Discrete Applied Mathematics 161 (2013) 369–376

Contents lists available at SciVerse ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Harmonious chromatic number of directed graphs

Keith J. Edwards *

School of Computing, University of Dundee, Dundee, DD1 4HN, UK

ARTICLE INFO

Article history:
Received 1 March 2012
Received in revised form 3 September 2012
Accepted 5 September 2012
Available online 25 September 2012

Keywords:
Harmonious colouring
Complete colouring
Directed graph

ABSTRACT

We consider the extension to directed graphs of the concepts of harmonious colouring and complete colouring. We give an upper bound for the harmonious chromatic number of a general directed graph, and show that determining the exact value of the harmonious chromatic number is NP-hard for directed graphs of bounded degree (in fact graphs with maximum indegree and outdegree 2); the complexity of the corresponding undirected case is not known. We also consider complete colourings, and show that in the directed case the existence of a complete colouring is NP-complete. We also show that the interpolation property for complete colourings fails in the directed case.

1. Introduction

A *harmonious colouring* of a simple undirected graph G is a proper vertex colouring such that each pair of colours appears together on at *most* one edge. The *harmonious chromatic number* $h(G)$ is the least number of colours in such a colouring.

Similarly, *complete colouring* of a simple undirected graph G is a proper vertex colouring such that each pair of colours appears together on at *least* one edge. The *achromatic number* $\psi(G)$ is the greatest number of colours in such a colouring.

These concepts were introduced by Hopcroft and Krishnamoorthy [11] and Harary et al. [8] respectively. For surveys of the harmonious chromatic number see [4,13], for the achromatic number see [4,12].

In this paper we consider the extension of these concepts to directed graphs. The directed harmonious chromatic number has previously been considered by Hegde and Priya Castelino [10,9], who gave results on colourings of directed and alternately oriented paths and cycles.

We first give an upper bound for the harmonious chromatic number of a general directed graph. We then show that determining the exact value of the harmonious chromatic number is NP-hard for directed graphs of bounded degree (in fact graphs with maximum indegree and outdegree 2). Although it is known that determining the harmonious chromatic number of an undirected graph is NP-complete for several restricted classes of graphs such as interval and permutation graphs [1,2] and trees [6], the case of undirected graphs of bounded degree is open.

Next we consider complete colourings, and show that whereas in the undirected case the existence of a complete colouring is automatic (since any vertex colouring with the minimum possible number of colours is complete), in the directed case this is NP-complete. We also show that a key property of complete colourings of undirected graphs fails in the directed case.

2. Notation and terminology

In this section we give formal definitions of the terms harmonious colouring, exact colouring and detachment, and introduce some further notation and terminology.

* Fax: +44 1382 385509.
E-mail address: kjedwards@dundee.ac.uk.

0166-218X/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2012.09.003

A vertex w is an in-neighbour (resp. out-neighbour) of v if $w \to v$ (resp. $v \to w$) is an edge of the graph. The indegree (resp. outdegree) of v is the number of in-neighbours (resp. out-neighbours) that it has.

Harmonious colouring

A harmonious (resp. exact) colouring is a function f from the set $V(G)$ of vertices of G to a colour set C such that (i) for any edge $x \to y$ of G, $f(x) \neq f(y)$, and (ii) for each ordered pair (c, c') of distinct colours, there is at most one (resp. exactly one) edge $x \to y$ such that x has colour c and y has colour c'. The *harmonious chromatic number* $h(G)$ is the *least* number of colours in a harmonious colouring of G.

Definition of $Q(m)$

Let m be a positive real number. Then, as in [4] we define $Q(m)$ to be the least integer k such that $\binom{k}{2} \geq m$.

Then

$$Q(m) = \left\lceil \frac{1 + \sqrt{8m+1}}{2} \right\rceil.$$

By comparing the number of directed edges of G with the number of colour pairs, it follows immediately that for any graph G with m edges, $h(G) \geq Q(m/2)$.

Detachment and amalgamation

If X, Y are disjoint sets of vertices in a directed graph, let $E(X \to Y)$ be the set of edges from a vertex in X to a vertex in Y.

Let G be a directed graph with vertex set $v_1, \ldots, v_n$. Then a directed graph H is a *detachment* of G if the vertex set $V(H)$ can be partitioned into n sets $V_1, \ldots, V_n$ such that (i) for each i, V_i is an independent set, and (ii) for each i, j, $i \neq j$,

$$\left|E(V_i \to V_j)\right| = \begin{cases} 1 & \text{if } v_i \to v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

In the case that G is the complete (bi-)directed graph K_n, condition (ii) simply becomes $\left|E(V_i \to V_j)\right| = 1$ for each i, j, $i \neq j$.

We say that G is an amalgamation of H if and only if H is a detachment of G.

Remarks. (1) It is immediate that any directed graph has a harmonious colouring, since we can simply give each vertex a distinct colour. Thus the existence of the harmonious chromatic number is also immediate.

(2) A directed graph H has an exact n-colouring if and only if it is a detachment of the complete bidirected graph K_n; the colour classes correspond to the vertex sets $V_1, \ldots, V_n$.

Partial harmonious colouring

A partial harmonious colouring of a directed graph G is a colouring of a subset of the vertices of G which is a harmonious colouring of the graph induced by the coloured vertices, and is such that no uncoloured vertex has two or more coloured in-neighbours or out-neighbours with the same colour.

We will say that an ordered pair (s, s') of colours *occurs on* or is *used on* an edge $e = x \to y$ if x has colour s and y has colour s'.

We say that an edge is coloured if both of its endpoints are coloured, otherwise we say it is uncoloured.

Degree sums

Suppose that we have a (possibly partial) vertex colouring of a directed graph G, and let S_c be the set of vertices coloured with some colour c. Then we define the indegree (resp. outdegree) sum of c, denoted $ds^-(c)$ (resp. $ds^+(c)$), to be $\sum_{v \in S_c} d^-(v)$ (resp. $\sum_{v \in S_c} d^+(v)$).

Bidegree and bidegree sum

We will refer to the pair $(d^-(v), d^+(v))$ as the bidegree of v, denoted $b(v)$. Similarly the pair $(ds^-(c), ds^+(c))$ is the bidegree sum of the colour c.

3. Upper bound

We first give a general upper bound for the harmonious chromatic number in terms of the number of edges and the maximum degree. The main idea, as in [5], is to colour the vertices sequentially while controlling the indegree and outdegree sums of the colours. However to avoid double counting we instead control the number of vertices of a given colour which have a particular indegree and outdegree.

Denote by $\Delta(G)$ and $\delta(G)$ respectively the maximum and minimum values, over the vertices v of G, of $d^-(v) + d^+(v)$. Define $\hat{\delta}(G)$ to be the maximum value, over all induced subgraphs G' of G, of $\delta(G')$.

Theorem 3.1. *Let G be a directed graph with m edges, and let $\Delta = \Delta(G)$ and $\hat{\delta} = \hat{\delta}(G)$. Then*

$$h(G) \leq 2\sqrt{\hat{\delta} m} + \frac{\hat{\delta}\Delta(\Delta+1)^2}{2} + \Delta(\hat{\delta} - 1) - \hat{\delta}(\hat{\delta} - 2) + 1.$$

Proof. Order the vertices of G as $v_1, \ldots, v_n$ so that for each j, there are at most $\hat{\delta}$ edges (of either orientation) joining v_j to the vertices $v_1, \ldots, v_{j-1}$. Partition the vertex set $V(G)$ into subsets R_{ij} with common indegree i and outdegree j, i.e.

$$R_{ij} = \{v \in V(G) \mid d^-(v) = i, d^+(v) = j\}.$$

Note that the total indegree and outdegree sums are both m. Also note that

$$\sum_{v \in R_{ij}} d^-(v) = i|R_{ij}|$$
$$\sum_{v \in R_{ij}} d^+(v) = j|R_{ij}|.$$

Let $T = \sqrt{m/\hat{\delta}}$, and define integers n_{ij} by

$$n_{ij} = \left\lceil \frac{T|R_{ij}|}{m} \right\rceil.$$

Note that $n_{ij} \geq 1$ provided $R_{ij} \neq \emptyset$. Also let $P = T + \frac{\Delta(\Delta+1)^2}{2}$.

We will colour the graph so that the number of vertices in R_{ij} with a given colour c is at most n_{ij}. Then the total indegree of the vertices coloured c, $ds^-(c)$, satisfies

$$\begin{aligned} ds^-(c) &\leq \sum_{i=0}^{\Delta} \sum_{j=0}^{\Delta} i n_{ij} \\ &= \sum_{i=0}^{\Delta} \sum_{j=0}^{\Delta} i \left\lceil \frac{T|R_{ij}|}{m} \right\rceil \\ &< \sum_{i=0}^{\Delta} \sum_{j=0}^{\Delta} i \left(\frac{T|R_{ij}|}{m} + 1 \right) \\ &= \frac{T}{m} \sum_{i=0}^{\Delta} \sum_{j=0}^{\Delta} i|R_{ij}| + \sum_{i=0}^{\Delta} \sum_{j=0}^{\Delta} i \\ &= T + \frac{\Delta(\Delta+1)^2}{2} \\ &= P, \end{aligned}$$

and similarly for the total outdegree $ds^+(c)$.

Now let $\lambda = 2\sqrt{\hat{\delta} m} + \frac{\hat{\delta}\Delta(\Delta+1)^2}{2} + \Delta(\hat{\delta} - 1) - \hat{\delta}(\hat{\delta} - 2) + 1$. Suppose that we have coloured vertices $v_1, \ldots, v_t$ using at most λ colours to give a partial harmonious colouring of G, with the property that the number of vertices in R_{ij} with a given colour c is at most n_{ij}. We now show that we can extend this colouring to the vertex v_{t+1}. To do this we bound the number of colours which cannot be used on v_{t+1}. Let A^+ and A^- be the sets of out-neighbours and in-neighbours of v_{t+1} among $v_1, \ldots, v_t$, and let $x = |A^+| + |A^-|$, so $x \leq \hat{\delta}$. Then the colours which cannot be used for v_{t+1} are as follows:

1. We cannot use any of the colours used on $A^+ \cup A^-$; there are at most x of these.
2. If colour c is used on a vertex in A^+, then we need to exclude any colour used on an in-neighbour of some vertex coloured c, similarly if c is used on a vertex in A^-, then we need to exclude any colour used on an out-neighbour of some vertex coloured c. There are at most x possible such colours c, and for each c we need to exclude $ds^-(c)$ or $ds^+(c)$ colours, i.e. less than P colours for each c. Thus less than xP colours are excluded in this way.
3. If w is an uncoloured in-neighbour of v_{t+1}, we must exclude any colour used on an out-neighbour of w, and similarly for colours used on an in-neighbour of an uncoloured out-neighbour of v_{t+1}. The sum of the number of uncoloured in- and out-neighbours of v_{t+1} is $\Delta - x$, and each has at most $\hat{\delta}$ out- or in-neighbours among $v_1, \ldots, v_{t+1}$, one of which is v_{t+1}, hence we must exclude at most $(\Delta - x)(\hat{\delta} - 1)$ colours this way.
4. If $v_{t+1} \in R_{ij}$, we must exclude a colour c if colouring vertex v_{t+1} with colour c would result in more than n_{ij} vertices in R_{ij} having colour c. This can only happen if there are already at least n_{ij} vertices in R_{ij} with colour c. But the number of such colours c can be at most $|R_{ij}|/n_{ij} \leq m/T$. So we exclude at most m/T colours this way.

Thus the total number U of excluded colours satisfies

$$U < x + xP + (\Delta - x)(\hat{\delta} - 1) + \frac{m}{T}.$$

Note that the coefficient of x is $1 + P - \hat{\delta} + 1$, and we have

$$1 + P - \hat{\delta} + 1 \geq P - \Delta + 2 \geq 0$$

since $\Delta \geq 1$, hence we can replace x by $\hat{\delta}$ to obtain

$$\begin{aligned} U &< \hat{\delta}P + \Delta(\hat{\delta} - 1) - \hat{\delta}(\hat{\delta} - 2) + \frac{m}{T} \\ &= \hat{\delta}\left(T + \frac{\Delta(\Delta + 1)^2}{2}\right) + \frac{m}{T} + \Delta(\hat{\delta} - 1) - \hat{\delta}(\hat{\delta} - 2). \end{aligned}$$

Substituting $T = \sqrt{m/\hat{\delta}}$, we obtain

$$U < 2\sqrt{\hat{\delta}m} + \hat{\delta}\frac{\Delta(\Delta + 1)^2}{2} + \Delta(\hat{\delta} - 1) - \hat{\delta}(\hat{\delta} - 2) = \lambda - 1.$$

Hence there is at least one colour available to extend the colouring to v_{t+1}. Continuing in this way, we obtain $h(G) \leq \lambda$. □

Corollary 3.2. *Let G be a directed graph with m edges and maximum degree Δ. Then*

$$h(G) \leq 2\sqrt{\Delta m} + \frac{1}{2}\Delta(\Delta + 2)(\Delta^2 + 1) + 1.$$

Proof. This follows from Theorem 3.1 by noting that $\hat{\delta}(G) \leq \Delta(G)$. □

Corollary 3.3. *Let T be an oriented tree with m edges and maximum degree Δ. Then*

$$h(G) \leq 2\sqrt{m} + \frac{1}{2}\Delta(\Delta^2 + 2\Delta + 3) + 1.$$

Proof. For an oriented tree, we have $\hat{\delta} = 1$, and the result follows from Theorem 3.1. □

There are two easy lower bounds for $h(G)$, namely $Q(m/2)(\approx \sqrt{m})$ and $1 + \max\{\Delta^-, \Delta^+\}$ (since a vertex and its out- or in-neighbours must have distinct colours). We might hope therefore for an upper bound of the form $A\sqrt{m} + B\Delta$, where A, B are (small) constants. However this is not possible as the following example shows.

Lemma 3.4. *For any constants A, B, there exist graphs G with $m(= |E(G)|)$ arbitrarily large, such that $h(G) > A\sqrt{m} + B\Delta$.*

Proof. Consider a projective plane of order p, with set P of points and set L of lines. Construct a directed graph G with vertex set $P \cup L$, and an edge from vertex $v \in P$ to vertex $\ell \in L$ if and only if the point v lies on the line ℓ in the projective plane. Thus $|P| = |L| = p^2 + p + 1$, and for $v \in P$, $d^+(v) = p + 1$, $d^-(v) = 0$, while for $\ell \in L$, $d^+(\ell) = 0$, $d^-(\ell) = p + 1$. Thus $\Delta(G) = \hat{\delta}(G) = p + 1$.

Note that in any harmonious colouring of G, no two vertices v, w of P can have the same colour, since otherwise, taking ℓ to be the unique line containing v and w, the same ordered pair of colours would occur on the two edges $v \to \ell$ and $w \to \ell$. Hence $h(G) \geq |P|$, so $h(G) \geq p^2 + 2p + 1$.

On the other hand, $m = (p + 1)(p^2 + p + 1)$ so that $A\sqrt{m} + B\Delta$ is $O(p^{3/2})$. □

4. Bounded degree graphs

We now show that determining the harmonious chromatic number of a directed graph is NP-hard, even when the graph has maximum indegree and outdegree at most 2. The complexity of the problem for bounded degree undirected graphs is not known.

DIRECTED HARMONIOUS CHROMATIC NUMBER

Instance. Directed graph G with m edges, positive integer k.
Question. Is $h(G) \leq k$?

Theorem 4.1. *The problem DIRECTED HARMONIOUS CHROMATIC NUMBER is NP-complete, even for directed graphs with maximum indegree and outdegree at most 2.*

Proof. We reduce from the strongly NP-complete problem BIN PACKING [7]:

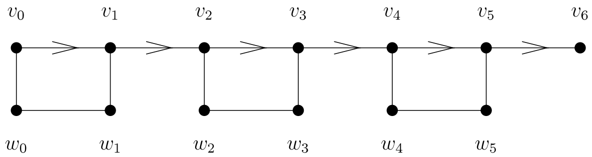
BIN PACKING

Instance. Finite set U of items, a size $s(u) \in Z^+$ for each $u \in U$, a positive integer bin capacity B, and a positive integer K.
Question. Is there a partition of U into disjoint sets $U_1, \ldots, U_K$ such that for $1 \leq i \leq K$, $\sum_{u \in U_i} s(u) \leq B$?

Let Π be an instance of BIN PACKING, with set U, integers B and K, and sizes $s(u)$, $u \in U$. First note that we can assume that B and each $s(u)$ are even, for otherwise we could double them all. We can also assume that $\sum_{u \in U} s(u) = KB$, for otherwise the problem is trivial.

We construct an instance of DIRECTED HARMONIOUS CHROMATIC NUMBER. The directed graph G has a number of components, as follows. (1) There are K components $O_1, \ldots, O_K$ which are oriented cycles of length B. (2) For each $u \in U$, there is a component G_u with $2s(u) + 1$ vertices $v_0, \ldots, v_{s(u)}, w_0, \ldots, w_{s(u)-1}$. There is a directed path on $v_0, \ldots, v_{s(u)}$, i.e. for each $i = 0, \ldots, s(u) - 1$, there is an edge $v_i \to v_{i+1}$. Also for each $i = 0, \ldots, \frac{1}{2}s(u) - 1$, there is a bidirected path $v_{2i} \leftrightarrow w_{2i} \leftrightarrow w_{2i+1} \leftrightarrow v_{2i+1}$. Thus G_u has $4s(u)$ directed edges. (3) There is a bidirected cycle C of length $\binom{KB}{2} - \frac{5}{2}KB$.

The graph G_u, where $s(u) = 6$, is shown below. (An undirected edge represents two directed edges, one in each direction.)



Note that G has exactly $KB(KB - 1)$ directed edges, so that $m = KB(KB - 1)$ and $Q(m/2) = KB$. We set the constant k equal to KB. Then we know that $h(G) \geq KB$, so it suffices to show that Π has a solution if and only if G has a harmonious colouring with KB colours, i.e. an exact colouring.

Now consider the bidegrees of the vertices of G. There are KB vertices of bidegree $(1, 1)$ in the directed cycles O_i. In G_u every vertex has bidegree $(2, 2)$, except that the path-start vertices v_0 have $b(v_0) = (1, 2)$ and the path-end vertices have $b(v_{s(u)}) = (1, 0)$. Finally in the bidirected cycle every vertex has bidegree $(2, 2)$.

First suppose that G has an exact colouring with KB colours. Then every colour must have bidegree sum $(KB - 1, KB - 1)$, where $KB - 1$ is odd. Thus every colour must be used (1) on an equal number of path-starts and path-ends (otherwise the indegree sum and outdegree sum will differ); and (2) on exactly one vertex from the directed cycles (otherwise the degree sums will be even).

Therefore the KB vertices in the directed cycles all have distinct colours.

Now every edge of a directed cycle uses an ordered colour pair (c, c') say. Hence there must also be a unique directed edge using the ordered pair (c', c), and this can only be in a component G_u. Since the total length of the directed cycles equals the total length of the directed paths, there is a one-to-one pairing between the directed cycle edges and the directed path edges. Furthermore it easy to see that for each u, all the directed edges of G_u are paired with edges of the same directed cycle (otherwise the same colour would have to occur on two distinct directed cycles). Thus each component G_u is associated with a unique directed cycle, and this gives a partition $U_1, \ldots, U_K$ of U such that every directed edge in G_u, $u \in U_i$ is paired with an edge of O_i. Thus $\sum_{u \in U_i} s(u) = B$, and the instance of BIN PACKING has a solution.

Conversely, if the instance of BIN PACKING has a solution, then we can identify path-start and path-end vertices of the components G_u to form K graphs $J_1, \ldots, J_K$ consisting of a directed cycle of length B with pendant undirected paths of length 3. Identify the vertices of J_i with the vertices of O_i to form K bidirected graphs of even degree. It is not hard to see that these graphs, together with the bidirected cycle, can be amalgamated to form the complete bidirected graph K_{KB}. Colouring this with the KB distinct colours and then reversing the vertex identifications gives the required colouring. □

As a corollary, we obtain the following result for the detachment problem:
DIRECTED DETACHMENT

Instance. Directed graphs D_1, D_2.
Question. Is D_2 a detachment of D_1?

Theorem 4.2. *The problem DIRECTED DETACHMENT is NP-complete, even when G is complete and H has maximum indegree and outdegree at most* 2.

Proof. This is almost immediate. We see from the above that DIRECTED HARMONIOUS CHROMATIC NUMBER is NP-complete when restricted to an instance where G has maximum indegree and outdegree at most 2 and $k = Q(m/2)$. Given such an instance, take $D_2 = G$ and D_1 to be the complete bidirected graph on $Q(m/2)$ vertices. Then D_2 is a detachment of D_1 if and only if $h(G) = Q(m/2)$, and the result follows. □

5. Complete colourings

Analogous to harmonious colouring and harmonious chromatic number are the (older) concepts of complete colouring and achromatic number [8]. For an undirected graph G, a complete colouring is a function f from the set $V(G)$ of vertices of G to a colour set C such that for any edge (x, y) of G, $f(x) \neq f(y)$, and for each pair of distinct colours c, c', there is at least one edge (x, y) such that x has colour c and y has colour c'. The *achromatic number* $\psi(G)$ is the *greatest* number of colours in a complete colouring of G.

These definitions readily extend to directed graphs: we define a complete colouring to be a function f from the set $V(G)$ of vertices of G to a colour set C such that (i) for any edge $x \to y$ of G, $f(x) \neq f(y)$, and (ii) for each ordered pair (c, c') of distinct colours, there is at least one edge $x \to y$ such that x has colour c and y has colour c'.

For an undirected graph the existence of some complete colouring is almost immediate, since any colouring with the minimum possible number $\chi(G)$ of colours is automatically a complete colouring (otherwise we could combine two colour classes to obtain a colouring with fewer colours).

However a directed graph is not guaranteed to have any complete colouring (the simplest example is a single directed edge). In fact determining whether one exists is an NP-complete problem.

We define the problem COMPLETE COLOURING OF ORIENTED GRAPH as follows:
COMPLETE COLOURING OF ORIENTED GRAPH

Instance. Oriented graph D.
Question. Does D have a complete colouring?

Theorem 5.1. *The problem COMPLETE COLOURING OF ORIENTED GRAPH is NP-complete.*

Proof. We reduce from the strongly NP-complete problem 3-PARTITION [7]:
3-*PARTITION*

Instance. Set A of $3m$ elements, a bound $B \in Z^+$, and a size $s(a) \in Z^+$ for each $a \in A$ such that $B/4 < s(a) < B/2$, and such that $\sum_{a \in A} s(a) = mB$.
Question. Can A be partitioned into m disjoint sets $A_1, \ldots, A_m$ such that for $1 \le i \le m$, $\sum_{a \in A_i} s(a) = B$?

Note that necessarily each A_i contains exactly three elements.

Let Π be an instance of 3-PARTITION, with set A, bound B and sizes $s(a)$, $a \in A$. First note that we can assume that $m \le 2B + 1$, for otherwise we can multiply B and each $s(a)$ by (for example) m without changing the set of instances which have a solution.

We construct an instance of COMPLETE COLOURING OF ORIENTED GRAPH. Let $N = 2B + 1$, so that $N \ge m$. Define a directed graph D with $|A| + N - m + 1$ components, as follows: (i) for each $a \in A$, there is an outwardly directed star with $s(a)$ leaves; (ii) there are $N - m$ outwardly stars with B leaves; (iii) there is a graph D_K on N vertices $v_1, \ldots, v_N$ with the following edges: for each i such that $1 \le i \le N$, and $j = 1, \ldots, B$, there is an edge $v_i \to v_{i+j}$ (where the addition is understood cyclically).

Note that since $N = 2B + 1$, every pair of distinct vertices of D_K is joined by one oriented edge. Also note that D has $NB + (N - m)B + mB = N(N - 1)$ edges, and there are no bidirected edges, so D is an oriented graph.

Suppose first that the graph D has a complete colouring. Then every vertex of D_K must have a distinct colour, or the colouring would not be proper. Hence there are at least N colours, and since there are $N(N - 1)$ edges, there must be exactly N colours, with every (ordered) colour pair occurring on exactly one edge. Let the colours be $c_1, \ldots, c_N$, and assume that vertex v_i receives colour c_i.

For each i, let M_i be the set of colours $c \neq c_i$ for which the ordered colour pair (c_i, c) does not occur on D_K (i.e. there is no edge $u \to v$ in D_K, with u, v coloured c_i, c respectively). Then $|M_i| = B$ for each i. The missing colour pairs must occur on the stars, and it is easy to see that the centres of the stars with B leaves must all have distinct colours, and that for each such centre, coloured c_i, the leaves must be coloured with the elements of M_i. For the remaining m colours, which we may assume are c_i, $i = 1, \ldots, m$, there must be exactly three stars with centres coloured c_i, with a total of B leaves coloured with the colours in M_i. This immediately gives a solution to the instance Π of 3-PARTITION.

Conversely, given a solution to Π, it is easy to identify the vertices of the stars with vertices of D_K in such a way as to form a bi-directed complete graph on N vertices, which immediately gives the necessary complete colouring of D. □

Remark 5.1. If a directed graph G has at least $2\Delta(G)^4$ edges, then G has a complete colouring with $\Delta(G) + 1$ colours. We first find sufficient edges which are at distance at least 2 from one another, and use all the colour pairs on these edges, then colour the rest of the graph by a greedy method.

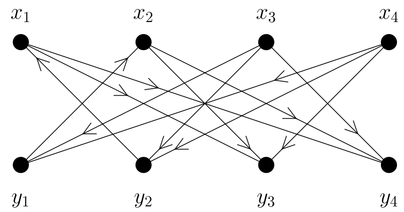
5.1. Non-interpolation

In the case of undirected graphs there is an interpolation result due to Harary et al. concerning the number of colours which can occur in a complete colouring. If an undirected graph G has a complete colouring with r colours, and another with s colours, $r > s$, then it has a complete colouring with t colours whenever $r \le t \le s$ [8]. Thus since $\chi(G)$, $\psi(G)$ respectively are the least and greatest number of colours for which a complete colouring of G exists, then G has a complete colouring with t colours if and only if $\chi(G) \le t \le \psi(G)$.

We show below that this interpolation result also fails in the directed case, by giving a simple example.

Theorem 5.2. *Let $r \ge 4$ be a positive integer. There exists a directed (in fact oriented) graph D_r which has a complete 2-colouring and a complete r-colouring, but no complete t-colouring for any t with $r > t > 2$.*

Proof. Define a bipartite (undirected) graph with vertex sets $X = \{x_1, \ldots, x_r\}$ and $Y = \{y_1, \ldots, y_r\}$, and edges as follows: for each $i, j, i \neq j$, there is an edge (x_i, y_j). Now obtain a directed graph D by orienting all edges from X to Y except for the edges (x_1, y_2), (x_2, y_1), which are oriented from Y to X. Thus D_4 is the graph shown below.



We obtain a complete 2-colouring c_2 of D by setting $c_2(x_i) = 1$, $c_2(y_i) = 2$ for each i. Then c_2 is clearly proper, and the edges $x_1 \to y_3$ and $y_1 \to x_2$ show that it is complete.

We can also obtain a complete r-colouring c_r by setting $c_r(x_i) = c_r(y_i) = i$ for each i. Again this is clearly proper, and for each $i, j, i \neq j$, the four vertices x_i, x_j, y_i, y_j induce two edges which use the ordered pairs (i, j) and (j, i), so c_r is complete.

However D has no complete t-colouring if $r > t > 2$. To see this, suppose that c_t is a complete t-colouring. Note (i) there cannot be two colours i, j which occur only on X or only on Y, for then the ordered pairs (i, j), (j, i) are not used on any edge; (ii) some colour does not occur on both X and Y, for if all colours occur on both sides, then (since $t < r$) some colour occurs twice on X and at least once on Y, and the colouring is not proper.

So suppose that colour 1 occurs only on X. Since the only vertices of X which have positive indegrees are x_1 and x_2, then it is clear that no complete t-colouring can exist if $t > 3$. So suppose $t = 3$, then since $d^-(x_1) = d^-(x_2) = 1$, we must have $c_3(x_1) = c_3(x_2) = 1$ and $\{c_3(y_1), c_3(y_2)\} = \{2, 3\}$, otherwise the colour pairs (2, 1), (3, 1) cannot both occur. But by (i), at least one of the colours 2 or 3 occurs on X, so the colouring is not proper. The case when some colour occurs only on Y is similar. $\square$

Corollary 5.3. *Let r, s be positive integers with $r \geq 2$ and $s \geq r + 2$. There exists a directed graph $D_{r,s}$ which has a complete r-colouring and a complete s-colouring, but no complete t-colouring for any t with $s > t > r$.*

Proof. By Theorem 5.2, the graph D_{s-r+2} has complete k-colourings only when $k = 2$ or $k = s - r + 2$. Take a complete bidirected graph K_{r-2} and join (in both directions) every vertex of K_{r-2} to every vertex of D_{s-r+2} to form $D_{r,s}$. Then it is clear that $D_{r,s}$ has a complete k-colouring only when $k = r$ or $k = s$. $\square$

Remark 5.2. For undirected graphs H and G, where H is an induced subgraph of G, we have $\psi(H) \leq \psi(G)$. However, for directed graphs, it is easy to find examples to show that any of the following can hold: (i) $\psi(H) > \psi(G)$; and (ii) H has some complete colouring but G does not; G has some complete colouring but H does not.

Remark 5.3. A pseudocomplete colouring [3] is like a complete colouring but without the requirement that the colouring be proper, and the pseudoachromatic number $\psi_s(G)$ of a graph G is the largest number of colours in a pseudocomplete colouring of G. It is easy to see that pseudocomplete colourings exist for any directed graph and satisfy the interpolation property, and that the pseudoachromatic number is monotonic with respect to subgraph inclusion (not merely induced subgraphs).

6. Conclusions and open problems

There is a considerable gap between the upper bound on the harmonious chromatic number of a general directed graph given in Theorem 3.1 and the known lower bounds. It is natural to ask if the upper bound could be improved, particularly the leading term $2\sqrt{\hat{\delta}m}$. The example of Lemma 3.4 shows that the coefficient of $\sqrt{\hat{\delta}m}$ could not be reduced below 2 in general, however it seems possible that the factor of 2 could be removed by an improved technique.

As already mentioned, the complexity of determining the harmonious chromatic number of undirected graphs of bounded degree is unknown, and the corresponding problem for oriented graphs is also open (note that the proof of Theorem 4.1 relies on the fact that most edges are bidirected). It is also not known if it is NP-hard to determine the harmonious chromatic number of a regular undirected graph, although this seems very likely.

References

[1] K. Asdre, K. Ioannidou, S.D. Nikolopoulos, The harmonious coloring problem is NP-complete for interval and permutation graphs, Discrete Appl. Math. 155 (2007) 2377–2382.
[2] K. Asdre, S.D. Nikolopoulos, NP-completeness results for some problems on subclasses of bipartite and chordal graphs, Theoret. Comput. Sci. 381 (2007) 248–259.
[3] V.N. Bhave, On the pseudoachromatic number of a graph, Fund. Math. 102 (1979) 159–164.
[4] K.J. Edwards, The harmonious chromatic number and the achromatic number, in: R.A. Bailey (Ed.), Surveys in Combinatorics 1997 (Invited papers for 16th British Combinatorial Conference), Cambridge University Press, Cambridge, 1997, pp. 13–47.
[5] K.J. Edwards, C.J.H. McDiarmid, New upper bounds on harmonious colorings, J. Graph Theory 18 (1994) 257–267.

[6] K.J. Edwards, C.J.H. McDiarmid, The complexity of harmonious colouring for trees, Discrete Appl. Math. 57 (1995) 133–144.
[7] M.R. Garey, D.S. Johnson, Computers and Intractability, Freeman, 1980.
[8] F. Harary, S.T. Hedetniemi, G. Prins, An interpolation theorem for graphical homomorphisms, Port. Math. 26 (1967) 453–462.
[9] S.M. Hegde, L. Priya Castelino, Harmonious colorings of digraphs, Ars Combin. (in press).
[10] S.M. Hegde, L. Priya Castelino, Further results on harmonious coloring of digraphs, AKCE Int. J. Graphs Combin. 8 (2011) 151–159.
[11] J.E. Hopcroft, M.S. Krishnamoorthy, On the harmonious coloring of graphs, SIAM J. Algebraic Discrete Methods 4 (1983) 306–311.
[12] F. Hughes, G. MacGillivray, The achromatic number of graphs: a survey and some new results, Bull. Inst. Combin. Appl. 19 (1997) 27–56.
[13] B.J. Wilson, Line-distinguishing and harmonious colourings, in: R. Nelson, R.J. Wilson (Eds.), Graph Colourings (Proceedings of a Conference on Graph Colourings, Milton Keynes, 1988), in: Pitman Research Notes in Mathematics, vol. 218, Longman Scientific & Technical, Essex, 1990, pp. 115–133.