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SEPARABLE CATEGORIES

Steven M. AMGOTT
Department of Mathematics, Villanova University, Villanova, PA 19085, USA

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Introduction

A *ringoid* is a small (pre) additive category. A ring can be viewed as a ringoid with only one object. Many facts about rings generalize to yield facts about ringoids. The advantage of working in the general case is that there are techniques and constructions available which are not available for rings. One such construction is the notion of an idempotent completion, due to Freyd, discussed in Section 1. The idempotent completion of a ring, while still a ringoid, is a ring only for the zero ring.

If R is a commutative ring, then an R-algebra A is *R-separable* if A is projective as an $A^e = A \otimes_R A^{op}$-module (i.e., as a two-sided A-module). In this paper we consider not only the $\mathbb{Z}$-separability of a monoid ring $\mathbb{Z}G$, but we also define and consider that of a ringoid $\mathbb{Z}\mathbf{C}$ where $\mathbf{C}$ is a small category. Since separability is preserved by Morita equivalence, it turns out that we can pass to the case where $\mathbf{C}$ is skeletal and idempotent complete.

A small category $\mathbf{C}$ is *splintered* if it satisfies four conditions:

(i) $\mathbf{C}(p,q)$ is finite for all $p,q \in \operatorname{ob} \mathbf{C}$.

(ii) All isomorphisms in $\mathbf{C}$ are identities.

(iii) Every morphism in $\mathbf{C}$ factors as a retraction (split epimorphism) followed by a coretraction (split monomorphism).

(iv) Let $p,q \in \operatorname{ob} \mathbf{C}$ with p a retract of q. Let $\alpha_1, \dots, \alpha_m$ be the coretractions from p to q and let $\beta_1, \dots, \beta_n$ be the retractions from q to p. Let $M(p,q)$ be the matrix with 1 in position (i,j) if $\beta_i \alpha_j = 1_p$, and 0 otherwise. Then $m=n$ and $M(p,q)$ is invertible over $\mathbb{Z}$.

In this paper we show that for a small idempotent complete skeletal category $\mathbf{C}$, $\mathbb{Z}\mathbf{C}$ is $\mathbb{Z}$-separable if and only if $\mathbf{C}$ is splintered, and equivalently, that $\mathrm{Ab}^{\mathbf{C}}$ is equivalent to a product of copies of Ab. As a corollary we have proven a conjecture of B. Mitchell that $\mathbb{Z}\mathbf{C}$ is $\mathbb{Z}$-separable if and only if it is Morita equivalent to $\mathbb{Z}\mathbf{D}$ for some discrete category $\mathbf{D}$.

In Section 1 we give definitions and constructions. In Section 2 we show some general results on separability. Section 3 contains lemmas necessary for the main result, Section 4 has a short discussion about semigroups, and Section 5 is on splintered categories. The main results are in Section 6.

1. Definitions and constructions

Let R be a commutative ring with identity. An *R-category* $\mathscr{C}$ is a category equipped with an R-module structure on each Hom set so that composition is R-bilinear. An *R-algebroid* is a small R-category. If $\mathscr{C}$ and $\mathscr{D}$ are R-categories, then a functor $T: \mathscr{C} \to \mathscr{D}$ is an *R-functor* if the map $\mathscr{C}(p,q) \to \mathscr{D}(Tp, Tq)$ is R-linear for each p and q in ob $\mathscr{C}$.

If $\mathscr{C}$ is an R-algebroid, then its *enveloping algebroid* is $\mathscr{C}^{\mathrm{e}} = \mathscr{C} \otimes_R \mathscr{C}^{\mathrm{op}}$. There is an epimorphism (multiplication)

$$\mu: \bigoplus_{p \in \mathrm{ob}\, \mathscr{C}} (\mathscr{C}(p, \cdot) \otimes_R \mathscr{C}(\cdot, p)) \to \mathscr{C}(\cdot, \cdot)$$

of $\mathscr{C}^{\mathrm{e}}$-modules given on basis elements by

$$\mu(1_p \otimes 1_p) = 1_p.$$

The *R-Hochschild dimension of* $\mathscr{C}$, $\dim_R \mathscr{C}$, is the projective dimension of $\mathscr{C} = \mathscr{C}(\cdot, \cdot)$ as a $\mathscr{C}^{\mathrm{e}}$-module.

Let $\mathbf{C}$ be a small category. We denote by $R\mathbf{C}$ the R-algebroid whose objects are those of $\mathbf{C}$, and where $R\mathbf{C}(p,q)$ is the free R-module on $\mathbf{C}(p,q)$, with the obvious laws of composition. We will denote $\dim_R R\mathbf{C}$ by $\dim_R \mathbf{C}$.

Two R-algebroids $\mathscr{C}$ and $\mathscr{D}$ are *R-Morita equivalent* if there is an R-equivalence of categories

$$\mathrm{Mod}\, R^{\mathscr{C}} \simeq \mathrm{Mod}\, R^{\mathscr{D}}.$$

This can be shown to be equivalent to the existence of bimodules

$${}_{\mathscr{D}}P_{\mathscr{C}} \quad \text{and} \quad {}_{\mathscr{C}}Q_{\mathscr{D}},$$

where the action of R is the same on the left and right sides, with

$$P \otimes_{\mathscr{C}} Q \simeq \mathscr{D} \quad \text{and} \quad Q \otimes_{\mathscr{D}} P \simeq \mathscr{C}$$

as bimodules. Then

$$\mathrm{Mod}\, R^{\mathscr{C} \otimes_R \mathscr{C}^{\mathrm{op}}} \xrightarrow[P \otimes_{\mathscr{C}} \cdot]{\simeq} \mathrm{Mod}\, R^{\mathscr{D} \otimes_R \mathscr{C}^{\mathrm{op}}} \xrightarrow[\cdot \otimes_{\mathscr{C}} Q]{\simeq} \mathrm{Mod}\, R^{\mathscr{D} \otimes_R \mathscr{D}^{\mathrm{op}}},$$

and the composite takes the bimodule $\mathscr{C}$ to

$$P \otimes_{\mathscr{C}} \mathscr{C} \otimes_{\mathscr{C}} Q = P \otimes_{\mathscr{C}} Q = \mathscr{D}.$$

Therefore, if $\mathscr{C}$ and $\mathscr{D}$ are R-Morita equivalent, then $\dim_R \mathscr{C} = \dim_R \mathscr{D}$.

Example 1. A category is *idempotent complete* if every idempotent θ splits, that is if $\theta=\alpha\beta$ with $\beta\alpha$ an identity. A category $\hat{\mathbf{C}}$ is an idempotent completion for a subcategory $\mathbf{C}$ of $\hat{\mathbf{C}}$ if $\hat{\mathbf{C}}$ is idempotent complete and if every object of $\hat{\mathbf{C}}$ is a retract of one in $\mathbf{C}$. It is shown in section 1 of [6] that the inclusion functor $\mathbf{C}\to\hat{\mathbf{C}}$ induces an equivalence of categories

$$\mathbf{A}^{\hat{\mathbf{C}}}\to\mathbf{A}^{\mathbf{C}},$$

for any idempotent complete category $\mathbf{A}$, when $\mathbf{C}$ is small, and that idempotent completions are unique up to equivalence. Given a category $\mathbf{C}$, let $\hat{\mathbf{C}}$ be the category whose objects are idempotents θ in $\mathbf{C}$, and where a morphism $\theta\to\theta'$ is a triple (θ',x,θ), where x is a morphism of $\mathbf{C}$ satisfying $\theta'x=x=x\theta$. Composition is inherited from $\mathbf{C}$. Then $\hat{\mathbf{C}}$ is an idempotent completion of $\mathbf{C}$. (This construction, due to Freyd, can also be found in section 1 of [6].) Therefore, if $\hat{\mathscr{C}}$ is an idempotent completion of $\mathscr{C}$, then $\dim_R \mathscr{C}=\dim_R \hat{\mathscr{C}}$. If $\mathbf{C}$ is a small category, then the category of R-functors $(\operatorname{Mod} R)^{R\mathbf{C}}$ is equivalent to the category of all functors $(\operatorname{Mod} R)^{\mathbf{C}}$. Therefore, if $\hat{\mathbf{C}}$ is an idempotent completion of $\mathbf{C}$, then $R\mathbf{C}$ and $R\hat{\mathbf{C}}$ are R-Morita equivalent, and so $\dim_R \mathbf{C}=\dim_R \hat{\mathbf{C}}$.

Example 2. Let $\mathscr{C}$ be an R-algebroid with only a finite number of objects. Define $[\mathscr{C}]$ to be the set of matrices (α_{qp}) with $\alpha_{qp}\in\mathscr{C}(p,q)$. These can be added and multiplied using addition and composition in $\mathscr{C}$ to make $[\mathscr{C}]$ into an R-algebra. It is shown in section 7 of [6] that $\mathscr{C}$ and $[\mathscr{C}]$ are R-Morita equivalent, so $\dim_R \mathscr{C}=\dim_R [\mathscr{C}]$.

It is easily shown (see section 12 of [6]) that if $\mathscr{C}$ is an R-algebroid, then $\dim_R \mathscr{C}=\dim_R \mathscr{C}^{\mathrm{op}}$. Also if $\mathscr{C}$ is the coproduct of R-algebroids $\mathscr{C}_i$ (disjoint union with 0 morphisms added between objects of different categories), then $\dim_R \mathscr{C}=\sup_i \dim_R \mathscr{C}_i$.

An R-algebroid $\mathscr{C}$ is *R-separable* if $\mathscr{C}$ is projective as a $\mathscr{C}^{\mathrm{e}}$-module. When $\mathscr{C}=R\mathbf{C}$ we shall simply say that $\mathbf{C}$ is an R-separable category. This is, of course, equivalent to $\dim_R \mathscr{C}=0$ and also to the splitting of the multiplication map μ as a map of $\mathscr{C}^{\mathrm{e}}$-modules. If λ is a splitting for μ, set $e_q=\lambda(1_q)$ for each $q\in\operatorname{ob}\mathscr{C}$. This gives a family

$$\Big\{e_q\in\bigoplus_{p\in\operatorname{ob}\mathscr{C}}\mathscr{C}(p,q)\otimes_R\mathscr{C}(q,p)\Big\}_{q\in\operatorname{ob}\mathscr{C}}.$$

Conversely, given such a family, define λ by $\lambda_{q,r}(\gamma)=(\gamma\otimes 1_q)e_q$ for $\gamma\in\mathscr{C}(q,r)$. Then naturality of λ is equivalent to

$$(1)\qquad (\gamma\otimes 1_q)e_q=(1_r\otimes\gamma)e_r,$$

and the condition that λ split μ is equivalent to

$$(2)\qquad \mu e_q=1_q$$

using naturality of μ. Therefore $\mathscr{C}$ is R-separable if and only if there is such a family $\{e_q\}$ satisfying (1) and (2). We call such a family an *R-separability set* for $\mathscr{C}$. When the ring R is apparent we shall omit it from the notation and simply say *separable* and *separability set*. If $\mathscr{C}$ is an algebra, the element $e \in \mathscr{C}^e$ is then idempotent and is called a *separability idempotent* for $\mathscr{C}$ (see page 40 of [4]). When $\mathscr{C} = R\mathbf{C}$, the categories $(\text{Mod}\, R)^{R\mathbf{C} \otimes_R R\mathbf{C}^{\text{op}}}$ and $(\text{Mod}\, R)^{R(\mathbf{C} \times \mathbf{C}^{\text{op}})}$ of R-functors are equivalent (each is equivalent to the category of all functors $(\text{Mod}\, R)^{\mathbf{C} \times \mathbf{C}^{\text{op}}}$), so the family $\{e_q\}$ takes on the form

$$e_q = \sum r^q_{g,h}(g,h),$$

where the sum is indexed by all composable pairs (g,h) with $\text{dom}\, h = \text{cod}\, g = q$. Then (1) and (2) above become

$$(1') \qquad \sum r^q_{g,h}(\lambda g, h) = \sum r^p_{g,h}(g, h\lambda)$$

where $\gamma: q \to p$ in $\mathbf{C}$, and

$$(2') \qquad \sum r^q_{g,h} gh = 1_q$$

with $r^q_{g,h} \in R$. We shall use this notation in the remainder of this paper.

2. General results

Let $\mathscr{C}$ be an R-algebroid. An R-functor $M: \mathscr{C} \to \text{Mod}\, R$ is *R-projective* (resp. *R-finitely generated*) if $M(p)$ is a projective (resp. finitely generated) R-module for all $p \in \text{ob}\, \mathscr{C}$. $\mathscr{C}$ is an *R-projective* (resp. *R-finitely generated*) *R-algebroid* if it is R-projective (resp. R-finitely generated) as a functor $\mathscr{C}^e \to \text{Mod}\, R$ (i.e., if $\mathscr{C}(p,q)$ is a projective (resp. finitely generated) R-module for all $p,q \in \text{ob}\, \mathscr{C}$). The proof of the result of Villamayor and Zelinski on page 47 of [4] can be generalized to show that if $\mathscr{C}$ is an R-projective separable R-algebroid, then it is R-finitely generated. However, we will only prove and use a special case.

Proposition 2.1 *Suppose* $\mathbf{C}$ *is a small R-separable category. Then for each p and p′ in* $\mathbf{C}$, $\mathbf{C}(p,p')$ *is finite.*

Proof. Let $\{e_q\}$ be a separability set for $R\mathbf{C}$ and let $\gamma \in \mathbf{C}(p,p')$. Then

$$\gamma = \gamma\mu(e_p) = \mu((\gamma \otimes 1)e_p) = \sum_{(g,h)} r^p_{g,h}\gamma gh = \sum_{(g',h)} \sum_{\gamma g = g'} r^p_{g,h} g'h.$$

But

$$\sum r^p_{g,h}(\gamma g, h) = \sum r^{p'}_{g',h'}(g', h'\gamma).$$

Deleting superscripts, if $\sum_{\gamma g = g'} r_{g,h} \neq 0$ for a pair (g', h), then there is a term on the right hand side with $r_{g',h'} \neq 0$ for some h'. Therefore the sum in the first equation can be viewed as being over the fixed finite set

$$\{(g',h) \mid r_{g',h'} \neq 0 \text{ and } r_{g,h} \neq 0 \text{ for some } g \text{ and } h'\},$$

so $\mathbf{C}(p,p')$ is finite.

Lemma 2.2. *If $\mathscr{C}$ is a separable R-algebroid and S is a commutative R-algebra, then $\mathscr{C} \otimes_R S$ is S-separable.*

Proof. The diagram

$$\begin{array}{ccc} \bigoplus_p (\mathscr{C}(p,\cdot) \otimes_R S) \otimes_S (\mathscr{C}(\cdot,p) \otimes_R S) & \xrightarrow{\mu_{\mathscr{C}\otimes_R S}} & \\ \Big\downarrow \wr & & \mathscr{C}(\cdot,\cdot) \otimes_R S \\ \Big(\bigoplus_p (\mathscr{C}(p,\cdot) \otimes_R \mathscr{C}(\cdot,p))\Big) \otimes_R S & \xrightarrow{\mu_{\mathscr{C}} \otimes 1_S} & \end{array}$$

commutes by associativity of the tensor product, so if $\mu_{\mathscr{C}}$ splits, then so does $\mu_{\mathscr{C}\otimes_R S}$.

Note that the above shows that if $\mathbf{C}$ is $\mathbb{Z}$-separable, then $\mathbf{C}$ is R-separable for any commutative ring R.

Lemma 2.3. *If $\mathscr{C}$ is a separable R-algebroid and $M \in (\mathrm{Mod}\, R)^{\mathscr{C}}$ is R-projective, then M is projective as a $\mathscr{C}$-module.*

Proof. $\mathscr{C}(p,\cdot) \otimes_R : \mathrm{Mod}\, R \to (\mathrm{Mod}\, R)^{\mathscr{C}}$ is left adjoint to the exact functor $M \mapsto M(p)$. Since $M(p)$ is a projective R-module, the $\mathscr{C}$-module

$$\mathscr{C}(p,\cdot) \otimes_R \mathscr{C}(\cdot,p) \otimes_{\mathscr{C}} M = \mathscr{C}(p,\cdot) \otimes_R M(p)$$

is projective for all p. Hence the $\mathscr{C}$-module

$$\Big(\bigoplus_p \mathscr{C}(p,\cdot) \otimes_R \mathscr{C}(\cdot,p)\Big) \otimes_{\mathscr{C}} M$$

is projective. But $\mathscr{C}(\cdot,\cdot)$ is a retract of $\bigoplus(\mathscr{C}(p,\cdot) \otimes_R \mathscr{C}(\cdot,p))$ as a $\mathscr{C}^{\mathrm{e}}$-module, so $M = \mathscr{C}(\cdot,\cdot) \otimes_{\mathscr{C}} M$ is a projective $\mathscr{C}$-module.

Proposition 2.4. *Let A be an R-algebra, finitely generated as an R-module. Then the following are equivalent:*

(a) *A is R-separable.*

(b) *$A \otimes_R K$ is K-separable for all commutative R-algebras K.*

(c) *$A \otimes_R K$ is semisimple for all fields K which are R-algebras.*

(d) *$A/\mathfrak{m}A$ is $R/\mathfrak{m}$-separable for all maximal ideals $\mathfrak{m}$ of R.*

Proof. (a) ⇒ (b) follows from Lemma 2.2.

(b) ⇒ (c) follows from Lemma 2.3.

(c) ⇒ (d). With $K=R/\mathfrak{m}$ in (c) we see that $A/\mathfrak{m}A \cong A\otimes R/\mathfrak{m}$ is classically separable, and so (d) holds by Theorem 2.5 in [4, p. 50].

(d) ⇒ (a). See Theorem 7.1 in [4, p. 72].

Corollary 2.5. *Let $\mathscr{C}$ be a separable R-finitely generated R-algebroid. If $\mathscr{D}$ is a full subalgebroid of $\mathscr{C}$ with only a finite number of objects, then $\mathscr{D}$ is R-separable.*

Proof. By Lemmas 2.2 and 2.3, $\mathscr{C}\otimes_R K$ is semisimple for all fields K which are R-algebras. By [6, section 4] any full subalgebroid of a semisimple algebroid is semisimple, so $\mathscr{D}\otimes_R K$ is semisimple. Since $\mathscr{D}$ has only a finite number of objects, the ring $[\mathscr{D}]\otimes_R K=[\mathscr{D}\otimes_R K]$ is semisimple. Therefore, by the above proposition, $[\mathscr{D}]$, and hence $\mathscr{D}$, is R-separable.

3. Lemmas

The following lemma is well known and was shown by E.H. Moore in [8].

Lemma 3.1. *Let G be a finite monoid. Then every element of G has an idempotent power.*

Proof. If $g\in G$, then $g^r=g^{r+s}$ for some r and s. Then

$$g^r=g^{r+s}=g^{r+2s}=\cdots=g^{2r+t}$$

for some t. Therefore

$$g^{r+t}=g^{2r+2t}=(g^{r+t})^2.$$

Lemma 3.2. *Let* **C** *be a category and let $q\in \mathrm{ob}\,\mathbf{C}$ with $\mathbf{C}(q,q)$ finite. Then any monomorphism (epimorphism) in $\mathbf{C}(q,q)$ is an automorphism. In particular, any elemen of a finite monoid which is either left or right invertible has a two-sided inverse.*

Proof. Let $\alpha\in\mathbf{C}(q,q)$ be a monomorphism. Then the sets $\{\alpha g\mid g\in\mathbf{C}(q,q)\}$ anc $\mathbf{C}(q,q)$ have the same number of elements, so they are equal. Therefore, there is a g with $\alpha g=1_q$. Then α is both a retraction and a monomorphism, hence an isomor phism.

Lemma 3.3. *Suppose G is an R-separable monoid. Let H be a submonoid of G clos ed to factors (that is, if $gh\in H$ with g and h in G, then g and h are in H). Then H is R-separable.*

Proof. Let $e=\sum r_{g,h}(g,h)$ be a separability idempotent for RG. Since $gh\in H$ if and only if $g\in H$ and $h\in H$, the right hand side of

$$1=\sum r_{g,h}gh$$

splits into two sums, one over $\{(g,h)\mid g\in H \text{ and } h\in H\}$ and one over $\{(g,h)\mid g\notin H \text{ and } h\notin H\}$. Since $1\in H$, the second sum is zero.

Next, let $k\in H$. Consider the equation

$$\sum r_{g,h}(kg,h)=\sum r_{g,h}(g,hk).$$

Since $kg\in H$ if and only if $g\in H$ and $hk\in H$ if and only if $h\in H$, the only terms of the form $r(h_1,h_2)$ with h_1 and h_2 in H occur with both g and h in H. Therefore, the above equation holds when the sums are over $(g,h)\in H\times H$. As a result

$$\sum_{(g,h)\in H\times H} r_{g,h}(g,h)$$

is a separability idempotent for RH.

The next lemma is a well-known generalization of Maschke's Theorem and can easily be proven using 2.1 and separability idempotents.

Lemma 3.4. *Let G be a group. The RG is R-separable if and only if G is finite and $|G|$, the order of G, is invertible in R.*

Corollary 3.5. *Let G be an R-separable monoid and let H be the subgroup of invertible elements of G. Then $|H|$ is invertible in R.*

Proof. If $gh\in H$, then g has a right inverse and h has a left inverse. Since G is finite by Proposition 2.1, by Lemma 3.2 g and h are invertible and hence in H. By Lemma 3.3, H is then R-separable and the result follows from Lemma 3.4.

Corollary 3.6. *If $\mathbb{Z}\mathbf{C}$ is a separable ringoid, then the only automorphisms of $\mathbf{C}$ are identities.*

Proof. By Corollary 2.5, for each $q\in \operatorname{ob}\mathbf{C}$, $\mathbb{Z}\mathbf{C}(q,q)$ is a separable monoid ring. The rest follows from Corollary 3.5.

In [5], an argument by B. Mitchell following Corollary 7.3, p. 880, showed the following result as a corollary to a theorem of Laudal.

Proposition 3.7. *Let $\mathbf{C}$ be a small connected $\mathbb{Z}$-separable idempotent complete category. Then $\mathbf{C}$ has a zero object.*

4. Rees semigroups

Let G be a group. Let G^0 be the semigroup obtained by adjoining a zero element to G. Let P be an $n\times m$ matrix with entries in G^0. The *Rees matrix semigroup* $\mathcal{M}^0(G; m,n,P)$ is the set of all $m\times n$ matrices over G^0 with at most one non-zero entry, with multiplication

$$A\cdot B=APB.$$

Let R be a ring and S be a semigroup with zero element z. Set

$$R_0S=RS/Rz.$$

If $S=\mathcal{M}^0(G; m,n,P)$, then R_0S consists of all $m\times n$ matrices with entries in RG, with the above multiplication. The following result was intrinsic in the proof of Theorem 4.7 in [9] and can be found, with R a field, as Lemma 5.18 in [3].

Lemma 4.1. *Let $S=\mathcal{M}^0(G; m,n,P)$. Then R_0S has an identity if and only if $m=n$ and P is invertible over RG.*

Proof. Let E be an identity in R_0S. Then for any matrix A in R_0S we have $EPA=A$ and $APE=A$. It is easily seen that as a result $EP=I_m$ and $PE=I_n$. Therefore $m=n$ and P is invertible over RG. Conversely, it is clear that if $m=n$ and P is invertible over RG, then P^{-1} is an identity in R_0S.

5. Splintered categories

Recall the definition of a splintered category from the introduction.

First we note that (ii) is equivalent to **C** being skeletal with only identities as automorphisms. Also if θ is an idempotent in a category **C** satisfying (iii), then, writing $\theta=\alpha\beta$ as in (iii), we have that $\alpha\beta\alpha\beta=\alpha\beta$. Since α is a monomorphism and β is an epimorphism, $\beta\alpha$ is an identity. Therefore, θ splits and hence **C** is idempotent complete.

Lemma 5.1. *If* **C** *is a skeletal category with* $\mathbf{C}(q,q)$ *finite, then the set of retracts of q is finite.*

Proof. Let $\{p_i\}_{i\in I}$ be the set of retracts of q and choose $\alpha_i: p_i\to q$ and $\beta_i: q\to p$ with $\beta_i\alpha_i=1_{p_i}$. If $\alpha_i\beta_i=\alpha_j\beta_j$, then

$$\beta_i\alpha_j\beta_j\alpha_i=\beta_i\alpha_i\beta_i\alpha_i=1_{p_i},$$

so p_i is a retract of p_j. By symmetry p_j is a retract of p_i, so $i=j$. Since $\mathbf{C}(q,q)$ i finite, there can only be a finite number of distinct $\alpha_i\beta_i$, so I is finite.

Recall that Lemma 3.2 says that if $\mathbf{C}(q,q)$ is finite, then any monomorphism or epimorphism α in $\mathbf{C}(q,q)$ is an automorphism. Therefore, if (ii) also holds, $\alpha = 1_q$.

Proposition 5.2. *If a small skeletal category* $\mathbf{C}$ *satisfies* (iii) *and if all retractions which are endomorphisms are identities, then the factorization in* (iii) *is unique.*

Proof. Suppose α_1 and α_2 are coretractions and β_1 and β_2 are retractions with $\alpha_1\beta_1 = \alpha_2\beta_2$. Choose β with $\beta\alpha_1$ the identity. Then $\beta\alpha_2\beta_2 = \beta\alpha_1\beta_1 = \beta_1$. Write $\beta\alpha_2 = \bar{\alpha}\bar{\beta}$ as in (iii). Then $\bar{\alpha}\bar{\beta}\beta_2 = \beta_1$, so $\bar{\alpha}$ is both a retraction and a coretraction, and hence an isomorphism. Then by the hypotheses $\bar{\alpha}$ is an identity, so $\operatorname{dom}\alpha_1$ is a retract of $\operatorname{dom}\alpha_2$. By symmetry, $\operatorname{dom}\alpha_2$ is a retract of $\operatorname{dom}\alpha_1$, so they are equal. Then $\bar{\beta}$ is an endomorphism, and hence an identity, so $\beta_1 = \beta_2$, and hence $\alpha_1 = \alpha_2$.

If $\mathbf{C}$ is any category, we may preorder $\operatorname{ob}\mathbf{C}$ by $p \le q$ if p is a retract of q. If $\mathbf{C}$ is a splintered category, then this is a partial order. For $q \in \operatorname{ob}\mathbf{C}$, we define the *height of* q to be the sup of lengths n of chains $p_0 < p_1 < \cdots < p_n = q$ of retracts of q.

Example 1. Let $\mathbf{C}$ be the category with objects z, p, and q with z a zero object, and with morphisms defined as follows. Let M be an $n \times m$ matrix with entries $m_{ij} \in \{0,1\}$. We take the free pointed category with generators $\alpha_1, \dots, \alpha_m : p \to q$ and $\beta_1, \dots, \beta_n : q \to p$, subject to relations $\beta_i\alpha_j = m_{ij}$ (viewing 1 as 1_p). It can be shown using only these relations that the morphisms $\alpha_i\beta_j$ are distinct endomorphisms of q. If fact, $\mathbf{C}$ is the skeletalization of the idempotent completion of the monoid obtained by adding an identity to the Rees semigroup $\mathcal{M}^0(1; m, n, M)$. If $m = n$ and M is invertible over $\mathbb{Z}$, then $\mathbf{C}$ is splintered.

Example 2. Let P be a poset in which, for each $q \in P$, the set $\{p \mid p \le q\}$ forms a lower semilattice (i.e., each pair of elements $\{r, s\}$, with $r \le q$ and $s \le q$, has an inf). Let $\mathbf{P}$ be the category whose objects are the elements of P and where $\mathbf{P}(p,q) = \{r_{qp} \mid r \le p \text{ and } r \le q\}$. Composition is given by $s_{rq}t_{qp} = u_{rp}$ where $u = \inf(s,t)$. Then $\mathbf{P}$ satisfies (ii), (iii), and (iv), and each Hom set has at most one retraction and at most one coretraction. We note that if $\mathbf{C}$ is a small skeletal category satisfying (iii) in which each Hom set has at most one retraction and at most one coretraction, then $\mathbf{C}$ is of the above form. If P is downward finite, then $\mathbf{P}$ is splintered.

Example 3. Let Δ_* be the category whose objects are finite totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$, $n \ge 0$, and where a morphism from $[n]$ to $[m]$ is a non-decreasing function taking n to m. When $[p]$ is a retract of $[q]$, the set of coretractions from $[p]$ to $[q]$ and the set of retractions from $[q]$ to $[p]$ can each be ordered so that the matrix $M([p],[q])$ is upper triangular, with diagonal entries 1. Since (i) to (iii) are also satisfied, Δ_* is splintered.

6. Main results

Let **C** be a small category and let $q \in \text{ob}\,\mathbf{C}$. Denote by $c(q)$ the set of coretractions (split monomorphisms) with codomain q, and by $r(q)$ the set of retractions (split epimorphisms) with domain q. If **A** is any pointed category with coproducts, then there is a functor

$$T: \mathop{\mathsf{X}}_{p \in \text{ob}\,\mathbf{C}} \mathbf{A} \to \mathbf{A}^{\mathbf{C}}$$

given on objects by

$$T(A_p)(q) = \bigoplus_{\alpha \in c(q)} A_{\text{dom}\,\alpha}.$$

If $\gamma: q \to q'$ in **C**, then

$$T(A_p)(\gamma): \bigoplus_{\alpha \in c(q)} A_{\text{dom}\,\alpha} \to \bigoplus_{\alpha \in c(q')} A_{\text{dom}\,\alpha}$$

is defined by

$$\begin{aligned} T(A_p)(\gamma)u_\alpha &= u_{\gamma\alpha} \quad \text{if } \gamma\alpha \text{ is a coretraction,} \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

where the u_α are the coproduct injections. That $T(A_p)$ is a functor is verified by noting that if $\gamma\alpha$ is not a coretraction, then neither is $\gamma'\gamma\alpha$. If $(f_p): (A_p) \to (A_p')$ is in $\mathsf{X}_{p \in \text{ob}\,\mathbf{C}}\,\mathbf{A}$, then $T(f_p)$ is defined by

$$T(f_p)_q = \bigoplus_{\alpha \in c(q)} f_{\text{dom}\,\alpha}.$$

This is easily seen to be a natural transformation. Note that the functor T is always faithful. (This functor was described by *M.* André and mentioned by B. Mitchell in [6]).

Theorem 6.1. *Let* **C** *be a small idempotent complete skeletal category. Then the following are equivalent:*

(a) **C** *is splintered.*

(b) *The functor* $T: \mathsf{X}_{p \in \text{ob}\,\mathbf{C}}\, \mathscr{A} \to \mathscr{A}^{\mathbf{C}}$ *defined above is an equivalence for all idempotent complete additive categories* $\mathscr{A}$.

(c) $\text{Ab}^{\mathbf{C}}$ *is equivalent to a product of copies of* Ab.

(d) **C** *is* $\mathbb{Z}$*-separable.*

We immediately obtain the following results for any small category **C**.

Corollary 6.2. **C** *is* $\mathbb{Z}$*-separable if and only if the skeletalization of its idempotent completion is splintered.*

Corollary 6.3. **C** *is* $\mathbb{Z}$*-separable if and only if* $\mathbb{Z}$**C** *is Morita equivalent to* $\mathbb{Z}$**D** *for some discrete category* **D**.

Proof of Theorem 6.1. (a) ⇒ (b). Let $B \in \mathscr{A}^{\mathbf{C}}$. By induction on height q we define an object $A_q \in \mathscr{A}$ and morphisms

$$B(q) \underset{u_q}{\overset{\pi_q}{\rightleftarrows}} A_q$$

such that $\pi_q u_q$ is the identity, $B(\beta)u_q = 0$ for non-identity retractions β, and $\pi_q B(\alpha) = 0$ for non-identity coretractions α. If q has height 0, set $\pi_q = u_q = 1_{B(q)}$. Assume A_p, π_p, and u_p have been defined for all $p < q$ as above. Define

$$\bigoplus_{\substack{\alpha \in c(q) \\ \alpha \neq 1_q}} A_{\operatorname{dom}\alpha} \xrightarrow{\bar{u}_q} B(q) \xrightarrow{\bar{\pi}_q} \mathop{\mathsf{X}}_{\substack{\beta \in r(q) \\ \beta \neq 1_q}} A_{\operatorname{cod}\beta}$$

by

$$\bar{u}_q u_\alpha = B(\alpha) u_{\operatorname{dom}\alpha} \quad \text{and} \quad \pi_\beta \bar{\pi}_q = \pi_{\operatorname{cod}\beta} B(\beta).$$

If $\alpha \in c(q)$ and $\beta \in r(q)$ are not identities, then we have that

$$\pi_\beta \bar{\pi}_q \bar{u}_q u_\alpha = \pi_{\operatorname{cod}\beta} B(\beta) B(\alpha) u_{\operatorname{dom}\alpha}.$$

If $\beta\alpha$ is an identity, then this is 1. If not, write $\beta\alpha = \bar{\alpha}\bar{\beta}$ as in (iii). The above becomes

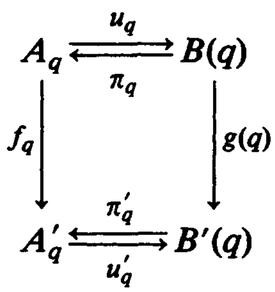
$$\pi_{\operatorname{cod}\bar{\alpha}} \beta(\bar{\alpha}) B(\bar{\beta}) u_{\operatorname{dom}\bar{\beta}}$$

which, since either $\bar{\alpha}$ or $\bar{\beta}$ is not an identity, is 0 by induction. Therefore, $\bar{\pi}_q \bar{u}_q$ is the matrix whose non-zero entries are in blocks on the diagonal which are the matrices $M(p, q)$ of condition (iv), and hence is invertible. Define u_q and π_q so that we have a coproduct decomposition

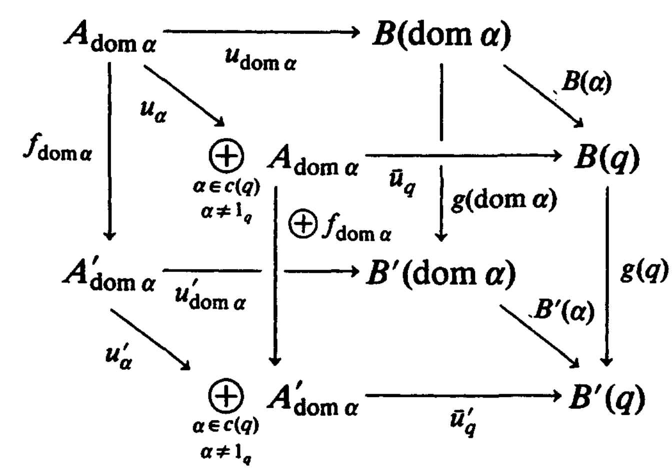
$$\begin{array}{ccc}
\bigoplus_{\substack{\alpha \in c(q) \\ \alpha \neq 1_q}} A_{\operatorname{dom}\alpha} & \overset{\bar{u}_q}{\searrow} & \\
\Big\| \wr & & B(q) \underset{u_q}{\overset{\pi_q}{\rightleftarrows}} A_q \\
\bigoplus_{\substack{\beta \in r(q) \\ \beta \neq 1_q}} A_{\operatorname{cod}\beta} & \underset{\bar{\pi}_q}{\swarrow} &
\end{array}$$

To see that $\pi_q B(\alpha) = 0$ for all non-identity coretractions α, note that $B(\alpha)u_{\operatorname{dom}\alpha}$ and $B(\alpha)\bar{u}_{\operatorname{dom}\alpha}$ factor through $\bar{u}_q$. Dually $B(\beta)u_q = 0$ for all non-identity retractions β. Finally, using these and (iii), it is easy to see that the coproduct decomposition gives a natural transformation $B \simeq T(A_p)$.

As we noted earlier T is always faithful, so all that remains is to show that it is also full. Suppose that $g: B \to B'$ is a natural transformation between functors in $\mathscr{A}^{\mathbf{C}}$. We define inductively morphisms $f_q: A_q \to A'_q$ so that the diagram



commutes in the obvious sense. Then $g \simeq T(f_p)$. Assume the diagram commutes for $p<q$ and consider the cube



where $\bar{u}_q$ is defined earlier. All faces commute save possibly the front face, so that face must also commute by the universal property of the coproduct. Dually we get commutativity with the projections $\bar{\pi}_q$ and $\bar{\pi}'_q$, so $\bigoplus f_{\mathrm{dom}\,\alpha}$ splits off from $g(q)$, defining f_q.

(b)⇒(c)⇒(d) is obvious.

(d)⇒(a). (i) This follows from Proposition 2.1 (Villamayor–Zelinski).

(ii) This is Corollary 3.6.

(iii) By Lemma 5.1, if $q \in \mathbf{C}$, then the height of q is finite. By Proposition 3.7, each connected component of $\mathbf{C}$ has a zero object. Let $\gamma : p \to q$. We proceed by induction on height p + height q. If this is 0 or 1, then either p or q is the zero object in its connected component of $\mathbf{C}$, and hence γ is either a retraction or a coretraction. By induction it is sufficient to show that if γ is neither a retraction nor a coretraction, then it must factor through some object s with $s<p$ and $s<q$. Since $\mathbf{C}$ is idempotent complete, it is sufficient to find an idempotent θ not an identity with either $\theta\gamma = \gamma$ or $\gamma\theta = \gamma$, since this idempotent then splits. In view of Lemma 3.1 we need only do this for endomorphism θ.

Consider the full subcategory $\mathbf{D}$ of $\mathbf{C}$ whose objects are those objects s of $\mathbf{C}$ with either s a retract of p or s a retract of q. Again, by Lemma 5.1, $\mathbf{D}$ has only a finite number of objects, so by Corollary 2.5, $\mathbf{D}$ is $\mathbb{Z}$-separable. Let

$$\{e_s = \sum r^s_{g,h}(g,h)\}$$

be a separability set for $\mathbb{Z}\mathbf{D}$. Consider the equation

$$\sum r^p_{g,h}(\gamma g, h) = \sum r^q_{g,h}(g, h\gamma).$$

If p is not a retract of q, then there is no h with $h\gamma = 1_p$. In order that $\mu e_p = 1_p$ by (ii) we must have $r^p_{1_p,1_p} = 1$. Therefore, comparing coefficients of $(\gamma, 1_p)$, we see that there must be a $g \neq 1_p$ with $\gamma g = \gamma$. Similarly if q is not a retract of p, then there is an $h \neq 1_q$ with $h\gamma = \gamma$.

(iv) Choose p and q in ob $\mathbf{C}$. Let $\sim$ be the congruence relation on $\mathbf{C}$ generated by $\gamma \sim 0_{rs}$ if $\gamma : r \to s$ factors through a proper retract of p. Let $\hat{\mathbf{C}}$ be the skeletalization of the quotient category $\mathbf{C}/\sim$. In $\hat{\mathbf{C}}$ the object p has height 1, and the matrix of condition (iv) in $\hat{\mathbf{C}}$ is the matrix $M(p,q)$ in $\mathbf{C}$. Clearly $\hat{\mathbf{C}}$ is separable, so without loss of generality we may reduce to the case that p has height 1 in $\mathbf{C}$.

Consider the semigroup $S \subsetneq \mathbf{C}(q,q)$ consisting of the morphisms $\alpha_i\beta_j$ together with the zero morphism, where $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ are as in (iv). Then $S \simeq \mathcal{M}^0(1; m, n, P)$ under the identification

$$\alpha_i\beta_j \leftrightarrow e_{ij}$$

where e_{ij} is the matrix whose (i,j) entry is 1 and whose other entries are 0, with $P = M(p,q)$. S is closed under left and right multiplication by elements of $\mathbf{C}(q,q)$ since p has height 1, so $(\mathbb{Z}/r\mathbb{Z})S$ is an ideal of $(\mathbb{Z}/r\mathbb{Z})\mathbf{C}(q,q)$. If r is a prime, then the later is semisimple by Proposition 2.4. Therefore, by the Wedderburn-Artin structure theorem, so are $(\mathbb{Z}/r\mathbb{Z})S$, and hence $(\mathbb{Z}/r\mathbb{Z})_0S$. As a result $(\mathbb{Z}/r\mathbb{Z})_0S$ has an identity, so by Lemma 4.1, $m = n$ and P is invertible over $\mathbb{Z}/r\mathbb{Z}$. This is true for all primes r, so $\det P$ is not divisible by any prime, hence it is ± 1. Therefore, $P = M(p,q)$ is invertible over $\mathbb{Z}$.

In [12] J. Shapiro proved for a finite semigroup S that S is separable if and only if there is a chain of ideals $\phi = S_0 \subset S_1 \subset \cdots \subset S_n = S$ with $\mathbb{Z}/p\mathbb{Z}(S_i/S_{i+1})$ $\mathbb{Z}/p\mathbb{Z}$-separable for each prime p and each i, $0 \leq i \leq n$. C. Cheng obtained this as Corollary 12 to Theorem B in [2], which stated that if S is a semigroup with a zero element such that $\mathbb{Z}S$ has an identity, then the following are equivalent.

(1) $\mathbb{Z}S$ is separable.

(2) S is finite with a principal series $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = S$ such that $S_i/S_{i-1} \simeq \mathcal{M}^0(1; m_i, m_i, P_i)$ where P_i is invertible over $\mathbb{Z}$.

(3) $\mathbb{Z}S \simeq (\mathsf{X}_{i=1}^{n-1} M_{m_i}(\mathbb{Z})) \times \mathbb{Z}$.

We proceed to show the connection between this and our main result by showing that Corollary 6.2 yields (1)$\Rightarrow$(2).

Let S be separable and let M be the monoid obtained by adjoining an identity to S. Then M is separable, so the skeletalization $\mathbf{C}$ of its idempotent completion is splintered with a maximal element q. Let $p_1, p_2, \ldots, p_{n+1} = q$ be the objects of $\mathbf{C}$ with $\mathrm{ht}\, p_1 \leq \mathrm{ht}\, p_2 \leq \cdots \leq \mathrm{ht}\, p_{n+1}$. Let S_k be the set of all endomorphisms of q which factor through some p_j with $j \leq k$. Then $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_n = S$ $(M = S_{n+1} = S \cup \{1\})$ are all subsemigroups of S. As in the proof of condition (iv) above it is easy to see that

$$S_k/S_{k-1} = \{\alpha_i\beta_j \mid \alpha_i : p_k \to q \text{ and } \beta_j : q \to p_k \text{ are as in (iv)}\}.$$

Therefore S_k/S_{k-1} is isomorphic to $\mathscr{M}^0(1; m, m, P)$ with P the matrix $M(p_k, q)$ of condition (iv), hence invertible over $\mathbb{Z}$.

It should be noted that Cheng's Theorem B requires the notion of 0-simple semigroups and a Theorem of Rees characterizing such, which our result does not use. However, Theorem B does allow for coefficient rings other than $\mathbb{Z}$. A generalization of our results to coefficient rings other than $\mathbb{Z}$ will form part of a forthcoming paper by B. Mitchell, [7], and will use techniques similar to those in this paper rather than results of Rees.

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