situations, the obtained criterion allows explicit calculations with a minimum of computing power: to determine whether or not the first syzygy module of the transpose has a free summand, one needs a presentation matrix for the syzygy module and a procedure to check whether or not one of the rows of the matrix is a linear combination of the remaining rows.[1]

In Section 6 we give a criterion for the 1-torsion submodule to be a direct summand. This is done in a greater generality: the ring is two-sided noetherian but not necessarily semiperfect. Our methods do not impose any significant restrictions on the rings in question: there is no assumption on the characteristic, the ring does not have to be commutative or a domain, nilpotent elements are allowed, etc. For that reason, it is to be hoped that a proof of the Platte–Scheja conjecture, if at all possible, can be obtained by some sort of a dimension–reduction procedure. As we mentioned above, in dimension zero the 1-torsion submodule cannot be a direct summand!

The author is grateful to the referee, whose comments strengthened the original version of Proposition 12 and also led to Proposition 8 (and some of its consequences).

2. Notation and preliminaries

Throughout this paper all rings will be assumed to be associative with identity and all modules to be unital. In this section we recall some basic facts from module theory. Most of this material, in one form or another, can be found in [1–3]. Let Λ be a ring and M a (left) Λ-module with a *finite* projective presentation

$$P_1 \xrightarrow{\partial} P_0 \xrightarrow{p} M \longrightarrow 0.$$

If finite Λ-modules admit projective covers (i.e., Λ is semiperfect), we shall automatically assume that the presentation above is minimal. The first syzygy module ΩM of M is defined as the kernel of the map $P_0 \to M$. The transpose $\operatorname{Tr} M$ of M is defined by the exact sequence

[1] A careful reader may add that one needs a presentation matrix of the original module to begin with.

$$0 \longrightarrow M^* \xrightarrow{p^*} P_0^* \xrightarrow{\partial^*} P_1^* \xrightarrow{\omega} \operatorname{Tr} M \longrightarrow 0,$$

where $(-)^*$ stands for the functor $\operatorname{Hom}_\Lambda(-, \Lambda)$. The finiteness assumption on the projective presentation of M implies that the beginning of the above sequence is a *finite* projective presentation of $\operatorname{Tr} M$. If finite Λ-modules admit projective covers, then both ΩM and $\operatorname{Tr} M$ are defined uniquely up to isomorphism because of our convention that projective presentations be minimal. In general, however, both ΩM and $\operatorname{Tr} M$ are only defined up to projective equivalence.

The following operation on Λ-modules will be of fundamental importance to us.

Definition 1. $\lambda M := \Omega \operatorname{Tr} M$.

In the above notation, $\lambda M = \operatorname{Ker} \omega = \operatorname{Im} \partial^* \simeq \operatorname{Coker} p^*$, which shows that while λM is still defined up to projective equivalence, its isomorphism class does not depend on the choice of P_1.

Lemma 2. *Let Λ be a semiperfect ring and N a submodule of a finitely generated projective Λ-module P. Then N is superfluous in P if and only if N and P have no common nonzero projective summands.*

The following consequence of this result is of main interest to us.

Proposition 3. *Let Λ be a semiperfect ring, M a finitely presented Λ-module with minimal projective presentation $P_1 \xrightarrow{\partial} P_0 \to M \to 0$, and*

$$0 \longrightarrow M^* \longrightarrow P_0^* \xrightarrow{\partial^*} P_1^* \xrightarrow{\omega} \operatorname{Tr} M \longrightarrow 0$$

the corresponding (*augmented on the left*) *finite presentation for* $\operatorname{Tr} M$. *Then*:

a) *The map $\omega : P_1^* \to \operatorname{Tr} M$ is a projective cover.*
b) *M is stable if and only if the above presentation of* $\operatorname{Tr} M$ *is minimal.*
c) *M is stable if and only if $P_0^* \to \lambda M$ is a projective cover.*
d) *If Q is a maximal projective direct summand of M, then Q^* is a maximal common direct summand of M^* and P_0^*.*
e) $\operatorname{Tr} M$ *is stable.*
f) $\operatorname{Tr} M$ *is zero if and only if M is projective.*

The proof consists of standard arguments and is left to the reader.

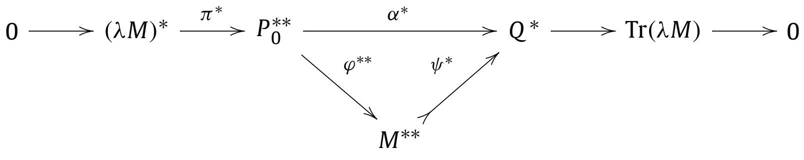
For any Λ-module M, the kernel $t(M)$ of the canonical map $e_M : M \to M^{**}$ will be called the 1-torsion submodule of M. The image of e_M can be computed as follows.

Lemma 4. *Let Λ be a semiperfect ring and M a finitely presented Λ-module such that M^* is finitely generated. If M is stable, then the image of the canonical map $e_M : M \to M^{**}$ is isomorphic to $\lambda^2 M$. If $M \simeq \underline{M} \coprod Q$, where $\underline{M}$ is stable and Q is projective, then the image of e_M is isomorphic to $\lambda^2 M \coprod Q \simeq \lambda^2 \underline{M} \coprod Q$.*

Proof. The second part of the lemma immediately follows from the first. To prove the first part, we start with a minimal presentation $P_1 \to P_0 \xrightarrow{\varphi} M \to 0$. Since double dual is a natural transformation, we have $\operatorname{Im}(e_M) = \operatorname{Im}(\varphi^{**})$. Applying $\operatorname{Hom}(-, \Lambda)$ to the minimal presentation above, we have an exact sequence

$$0 \longrightarrow M^* \xrightarrow{\varphi^*} P_0^* \xrightarrow{\pi} \lambda M \longrightarrow 0\ . \tag{A}$$

Let $\psi : Q \to M^*$ be a projective cover (it exists since M^* is, by assumption, finitely generated) and $\alpha := \varphi^*\psi$. Assuming that M is stable, we have, by Proposition 3, a minimal presentation $Q \xrightarrow{\alpha} P_0^* \xrightarrow{\pi} \lambda M \to 0$. Applying to it the functor $\mathrm{Hom}(-, \Lambda)$ we have a commutative diagram with an exact top row:



(B)

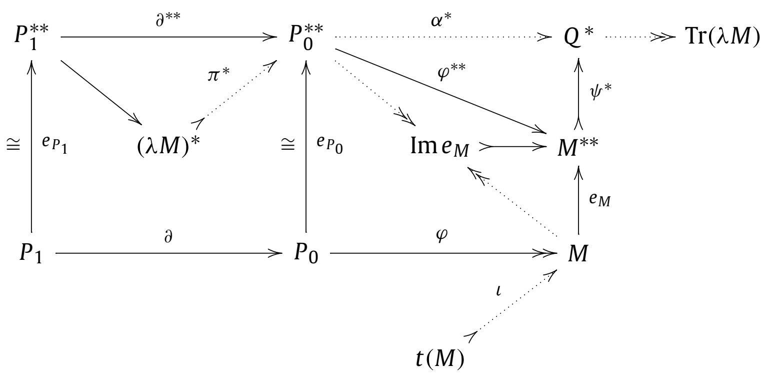
By the left-exactness of the Hom-functor, ψ^* is a monomorphism and therefore $\mathrm{Im}(\alpha^*) \simeq \mathrm{Im}(\varphi^{**})$. By Proposition 3, a), the map $Q^* \to \mathrm{Tr}(\lambda M)$ is a projective cover and thus $\mathrm{Im}(\alpha^*) \simeq \Omega(\mathrm{Tr}(\lambda M)) = \lambda^2 M$. This finishes the proof of the lemma. □

3. The main theorem and first applications

Our goal in this section is to give a necessary and sufficient condition for the 1-torsion submodule to contain a minimal generator of the ambient module. The ring Λ will be semiperfect and two-sided noetherian.

Theorem 5. *Let Λ be a two-sided noetherian semiperfect ring and M a finitely generated Λ-module. Then the* 1*-torsion submodule $t(M)$ contains a minimal generator of M if and only if λM has a nonzero projective summand.*

Proof. The constructions (and notation) used above are collected in the following commutative diagram:



(C)

where the complexes consisting of dotted arrows are exact (assuming that the epimorphisms are followed by maps to the zero module and the monomorphisms are preceded by maps from the zero module). The two shorter complexes of such type give rise to the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega M & = & \Omega M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (\lambda M)^* & \xrightarrow{\pi^*} & P_0^{**} & \longrightarrow & \operatorname{Im} e_M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi e_{P_0}^{-1}} & & \| & & \\
0 & \longrightarrow & t(M) & \xrightarrow{\iota} & M & \longrightarrow & \operatorname{Im} e_M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{D}
$$

The map $\varphi e_{P_0}^{-1}$, being the composition of an isomorphism and a projective cover, is an isomorphism modulo the Jacobson radical J of Λ. By Nakayama's lemma, $t(M)$ contains a minimal generator of M if and only if $t(M)$ is not contained in JM. Reducing the south-west square modulo J, we see that this happens precisely when $(\lambda M)^*$ is not contained in JP_0^{**}. In view of Proposition 3, a), this is equivalent to saying that the map α^* in diagram (C) is not a minimal presentation of $\operatorname{Tr}(\lambda M)$. By Proposition 3, c), (with λM in place of M) this is equivalent to saying that λM has a nonzero projective summand. □

Corollary 6. *Under the assumptions of Theorem* 5*, the* 1*-torsion submodule $t(M)$ contains a minimal generator of M if and only if $(\lambda M)^*$ and P_0^{**} have a common nonzero projective summand under the map π^*.*

Proof. This is just a reformulation of the theorem. In view of Lemma 2, the "only if" part was already shown at the end of the proof of the theorem. Suppose now that there is a common nonzero projective summand. Then sequence (B) is not a minimal presentation of $\operatorname{Tr}(\lambda M)$ and we are done by Proposition 3, b). □

The short exact sequence $0 \to \Omega M \to (\lambda M)^* \to t(M) \to 0$ yields the following.

Corollary 7. *Under the assumptions of Theorem* 5*, if $t(M) = 0$, then ΩM and $(\lambda M)^*$ are isomorphic. If Λ is artinian, then the converse is true.*

Proof. The first assertion is immediate. The second follows from the fact than an injective endomorphism of a module of finite length is an isomorphism. □

Our next goal is to find classes of rings over which the 1-torsion submodule of an arbitrary finitely generated module cannot reach the top. We begin by establishing a simple criterion. Let Λ be a (not necessarily noetherian) ring with Jacobson radical J. We shall say that Λ has *low* 1*-torsion* if the 1-torsion submodule $t(M)$ of an arbitrary (not necessarily finitely generated) Λ-module M is contained in JM.

Proposition 8. *A semilocal ring has low* 1*-torsion if and only if every simple module is* 1*-torsion free.*

Proof. The "only if" part is immediate. Assume now that every simple is 1-torsion free. For an arbitrary module M we have an exact sequence

$$0 \to JM \to M \to M/JM \to 0.$$

If $m \in t(M)$, then any linear functional $M \to \Lambda$ vanishes on m. Therefore any linear functional on M/JM vanishes on the image of m in M/JM. Since M/JM is semisimple, our assumption implies that the image of m is zero, i.e., $m \in JM$. $\square$

Proposition 9. *Suppose Λ is a local (not necessarily noetherian) ring with a nonzero socle. Then Λ has low 1-torsion. If, in addition, Λ is semiperfect, then the 1-torsion submodule of any finitely generated Λ-module does not contain minimal generators of the module.*

Proof. Let S be the unique simple module Λ/J. For the first claim it suffices to show that S is 1-torsion free. Suppose this is not true. Then $t(S) = S$ and λS must have a projective summand: in our case $\lambda S \simeq X \coprod \Lambda$. Let $\Lambda^n \to \Lambda \to S \to 0$ be a minimal projective presentation of S. Dualizing into Λ, we have a short exact sequence $0 \to \mathrm{Hom}(S,\Lambda) \to \Lambda_\Lambda \to X \coprod \Lambda_\Lambda \to 0$, which yields, via the composition with the projection to the second summand, a surjective endomorphism of Λ_Λ. Since Λ is local, it has IBN. As a consequence, that endomorphism must be an isomorphism. This implies that $\mathrm{Hom}(S,\Lambda) = 0$. But, by assumption, the socle of Λ is nonzero and therefore S embeds in Λ, a contradiction. Thus S is indeed 1-torsion free and Λ has low 1-torsion. The second claim now follows immediately. $\square$

Corollary 10. *Any local artin algebra has low 1-torsion.*

Corollary 11. *Any commutative local ring of depth zero has low 1-torsion.*

Recall that a commutative ring is semiperfect if and only if it is a finite direct product of commutative local rings. Therefore, by the Krull–Akizuki theorem, commutative artinian rings are semiperfect. The next result provides examples of low 1-torsion for nonlocal rings.

Proposition 12. *Let A be a commutative artinian ring. Then the 1-torsion submodule of a finitely generated A-module does not contain minimal generators of the module.*

Proof. By Theorem 5, it suffices to show that the first syzygy module ΩM of any finitely generated A-module M has no nonzero projective summands. Suppose that this is not the case. We then have a short exact sequence $0 \to \underline{\Omega M} \coprod P_1 \to P_0 \to M \to 0$, where P_1 is a nonzero projective and $P_0 \to M$ is a projective cover. Since P_1 is superfluous in P_0, the short exact sequence $0 \to P_1 \to P_0 \to X \to 0$ is not split, i.e., $\mathrm{Ext}^1(X, P_1) \neq 0$. Therefore, there is a maximal ideal $\mathfrak{m}$ of A such that $\mathrm{Ext}^1_A(X, P_1)_\mathfrak{m} = \mathrm{Ext}^1_{A_\mathfrak{m}}(X_\mathfrak{m}, P_{1\mathfrak{m}}) \neq 0$. But then $\operatorname{proj.dim.} X_\mathfrak{m} = 1$, contrary to the Auslander–Buchsbaum formula. $\square$

As a consequence of the proof of Theorem 5, we can now quantify the extent to which the 1-torsion submodule $t(M)$ "penetrates" the top of M (i.e., M/JM).

Definition 13. Let $\mathsf{T}(M)$ be the submodule of M generated by the elements of $t(M)$ not contained in JM.

Proposition 14. *Suppose $\lambda M \simeq \underline{\lambda M} \coprod S$, where $\underline{\lambda M}$ is stable and S is projective. Then, in the notation of Theorem 5, $\varphi e_{P_0}^{-1}|_{S^*} : S^* \to \mathsf{T}(M)$ is a projective cover.*

Proof. Proposition 3, d) shows that S^* is a maximal common projective summand of $(\lambda M)^*$ and P_0^{**}. Reducing modulo J the south-west commutative square in diagram (D), we see that the statement of this corollary correctly identifies $\mathsf{T}(M)$ as the image of the restriction map in question. That this map is a projective cover follows from the fact that it is an isomorphism modulo the radical. $\square$

Suppose Λ is a local ring. Then the top of any finitely generated projective Λ-module becomes a vector space over the residue skew field $\Lambda/J\Lambda$ and we have:

Proposition 15. *If Λ is a two-sided noetherian local ring, then the dimension of the vector subspace of the top of M generated by the image of $\mathsf{T}(M)$ (equivalently, by the image of $t(M)$) equals the rank of a maximal projective summand of λM.*

Remarks. a) The last proposition can be quickly proved by an argument which does not appeal to Theorem 5. Let f-rank M denote the rank of a maximal projective (i.e., free) summand of M and $b(M)$ the minimal number of generators of M. Using the definition of the operator λ and Proposition 3, d), we have

$$b(\lambda M) = b(M) - \text{f-rank}\, M.$$

Applying this formula twice, we have

$$b\big(\lambda^2 M\big) = b(M) - \text{f-rank}\,\lambda M - \text{f-rank}\, M.$$

Lemma 4 gives rise to a short exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow \lambda^2 M \coprod \Lambda^{\text{f-rank}\, M} \longrightarrow 0,$$

which shows that $b(\mathsf{T}(M)) = b(M) - b(\lambda^2 M) - \text{f-rank}\, M$. In view of the previous formula, this equals f-rank λM.

b) For any ring Λ and any superfluous epimorphism $f : M \to N$ of finite Λ-modules we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & t(M) & \xrightarrow{i_M} & M & \xrightarrow{e_M} & M^{**} \\
 & & \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^{**}} \\
0 & \longrightarrow & t(N) & \xrightarrow{i_N} & N & \xrightarrow{e_N} & N^{**}
\end{array}$$

Let J be the radical of Λ and suppose that $i_M \otimes \Lambda/J \neq 0$. Since $f \otimes \Lambda/J$ is an isomorphism, we have that $i_N \otimes \Lambda/J \neq 0$.

4. Applications to local algebra

We can now offer another perspective on the results of Reiffen–Vetter and Scheja on hypersurface algebras.

Proposition 16. *Let R be a commutative noetherian local ring, $a_1, \dots, a_n$, where $n \geqslant 1$, elements of R generating a nonzero proper ideal $\mathfrak{a} \subsetneq R$, and M an R-module with presentation*

$$R \xrightarrow{[a_1, a_2, \dots, a_n]^T} R^n \longrightarrow M \longrightarrow 0.$$

Then $\mathsf{T}(M)$ is nonzero if and only if $\mathfrak{a}$ is a principal ideal generated by a nonzerodivisor. In that case, the 1-torsion submodule $t(M)$ is a direct summand of M.[2]

[2] Under an additional assumption that the ideal $\mathfrak{a}$ contains a nonzerodivisor, this result was also proved in [12], Hilfsatz (9.10).

Proof. If $\mathfrak{a}$ is a principal ideal generated by $a \in R$, then for $i = 1, \ldots, n$ there are elements $b_i \in R$ and $c_i \in R$ such that $a_i = b_i a$ and $a = \sum_{i=1}^{n} c_i a_i$. Therefore $(1 - \sum_{i=1}^{n} c_i b_i)a = 0$. Since R is local and $(a) = \mathfrak{a} \neq 0$, one of the b_i (and the corresponding c_i) must be a unit. Thus one of the a_i generates $\mathfrak{a}$ and M has a presentation

$$R \xrightarrow{[a0\ldots0]^T} R^n \longrightarrow M \longrightarrow 0.$$

This shows that $M \simeq R/(a) \coprod R^{n-1}$. By assumption, a is neither the zero element nor a unit. Therefore the obtained presentation is minimal and $\lambda M \simeq \mathfrak{a} = (a)$. When a is a nonzerodivisor, this module is isomorphic to R, showing that $\mathsf{T}(M) \neq (0)$. In that case $(R/(a))^* \simeq \operatorname{Ann} a = (0)$ and thus $t(M) = R/(a) = \mathsf{T}(M)$.

To prove the other implication, we may assume that $\mathfrak{a}$ is minimally generated by $a_1, \ldots, a_n$, thus making the defining presentation of M minimal. By Theorem 5, $\lambda M \simeq \mathfrak{a}$ has a nonzero projective summand, say, $\mathfrak{a} \simeq \mathfrak{a}_1 \oplus \mathfrak{a}_2$ with $\mathfrak{a}_2$ isomorphic to R. Since R is commutative, $\mathfrak{a}_1\mathfrak{a}_2$ is contained in both $\mathfrak{a}_1$ and $\mathfrak{a}_2$ and is therefore zero. Since a nonzero element cannot annihilate the identity of R, we must have $\mathfrak{a}_1 = (0)$ and therefore $\mathfrak{a} \simeq R$. This shows that $\mathfrak{a}$ is principal and generated by a nonzerodivisor. $\square$

Example. Let $R := k[|x, y|]/(x^6 - x^2y^3 - y^5)$, where k is a field of characteristic 0. The extension of the Jacobian ideal $(6x^5 - 2xy^3, -3x^2y^2 - 5y^4)$ of this curve to R is nonprincipal, and therefore there are no torsion elements among minimal generators of the module of differentials, i.e., $\mathsf{T}(D_k(R)) = 0$. Assume now that $\operatorname{char} k = 2$. Then the extension of the Jacobian ideal $(x^2y^2 + y^4)$ is generated by a nonzerodivisor and, therefore, the torsion submodule of the module of differentials reaches the top of the module. In this case, $\lambda D_k(R)$ is free of rank one, since it is isomorphic to the ideal generated by the image of the nonzero partial derivative. Consequently, $\mathsf{T}(D_k(R))$ is a nontrivial cyclic module.

Assume once again that R is a commutative noetherian local ring. As another application of Theorem 5, we shall show that if the transpose of the module M is of large enough depth, then $\mathsf{T}(M) = 0$. First we recall an auxiliary result (see [4], Lemma 4.7; see also [9], Proposition 3 for a proof inspired by the present paper).

Lemma 17. *Let N be a finitely generated R-module. Then $\Omega^i N$ has no nonzero free summands for $i >$* $\max(\operatorname{depth} R - \operatorname{depth} M, 0)$.

Combining this with Theorem 5 and recalling that $\lambda M = \Omega \operatorname{Tr} M$, we have:

Proposition 18. *Let M be a finitely generated R-module such that* $\operatorname{depth} \operatorname{Tr} M \geqslant \operatorname{depth} R$. *Then the* 1*-torsion submodule of M contains no minimal generators of M.*

5. Further applications to local algebra: 1-torsion and Tate–Vogel cohomology

For finite modules over a commutative local ring, the absence of free summands can be detected by the vanishing of the ξ-invariant, introduced by the author in [7]. This nonnegative integer is the dimension of the kernel of the natural transformation from the cohomology of the module with coefficients in the residue field to the Tate–Vogel cohomology of the same pair. The details of the construction can be found in the above reference. Since in this paper we are only interested in applications, we provide a very simple equivalent definition of the ξ-invariant.

Definition 19. Let R be a commutative noetherian local ring and M a finite R-module. Let $V(M, k)$ denote the vector subspace of $\operatorname{Hom}_R(M, k)$ consisting of bounded maps, i.e., the maps that admit a lifting with only finitely many nonzero components to some projective resolutions of M and k. We set $\xi(M) := \dim V(M, k)$ and $\xi^n(M) := \dim V(\Omega^n M, k)$. We also set $\xi^0(M) := \xi(M)$.

Immediately from the definition we deduce that ξ is additive on direct sums and, for a module of *finite* projective dimension, coincides with the zeroth Betti number. In particular, $\xi(R^n) = n$. Consequently, if $\xi(M) = 0$, then M cannot have a nonzero projective summand. Taking account of Theorem 5, we have:

Proposition 20. *Let R be a commutative noetherian local ring and M a finite R-module. If $\xi(\lambda M) = 0$, then the 1-torsion submodule $t(M)$ does not contain minimal generators of M.*

In order to make this useful we need to be able to compute the ξ-invariant. In general this is difficult. But in some situations (see [7,8]) this invariant has been computed. A case of interest to us is provided by the following result (see Theorem 3.1, [7]).

Theorem 21. *Let $(S, \mathfrak{m}, k)$ be a commutative noetherian local ring, $x \in \mathfrak{m}$ an S-regular element, $R := S/(x)$, and M a finite R-module. If $x \in \mathfrak{m}\mathrm{Ann}_S M$, then $\xi^i(M) = 0$ for all i.*

Remark. If $x \in \mathfrak{m}\mathrm{Ann}_S M$, then, clearly, the same condition holds if M is replaced by any of its quotient modules.

Proposition 22. *Let $(S, \mathfrak{m}, k)$ be a commutative noetherian local ring, $x \in \mathfrak{m}$ an S-regular element, $R := S/(x)$, and M a finite R-module. If $x \in \mathfrak{m}\mathrm{Ann}_S(\lambda_R M)$, then the 1-torsion submodule $t(M)$ does not contain minimal generators of M.*

6. 1-Torsion as a direct summand

In this section we shall give a necessary and sufficient condition for a finitely generated module to have its 1-torsion submodule as a direct summand. This will be done in a more general context than we have been working in so far: the ring will be two-sided noetherian but not necessarily semiperfect.

As a motivating example, we consider first the "hypersurface" module of Proposition 16. Let R be a commutative *domain*, $\mathfrak{a} := (a_1, \dots, a_n)$ a *nonzero* ideal of R, and M a module with presentation

$$0 \longrightarrow R \xrightarrow{[a_1,\dots,a_n]^T} R^n \xrightarrow{\varphi} M \longrightarrow 0\ .$$

Problem 1. Describe the torsion submodule $t(M)$ of M.

Solution. Suppose $x \in M$ is torsion: there is $\mu \neq 0$ in R such that $\mu x = 0$. Choose $x_1, \dots, x_n \in R$ such that $\varphi((x_1, \dots, x_n)^T) = x$; then $\mu x_i = \lambda a_i$, $i = 1, \dots, n$, for some $\lambda \in R$. In other words, $x_i = (\lambda/\mu) a_i$, $i = 1, \dots, n$, in the field of quotients K of R. Since each x_i is in R, we must have $(\lambda/\mu) \in (R : \mathfrak{a})$. As a result, $(x_1, \dots, x_n)^T$ is in the image of the R-linear map

$$f : (R : \mathfrak{a}) \to R^n : \lambda/\mu \mapsto (\lambda/\mu)(a_1, \dots, a_n)^T.$$

Conversely, it is immediate that any element in the image of f gives rise, after applying φ, to a torsion element of M. The canonical inclusions $R \to (R : \mathfrak{a})$ and $\iota : t(M) \to M$ now become parts of a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R & \longrightarrow & (R:\mathfrak{a}) & \dashrightarrow & t(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & R & \xrightarrow{[a_1,\dots,a_n]^T} & R^n & \xrightarrow{\varphi} & M & \longrightarrow & 0
\end{array}$$

This diagram describes both the torsion submodule $t(M)$ and its embedding in M.

Corollary 23. *M is torsion-free if and only if $(R:\mathfrak{a}) = R$.*

Completing the columns of the above diagram we also have the following description of $\operatorname{Im} e_M \simeq \operatorname{Coker}\iota$:

Corollary 24. *The sequence*

$$0 \longrightarrow (R:\mathfrak{a}) \xrightarrow{f} R^n \xrightarrow{e_M\varphi} \operatorname{Im} e_M \longrightarrow 0$$

is exact.

The next problem appears as an exercise in (see [5], Ch. VII, §1, Ex. 32).

Problem 2. Under the above assumptions, show that $t(M)$ is a direct summand of M if and only if

$$\mathfrak{a}(R:\mathfrak{a}) + \big(R:(R:\mathfrak{a})\big) = R.$$

Lemma 25. $(R:(R:\mathfrak{a})) \subseteq R$.

Proof. Since $(R:\mathfrak{a}) \supseteq R$, we have $(R:(R:\mathfrak{a})) \subseteq (R:R) = R$. □

The lemma shows that the left-hand side of the desired equality is contained in R. Thus we have to show that the torsion is a direct summand if and only if the identity of R belongs to the left-hand side.

First, assume that the embedding $\iota : t(M) \to M$ admits a splitting $p : M \to t(M)$. Using the lifting property of the projective resolution of M, we obtain a commutative diagram of R-linear maps with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \longrightarrow & (R:\mathfrak{a}) & \longrightarrow & t(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & R & \xrightarrow{[a_1,\dots,a_n]^T} & R^n & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & R & \longrightarrow & (R:\mathfrak{a}) & \longrightarrow & t(M) & \longrightarrow & 0
\end{array}$$

We now examine the maps g and σ. Let $g(e_i) := b_i \in (R:\mathfrak{a})$, $i = 1, \dots, n$, where e_i is the ith standard basis vector. The commutativity of the south-west square implies then that $\sigma = \sum a_i b_i \in \mathfrak{a}(R:\mathfrak{a})$.

Since $\mathrm{Id}_{t(M)} - p\iota$ is the zero map, there exists an R-linear map $h : (R : \mathfrak{a}) \to R$ such that $(1-\sigma)\cdot r = h(r)$ for any $r \in R$. The map h can be computed explicitly. Indeed, if $\lambda/\mu \in (R : \mathfrak{a})$, then $\mu \cdot h(\lambda/\mu) = h(\lambda) = (1-\sigma)\cdot\lambda$ and, therefore, $h(\lambda/\mu) = (1-\sigma)\cdot\lambda/\mu$. Since the image of h is in R, we must have $(1-\sigma) \in (R : (R : \mathfrak{a}))$ thus obtaining the desired decomposition of the identity: $1 = \sigma + (1-\sigma)$.

Conversely, suppose $1 = \sigma + (1-\sigma)$, where $\sigma \in \mathfrak{a}(R : \mathfrak{a})$ and $1-\sigma \in (R : (R : \mathfrak{a}))$. Writing σ as $\sum a_i b_i$ with all $b_i \in (R : \mathfrak{a})$, and setting $g(e_i) := b_i$ for each i, we recover the above diagram. By construction, $\mathrm{Id}_{(R:\mathfrak{a})} - gf$ is multiplication by $1-\sigma$, the latter being an element of $(R : (R : \mathfrak{a}))$. Therefore, $\mathrm{Id}_{(R:\mathfrak{a})} - gf$ factors through R, showing that $\mathrm{Id}_{t(M)} - p\iota = 0$. This solves Problem 2.

Remark. Using Corollary 24 we can provide an alternative solution to Problem 2. The 1-torsion submodule of M is a direct summand if and only if the canonical map $e_M : M \to \mathrm{Im}\, e_M$ is a split epimorphism. Suppose there is a map $i : \mathrm{Im}\, e_M \to M$ such that $e_M i = \mathrm{Id}_{\mathrm{Im}\, e_M}$. Lifting i by maps k and l, we have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (R : \mathfrak{a}) & \xrightarrow{f} & R^n & \xrightarrow{e_M\varphi} & \mathrm{Im}\, e_M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & R & \xrightarrow{[a_1,\ldots,a_n]^T} & R^n & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j} & & \| & & \downarrow{\scriptstyle e_M} & & \\
0 & \longrightarrow & (R : \mathfrak{a}) & \xrightarrow{f} & R^n & \xrightarrow{e_M\varphi} & \mathrm{Im}\, e_M & \longrightarrow & 0
\end{array}$$

Here j is the canonical inclusion. As $\mathrm{Id}_{\mathrm{Im}\, e_M} = e_M i$, there is a map $h : R^n \to (R : \mathfrak{a})$ such that $fh = \mathrm{Id}_{R^n} - k$. This is equivalent to saying that $hf = \mathrm{Id}_{(R:\mathfrak{a})} - jl$. Let $h(e_i) := b_i \in (R : \mathfrak{a})$, where e_i is the ith standard basis vector. Then hf is just multiplication by $\sigma := \sum a_i b_i \in \mathfrak{a}(R : \mathfrak{a})$ and therefore $\mathrm{Id}_{(R:\mathfrak{a})} - hf$ is multiplication by $1-\sigma$. On the other hand, since the latter factors through R as the composition jl, the image of this map must be in R. Consequently, $1-\sigma \in (R : (R : \mathfrak{a}))$.

Conversely, suppose there is $\sigma \in \mathfrak{a}(R : \mathfrak{a})$ such that $1-\sigma \in (R : (R : \mathfrak{a}))$. Our immediate goal is to recover the triple-decker diagram above. Let $\sigma = \sum_1^n a_i b_i$, with each b_i in $(R : \mathfrak{a})$. We first define a map $h : R^n \to (R : \mathfrak{a})$ by setting $h(e_i) := b_i$ for each i. This allows to define a map $k : R^n \to R^n$ by setting $k := \mathrm{Id}_{R^n} - fh$, and a map $c : (R : \mathfrak{a}) \to (R : \mathfrak{a})$ by setting $c := \mathrm{Id}_{(R:\mathfrak{a})} - hf$. Since $hf = \sigma$, the last map is just multiplication by $1-\sigma \in (R : (R : \mathfrak{a}))$, and therefore its image must be in R. In other words, c factors through R, i.e., $c = jl$, where j is the canonical inclusion $R \to (R : \mathfrak{a})$ and l is a map $(R : \mathfrak{a}) \to R$. It is now straightforward to check that the pair l, k gives rise to a map $i : \mathrm{Im}\, e_M \to M$ and that $e_M i = \mathrm{Id}_{\mathrm{Im}\, e_M}$.

We now switch to a general context: Λ is a two-sided noetherian ring and M a finitely generated (left) Λ-module. Diagram (D) of Section 3 provides the following lifting of the canonical inclusion $\iota : t(M) \to M$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega M & \longrightarrow & (\lambda M)^* & \longrightarrow & t(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle e_{P_0}^{-1}\pi^*} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & P_0 & \xrightarrow{\varphi} & M & \longrightarrow & 0
\end{array}$$

Suppose now that ι admits a splitting $p : M \to t(M)$. Lifting p, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega M & \longrightarrow & (\lambda M)^* & \longrightarrow & t(M) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle e_{P_0}^{-1}\pi^*} & & \downarrow{\scriptstyle \iota} & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & P_0 & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & \Omega M & \longrightarrow & (\lambda M)^* & \longrightarrow & t(M) & \longrightarrow & 0
\end{array}$$

Since $\mathrm{Id}_{t(M)} - p\iota$ is the zero map, there exists a Λ-linear map $h : (\lambda M)^* \to \Omega M$ such that $\mathrm{Id}_{(\lambda M)^*} - g e_{P_0}^{-1}\pi^*$ factors through h.

Conversely, given Λ-linear maps $g : P_0 \to (\lambda M)^*$ and $h : (\lambda M)^* \to \Omega M$ such that $\mathrm{Id}_{(\lambda M)^*} - g e_{P_0}^{-1}\pi^*$ factors through h, define $\sigma : \Omega M \to \Omega M$ by setting $\sigma := \mathrm{Id}_{\Omega M} - h|_{\Omega M}$. It is then clear that g and σ are part of a commutative square as above, and therefore they give rise to a map $p : M \to t(M)$. It is also clear that $p\iota = \mathrm{Id}_{t(M)}$. Thus we have:

Proposition 26. *Let Λ be a two-sided noetherian ring and M a finitely generated (left) Λ-module. Then the 1-torsion submodule $t(M)$ is a direct summand of M if and only if there is a Λ-linear map $g : P_0 \to (\lambda M)^*$ such that $\mathrm{Id}_{(\lambda M)^*} - g e_{P_0^{-1}}\pi^*$ admits a lifting $h : (\lambda M)^* \to \Omega M$.*

Similar to the remark on p. 2605, we can give an alternative criterion for 1-torsion being a direct summand. Suppose the canonical map $M \to \mathrm{Im}\, e_M$ admits a splitting $i : \mathrm{Im}\, e_M \to M$. Augmenting the notation of Theorem 5, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\lambda M)^* & \xrightarrow{e_{P_0}^{-1}\pi^*} & P_0 & \xrightarrow{e_M\varphi} & \mathrm{Im}\, e_M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \Omega M & \xrightarrow{\nu} & P_0 & \xrightarrow{\varphi} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle j} & & \| & & \downarrow{\scriptstyle e_M} & & \\
0 & \longrightarrow & (\lambda M)^* & \xrightarrow{e_{P_0}^{-1}\pi^*} & P_0 & \xrightarrow{e_M\varphi} & \mathrm{Im}\, e_M & \longrightarrow & 0
\end{array}$$

where the maps l and k are some liftings of the map i. (Once again, for the sake of simplicity, we have slightly abused the notation for the maps going into the south-east corner.) Since $\mathrm{Id}_{\mathrm{Im}\, e_M} - e_M i = 0$, there is a Λ-linear map $h : P_0 \to (\lambda M)^*$ such that $\mathrm{Id}_{P_0} - k = e_{P_0}^{-1}\pi^* h$. This implies that $\mathrm{Id}_{(\lambda M)^*} - jl = h e_{P_0}^{-1}\pi^*$.

Conversely, suppose there are Λ-linear maps $l : (\lambda M)^* \to \Omega M$ and $h : P_0 \to (\lambda M)^*$ such that $\mathrm{Id}_{(\lambda M)^*} - jl = h e_{P_0}^{-1}\pi^*$. Define $k := \mathrm{Id}_{P_0} - e_{P_0}^{-1}\pi^* h$. It is then clear that k and l are part of a commutative square as above and therefore they give rise to a map $i : \mathrm{Im}\, e_M \to M$. It is also clear that $e_M i = \mathrm{Id}_{\mathrm{Im}\, e_M}$. Thus we have proved:

Proposition 27. *Let Λ be a two-sided noetherian ring and M a finitely generated (left) Λ-module. Then the 1-torsion submodule $t(M)$ is a direct summand of M if and only if there is a Λ-linear map $l : (\lambda M)^* \to \Omega M$ such that $\mathrm{Id}_{(\lambda M)^*} - jl$ admits an extension $h : P_0 \to (\lambda M)^*$.*

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