JOURNAL OF APPROXIMATION THEORY 25, 50–55 (1979)

Approximation of Functions with a Positive Derivative

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Communicated by Richard S. Varga

Received June 24, 1977

1. INTRODUCTION

Let a real function f be continuous on $[-1, 1]$ ($f \in C[-1, 1]$). For each nonnegative integer n, let Π_n *denote the set of algebraic polynomials with real coefficients of degree n or less*. The *degree of approximation for f* is defined by

$$E_n(f) = \inf\{\|f - p\|; p \in \pi_n\},$$

where $\|\cdot\|$ denotes the uniform norm on $[-1, 1]$. Given f as above and an integer $n \geqslant 0$, let p_n be the *unique polynomial in Π_n of best approximation to f on* $[-1, 1]$. That is

$$\|f - p_n\| = E_n(f).$$

The problem we study is the following:

Under what condition on a function $f \in C[-1, 1]$ for which there is $\delta > 0$ such that

$$(f(x_2) - f(x_1))/(x_2 - x_1) \geqslant \delta \quad \text{for all } x_1, x_2 \in [-1, 1] \text{ with } x_1 \neq x_2, \tag{1.1}$$

is it true that for n sufficiently large, p_n is increasing on $[-1, 1]$?

Roulier [6] has shown that $f \in C^2[-1, 1]$ is such a condition. See also Roulier [5]. Furthermore, in [7] Roulier has shown that *for any given modulus of continuity ω there is an $f \in C[-1, 1]$ satisfying* (1.1) *and a constant $K > 0$ so that*

$$\omega(h) \leqslant \omega(f, h) \leqslant K\omega(h) \qquad \text{for} \quad 0 \leqslant h \leqslant 1 \tag{1.2}$$

and yet infinitely many of the p_n are not increasing on $[-1, 1]$. For extensions of these results to generalized convex function see Passow and Roulier [4].

* This author's research was supported in part by NSF Grant No. MCS 76-04033.

0021-9045/79/010050-06$02.00/0

There is a gap between the two results mentioned above. If $f \in C^1[-1, 1]$ and satisfies (1.1) (or, equivalently, $f'(x) \geqslant \delta > 0$ for $x \in [-1, 1]$), may we conclude that p_n is increasing for n sufficiently large?

In [7] Roulier has conjectured that the answer is negative. In [8] Tzimbalario has claimed to have verified this conjecture. He states the following theorem.

THEOREM (1.1). *Let $f \in C^1[-1, 1]$ with f' not in some* Lip α, $\alpha < 1$ *and $f'(x) \geqslant \delta > 0$ for $x \in [-1, 1]$. Then there are infinitely many positive integers n, for which the polynomial p_n of best approximation to f is not increasing.*

We note that this result is presented without proof. It is the purpose of this paper to provide a class of counterexamples to this theorem as well as to provide a correct theorem verifying that the afore-mentioned conjecture in [7] is true.

2. THE MAIN THEOREMS

THEOREM (2.1). *Let $0 < \alpha < 1$ be given. There exists $f \in C^1[-1, 1]$ for which*

$$f'(x) \geqslant \delta > 0 \qquad \text{on } [-1, 1] \tag{2.1}$$

and

$$f' \in \operatorname{Lip} \alpha \qquad \text{but} \quad f' \notin \operatorname{Lip}(\alpha + \epsilon) \qquad \text{for any } \epsilon > 0, \tag{2.2}$$

such that there are infinitely many n for which p_n is not increasing on $[-1, +1]$.

THEOREM (2.2). *Let $0 < \alpha < 1$ be given. There exists $f \in C^1[-1, 1]$ for which* (2.1) *and* (2.2) *hold and such that p_n is increasing for n sufficiently large.*

The proof of Theorem (2.1) will follow by a lemma of Tzimbalario [8] which modifies a theorem of Kadec [2] involving the location of the deviation points in Chebyshev approximation, and by a theorem of Bernstein [1] involving lower bounds for the degree of approximation of certain functions.

The following lemma is stated by Tzimbalario [8]. Since it is not proven in [8], we will provide a brief sketch of the proof which is an easy modification of a proof of a theorem of Kadec [2].

LEMMA 1 (Tzimbalario). *Let $f \in C[-1, 1]$, f not a polynomial. For each n let $p_n \in \Pi_n$ be the polynomial of best approximation to f as above. Let $-1 \leqslant x_{0,n} < x_{1,n} < \cdots < x_{n+1,n} \leqslant 1$ be a Chebyshev alternation for $f - p_n$. Then*

$$\liminf_{n\to\infty}(1 + x_{k,n})(n/\ln n)^2 \leqslant C(k) \tag{2.3}$$

and

$$\liminf_{n\to\infty}(1 - x_{n+1-k,n})(n/\ln n)^2 \leqslant C(k), \tag{2.4}$$

where $C(k)$ is a constant depending only on k and f.

For the proof of this lemma, it is sufficient to establish that

$$\liminf_{n\to\infty} t_{k,n}(n/\ln n) \leqslant D_k \tag{2.5}$$

and that

$$\liminf_{n\to\infty}(\pi - t_{n+1-k,n})(n/\ln n) \leqslant D_k , \tag{2.6}$$

where $0 \leqslant t_{0,n} < t_{1,n} < \cdots < t_{n+1,n} \leqslant \pi$ are $n + 2$ alternation points in the best approximation by trigonometric polynomials to the function $g(t) = f(\cos t)$, and where the constants $D_k \geqslant 0$ depend on f and k.

To establish (2.5) and (2.6), one observes in [2, proof of Lemma 2] that the left-most inequality in expression (8) may be used in (11) in the proof of the main theorem. One then compares the divergent series of the form $\sum_{n=1}^{\infty} 1/e^{t_{k,n}na_k}$ with the convergent series $\sum_{n=1}^{\infty} 1/n^2$ to get the desired result.

The following lemma is a special case of a theorem of Bernstein [1, page 175].

LEMMA 2. *Let* $h(x) = | x |^p$ *with* $2 < p \leqslant 3$. *Then there is a constant* $C_p > 0$ *for which*

$$E_n(h) \geqslant C_p/n^p \qquad \textit{for} \quad n = 1, 2, 3,\ldots . \tag{2.7}$$

The following lemma shows that if for a given nondecreasing $f \in C[-1, 1]$ we have $p_n'(x) \geqslant 0$ on $[-1, 1]$ for n sufficiently large, then some subsequence of $\{E_n(f)\}$ tends to zero faster than expected.

LEMMA 3. *Let* $f \in C[-1, 1]$ *and assume that* f *is nondecreasing on* $[-1, 1]$. *If* $p_n'(x) \geqslant 0$ *on* $[-1, 1]$ *for n sufficiently large, then there exists an increasing sequence of positive integers* $\{n_j\}_{j=1}^{\infty}$ *and a constant* $C > 0$ *depending only on* f *such that*

$$E_{n_j}(f) \leqslant C\omega(f, (\ln n_j/n_j)^2). \tag{2.8}$$

Proof. If f is a polynomial then the lemma is obviously true. So, we assume that f is not a polynomial. Let $-1 \leqslant x_{0,n} < x_{1,n} < \cdots < x_{n+1,n} \leqslant 1$ be a Chebyshev alternation for $f - p_n$. By (2.4) there is a sequence $\{n_j\}_{j=1}^{\infty}$ of positive integers and a constant $K > 0$ such that

$$1 - x_{n_j-1,n_j} \leqslant K(\ln n_j/n_j)^2$$

for $j = 1, 2,\ldots$. Now for each $j = 1, 2, 3,\ldots$ we may assume that

$$p_{n_j}(x_{n_j-1,n_j}) - f(x_{n_j-1,n_j}) = E_{n_j}(f)$$

and

$$f(x_{n_j,n_j}) - p_{n_j}(x_{n_j,n_j}) = E_{n_j}(f)$$

(otherwise, we use x_{n_j,n_j} and x_{n+_j1,n_j}). Thus

$$2E_{n_j}(f) = f(x_{n_j,n_j}) - f(x_{n_j-1,n_j}) - (p_{n_j}(x_{n_j,n_j}) - p_{n_j}(x_{n_j-1,}n_j)).$$

But the assumption that $p_n'(x) \geqslant 0$ for n sufficiently large allows us to conclude (by taking a subsequence if necessary) that

$$\begin{aligned} 2E_{n_j}(f) &\leqslant f(x_{n_j,n_j}) - f(x_{n_j-1,n_j}) \\ &\leqslant f(1) - f(x_{n_j-1,n_j}) \\ &\leqslant \omega(f, 1 - x_{n_j-1,n_j}) \\ &\leqslant \omega\left(f, K\left(\frac{\ln n_j}{n_j}\right)^2\right) \\ &\leqslant C_1\omega\left(f, \left(\frac{\ln n_j}{n_j}\right)^2\right) \end{aligned}$$

for $j = 1, 2, \ldots$.

Proof of Theorem (2.1). Let $0 < \alpha < 1$ be given and define

$$\begin{aligned} g(x) &= x^{1+\alpha} \qquad &\text{for} \quad x \geqslant 0, \\ &= -(-x)^{1+\alpha} \qquad &\text{for} \quad x < 0. \end{aligned}$$

Let $\delta > 0$ be given. Define

$$f(x) = g(x) + \delta x.$$

Then $f'(x) \geqslant \delta > 0$ on $[-1, +1]$, and $f' \in \mathrm{Lip}\,\alpha$ but not in $\mathrm{Lip}(\alpha + \epsilon)$ for $\epsilon > 0$. Moreover, if $h(x) = [1/(2 + \alpha)] \mid x \mid^{2+\alpha}$ we see that $g(x) = h'(x)$. Thus by Lemma 2 there is a constant $D > 0$ so that $E_n(h) \geqslant D/n^{2+\alpha}$ for $n = 1, 2, 3, \ldots$. But then

$$E_n(f) = E_n(g) \geqslant K/n^{1+\alpha} \tag{2.9}$$

for $n = 1, 2, 3, \ldots$ and where $K > 0$ is a constant depending only on f. Now assume that for n sufficiently large $p_n'(x) \geqslant 0$ on $[-1, 1]$, where $p_n \in \Pi_n$ is the polynomial of best approximation for f on $[-1, +1]$. By Lemma 3 and the fact that $f \in \mathrm{Lip}\ 1$ we have

$$E_{n_j}(f) \leqslant C\left(\frac{\ln n_j}{n_j}\right)^2 \qquad \text{for} \quad j = 1, 2, \ldots \tag{2.10}$$

Now by (2.9) and (2.10) we have,

$$\frac{n_j^{1-\alpha}}{(\ln n_j)^2} \leqslant \frac{C}{K} \qquad \text{for} \quad j = 1, 2, 3, \ldots$$

But this is a contradiction since the term on the left side of the above inequality tends to infinity. Hence there are infinitely many n for which $p_n'(x) < 0$ for some x in $[-1, 1]$.

Proof of Theorem (2.2). Let $0 < \alpha < 1$ be given. Define

$$f(x) = (1 + x)^{1+\alpha} + x.$$

Then $f \in C^1[-1, 1]$ and $f'(x) = 1 + (1 + \alpha)(1 + x)^\alpha \in \operatorname{Lip} \alpha$ but $f' \notin \operatorname{Lip}(\alpha + \epsilon)$ for any $\epsilon > 0$. Moreover, $f'(x) \geqslant 1$ on $[-1, 1]$. Form the function

$$g(t) = f(\cos t).$$

It is easy to see that g is a continuous 2π-periodic function and that g' and g'' are both continuous and 2π-periodic. Moreover $g'' \in \operatorname{Lip} 2\alpha$ if $\alpha \leqslant 1/2$, and $g'' \in \operatorname{Lip} 1$ if $\alpha > 1/2$.

Now let $\beta = \min(\alpha, 1/2)$.

Since the degree of approximation to f by algebraic polynomials of degree n is the same as the degree of approximation to g by trigonometric polynomials, we have by Jackson's theorem (see [3]),

$$E_n(f) \leqslant \frac{K}{n^{2(1+\beta)}}, \tag{2.11}$$

where K is a constant independent of n. Now for $n = 1, 2, 3,...$ let $p_n \in \Pi_n$ be the polynomial of best approximation to f on $[-1, 1]$ and define for $0 \leqslant h \leqslant 2$ a modulus of continuity $\omega(h) = Kh^\beta$. Then for each x in $[-1, 1]$ we have $|f(x) - p_n(x)| \leqslant (1/n^2)\, \omega(1/n^2) \leqslant \Delta_n(x)\, \omega(\Delta_n(x))$, where $\Delta_n(x) = \max((1/n)(1 - x^2)^{1/2}, 1/n^2)$. Now by a theorem of Steckin (see Lorentz [3, p. 74]) we have

$$|f'(x) - p_n'(x)| \leqslant M \sum_{k=n}^{\infty} \frac{1}{k}\, \omega\left(\frac{1}{k}\right) = MK \sum_{k=n}^{\infty} \frac{1}{k^{\beta+1}}.$$

Hence $\|f' - p_n'\| \to 0$ as $n \to \infty$. Thus for n sufficiently large $p_n'(x) > 0$ on $[-1, 1]$. This completes the proof of Theorem (2.2).

3. Conclusions

Clearly, then Theorem (2.1) verifies that the aforementioned conjecture is true, and Theorem (2.2) provides counterexamples to Theorem (1.1).

This approach may also be used to fill gaps in the generalized convex approximation studied by Passow and Roulier [4].

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