Matricial Norms and the Zeros of Polynomials

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A matricial norm \([1]\) is a mapping \(\mu\) from the algebra \(M_n\) of complex \(n \times n\) matrices into the set \(M_{k^+}\) of nonnegative \(k \times k\) matrices and which satisfies the following axioms:

\[
\begin{align*}
(i) & \quad \mu(xA) = |x|\mu(A) \quad \forall x \in \mathbb{C}, \quad \forall A \in M_n; \\
(ii) & \quad \mu(A + B) \leq \mu(A) + \mu(B) \quad \forall A, B \in M_n; \\
(iii) & \quad \mu(AB) \leq \mu(A)\mu(B) \quad \forall A, B \in M_n; \\
(iv) & \quad \mu(A) \neq 0 \quad \text{if} \quad A \neq 0.
\end{align*}
\]

Here \(\mathbb{C}\) denotes the complex field. The set \(M_{k^+}\) is partially ordered componentwise, i.e., \((a_{ij}) \leq (b_{ij})\) if and only if \(a_{ij} \leq b_{ij}\) for all \(i, j = 1, \ldots, n\). If \(k = 1\), then \(\mu\) is a matrix norm \([2]\). Denoting by \(r(A)\) the spectral radius of an \(n \times n\) matrix \(A\), it has been proved \([1, 5]\) that

\[
r(A) \leq r(\mu(A)),
\]

which generalizes a well-known property of matrix norms \([2]\).

A particular class of matricial norms can be obtained in the following manner. For an arbitrary complex \(p \times q\) matrix \(B = (b_{ij})\) denote

\[
\psi(B) = \max_{i=1, \ldots, p} (|b_{i1}| + |b_{i2}| + \cdots + |b_{iq}|);
\]

that is, \(\psi(B)\) is the row norm of \(B\). If

\[
A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & & & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix}
\]

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is an arbitrary but fixed partitioning of the \( n \times n \) matrix \( A \), where \( A_{11}, A_{22}, \ldots, A_{kk} \) are square matrices, then the mapping

\[
\phi: M_n \to M_k^+,
\]

\[
\phi(A) = (\phi(A_{ij}))_{i,j=1,\ldots,k} \quad (A \in M_n)
\]

(2)

is a matricial norm on \( M_n \) [1].

In this paper we will apply matricial norms to obtain upper bounds for the zeros of the polynomial

\[
f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0.
\]

where \( a_j \in \mathbb{C} \) (\( j = 0, 1, \ldots, n - 1 \)). We will denote by \( \rho(f) \) the largest of the absolute values of the zeros of \( f(z) \).

It is known that the zeros of \( f(z) \) are the eigenvalues of its companion matrix:

\[
F = \begin{bmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & 0 & -a_2 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & 1 & -a_{n-2} \\
0 & 0 & \cdots & 0 & -a_{n-1}
\end{bmatrix}
\]

Let \( k_0, k_1, \ldots, k_{n-2} \) be arbitrary positive numbers and denote \( D = \text{diag}(k_0, k_1, \ldots, k_{n-2}, 1) \in M_n \). The matrix

\[
D^{-1}FD = \begin{bmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
\frac{k_0}{k_1} & 0 & \cdots & 0 & -\frac{a_1}{k_1} \\
0 & \frac{k_1}{k_2} & \cdots & 0 & -\frac{a_2}{k_2} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & \frac{k_{n-3}}{k_{n-2}} & -\frac{a_{n-2}}{k_{n-2}} \\
0 & 0 & \cdots & \frac{k_{n-2}}{k_{n-1}} & -a_{n-1}
\end{bmatrix}
\]

(3)
has the same eigenvalues as \( F \), and so

\[ \rho(f) = r(F) = r(D^{-1}FD). \tag{4} \]

Let \( \phi: M_n \to M_2 \) denote the matricial norm given by (2) corresponding to the partitioning shown in (3). Then

\[ \phi(D^{-1}FD) = \begin{bmatrix} \beta & \gamma \\ k_{n-2} & |a_{n-1}| \end{bmatrix}, \tag{5} \]

where

\[ \beta = \max \left\{ \frac{k_0}{k_1}, \frac{k_1}{k_2}, \ldots, \frac{k_{n-3}}{k_{n-2}} \right\}, \tag{6} \]

\[ \gamma = \max \left\{ \frac{|a_0|}{k_0}, \frac{|a_1|}{k_1}, \ldots, \frac{|a_{n-2}|}{k_{n-2}} \right\}. \tag{7} \]

Applying inequality (1) to \( D^{-1}FD \), we obtain from (4)

\[ \rho(f) \leq r(\phi(D^{-1}FD)). \tag{8} \]

If \( \sigma \) is an arbitrary matrix norm on \( M_2 \), we obtain from (8)

\[ \rho(f) \leq \sigma(\phi(D^{-1}FD)). \tag{9} \]

Obviously, the upper bound given by (9) cannot be better than the one given by (8). We will take for \( \sigma \) either the column norm \( \sigma_1 \) or the Euclidean norm \( \sigma_2 \) defined, respectively, by

\[ \sigma_1(A) = \max\{|a_{11}| + |a_{21}|, |a_{12}| + |a_{22}|\}, \]

\[ \sigma_2(A) = (|a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2)^{1/2}, \]

where \( A = (a_{ij})_{i,j=1,2} \). Evaluating the right-hand sides of (8) and (9), we obtain

**Proposition 1.** If \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \), and \( k_0, k_1, \ldots, k_{n-2} \) are arbitrary positive numbers, then

\[ \rho(f) \leq \frac{1}{2} \left[ \beta + |a_{n-1}| + \sqrt{(|a_{n-1}| - \beta)^2 + 4\gamma k_{n-2}} \right], \tag{10} \]

\[ \rho(f) \leq \max\{\beta + k_{n-2}, \gamma + |a_{n-1}|\}. \tag{11} \]
\[ \rho(f) \leq \sqrt{\beta^2 + \gamma^2 + k_{n-2}^2 + |a_{n-1}|^2}, \]  

(12)

where \( \beta \) and \( \gamma \) are given by (6) and (7), respectively.

**Remark 1.** Since inequalities (11) and (12) have been obtained by applying a matrix norm to \( \phi(D^{-1}FD) \), they cannot give better bounds than (10), of which the right-hand side is the spectral radius of \( \phi(D^{-1}FD) \). We will see in Example 1 that even inequality (11) can yield a better upper bound for \( \rho(f) \) than the following known inequality [6]:

\[ \rho(f) \leq \max \left\{ \left| a_0 \right|, \left| k_0 + a_1 \right|, \left| k_1 + a_2 \right|, \ldots, \left| k_{n-3} + a_{n-2} \right|, \left| k_{n-2} + a_{n-1} \right| \right\}. \]

(13)

Incidentally, inequality (13) can be obtained by taking the row norm of the matrix \( D^{-1}FD \).

Taking \( k_0 = k_1 = \cdots = k_{n-2} = 1 \), we obtain

**Corollary 1.** If \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \), then

\[ \rho(f) \leq \frac{1}{2} \left[ 1 + |a_{n-1}| + \sqrt{(a_{n-1} - 1)^2 + 4M} \right], \]

(14)

\[ \rho(f) \leq \max\{2, |a_0| + |a_{n-1}|, |a_1| + |a_{n-1}|, \ldots, |a_{n-2}| + |a_{n-1}|\}. \]

(15)

\[ \rho(f) \leq \sqrt{2 + M^2 + |a_{n-1}|^2}, \]

(16)

where \( M = \max\{|a_0|, |a_1|, \ldots, |a_{n-2}|\} \).

**Remark 2.** Inequality (14) has been proved in a different way in [3].

**Remark 3.** Even the upper bound given by (15) can be better than Cauchy's [4]:

\[ \rho(f) \leq \max\{|a_0|, 1 + |a_1|, 1 + |a_2|, \ldots, 1 + |a_{n-1}|\}. \]

(17)

The latter is obtained from (13) by taking \( k_j = 1 \) \((j = 0, 1, \ldots, n - 2)\).

**Example 1.** Consider \( f(z) = z^3 - 0.5z^2 - 2z - 2 \). Inequality (17) gives \( \rho(f) \leq 3 \), while (14), (15), and (16) give the upper bounds 2.19, 2.5, and 2.5, respectively.

Assuming that \( a_j \neq 0 \) for all \( j = 0, 1, \ldots, n - 1 \), and taking \( k_j = a_{j+1} \) \((j = 0, 1, \ldots, n - 2)\) in Proposition 1, we obtain

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COROLLARY 2. If \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) with \( a_j \neq 0 \) \((j = 0, 1, \ldots, n - 1)\), then

\[
\rho(f) \leq \frac{1}{2} \beta' + |a_{n-1}| + \sqrt{(|a_{n-1}| - \beta')^2 + 4\gamma'|a_{n-1}|},
\]

\[
\rho(f) \leq |a_{n-1}| + \max \left\{ \left| \frac{a_0}{a_1} \right|, \left| \frac{a_1}{a_2} \right|, \ldots, \left| \frac{a_{n-2}}{a_{n-1}} \right| \right\},
\]

\[
\rho(f) \leq \sqrt{2|a_{n-1}|^2 + \beta'^2 + \gamma'^2},
\]

where

\[
\beta' = \max \left\{ \left| \frac{a_1}{a_2} \right|, \left| \frac{a_2}{a_3} \right|, \ldots, \left| \frac{a_{n-2}}{a_{n-1}} \right| \right\},
\]

\[
\gamma' = \max \left\{ \left| \frac{a_0}{a_1} \right|, \left| \frac{a_1}{a_2} \right|, \ldots, \left| \frac{a_{n-2}}{a_{n-1}} \right| \right\}.
\]

Remark 4. Corollary 2 can be modified for the case when some of the \( a_j \)'s are equal to zero. Then, in Proposition 1, one should take \( k_j = a_{j+1} \) if \( a_{j+1} \neq 0 \), and \( k_j = 1 \) if \( a_{j+1} = 0 \).

Remark 5. If \( \beta' = \gamma' \), then the right-hand sides of (18) and (19) coincide.

Remark 6. Inequality (19) can yield a better upper bound for \( \rho(f) \) than Kojima's [4]:

\[
\rho(f) \leq \max \left\{ \left| \frac{a_0}{a_1} \right|, 2 \left| \frac{a_1}{a_2} \right|, 2 \left| \frac{a_2}{a_3} \right|, \ldots, 2 \left| \frac{a_{n-2}}{a_{n-1}} \right|, 2|a_{n-1}| \right\}.
\]

The latter is obtained from (13) by taking \( k_j = |a_{j+1}| \) \((j = 0, 1, \ldots, n - 2)\).

Example 2. Consider \( f(z) = z^4 - 4z^3 + 3z^2 + 2z - 1 \). Inequality (21) gives \( \rho(f) \leq 8 \), while both (18) and (19) give \( \rho(f) \leq 4.75 \).

Let \( t \) be an arbitrary positive number and in Proposition 1 take \( k_j = t^{n-j-1} \) \((j = 0, 1, \ldots, n - 2)\). We obtain

PROPOSITION 2. If \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \), and \( t \) is an arbitrary positive number, then
\[ \rho(f) \leq \frac{1}{2} [t + |a_{n-1}| + \sqrt{(|a_{n-1}| - t)^2 + 4dt}], \]  
\[ \rho(f) \leq \max \left\{ 2t, \frac{|a_0|}{t^{n-1}} + a_{n-1}, \frac{|a_1|}{t^{n-2}} + a_{n-1}, \ldots, \frac{|a_{n-2}|}{t} + a_{n-1} \right\}, \]  
\[ \rho(f) \leq \sqrt{2t^2 + \delta^2 + |a_{n-1}|^2}, \]  

where

\[ \delta = \max \left\{ \frac{|a_0|}{t^{n-1}}, \frac{|a_1|}{t^{n-2}}, \ldots, \frac{|a_{n-2}|}{t}, |a_{n-1}| \right\}. \]

**Remark 7.** Inequality (23) can yield a better upper bound for \( \rho(f) \) than Wilf’s [6]:

\[ \rho(f) \leq \max \left\{ \frac{|a_0|}{t^{n-1}}, t + \frac{|a_1|}{t^{n-2}}, t + \frac{|a_2|}{t^{n-3}}, \ldots, t + \frac{|a_{n-2}|}{t}, t + |a_{n-1}| \right\}. \]

The latter is obtained from (13) by taking \( k_j = t^{n-j-1} \) (\( j = 0, 1, \ldots, n - 2 \)).

**Example 3.** Consider \( f(z) = z^4 + 2z^3 - 13z^2 - 38z - 24 \). Inequality (25) gives \( \rho(f) \leq 7.34 \), while (22) and (23) give \( \rho(f) \leq 6.14 \) and \( \rho(f) \leq 6.34 \), respectively.

Assuming that \( a_{n-1} \neq 0 \) and, in Proposition 2, taking \( t = |a_{n-1}| \), we obtain

**Corollary 3.** If \( f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) with \( a_{n-1} \neq 0 \), then

\[ \rho(f) \leq |a_{n-1}| + (\delta')^{1/2}, \]  
\[ \rho(f) \leq |a_{n-1}| + \max \left\{ |a_{n-1}|, \frac{\delta'}{|a_{n-1}|} \right\}, \]  
\[ \rho(f) \leq \sqrt{3|a_{n-1}|^2 + \delta'^2|a_{n-1}|^2}, \]  

where

\[ \delta' = \max \left\{ \frac{|a_0|}{|a_{n-1}|^{n-2}}, \frac{|a_1|}{|a_{n-1}|^{n-3}}, \ldots, \frac{|a_{n-2}|}{|a_{n-1}|}, \frac{|a_{n-1}|}{a_{n-2}} \right\}. \]
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Denote
\[ N = \max \{ |a_0|^{1/n}, |a_1|^{1/(n-1)}, \ldots, |a_{n-1}|^{1/(n-2)}, |a_n^{1/2}| \}. \] (29)

Then \(|a_j| \leq N^{n-j}\) \((j = 0, 1, \ldots, n - 2)\) with equality for at least one \(j\).

Taking, in Proposition 2, \(t = N\), we obtain

**Corollary 4.** If \(f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0\), then
\[
\rho(f) \leq \frac{1}{2} [N + |a_{n-1}| + \sqrt{(|a_{n-1}| - N)^2 + 4N^2}],
\] (30)
\[
\rho(f) \leq N + \max \{N, |a_{n-1}|\},
\] (31)
\[
\rho(f) \leq \sqrt{3N^2 + |a_{n-1}|^2},
\] (32)

where \(N\) is given by (29).

**Remark 8.** From inequality (31) we obtain Fujiwara's upper bound [4]:
\[
\rho(f) \leq 2 \max_{j=1, \ldots, n} |a_{n-j}|^{1/j},
\]
which is weaker than both (30) and (31) (see Remark 1).

**References**


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