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Modules cofinite with respect to an ideal

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1. Introduction

We continue the study of modules cofinite with respect to an ideal of a (noetherian commutative) ring. Hartshorne [8] introduced this class of modules, answering in negative a question of Grothendieck [6, Exposé XIII, Conjecture 1.1]. He asked if the modules $\mathrm{Hom}_A(A/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ always are finitely generated for every ideal $\mathfrak{a} \subset A$ and each finite A-module M. This is the case when $\mathfrak{a} = \mathfrak{m}$, the maximal ideal in a local ring, since the modules $\mathrm{H}^i_{\mathfrak{m}}(M)$ are artinian. Hartshorne defined a module M to be $\mathfrak{a}$-cofinite, if $\mathrm{Supp}_A M \subset \mathrm{V}(\mathfrak{a})$ and $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ is a finite module for all i. He proved that the local cohomology modules $\mathrm{H}^i_{\mathfrak{p}}(M)$ are $\mathfrak{p}$-cofinite for all finite modules M over a complete regular local ring A, when $\mathfrak{a} = \mathfrak{p}$ is a prime ideal of A, such that $\dim A/\mathfrak{p} = 1$. This result was later extended to more general local rings and one-dimensional ideals $\mathfrak{a}$ by Huneke and Koh in [9] and by Delfino in [3] until finally Delfino and Marley in [4] and K.-I. Yoshida in [22] proved that the local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite for all finite A-modules M, where the ideal $\mathfrak{a}$ of a local ring A, satisfies $\dim A/\mathfrak{a} = 1$.

Instead of requiring the finiteness of the modules $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ in the definition of $\mathfrak{a}$-cofiniteness, we showed in [16] that one could require the finiteness of the Koszul cohomology modules $\mathrm{H}^i(x_1, \ldots, x_n; M)$, where $x_1, \ldots, x_n$ are generators for $\mathfrak{a}$. Our proof used the change of rings principle of Delfino and Marley [4, Proposition 2]. They proved the change of rings principle using a spectral sequence argument. We are however avoiding the use of spectral sequences completely in this work, even if we can show some of our results with this technique. So we provide an elementary proof of the equivalence of the

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finiteness for all i of the modules $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$, $\mathrm{Tor}^A_i(A/\mathfrak{a}, M)$ and $\mathrm{H}^i(x_1, \ldots, x_n; M)$ in Theorem 2.1. This theorem is then used to deduce the change of rings principle (see 2.6).

In Section 3 we give various conditions for cofiniteness. Very useful, in particular in induction arguments, is the criterion given in 3.4. We prove there that for a module M with support in $\mathrm{V}(\mathfrak{a})$ to be $\mathfrak{a}$-cofinite, it is sufficient to find an element $x \in \mathfrak{a}$, such that $0 :_M x$ and M/xM are $\mathfrak{a}$-cofinite.

In [16] we showed that an artinian module M with support in $\mathrm{V}(\mathfrak{a})$ is $\mathfrak{a}$-cofinite if and only if $0 :_M \mathfrak{a}$ has finite length. We extend this result to the class of minimax modules in 4.3. A module is called a minimax module, when it has a finite submodule, such that the quotient by it is an artinian module [24]. We also show that if $\dim A = 1$, then each $\mathfrak{a}$-cofinite module is a minimax module and that all submodules and quotients of $\mathfrak{a}$-cofinite modules are $\mathfrak{a}$-cofinite.

In Section 5 we study the top local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(M)$ of a module M over a ring of finite Krull dimension d. We show that it is an artinian $\mathfrak{a}$-cofinite module if M is finite and when M is no longer finite but if $\mathrm{H}^d_{\mathfrak{a}}(M)$ is artinian it must be $\mathfrak{a}$-cofinite. In the proofs of these results we use the theory of asymptotic prime divisors, see [13]. A prime ideal $\mathfrak{p} \supset \mathfrak{a}$ is called a quintasymptotic prime divisor of the ideal $\mathfrak{a}$, if there is a minimal prime ideal $\mathfrak{q}$ in the completion $\widehat{A_{\mathfrak{p}}}$ of the local ring $A_{\mathfrak{p}}$, such that its maximal ideal $\mathfrak{p}\widehat{A_{\mathfrak{p}}}$ is minimal over $\mathfrak{a}\widehat{A_{\mathfrak{p}}} + \mathfrak{q}$. If $\mathfrak{p}$ is a quintasymptotic prime divisor of $\mathfrak{a}$, then $\mathfrak{p}$ is an asymptotic prime divisor of $\mathfrak{a}$ in the sense that it is an associated prime ideal of the integral closure $(\mathfrak{a}^n)^*$ of the powers $\mathfrak{a}^n$ for large n and therefore also an associated prime ideal for $\mathfrak{a}^n$ for all large n. There are just a finite number of asymptotic primes of an ideal $\mathfrak{a}$.

In 5.5 we decide when all local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ are artinian in the range $i \leqslant r$ or for all i, extending a result of Lescot [11], who treated the case when $\mathfrak{a}$ is the maximal ideal of a local ring.

In 6.5 we show that if $\mathfrak{a}$ is generated by a sequence which is filter-regular on the finite module M, then all local cohomology modules $\mathrm{H}^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite.

In the last section we deal with the problem, when the kernel (hence also the cokernel and the image) of a homomorphism between $\mathfrak{a}$-cofinite modules is again $\mathfrak{a}$-cofinite. Hartshorne [8] showed that this is the case, when $\mathfrak{a}$ is a one-dimensional prime ideal of a complete regular local ring. This was later generalized by Delfino and Marley [4] to the case of a one-dimensional prime ideal in any complete local ring. However it is not known to hold even for a one-dimensional prime ideal of a local ring which is not complete. After passing to the completion the extended ideal may no longer be prime! Our efforts to solve this problem for a one-dimensional ideal $\mathfrak{a}$ in any local ring has considerably delayed the publication of this paper. Many of our results did we obtain quite a time ago and have been presented at seminars at various universities and conferences. We did not succeed to answer the question, but we have succeeded to reduce the question (in order to get a positive answer) to the study of certain local cohomology modules in 7.12. Namely is $\Gamma_{\mathfrak{p}}(M)$ (or equivalently $\mathrm{H}^1_{\mathfrak{p}}(M)$) $\mathfrak{p}$-cofinite for all prime ideals $\mathfrak{p}$ minimal over $\mathfrak{a}$ whenever M is an $\mathfrak{a}$-cofinite module over a complete local ring A and $\dim A/\mathfrak{a} = 1$ in 7.12. We have also succeeded to prove positive solutions for the case $\dim A \leqslant 2$ in 7.4 and 7.11.

2. Equivalent conditions for cofiniteness with respect to an ideal

Theorem 2.1. *Let* $\mathfrak{a} = (x_1, \ldots, x_n)$ *be an ideal in* A *and let* M *be an* A*-module. Then the following conditions are equivalent*:

(i) $\mathrm{Ext}^i_A(A/\mathfrak{a}, M)$ *is a finite* A*-module for all* i.
(ii) $\mathrm{Tor}^A_i(A/\mathfrak{a}, M)$ *is a finite* A*-module for all* i.
(iii) *The Koszul cohomology modules* $\mathrm{H}^i(x_1, \ldots, x_n; M)$ *are finite* A*-modules for* $i = 0, \ldots, n$.

Definition 2.2. An A-module M, such that $\mathrm{Supp}_A(M) \subset \mathrm{V}(\mathfrak{a})$, and which satisfies the equivalent conditions in the above theorem is called cofinite with respect to the ideal $\mathfrak{a}$ or shorter $\mathfrak{a}$-cofinite.

For the proof of our theorem we need two lemmata. In proving these we use the observation that if $0 :_M \mathfrak{a}$ is a finite A-module, then so is $0 :_M \mathfrak{a}^n$ for each n. Similarly if $M/\mathfrak{a}M$ is a finite A-module, then also $M/\mathfrak{a}^n M$ is finite for all n.

Lemma 2.3. *Let* $\mathcal{C}$ *be a class of* A*-modules and* $\mathfrak{a}$ *an ideal such that the following conditions are fulfilled*:

(1) *If* $0 \to M' \to M \to M'' \to 0$ *is exact, where* M' *and* M *are in* $\mathcal{C}$, *then also* M'' *is in* $\mathcal{C}$.
(2) $0 :_M \mathfrak{a}$ *is a finite* A*-module for every* M *in* $\mathcal{C}$.
(3) *Every finite* A*-module* M *such that* $\mathrm{Supp}\, M \subset \mathrm{V}(\mathfrak{a})$ *is in* $\mathcal{C}$.

Let $X^\bullet : 0 \to X^0 \to X^1 \to X^2 \to \cdots$ *be a cochain complex with modules in* $\mathcal{C}$, *such that for each* i *there is* n *with* $\mathfrak{a}^n\, \mathrm{H}^i(X^\bullet) = 0$. *Then* $\mathrm{H}^i(X^\bullet)$ *is a finite* A*-module for each* i.

Proof. Let B^i, respectively Z^i, be the modules of coboundaries and cocycles, so the cohomology modules are $H^i = Z^i/B^i$. Assume that B^i belongs to $\mathcal{C}$ for a certain i. Then $C^i = X^i/B^i$ also belongs to $\mathcal{C}$ by (1). Take n such that $\mathfrak{a}^n H^i = 0$. Then $H^i \subset 0 :_{C^i} \mathfrak{a}^n$, which is finite by (2) and therefore H^i is finite and thus belongs to $\mathcal{C}$, by (3). By (1) again and the exact sequence

$$0 \to H^i \to C^i \to B^{i+1} \to 0,$$

we get that B^{i+1} is also in $\mathcal{C}$ and we can continue by induction. □

The second lemma is dual to the previous one.

Lemma 2.4. *Let* $\mathfrak{a}$ *be an ideal of* A *and suppose* $\mathcal{C}$ *is a class of* A*-modules with the following properties*:

(1) *If* $0 \to M' \to M \to M'' \to 0$ *is exact and both* M *and* M'' *are in* $\mathcal{C}$*, then also* M' *is in* $\mathcal{C}$*.*
(2) $M/\mathfrak{a}M$ *is a finite* A*-module for each* M *in* $\mathcal{C}$*.*
(3) *Every finite* A*-module* M *such that* $\operatorname{Supp} M \subset \mathrm{V}(\mathfrak{a})$ *is in* $\mathcal{C}$*.*

Let now $X_\bullet : \cdots \to X_2 \to X_1 \to X_0 \to 0$ *be a chain complex of modules in* $\mathcal{C}$*, such that for each* i *there is* n *such that* $\mathfrak{a}^n \mathrm{H}_i(X_\bullet) = 0$*. Then* $\mathrm{H}_i(X_\bullet)$ *is a finite* A*-module for each* i*.*

Proof. Let Z_i, respectively B_i, be the modules of cycles and boundaries, so $H_i = Z_i/B_i$ is the ith homology module. Put also $C_i = X_i/Z_i$. Assume that Z_i is in $\mathcal{C}$ for a certain i. Take n such that $\mathfrak{a}^n H_i = 0$. Consider the exact sequence $0 \to C_{i+1} \to Z_i \to H_i \to 0$.

First we conclude that H_i is a homomorphic image of $Z_i/\mathfrak{a}^n Z_i$, which is a finite A-module by (2). Hence H_i is finite and therefore in $\mathcal{C}$ by (3). The exact sequence above and (1) implies that C_{i+1} is in $\mathcal{C}$ and by (1) again so is Z_{i+1} and we can proceed by induction. □

Proof of Theorem 2.1. Let $F_\bullet \to A/\mathfrak{a}$ be a resolution of $A/\mathfrak{a}$ consisting of finite free A-modules. We apply 2.3 and 2.4 to the complexes

$$\operatorname{Hom}_A(F_\bullet, M), \quad F_\bullet \otimes_A M, \quad K^\bullet(x_1, \dots, x_n; M), \quad K_\bullet(x_1, \dots, x_n; M).$$

In addition we use that $\mathrm{H}^i(x_1, \dots, x_n; M) \cong \mathrm{H}_{n-i}(x_1, \dots, x_n; M)$ for every i. □

Corollary 2.5. *If* M *satisfies the conditions of* 2.1 *and* N *is a finite* A*-module with* $\operatorname{Supp}_A(N) \subset \mathrm{V}(\mathfrak{a})$*, then* $\operatorname{Ext}^i_A(N, M)$ *and* $\operatorname{Tor}^A_i(N, M)$ *are finite* A*-modules for all* i*.*

Proof. Apply 2.3 and 2.4 to the complexes $\operatorname{Hom}_A(F_\bullet, M)$, and $F_\bullet \otimes_A M$, where $F_\bullet \to N$ is a resolution of N consisting of finite free modules. □

Corollary 2.6 (Change of rings principle). *Let* $\varphi : A \to B$ *be a homomorphism between noetherian rings and suppose that* $\mathfrak{a}$ *is an ideal of* A *such that* $B/\mathfrak{a}B$ *is a finite* A*-module. Then a* B*-module* M *is cofinite with respect to the extended ideal* $\mathfrak{a}B$ *if and only if* M *considered as an* A*-module is cofinite with respect to* $\mathfrak{a}$*.*

Proof. Let $\mathfrak{a} = (x_1, \dots, x_n)$, so $\mathfrak{a}B = (\phi(x_1), \dots, \phi(x_n))$. The B-module $\mathrm{H}^i(\phi(x_1), \dots, \phi(x_n); M)$ is annihilated by $\mathfrak{a}B$ and considered as a module over A it is isomorphic to $\mathrm{H}^i(x_1, \dots, x_n; M)$. Therefore the assertion follows from our hypothesis that $B/\mathfrak{a}B$ is a finite A-module. □

Submodules of $\mathfrak{a}$-cofinite modules are seldom $\mathfrak{a}$-cofinite. Direct summands are of course and more generally:

Proposition 2.7. *If* M *is* $\mathfrak{a}$*-cofinite and* N *is a pure submodule of* M*, then* N *and* M/N *are also* $\mathfrak{a}$*-cofinite.*

Proof. In fact, P.M. Cohns characterization of purity (see [17, Theorem 3.65]) can be restated as follows.

$N \subset M$ is pure if and only if for any chain complex $F_\bullet$ consisting of finite free modules, the induced cochain map $\alpha : \mathrm{Hom}_A(F_\bullet, N) \to \mathrm{Hom}_A(F_\bullet, M)$ has the property that if $f \in \mathrm{Hom}_A(F_\bullet, N)$ is taken by α to a coboundary in $\mathrm{Hom}_A(F_\bullet, M)$, then f already is a coboundary in $\mathrm{Hom}_A(F_\bullet, N)$. Equivalently the maps $\mathrm{H}^i(\alpha) : \mathrm{H}^i(\mathrm{Hom}_A(F_\bullet, N)) \to \mathrm{H}^i(\mathrm{Hom}_A(F_\bullet, M))$ are injective for all i. Let now $F_\bullet$ be a resolution of $A/\mathfrak{a}$ consisting of finite free modules. Consequently, if N is a pure submodule of M, then

$$0 \to \mathrm{Ext}^i_A(A/\mathfrak{a}, N) \to \mathrm{Ext}^i_A(A/\mathfrak{a}, M) \to \mathrm{Ext}^i_A(A/\mathfrak{a}, M/N) \to 0$$

is exact for all i. Hence if M is $\mathfrak{a}$-cofinite, then so are N and M/N. □

3. Criteria for cofiniteness

We begin with a homological lemma.

Lemma 3.1. *Let S and T be additive functors between the abelian categories $\mathcal{A}$ and $\mathcal{B}$ and let $\mathcal{S}$ be a Serre subcategory of $\mathcal{B}$, i.e., $\mathcal{S}$ is closed under taking subobjects, quotients and extensions.*

Suppose that every exact sequence $0 \to X' \xrightarrow{u} X \xrightarrow{v} X'' \to 0$ gives rise to an exact sequence

$$SX' \xrightarrow{Su} SX \xrightarrow{Sv} SX'' \to TX' \xrightarrow{Tu} TX \xrightarrow{Tv} TX''.$$

If $f : M \to N$ is a morphism in $\mathcal{A}$ such that $T\,\mathrm{Ker}\, f$ and $S\,\mathrm{Coker}\, f$ are in $\mathcal{S}$, then also $\mathrm{Ker}\, Tf$ and $\mathrm{Coker}\, Sf$ are in $\mathcal{S}$. If in addition $Tf = 0$ (respectively $Sf = 0$), then TM (respectively SN) is in $\mathcal{S}$.

Proof. We have two exact sequences

$$0 \to K \to M \xrightarrow{g} I \to 0 \quad \text{and} \quad 0 \to I \xrightarrow{h} N \to C \to 0,$$

where $K = \mathrm{Ker}\, f$, $I = \mathrm{Im}\, f$, $C = \mathrm{Coker}\, f$ and $f = h \circ g$.

We get exact sequences

$$SK \to SM \xrightarrow{Sg} SI \to TK \to TM \xrightarrow{Tg} TI,$$

$$SI \xrightarrow{Sh} SN \to SC \to TI \xrightarrow{Th} TN \to TC.$$

Hence $\mathrm{Coker}\, Sg$ is a subobject of TK and thus in $\mathcal{S}$ and $\mathrm{Ker}\, Tg$ is a quotient of TK and therefore also in $\mathcal{S}$. In a similar way, we use the second exact sequence to deduce that

$\operatorname{Ker} Th$ and $\operatorname{Coker} Sh$ are in $\mathcal{S}$. Since $Tf = Th \circ Tg$ and $Sf = Sh \circ Sg$, there are exact sequences

$$0 \to \operatorname{Ker} Tg \to \operatorname{Ker} Tf \to \operatorname{Ker} Th$$

and

$$\operatorname{Coker} Sg \to \operatorname{Coker} Sf \to \operatorname{Coker} Sh \to 0.$$

Hence $\operatorname{Ker} Tf$ and $\operatorname{Coker} Sf$ are in $\mathcal{S}$. □

Corollary 3.2. *Let (T^i) be a connected sequence of functors between $\mathcal{A}$ and $\mathcal{B}$ and let $f : M \to N$ be a morphism in $\mathcal{A}$. If for a certain i, $T^i \operatorname{Coker} f$ and $T^{i+1} \operatorname{Ker} f$ both belong to $\mathcal{S}$, then $\operatorname{Coker} T^i f$ and $\operatorname{Ker} T^{i+1} f$ also belong to $\mathcal{S}$. If for all i, $T^i \operatorname{Ker} f$ and $T^i \operatorname{Coker} f$ belong to $\mathcal{S}$ and $T^i f = 0$, then $T^i M$ and $T^i N$ belong to $\mathcal{S}$ for all i.*

Corollary 3.3. *Let $f : M \to N$ be an A-linear map. If for all i, the A-modules $\operatorname{Ext}^i_A(A/\mathfrak{a}, \operatorname{Ker} f)$ and $\operatorname{Ext}^i_A(A/\mathfrak{a}, \operatorname{Coker} f)$ are finite, then $\operatorname{Ker} \operatorname{Ext}^i_A(A/\mathfrak{a}, f)$ and $\operatorname{Coker} \operatorname{Ext}^i_A(A/\mathfrak{a}, f)$ are also finite for all i.*

The following corollary is a very useful criterion in order to decide whether a module is cofinite, especially in induction arguments.

Corollary 3.4. *Suppose $x \in \mathfrak{a}$ and $\operatorname{Supp}_A M \subset \mathrm{V}(\mathfrak{a})$. If $0 :_M x$ and M/xM are both $\mathfrak{a}$-cofinite, then M must also be $\mathfrak{a}$-cofinite.*

Proof. Apply 3.3 to the map $f = x1_M$. Since $x \in \mathfrak{a}$, $\operatorname{Ext}^i_A(A/\mathfrak{a}, f) = 0$ for all i. □

Here is a generalization to a matrix situation.

Corollary 3.5. *Let $u : F \to G$ be a homomorphism between nonzero finite free modules, such that $u(F) \subset \mathfrak{a}G$, i.e., after choosing bases in E and F, the matrix of u has its elements in $\mathfrak{a}$. Put $f = \operatorname{Hom}(u, M)$, and suppose that $\operatorname{Supp}_A M \subset \mathrm{V}(\mathfrak{a})$. If $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are both $\mathfrak{a}$-cofinite, then M must also be an $\mathfrak{a}$-cofinite module.*

Proof. We can write $f = \sum_{j,k} a_{j,k} f_{j,k}$ with some elements $a_{j,k} \in \mathfrak{a}$ and some maps $f_{j,k} : \operatorname{Hom}_A(G, M) \to \operatorname{Hom}_A(F, M)$. Hence for all i,

$$\operatorname{Ext}^i_A(A/\mathfrak{a}, f) = \sum_{j,k} a_{j,k} \operatorname{Ext}^i_A(A/\mathfrak{a}, f_{j,k}) = 0,$$

since $a_{j,k} \in \mathfrak{a}$ for all j, k. □

Corollary 3.6. *Let M be an A-module with $\operatorname{Supp}_A(M) \subset \mathrm{V}(\mathfrak{a})$. Suppose the endomorphism $f \in \operatorname{End}_A(M)$, satisfies a polynomial equation $f^n + a_1 f^{n-1} + \cdots + a_n = 0$, where $a_j \in \mathfrak{a}$, $1 \leqslant j \leqslant n$. If both $\operatorname{Ker} f$ and $\operatorname{Coker} f$ are $\mathfrak{a}$-cofinite, then M is $\mathfrak{a}$-cofinite.*

Proof. $\operatorname{Ext}^i_A(A/\mathfrak{a}, f^n) = \sum_{j=1}^n -a_j \operatorname{Ext}^i_A(A/\mathfrak{a}, f^{n-j}) = 0$ for all i. Note also that if u is an endomorphism on some module X, then $\operatorname{Ker} u$ is finite if and only if $\operatorname{Ker} u^n$ is finite for all n, if and only if $\operatorname{Ker} u^n$ is finite for some n. There is an analogous statement concerning the cokernels of u and its powers. □

Proposition 3.7. *If* $\operatorname{Supp}_A(M) \subset \mathrm{V}(\mathfrak{a})$, $\mathfrak{b} \subset \mathfrak{a}$ *and* $\operatorname{Ext}^i_A(A/\mathfrak{b}, M)$ *is* $\mathfrak{a}$*-cofinite for all* i*, then* M *is* $\mathfrak{a}$*-cofinite.*

Proof. Let $\bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $\bar{A} = A/\mathfrak{b}$. By the change of rings principle 2.6, the $\bar{A}$-modules $\operatorname{Ext}^i_A(A/\mathfrak{b}, M)$ are cofinite with respect to $\bar{\mathfrak{a}}$. Let now $0 \to M \to E^0 \xrightarrow{\partial^0} E^1 \xrightarrow{\partial^1} E^2 \xrightarrow{\partial^2} \cdots$ be an injective resolution of the A-module M. We split this into short exact sequences $0 \to M^i \to E^i \to M^{i+1} \to 0$, where $M^i = \operatorname{Ker} \partial^i$, $i = 0, 1, 2, \ldots$.

Observe that for each $i \geqslant 0$,

$$\operatorname{Ext}^{i+1}_A(A/\mathfrak{a}, M) \cong \operatorname{Ext}^1_A\big(A/\mathfrak{a}, M^i\big) \quad \text{and}$$
$$\operatorname{Ext}^{i+1}_A(A/\mathfrak{b}, M) \cong \operatorname{Ext}^1_A\big(A/\mathfrak{b}, M^i\big).$$

We first show by induction on i, that $\operatorname{Ext}^j_{\bar{A}}(\bar{A}/\bar{\mathfrak{a}}, 0 :_{M^i} \mathfrak{b})$ is a finite $\bar{A}$-module for all $j \geqslant 1$. Since $M^0 \cong M$ and $0 :_M \mathfrak{b}$ is an $\bar{\mathfrak{a}}$-cofinite $\bar{A}$-module, this is evidently true for $i = 0$. Suppose that this is true for a certain i. Consider the $\bar{A}$-module homomorphism $f_i : 0 :_{E^i} \mathfrak{b} \to 0 :_{M^{i+1}} \mathfrak{b}$. Since $\operatorname{Ker} f_i = 0 :_{M^i} \mathfrak{b}$ and $\operatorname{Coker} f_i \cong \operatorname{Ext}^1_A(A/\mathfrak{b}, M^i) \cong \operatorname{Ext}^{i+1}_A(A/\mathfrak{b}, M)$, the $\bar{A}$-modules $\operatorname{Ext}^{j+1}_{\bar{A}}(\bar{A}/\bar{\mathfrak{a}}, \operatorname{Ker} f_i)$ and $\operatorname{Ext}^j_{\bar{A}}(\bar{A}/\bar{\mathfrak{a}}, \operatorname{Coker} f_i)$ are finite for all $j \geqslant 0$. Therefore 3.2 implies that $\operatorname{Coker} \operatorname{Ext}^j_{\bar{A}}(\bar{A}/\bar{\mathfrak{a}}, f_i)$ is a finite $\bar{A}$-module for all $j \geqslant 0$. For $j = 0$, we get that $\operatorname{Ext}^1_A(A/\mathfrak{a}, M^i) \cong \operatorname{Ext}^{i+1}_A(A/\mathfrak{a}, M)$ is finite. Since $0 :_{E^i} \mathfrak{b}$ is an injective $\bar{A}$-module, $\operatorname{Ext}^j_{\bar{A}}(\bar{A}/\bar{\mathfrak{a}}, 0 :_{E^i} \mathfrak{b}) = 0$ for $j \geqslant 1$. Hence $\operatorname{Coker} \operatorname{Ext}^j_{\bar{A}}(\bar{A}/\bar{\mathfrak{a}}, f_i) \cong \operatorname{Ext}^j_{\bar{A}}(\bar{A}/\bar{\mathfrak{a}}, 0 :_{M^{i+1}} \mathfrak{b})$ when $j \geqslant 1$. The induction argument is ready and at the same time the proof shows that $\operatorname{Ext}^j_A(A/\mathfrak{a}, M)$ is finite for all $j \geqslant 1$. For $j = 0$ this is clear, since $\operatorname{Hom}_A(A/\mathfrak{a}, M) \cong \operatorname{Hom}_A(A/\mathfrak{a}, \operatorname{Hom}_A(A/\mathfrak{b}, M))$. Hence M is $\mathfrak{a}$-cofinite. □

Corollary 3.8. *Let* $x_1, \ldots, x_r$ *be elements in* $\mathfrak{a}$ *and let* M *be an* A*-module with support in* $\mathrm{V}(\mathfrak{a})$*. If for all* i *the modules* $\mathrm{H}^i(x_1, \ldots, x_r; M)$ *are* $\mathfrak{a}$*-cofinite, then* M *must be* $\mathfrak{a}$*-cofinite.*

Proof. We apply the change of rings principle 2.6 to the surjective homomorphism $\varphi : B \to A$, where $B = A[X_1, \ldots, X_r]$, defined by $\varphi(X_i) = x_i$ for $i = 1, \ldots, r$. M becomes a B-module with support in $\mathfrak{a}B + (X_1, \ldots, X_r)$ and this ideal is mapped onto $\mathfrak{a}$ by φ. Since the sequence $X_1, \ldots, X_r$ is regular on B, we get for all i that

$$\mathrm{H}^i(x_1, \ldots, x_r; M) \cong \mathrm{H}^i(X_1, \ldots, X_r; M) \cong \operatorname{Ext}^i_B\big(B/(X_1, \ldots, X_r), M\big). \qquad \square$$

Proposition 3.9. *Let* $\mathcal{S}$ *be a full subcategory of the category of* A*-modules closed under taking kernels, cokernels and extensions and let* n *be a natural number. If* M *is an* A*-module,* *such that* $\operatorname{Ext}_A^j(A/\mathfrak{a}, \mathrm{H}_\mathfrak{a}^i(M))$ *belongs to* $\mathcal{S}$ *for all* i *and all* j (*respectively for* $i \leqslant n$ *and all* j), *then* $\operatorname{Ext}_A^i(A/\mathfrak{a}, M)$ *belongs to* $\mathcal{S}$ *for all* i (*respectively for all* $i \leqslant n$).

Proof. The case $n = 0$ is clear, so let $n > 0$ and we do induction on n. We first reduce to the case $\Gamma_\mathfrak{a}(M) = 0$. This is possible, since if we let $\bar{M} = M/\Gamma_\mathfrak{a}(M)$, we have the long exact sequence

$$\begin{aligned} \cdots &\to \operatorname{Ext}_A^{i-1}(A/\mathfrak{a}, \bar{M}) \\ &\to \operatorname{Ext}_A^i\big(A/\mathfrak{a}, \Gamma_\mathfrak{a}(M)\big) \to \operatorname{Ext}_A^i(A/\mathfrak{a}, M) \to \operatorname{Ext}_A^i(A/\mathfrak{a}, \bar{M}) \\ &\to \operatorname{Ext}_A^{i+1}\big(A/\mathfrak{a}, \Gamma_\mathfrak{a}(M)\big) \to \cdots \end{aligned}$$

and isomorphisms

$$\mathrm{H}_\mathfrak{a}^i(\bar{M}) \cong \begin{cases} 0, & \text{if } i = 0, \\ \mathrm{H}_\mathfrak{a}^i(M), & \text{if } i > 0. \end{cases}$$

So let us assume that $\Gamma_\mathfrak{a}(M) = 0$. Let E be an injective hull of M and put $L = E/M$. Then also $\Gamma_\mathfrak{a}(E) = 0$ and $\operatorname{Hom}_A(A/\mathfrak{a}, E) = 0$, and we therefore get isomorphisms $\mathrm{H}_\mathfrak{a}^i(L) \cong \mathrm{H}_\mathfrak{a}^{i+1}(M)$ and $\operatorname{Ext}_A^i(A/\mathfrak{a}, L) \cong \operatorname{Ext}_A^{i+1}(A/\mathfrak{a}, M)$ for all $i \geqslant 0$. □

Corollary 3.10. *If* $\mathrm{H}_\mathfrak{a}^i(M)$ *is* $\mathfrak{a}$*-cofinite for all* i (*respectively for all* $i \leqslant n$), *then* $\operatorname{Ext}_A^i(A/\mathfrak{a}, M)$ *is a finite* A*-module for each* i (*respectively for* $i \leqslant n$).

The next result has been shown using a spectral sequence argument by T. Marley and J. Vassilev in [12, Proposition 2.5] under the assumption that M is finite. We give a direct proof with a weaker assumption on M which is needed in our applications.

Proposition 3.11. *Let* M *be a module, such that* $\operatorname{Ext}_A^i(A/\mathfrak{a}, M)$ *is a finite* A*-module for every* i, *for example* M *might be a finite* A*-module. If* s *is a number, such that* $\mathrm{H}_\mathfrak{a}^i(M)$ *is* $\mathfrak{a}$*-cofinite for all* $i \neq s$, *then this is the case also when* $i = s$.

Proof. We use induction on s. Let $\bar{M} = M/\Gamma_\mathfrak{a}(M)$. Then

$$\mathrm{H}_\mathfrak{a}^i(\bar{M}) \cong \begin{cases} 0, & \text{if } i = 0, \\ \mathrm{H}_\mathfrak{a}^i(M), & \text{if } i > 0. \end{cases}$$

If $s = 0$, then $\mathrm{H}_\mathfrak{a}^i(\bar{M})$ is $\mathfrak{a}$-cofinite for all i, so by 3.10, $\operatorname{Ext}_A^i(A/\mathfrak{a}, \bar{M})$ is a finite A-module for every i. Therefore the exactness of $0 \to \Gamma_\mathfrak{a}(M) \to M \to \bar{M} \to 0$ implies that $\operatorname{Ext}_A^i(A/\mathfrak{a}, \Gamma_\mathfrak{a}(M))$ is finite for all i, that is, $\Gamma_\mathfrak{a}(M)$ is $\mathfrak{a}$-cofinite. Suppose then that $s > 0$ and the case $s - 1$ is settled. Since $\Gamma_\mathfrak{a}(M)$ is $\mathfrak{a}$-cofinite, $\operatorname{Ext}_A^i(A/\mathfrak{a}, \Gamma_\mathfrak{a}(M))$ is finite and by hypothesis $\operatorname{Ext}_A^i(A/\mathfrak{a}, M)$ is finite. Hence $\operatorname{Ext}_A^i(A/\mathfrak{a}, \bar{M})$ is finite for all i. We may therefore assume that $\Gamma_\mathfrak{a}(M) = 0$. Let E be an injective hull of M and put $M_1 = E/M$. Then also $\Gamma_\mathfrak{a}(E) = 0$ and $\operatorname{Hom}_A(A/\mathfrak{a}, E) = 0$. Consequently $\operatorname{Ext}_A^i(A/\mathfrak{a}, M_1) \cong$

$\mathrm{Ext}_A^{i+1}(A/\mathfrak{a}, M)$ and $\mathrm{H}_\mathfrak{a}^i(M_1) \cong \mathrm{H}_\mathfrak{a}^{i+1}(M)$ for all $i \geqslant 0$ (including the case $i = 0$). The induction hypothesis applied to M_1, yields the $\mathfrak{a}$-cofiniteness of $\mathrm{H}_\mathfrak{a}^{s-1}(M_1)$. Hence $\mathrm{H}_\mathfrak{a}^s(M)$ which is isomorphic to it is $\mathfrak{a}$-cofinite. □

Corollary 3.12. *If M is a finite module such that $\mathrm{H}_\mathfrak{a}^i(M) = 0$ for all $i \neq s$, i.e.,* $\mathrm{depth}_\mathfrak{a} M = \mathrm{cd}(\mathfrak{a}, M) = s$, *then $\mathrm{H}_\mathfrak{a}^s(M)$ is $\mathfrak{a}$-cofinite.*

Corollary 3.13. *If M is finite and $\mathfrak{a}$ is generated by an M-regular sequence, then $\mathrm{H}_\mathfrak{a}^i(M)$ is $\mathfrak{a}$-cofinite for all i.*

Corollary 3.14. *If* $\mathrm{cd}\,\mathfrak{a} = 1$, *then $\mathrm{H}_\mathfrak{a}^i(M)$ is $\mathfrak{a}$-cofinite for all i and every finite A-module M.*

Proposition 3.15. *Suppose* $\mathrm{cd}\,\mathfrak{b} = 1$ *and $\mathfrak{a} \subset \mathfrak{b}$ and let M be a finite A-module. Then $\mathrm{H}_\mathfrak{b}^i(\mathrm{H}_\mathfrak{a}^j(M))$ is $\mathfrak{b}$-cofinite for all i and j.*

Proof. Since $\mathrm{cd}\,\mathfrak{b} = 1$, the ring $\mathrm{D}_\mathfrak{b}(A)$ is noetherian and $\mathfrak{b}\,\mathrm{D}_\mathfrak{b}(A) = \mathrm{D}_\mathfrak{b}(A)$, [19] or [2, p. 112]. Hence $\mathrm{H}_\mathfrak{b}^i(N) \cong \mathrm{H}_{\mathfrak{b}\,\mathrm{D}_\mathfrak{b}(A)}^i(N) = 0$ for all i and every $\mathrm{D}_\mathfrak{b}(A)$-module N. Since $L = \Gamma_\mathfrak{a}(M)$ is finite, $\mathrm{H}_\mathfrak{b}^i(L)$ is $\mathfrak{b}$-cofinite for all i by 3.14. Moreover $\mathrm{H}_\mathfrak{a}^i(M) \cong \mathrm{H}_\mathfrak{a}^i(M/L)$ for $i > 0$. We may therefore assume that $\Gamma_\mathfrak{a}(M) = 0$ and therefore also $\Gamma_\mathfrak{b}(M) = 0$. Consider the exact sequence $0 \to M \to \mathrm{D}_\mathfrak{b}(M) \to \mathrm{H}_\mathfrak{b}^1(M) \to 0$. We get the exact sequence $0 \to L_0 \to \mathrm{H}_\mathfrak{b}^1(M) \to \mathrm{H}_\mathfrak{a}^1(M) \to L_1 \to 0$ and isomorphisms $\mathrm{H}_\mathfrak{a}^i(M) \cong L_i$ for all $i \geqslant 2$, where $L_i = \mathrm{H}_\mathfrak{a}^i(\mathrm{D}_\mathfrak{b}(M))$, $i = 0, 1, 2, \ldots$. However L_i is a module over the ring $\mathrm{D}_\mathfrak{b}(A)$ and therefore as remarked above $\mathrm{H}_\mathfrak{b}^j(L_i) = 0$ for all i and j. Now we easily get that

$$\mathrm{H}_\mathfrak{b}^j\big(\mathrm{H}_\mathfrak{a}^i(M)\big) \cong \begin{cases} \mathrm{H}_\mathfrak{b}^1(M), & \text{if } j = 0, i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathrm{H}_\mathfrak{b}^1(M)$ is $\mathfrak{b}$-cofinite by 3.14, we are finished. □

Corollary 3.16. *If* $\mathrm{cd}\,\mathfrak{b} = 1$ *and $\mathfrak{a} \subset \mathfrak{b}$, then for every finite A-module M and every finite A-module N, such that* $\mathrm{Supp}_A N \subset \mathrm{V}(\mathfrak{b})$, *the modules* $\mathrm{Ext}_A^i(N, \mathrm{H}_\mathfrak{a}^j(M))$ *are finite for all i and j.*

Proof. Apply 3.10 and 2.5. □

Remark 3.17. Kawasaki [10] showed the conclusion in 3.16 assuming that the ideal $\mathfrak{b}$ satisfies the stronger condition that it is (up to radical) principal.

4. Minimax modules

Proposition 4.1. *Let M be a module with support in* $\mathrm{V}(\mathfrak{a})$. *M is artinian and $\mathfrak{a}$-cofinite if and only if $0 :_M \mathfrak{a}$ has finite length. If there is an element $x \in \mathfrak{a}$, such that $0 :_M x$ is artinian and $\mathfrak{a}$-cofinite, then M is artinian and $\mathfrak{a}$-cofinite.*

Proof. If $0 :_M \mathfrak{a}$ is artinian and $\operatorname{Supp}_A M \subset \mathrm{V}(\mathfrak{a})$, then M is artinian [14, Theorem 1.3]. The first assertion follows from the local case [16, Theorem 1.6]. There is also a direct proof in [15, Theorem 5.1].

If $L = 0 :_M x$ is $\mathfrak{a}$-cofinite, then $0 :_M \mathfrak{a} = 0 :_L \mathfrak{a}$ is finite. □

The same applies to a bigger class of modules, namely the class of minimax modules.

Definition 4.2 (see Zöschinger [24]). The A-module M is a minimax module, if there is a finite submodule N of M, such that M/N is artinian.

The class of minimax modules thus includes all finite and all artinian modules. Moreover, it is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of A-modules. Zöschinger has in [24,25] given many equivalent conditions for a module to be a minimax module. See also [18]. It was shown by T. Zink [23] and by Enochs [5] that a module over a complete local ring is minimax if and only if it is Matlis reflexive.

Proposition 4.3. *Let M be a minimax module with support in $\mathrm{V}(\mathfrak{a})$. Then M is $\mathfrak{a}$-cofinite if and only if $0 :_M \mathfrak{a}$ is finite. Moreover, if there is an element $x \in \mathfrak{a}$, such that $0 :_M x$ is $\mathfrak{a}$-cofinite, then M is $\mathfrak{a}$-cofinite.*

Proof. Let N be a finite submodule of M, such that $L = M/N$ is artinian and suppose that $0 :_M \mathfrak{a}$ is finite.

The exactness of

$$0 \to 0 :_N \mathfrak{a} \to 0 :_M \mathfrak{a} \to 0 :_L \mathfrak{a} \to \operatorname{Ext}^1_A(A/\mathfrak{a}, N)$$

implies that $0 :_L \mathfrak{a}$ is finite. Hence we get from 4.1 that L is $\mathfrak{a}$-cofinite, and therefore M is also $\mathfrak{a}$-cofinite. The second statement is proved as in Proposition 4.1. □

Corollary 4.4. *The class of $\mathfrak{a}$-cofinite minimax modules is closed under taking submodules, quotients and extensions, i.e., it is a Serre subcategory of the category of A-modules.*

If A is a local domain of dimension one with quotient field K, then K is a minimax module, since K/A is artinian. Hence if A is local and $\mathfrak{p}$ is a prime ideal, such that $\dim A/\mathfrak{p} = 1$, then $k(\mathfrak{p})$, the quotient field of the domain $A/\mathfrak{p}$ is a minimax module over A. Moreover, if $E\{\mathfrak{p}\}$ is the injective hull of the A-module $A/\mathfrak{p}$, then $0 :_{E\{\mathfrak{p}\}} \mathfrak{p}^n / 0 :_{E\{\mathfrak{p}\}} \mathfrak{p}^{n-1}$ is a minimax module for each $n \geqslant 1$, since it is a finite-dimensional vectorspace over $k(\mathfrak{p})$. Hence in this case the modules $0 :_{E\{\mathfrak{p}\}} \mathfrak{p}^n$ are minimax. If A is a one-dimensional local ring, then $E\{\mathfrak{p}\}$ is a minimax module for each minimal prime ideal $\mathfrak{p}$ of A. $E\{\mathfrak{p}\}$ is namely as an $A_\mathfrak{p}$-module isomorphic to the injective hull of the residue field $k(\mathfrak{p})$ of the artinian local ring $A_\mathfrak{p}$ and therefore $E\{\mathfrak{p}\} = 0 :_{E\{\mathfrak{p}\}} \mathfrak{p}^n$ for some n. It follows that if A is a one-dimensional local ring, then the class of minimax modules coincides with the class of modules of finite Goldie dimension. When A is an arbitrary noetherian ring, local or not, a module is minimax if and only if each of its quotients has finite Goldie dimension, [23] or [25, Anhang].

If $\dim A = 0$, then each $\mathfrak{a}$-cofinite module M is finite. In fact, if we take n such that $\mathfrak{a}^n = \mathfrak{a}^{n+1}$, then $M = 0 :_M \mathfrak{a}^n$.

We next describe the $\mathfrak{a}$-cofinite modules over a one-dimensional ring.

Proposition 4.5. *Let A be a noetherian ring of dimension one. An A-module M with support in $\mathrm{V}(\mathfrak{a})$ is $\mathfrak{a}$-cofinite if and only if $0 :_M \mathfrak{a}$ is a finite A-module. Each $\mathfrak{a}$-cofinite A-module is a minimax module. The class of $\mathfrak{a}$-cofinite modules is closed under taking submodules, quotients and extensions.*

Proof. If $\mathfrak{a}$ is nilpotent, say $\mathfrak{a}^n = 0$, then $M = 0 :_M \mathfrak{a}^n$, whenever $\operatorname{Supp}_A M \subset \mathrm{V}(\mathfrak{a})$. Suppose therefore that $\mathfrak{a}$ is not nilpotent. Take n such that $0 :_A \mathfrak{a}^n = \Gamma_{\mathfrak{a}}(A)$. Then $M/0 :_M \mathfrak{a}^n$ is a module over the ring $\bar{A} = A/\Gamma_{\mathfrak{a}}(A)$. Let $\bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $\bar{A}$. Then $\bar{\mathfrak{a}}$ contains an $\bar{A}$-regular element and therefore $\dim \bar{A}/\bar{\mathfrak{a}} = 0$. Suppose $\operatorname{Supp}_A M \subset \mathrm{V}(\mathfrak{a})$ and that $0 :_M \mathfrak{a}$ is finite. Then also $0 :_{\bar{M}} \bar{\mathfrak{a}} = 0 :_M \mathfrak{a}^{n+1}/0 :_M \mathfrak{a}^n$ is finite. Hence $\bar{M}$ is an $\bar{\mathfrak{a}}$-cofinite artinian $\bar{A}$-module and therefore also an $\mathfrak{a}$-cofinite artinian A-module. Since $0 :_M \mathfrak{a}^n$ is finite, M is $\mathfrak{a}$-cofinite and a minimax module. □

5. Artinian local cohomology modules

For another proof of the next result in the local case see [4, Theorem 3].

Proposition 5.1. *Let M be a finite module of dimension d over the noetherian ring A. For every ideal $\mathfrak{a}$ of A, the top local cohomology module $\mathrm{H}^d_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite artinian module.*

Proof. This is clear if $d = 0$. So let us assume that $d > 0$. As usual, replacing M with $M/\Gamma_{\mathfrak{a}}(M)$, we may assume that there is an M-regular element $x \in \mathfrak{a}$. Since $\dim M/xM < d$, we have that $\mathrm{H}^d_{\mathfrak{a}}(M/xM) = 0$ and by induction $\mathrm{H}^{d-1}_{\mathfrak{a}}(M/xM)$ is artinian and $\mathfrak{a}$-cofinite. From the exact sequence

$$\mathrm{H}^{d-1}_{\mathfrak{a}}(M/xM) \to \mathrm{H}^d_{\mathfrak{a}}(M) \xrightarrow{x} \mathrm{H}^d_{\mathfrak{a}}(M) \to 0,$$

4.1 and 4.4 implies that $\mathrm{H}^d_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite and artinian. □

Theorem 5.2. *Let A be a noetherian ring of finite Krull dimension d and let M be a module over A. If for each maximal ideal $\mathfrak{m}$ of A, the module $\mathrm{H}^d_{\mathfrak{a}}(M)_{\mathfrak{m}}$ is an artinian $A_{\mathfrak{m}}$-module, then $\mathrm{H}^d_{\mathfrak{a}}(M)$ is an artinian A-module, cofinite with respect to $\mathfrak{a}$. Moreover, each $\mathfrak{m} \in \operatorname{Supp}_A \mathrm{H}^d_{\mathfrak{a}}(M)$ is a (quint) asymptotic prime divisor of the ideal $\mathfrak{a}$.*

Proof. We first assume that A is a complete local ring and suppose that $\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$. Let $\mathfrak{p}$ be a coassociated prime ideal of $\mathrm{H}^d_{\mathfrak{a}}(M)$, so in particular $\mathrm{H}^d_{\mathfrak{a}}(M)/\mathfrak{p}\,\mathrm{H}^d_{\mathfrak{a}}(M) \neq 0$. But $\mathrm{H}^d_{\mathfrak{a}}(M)/\mathfrak{p}\,\mathrm{H}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{a}}(M/\mathfrak{p}M) \cong \mathrm{H}^d_{\mathfrak{a}+\mathfrak{p}/p}(M/\mathfrak{p}M)$. By the local Hartshorne–Lichtenbaum vanishing theorem [7] or [2, Chapter 8], the complete local ring $A/\mathfrak{p}$ has dimension d and the ideal $\mathfrak{a} + \mathfrak{p}$ is $\mathfrak{m}$-primary. Hence it follows from [16, 1.6] that the artinian module

$\mathrm{H}_{\mathfrak{a}}^d(M)$ must be $\mathfrak{a}$-cofinite and by [13, p. 1] $\mathfrak{m}$ is a quintasymptotic prime divisor of $\mathfrak{a}$. For the general case use [13, (1.1)a and (1.9)] to deduce that $\operatorname{Supp}_A \mathrm{H}_{\mathfrak{a}}^d(M)$ is contained in the set of quintasymptotic primes of $\mathfrak{a}$. Since this set is finite, $\operatorname{Supp}_A \mathrm{H}_{\mathfrak{a}}^d(M)$ consists of finitely many maximal ideals and for each such maximal ideal $\mathfrak{m}$, the $A_{\mathfrak{m}}$-module $\mathrm{H}_{\mathfrak{a}}^d(M)_{\mathfrak{m}}$ is artinian and cofinite with respect to $\mathfrak{a}A_{\mathfrak{m}}$. The assertion follows. □

Proposition 5.3. *Let $\mathfrak{b}$ be an ideal of a noetherian ring A, such that $A/\mathfrak{b}$ has finite Krull dimension d and let M be an A-module cofinite with respect to $\mathfrak{b}$. Then for each ideal $\mathfrak{a} \supset \mathfrak{b}$ the module $\mathrm{H}_{\mathfrak{a}}^d(M)$ is artinian and for each $\mathfrak{m} \in \operatorname{Supp}_A \mathrm{H}_{\mathfrak{a}}^d(M)$, $\mathfrak{m}/\mathfrak{b}$ is an asymptotic prime divisor of the ideal $\mathfrak{a}/\mathfrak{b}$ in the ring $A/\mathfrak{b}$.*

Proof. We first assume that $(A, \mathfrak{m})$ is local. If $d = 0$, then M is artinian, so let us assume that $d > 0$ and that we know the result for $d - 1$. By [16, Corollary 1.8], $L = \Gamma_{\mathfrak{m}}(M)$ is artinian and $\mathfrak{b}$-cofinite. Since $\mathrm{H}_{\mathfrak{a}}^d(M) \cong \mathrm{H}_{\mathfrak{a}}^d(M/L)$ and M/L is $\mathfrak{b}$-cofinite, we may assume that $\mathfrak{m} \notin \operatorname{Ass} M$. Since the set $\operatorname{Ass}_A(M)$ is finite, we can by prime avoidance take an element

$$x \in \mathfrak{m} \Big\backslash \Big(\bigcup_{\mathfrak{p} \in \operatorname{Ass} M} \mathfrak{p} \cup \bigcup_{\mathfrak{p} \in \operatorname{Min} A/\mathfrak{b}} \Big).$$

If we put $\mathfrak{c} = \mathfrak{b} + xA$, then $\dim A/\mathfrak{c} = d - 1$. From the exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, we get that M/xM is $\mathfrak{b}$-cofinite and therefore by the change of rings principle applied twice also cofinite with respect to $\mathfrak{c} = \mathfrak{b} + xA$. Since $\dim A/\mathfrak{c} = d - 1$, we get by induction that $\mathrm{H}_{\mathfrak{a}}^{d-1}(M/xM)$ is artinian. From the exactness of

$$\mathrm{H}_{\mathfrak{a}}^{d-1}(M/xM) \to \mathrm{H}_{\mathfrak{a}}^d(M) \xrightarrow{x} \mathrm{H}_{\mathfrak{a}}^d(M) \to 0$$

we obtain that $0 :_{\mathrm{H}_{\mathfrak{a}}^d(M)} x$ is artinian and therefore $\mathrm{H}_{\mathfrak{a}}^d(M)$ is artinian [14]. In order to show that $\mathfrak{m}/\mathfrak{b}$ is an asymptotic prime divisor of $\mathfrak{a}/\mathfrak{b}$ in case $\mathrm{H}_{\mathfrak{a}}^d(M) \neq 0$, we may assume that A is complete. Now $M = \bigcup_1^\infty 0 :_M \mathfrak{b}^n$, because M is assumed to be $\mathfrak{b}$-cofinite and therefore $\operatorname{Supp}_A M \subset \mathrm{V}(\mathfrak{b})$. Since local cohomology commutes with direct limits we thus get $\mathrm{H}_{\mathfrak{a}}^d(M) \cong \varinjlim \mathrm{H}_{\mathfrak{a}}^d(0 :_M \mathfrak{b}^n)$. Hence $\mathrm{H}_{\mathfrak{a}}^d(0 :_M \mathfrak{b}^n) \neq 0$ for some n. The Hartshorne–Lichtenbaum vanishing theorem implies that there is a prime ideal $\mathfrak{p} \supset \mathfrak{b}^n$, such that $\dim A/\mathfrak{p} = d$ and $\mathfrak{a} + \mathfrak{p}$ is $\mathfrak{m}$-primary. Hence $\mathfrak{m}/\mathfrak{b}$ is a quintasymptotic prime ideal over $\mathfrak{a}/\mathfrak{b}$. If A is not necessarily local, it follows that for each maximal ideal $\mathfrak{m}$ of A that $\mathrm{H}_{\mathfrak{a}}^d(M)_{\mathfrak{m}}$ is an artinian module over $A_{\mathfrak{m}}$ and if nonzero, that $\mathfrak{m}/\mathfrak{b}$ is a quintasymptotic prime of $\mathfrak{a}/\mathfrak{b}$. Since there are just finitely many quintasymptotic primes over an ideal, the support of $\mathrm{H}_{\mathfrak{a}}^d(M)$ consists of finitely many maximal ideals and for each such maximal ideal $\mathfrak{m}$ the $A_{\mathfrak{m}}$-module $\mathrm{H}_{\mathfrak{a}}^d(M)_{\mathfrak{m}}$ is artinian. Consequently $\mathrm{H}_{\mathfrak{a}}^d(M)$ must be an artinian A-module. □

If $0 :_M \mathfrak{a}$ is artinian, then so is $\Gamma_{\mathfrak{a}}(M)$, see [14]. In [15] we also investigated when $\mathrm{H}_{\mathfrak{a}}^i(M)$ is artinian in the range $i \leqslant r$, where r is fixed and the module M is finite. Next we will consider this problem for an arbitrary module M.

We first recall some basic facts about essential extensions.

If $f : E \to F$ is a homomorphism, such that $\operatorname{Ker} f$ is an essential submodule of E, then $\operatorname{Ker}\operatorname{Hom}_A(A/\mathfrak{a}, f)$ is an essential submodule of $\operatorname{Hom}_A(A/\mathfrak{a}, E)$ and $\operatorname{Ker}\Gamma_{\mathfrak{a}}(f)$ is an essential submodule of $\Gamma_{\mathfrak{a}}(E)$.

If $M \subset N$ is essential, then M is artinian if and only if N is artinian.

By induction this last fact is extended to a situation involving complexes:

Lemma 5.4. *Let $0 \to X^0 \to X^1 \to X^2 \to \cdots$ be a cochain complex with cocycles Z^i, coboundaries B^i and cohomology $H^i = Z^i/B^i$. If Z^i is an essential submodule of X^i for each i, then H^i is artinian for all $i \leqslant r$ if and only if X^i is artinian for all $i \leqslant r$.*

Proof. By induction on r. If H^r and X^{r-1} are artinian, then B^r is artinian, hence Z^r is artinian and therefore also its essential extension X^r is artinian. If X^r is artinian, then so is its subquotient H^r. □

Theorem 5.5. *Let $\mathfrak{a} = (x_1, \ldots, x_n)$ be an ideal of a noetherian ring A and let $0 \to M \to E^0 \to E^1 \to E^2 \to \cdots$ be a minimal injective resolution of the A-module M. The following conditions are equivalent*:

(i) $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is artinian for $i \leqslant r$.*
(ii) $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *is artinian for $i \leqslant r$.*
(iii) $\Gamma_{\mathfrak{a}}(E^i)$ *is artinian for $i \leqslant r$.*
(iv) *The set Λ of those prime ideals $\mathfrak{p} \supset \mathfrak{a}$, such that the Bass number $\mu^i(\mathfrak{p}, M) \neq 0$ for some $i \leqslant r$ is a finite subset of* $\operatorname{Max} A$ *and for all $\mathfrak{m} \in \Lambda$, $\mu^i(\mathfrak{m}, M)$ is finite when $i \leqslant r$.*
(v) $\mathrm{H}^i(x_1, \ldots, x_n; M)$ *is artinian for $i \leqslant r$.*

Proof. The equivalence of the first three conditions follows from 5.4 applied to the complexes

$$0 \to \Gamma_{\mathfrak{a}}\big(E^0\big) \to \Gamma_{\mathfrak{a}}\big(E^1\big) \to \Gamma_{\mathfrak{a}}\big(E^2\big) \to \cdots$$

and

$$0 \to \operatorname{Hom}_a\big(A/\mathfrak{a}, E^0\big) \to \operatorname{Hom}_A\big(A/\mathfrak{a}, E^1\big) \to \operatorname{Hom}_A\big(A/\mathfrak{a}, E^2\big) \to \cdots,$$

making use of the remarks above about essential extensions. Condition (iv) is more or less a restatement of (iii). In order to show the equivalence of (v) with the other conditions, we consider the surjective A-algebra homomorphism $\varphi : B \to A$, where $B = A[X_1, \ldots, X_n]$ and φ is defined by $\varphi(X_i) = x_i$ for $i = 1, \ldots, n$. Every A-module becomes in a natural way a B-module and as such it is artinian if and only if it is artinian as an A-module. Furthermore $X_1, \ldots, X_n$ is a B-regular sequence. Hence if $\mathfrak{b} = (X_1, \ldots, X_n)$, then

$$\operatorname{Ext}^i_B(B/\mathfrak{b}, M) \cong \mathrm{H}^i(X_1, \ldots, X_n; M) \cong \mathrm{H}^i(x_1, \ldots, x_n; M)$$

for all i. This shows that (v) is equivalent to (i). Note that $\mathrm{H}^i_{\mathfrak{b}}(M) \cong \mathrm{H}^i_{\mathfrak{a}}(M)$ for all i since φ maps $\mathfrak{b}$ onto $\mathfrak{a}$. □

Corollary 5.6. *Let $(A, \mathfrak{m})$ be a local ring. The following conditions are equivalent for an A-module M:*

(i) $\mathrm{H}^i_{\mathfrak{m}}(M)$ *is artinian for all i.*
(ii) *The Bass numbers $\mu^i(\mathfrak{m}, M)$ with respect to the maximal ideal are finite.*
(iii) *The Koszul cohomology modules $\mathrm{H}^i(x_1, \dots, x_r; M)$, $i = 0, \dots, r$, where $x_1, \dots, x_r$ are generators of $\mathfrak{m}$, are finite-dimensional vectorspaces over the residue field $A/\mathfrak{m}$.*

Remark 5.7. Belshoff and Wickham showed in [1], that if A is a complete local ring, that all Bass numbers $\mu^i(\mathfrak{m}, M)$ are finite for a module M if and only if M satisfies local duality.

6. Filter-regular sequences

We first give an appropriate extension of the notion of a sequence filter-regular on a module M to the case when M is not necessarily finitely generated. For filter-regularity on finite modules, see [20,21].

Definition 6.1. The element x is filter-regular on M, if $0 :_M x$ has finite length. The sequence $x_1, \dots, x_n$ is filter-regular on M, if x_j is filter-regular on $M/(x_1, \dots, x_{j-1})M$, for $j = 1, \dots, n$.

Remark 6.2. Since $0 :_M x = 0 :_{\Gamma_{xA}(M)} x$, the element x is filter-regular on M if and only if $\Gamma_{xA}(M)$ is artinian and cofinite with respect to the ideal xA, by 4.1.

We first state some elementary properties of filter-regular sequences.

Proposition 6.3. *Let $x_1, \dots, x_n$ be a sequence of elements in A and M an A-module.*

(a) *Let $1 \leqslant s \leqslant n$. The sequence $x_1, \dots, x_n$ is filter-regular on M if and only if $x_1, \dots, x_{s-1}$ is filter-regular on M and $x_s, \dots, x_n$ is filter-regular on $M/(x_1, \dots, x_{s-1})M$.*
(b) *The sequence $x_1, \dots, x_n$ is filter-regular on M if and only if it is filter-regular on $M/\Gamma_{x_1 A}(M)$. More generally, let N be an artinian and $x_1 A$-cofinite submodule of M. Then the sequence $x_1, \dots, x_n$ is filter-regular on M, if and only if it is filter-regular on M/N.*

Proof. Put $M_s = M/(x_1, \dots, x_{s-1})M$. Then (a) follows directly from the definition of filter-regularity, if we note that $M_s/(x_s, \dots, x_{j-1})M_s \cong M/(x_1, \dots, x_{j-1})M$ for $s \leqslant j \leqslant n$.

In order to prove the last assertion in (b), we use the exact sequence

$$0 \to 0 :_N x_1 \to 0 :_M x_1 \to 0 :_{M/N} x_1 \to N/x_1 N,$$

whose endterms have finite length, because N is assumed to be artinian and cofinite with respect to x_1A. Consequently $0:_M x_1$ has finite length if and only if $0:_{M/N} x_1$ has finite length. $\square$

Theorem 6.4. *If the sequence $x_1, \dots, x_n$ in the ideal $\mathfrak{a}$ is filter-regular on M, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is an $\mathfrak{a}$-cofinite artinian module for $i < n$.*

If in addition $\mathfrak{a}$ is generated by $x_1, \dots, x_n$, then the module $\mathrm{H}^n_{\mathfrak{a}}(M)$, too, is $\mathfrak{a}$-cofinite precisely when $M/\mathfrak{a}M$ is finite.

Proof. Since $\mathrm{H}^i_{\mathfrak{a}}(0:_M x_1) = 0$ for all $i > 0$, we can put together the two long exact sequences we get when we apply local cohomology to the two short exact sequences $0 \to 0:_M x_1 \to M \to x_1M \to 0$ and $0 \to x_1M \to M \to M/x_1M \to 0$. Thus we get the long exact sequence

$$\begin{aligned} 0 &\to \Gamma_{\mathfrak{a}}(0:_M x_1) \to \Gamma_{\mathfrak{a}}(M) \xrightarrow{x_1} \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M/x_1M) \\ &\to \mathrm{H}^1_{\mathfrak{a}}(M) \xrightarrow{x_1} \mathrm{H}^1_{\mathfrak{a}}(M) \to \mathrm{H}^1_{\mathfrak{a}}(M/x_1M) \\ &\to \mathrm{H}^2_{\mathfrak{a}}(M) \xrightarrow{x_1} \mathrm{H}^2_{\mathfrak{a}}(M) \to \mathrm{H}^2_{\mathfrak{a}}(M/x_1M) \to \cdots. \end{aligned}$$

Hence $0:_{\Gamma_{\mathfrak{a}}(M)} x_1$ has finite length, and therefore $\Gamma_{\mathfrak{a}}(M)$ is artinian and $\mathfrak{a}$-cofinite, by 4.1. Now $x_2, \dots, x_n$ is filter-regular on M/x_1M. We use induction and we suppose that $\mathrm{H}^i_{\mathfrak{a}}(M/x_1M)$ is an $\mathfrak{a}$-cofinite artinian module for $i < n-1$. From the long exact sequence above, we get that $0:_{\mathrm{H}^i_{\mathfrak{a}}(M)} x_1$ is $\mathfrak{a}$-cofinite artinian for $i < n$. Hence again 4.1 implies that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite artinian for $i < n$. Consider next the case when $\mathfrak{a}$ is generated by $x_1, \dots, x_n$. Use induction, our criterion in 3.4 and the exact sequence

$$0 \to \mathrm{H}^{n-1}_{\mathfrak{a}}(M)/x_1\,\mathrm{H}^{n-1}_{\mathfrak{a}}(M) \to \mathrm{H}^{n-1}_{\mathfrak{a}}(M/x_1M) \to \mathrm{H}^n_{\mathfrak{a}}(M) \xrightarrow{x_1} \mathrm{H}^n_{\mathfrak{a}}(M) \to 0.$$

Observe that $M/\mathfrak{a}M \cong M_1/\mathfrak{a}M_1$, where $M_1 = M/x_1M$. $\square$

Corollary 6.5. *Let M be an finite module. If the ideal $\mathfrak{a}$ is generated by a sequence, which is filter-regular on M, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i.*

7. When is the category of $\mathfrak{a}$-cofinite modules abelian?

Let $f : M \to N$ be a homomorphism between $\mathfrak{a}$-cofinite modules. If one of the modules $\operatorname{Ker} f$, $\operatorname{Coker} f$ and $\operatorname{Im} f$ is $\mathfrak{a}$-cofinite, then all of them are $\mathfrak{a}$-cofinite. If this is the case, we say that f is $\mathfrak{a}$-good. In order to decide whether f is $\mathfrak{a}$-good, it is often useful to reduce to the case that $\mathfrak{a}$ contains an A-regular element. This is done by taking n so large that $\Gamma_{\mathfrak{a}}(A) = 0:_A \mathfrak{a}^n$. If $\bar{\mathfrak{a}}$ is the image of $\mathfrak{a}$ in $\bar{A} = A/0:_A \mathfrak{a}^n$, then $\operatorname{depth}_{\bar{\mathfrak{a}}} \bar{A} > 0$. Replace f by the induced homomorphism $\bar{f}$ between the $\bar{\mathfrak{a}}$-cofinite $\bar{A}$-modules $\bar{M} = M/0:_M \mathfrak{a}^n$ and $\bar{N} = N/0:_N \mathfrak{a}^n$. Then f is $\mathfrak{a}$-good if and only if $\bar{f}$ is $\bar{\mathfrak{a}}$-good. There is namely an exact sequence

$$0 \to \operatorname{Ker} f_0 \to \operatorname{Ker} f \to \operatorname{Ker} \bar{f} \to \operatorname{Coker} f_0 \to \operatorname{Coker} f \to \operatorname{Coker} \bar{f} \to 0,$$

where $f_0 : 0 :_M \mathfrak{a}^n \to 0 :_N \mathfrak{a}^n$ is obtained from f by restriction. We get this exact sequence by applying the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 :_M \mathfrak{a}^n & \longrightarrow & M & \longrightarrow & M/0 :_M \mathfrak{a}^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \bar{f}} & & \\ 0 & \longrightarrow & 0 :_N \mathfrak{a}^n & \longrightarrow & N & \longrightarrow & N/0 :_N \mathfrak{a}^n & \longrightarrow & 0. \end{array}$$

Proposition 7.1. *Let M be $\mathfrak{a}$-cofinite, $\mathfrak{a} = \mathfrak{b} + xA$ and $\mathfrak{b} = (x_1, \dots, x_n)$. Then for each i, the module $\mathrm{H}^i(x_1, \dots, x_n; M)$ is $\mathfrak{a}$-cofinite.*

Proof. In the exact sequence

$$\mathrm{H}^i(x_1, \dots, x_n, x; M) \to \mathrm{H}^i(x_1, \dots, x_n; M) \xrightarrow{x} \mathrm{H}^i(x_1, \dots, x_n; M)$$
$$\to \mathrm{H}^{i+1}(x_1, \dots, x_n, x; M)$$

the outer terms are finite, since M is $\mathfrak{a}$-cofinite. Hence $0 :_L x$ and L/xL are finite, where $L = H^i(x_1, \dots, x_n; M)$. It follows from our criterion, 3.4 that L is $\mathfrak{a}$-cofinite. □

Lemma 7.2. *Let M be $\mathfrak{a}$-cofinite, $x \in \mathfrak{a}$, $\mathfrak{b} = (x_1, \dots, x_n)$, $\mathfrak{c} = \mathfrak{b} + xA$, and suppose that* $\dim A/\mathfrak{c} = 1$. *If* $\mathrm{H}^i(x_1, \dots, x_n, x; M)$ *is $\mathfrak{a}$-cofinite for all i, then* $\mathrm{H}^i(x_1, \dots, x_n; M)$ *is $\mathfrak{a}$-cofinite for all i.*

Proof. Let $\bar{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in $\bar{A} = A/\mathfrak{c}$. As a module over $\bar{A}$, $\mathrm{H}^i(x_1, \dots, x_n, x; M)$ is cofinite with respect to $\bar{\mathfrak{a}}$. Since submodules and quotients of $\bar{\mathfrak{a}}$-cofinite modules over the one-dimensional ring $\bar{A}$ are again $\bar{\mathfrak{a}}$-cofinite by 4.5, we can use the same long exact sequence as in the proof of 7.1. Thus the modules $0 :_L x$ and L/xL are $\mathfrak{a}$-cofinite, where $L = \mathrm{H}^i(x_1, \dots, x_r; M)$. Hence again by our criterion 3.4, L is $\mathfrak{a}$-cofinite. □

Lemma 7.3. *Let M be $\mathfrak{a}$-cofinite and suppose that $\mathfrak{b} = (x_1, \dots, x_n)$ is an ideal such that* $\dim A/(\mathfrak{b} + xA) \leqslant 1$ *for some $x \in \mathfrak{a}$. Then* $\mathrm{H}^i(x_1, \dots, x_n; M)$ *is $\mathfrak{a}$-cofinite for all i.*

Proof. Take elements $x_{n+1}, \dots, x_m$ in $\mathfrak{a}$, such that $x_{n+1} = x$ and $\mathfrak{a} + \mathfrak{b} = (x_1, \dots, x_m)$. Now use 7.2 repeatedly. □

Theorem 7.4. *Let A be a noetherian ring with* $\dim A \leqslant 2$ *and let $\mathfrak{a}$ be an ideal of A.*

If $f : M \to N$ is a homomorphism between the $\mathfrak{a}$-cofinite modules M and N, then $\operatorname{Ker} f$, $\operatorname{Coker} f$ *and* $\operatorname{Im} f$ *are $\mathfrak{a}$-cofinite, i.e., in our terminology f is $\mathfrak{a}$-good.*

More generally all (*co*)*homology modules of a* (*co*)*chain complex consisting of $\mathfrak{a}$-cofinite modules are $\mathfrak{a}$-cofinite.*

In particular if M is $\mathfrak{a}$-cofinite, then $\mathrm{H}^i(x_1, \dots, x_r; M)$ *is $\mathfrak{a}$-cofinite for all i and any elements $x_1, \dots, x_r$ of A.*

Proof. The other statements follows from the first one. As we remarked before, we may assume, that there is an A-regular element $x \in \mathfrak{a}$. The modules M/xM and $0 :_N x$ are $\mathfrak{a}$-cofinite by 7.3. Hence they are by 2.6 $\mathfrak{a}/(x)$-cofinite modules over the ring $\bar{A} = A/(x)$, which has dimension at most one. Let $I = \operatorname{Im} f$. Now I/xI is a homomorphic image of M/xM and $0 :_I x$ is a submodule of $0 :_N x$. These modules are therefore $\mathfrak{a}$-cofinite by 4.5 and 2.6. Hence by the criterion in 3.4, I is $\mathfrak{a}$-cofinite. Then also $\operatorname{Ker} f$ and $\operatorname{Coker} f$ must be $\mathfrak{a}$-cofinite. □

Proposition 7.5. *For an ideal $\mathfrak{a}$ of a noetherian ring A, let $\mathfrak{I}_0(\mathfrak{a})$, respectively $\mathfrak{I}_1(\mathfrak{a})$, be the set of ideals $\mathfrak{b}$ of A, such that $0 :_M \mathfrak{b}$, respectively $M/\mathfrak{b}M$, is $\mathfrak{a}$-cofinite for every $\mathfrak{a}$-cofinite module M. The sets $\mathfrak{I}_i(\mathfrak{a})$, $i = 0, 1$, have the properties*:

(a) $(0) \in \mathfrak{I}_i(\mathfrak{a})$ *and* $(1) \in \mathfrak{I}_i(\mathfrak{a})$ *and more generally* $(e) \in \mathfrak{I}_i(\mathfrak{a})$ *for every idempotent element $e \in A$.*
(b) *If $\mathfrak{b}, \mathfrak{c} \in \mathfrak{I}_i(\mathfrak{a})$, then $\mathfrak{b}\mathfrak{c} \in \mathfrak{I}_i(\mathfrak{a})$ and $\mathfrak{b} + \mathfrak{c} \in \mathfrak{I}_i(\mathfrak{a})$.*
(c) *If $\mathfrak{c} \supset \mathfrak{b}$, $\mathfrak{b} \in \mathfrak{I}_i(\mathfrak{a})$ and $\bar{\mathfrak{c}} \in \mathfrak{I}_i(\bar{\mathfrak{a}})$, where $\bar{\mathfrak{a}}$ and $\bar{\mathfrak{c}}$ are the images of the ideals $\mathfrak{a}$ and $\mathfrak{b}$ in the ring $A/\mathfrak{b}$, then $\mathfrak{c} \in \mathfrak{I}_i(\mathfrak{a})$.*
(d) *If* $\dim A/\mathfrak{b} \leqslant 2$, *then* $\mathfrak{b} \in \mathfrak{I}_i(\mathfrak{a})$.

Proof. (a) If $e^2 = e$, then $f = e1_M$ is an idempotent endomorphism on M. Hence $\operatorname{Ker} f = 0 :_M e$ and $\operatorname{Im} f = eM$ are direct summands of M.

(b) In order to prove the assertions concerning $\mathfrak{I}_0(\mathfrak{a})$ use the equalities $0 :_M \mathfrak{b}\mathfrak{c}/0 :_M \mathfrak{b} = 0 :_L \mathfrak{c}$, where $L = M/0 :_M \mathfrak{b}$ and $0 :_M (\mathfrak{b} + \mathfrak{c}) = 0 :_N \mathfrak{c}$, where $N = 0 :_M \mathfrak{b}$. In order to prove the assertions concerning $\mathfrak{I}_1(\mathfrak{a})$, use that an ideal $\mathfrak{d}$ belongs to $\mathfrak{I}_1(\mathfrak{a})$ if and only if $\mathfrak{d}M$ is $\mathfrak{a}$-cofinite and note that $M/(\mathfrak{b} + \mathfrak{c})M \cong P/\mathfrak{c}P$, where $P = M/\mathfrak{b}M$.

(c) Let $L = 0 :_M \mathfrak{b}$ and $N = M/\mathfrak{b}M$, which are $\bar{\mathfrak{a}}$-cofinite modules over $\bar{A} = A/\mathfrak{b}$. Use that $0 :_M \mathfrak{c} = 0 :_L \bar{\mathfrak{c}}$ and $\mathfrak{c}M/\mathfrak{b}M = \bar{\mathfrak{c}}N$.

(d) Since there are (not necessarily distinct) prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, such that $\mathfrak{b} \supset \mathfrak{p}_1 \cdots \mathfrak{p}_r$ and $\dim A/\mathfrak{p}_j \leqslant 2$ for $j = 1, \dots, r$, we are by (b), (c), and 7.4, reduced to the case $\mathfrak{b} = \mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal with $\dim A/\mathfrak{p} \leqslant 2$. If $\mathfrak{a} \subset \mathfrak{p}$, then $0 :_M \mathfrak{p}$ and $M/\mathfrak{p}M$ are finite modules. If $\mathfrak{a} \not\subseteq \mathfrak{p}$, take an element $x \in \mathfrak{a} \setminus \mathfrak{p}$. Then $\dim A/(\mathfrak{p} + xA) \leqslant 1$, so we may apply 7.3. □

Proposition 7.6. *Let $f : M \to N$ be a homomorphism between $\mathfrak{a}$-cofinite modules. If there is an ideal $\mathfrak{b}$, such that* $\dim A/\mathfrak{b} \leqslant 2$ *and* $\mathfrak{b} \operatorname{Ker} f = 0$ *or* $\mathfrak{b} \operatorname{Coker} f = 0$, *then f is $\mathfrak{a}$-good.*

Proof. Let $K = \operatorname{Ker} f$ and $C = \operatorname{Coker} f$. If $\mathfrak{b}K = 0$, we get the exact sequence $0 \to K \to 0 :_M \mathfrak{b} \to 0 :_N \mathfrak{b}$ and if $\mathfrak{b}C = 0$, we get the exact sequence $M/\mathfrak{b}M \to N/\mathfrak{b}N \to C \to 0$. Our assertion therefore follows from 7.5 and 7.4 □

Corollary 7.7. *Let $X^\bullet : 0 \to X^0 \xrightarrow{\partial_0} X^1 \xrightarrow{\partial_1} X^2 \to \cdots$ be a cochain complex consisting of modules cofinite with respect to $\mathfrak{a}$. Suppose there is an ideal $\mathfrak{b} \subset A$, with* $\dim A/\mathfrak{b} \leqslant 2$, *such that* $\mathfrak{b}\,\mathrm{H}^i(X^\bullet) = 0$ *for $i = 0, \dots, n$. Then* $\mathrm{H}^i(X^\bullet)$ *is $\mathfrak{a}$-cofinite for $i = 0, \dots, n$. A similar statement holds for the homology of chain complexes.*

Proof. Consider the exact sequences

$$0 \to \mathrm{H}^i(X^\bullet) \to \operatorname{Coker} \partial_{i-1} \to X^{i+1} \to \operatorname{Coker} \partial_i \to 0$$

and use induction. □

We are now able to strengthen 7.3.

Corollary 7.8. *Let M be $\mathfrak{a}$-cofinite, and suppose that the ideal $\mathfrak{b} = (x_1, \dots, x_r)$ satisfies $\dim A/\mathfrak{b} \leqslant 2$. Then the Koszul cohomology modules $\mathrm{H}^i(x_1, \dots, x_r; M)$, $i = 0, \dots, r$, are $\mathfrak{a}$-cofinite.*

In the proof of the next theorem, we need the lemma below, which for completeness we prove.

Lemma 7.9. *If $\Gamma_\mathfrak{a}(M) = 0$, then $\operatorname{Hom}_A(A/\mathfrak{a}, \mathrm{H}^1_\mathfrak{a}(M)) \cong \operatorname{Ext}^1_A(A/\mathfrak{a}, M)$.*

Proof. Let E be an injective hull of M and put $N = E/M$. Since $\Gamma_\mathfrak{a}(M) = 0$, also $\Gamma_\mathfrak{a}(E) = 0$ and $\operatorname{Hom}_A(A/\mathfrak{a}, E) = 0$. Therefore from the exact sequence $0 \to M \to E \to N \to 0$, we get the isomorphisms $\Gamma_\mathfrak{a}(N) \cong \mathrm{H}^1_\mathfrak{a}(M)$ and $\operatorname{Hom}_A(A/\mathfrak{a}, N) \cong \operatorname{Ext}^1_A(A/\mathfrak{a}, M)$. □

Theorem 7.10. *Let A be a noetherian ring with $\dim A \leqslant 2$ and let $\mathfrak{a} \subset A$ be an ideal and M an A-module. The following conditions are equivalent*:

(i) $\mathrm{H}^i_\mathfrak{a}(M)$ *is $\mathfrak{a}$-cofinite for all i.*
(ii) $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *is finite for all i.*
(iii) $\operatorname{Ext}^i_A(A/\mathfrak{a}, M)$ *is finite for $i \leqslant 2$.*

Proof. We need by 3.10 just show that (i) follows from (iii). Suppose that M satisfies (iii). If $\mathfrak{a}$ is nilpotent, then $\mathfrak{a}$-cofiniteness is the same as finiteness. If $\mathfrak{a}$ is nonnilpotent, take n such that $0 :_A \mathfrak{a}^n = \Gamma_\mathfrak{a}(A)$. There is $x \in \mathfrak{a}$ which is regular on $\bar{A} = A/\Gamma_\mathfrak{a}(A)$, and therefore $\dim \bar{A}/x\bar{A} \leqslant 1$. The module $\bar{M} = M/0 :_M \mathfrak{a}^n$ has a natural structure as a module over $\bar{A}$. Since $0 :_M \mathfrak{a}^n$ is finite, $\bar{M}$ must also satisfy (iii). The exact sequence $0 \to 0 :_M \mathfrak{a}^n \to M \to \bar{M} \to 0$ yields the exact sequence $0 \to 0 :_M \mathfrak{a}^n \to \Gamma_\mathfrak{a}(M) \to \Gamma_\mathfrak{a}(\bar{M}) \to 0$ and isomorphisms $H^i_\mathfrak{a}(M) \cong H^i_\mathfrak{a}(\bar{M})$ for $i \geqslant 1$. Thus replacing M by $\bar{M}$, we may assume that M is a module over $\bar{A}$. Let $L = \Gamma_\mathfrak{a}(N)$, where $N = 0 :_M x \subset M$. Since $0 :_L \mathfrak{a} = 0 :_M \mathfrak{a}$, which is finite, 4.5 implies that L is $\mathfrak{a}$-cofinite and therefore satisfies (ii). From the exact sequence $0 \to N \to M \to xM \to 0$, we get that $\operatorname{Ext}^1_A(A/\mathfrak{a}, N)$ is finite. Hence $\operatorname{Ext}^1_A(A/\mathfrak{a}, N/L)$ is finite. By 7.9 $\operatorname{Ext}^1_A(A/\mathfrak{a}, N/L) \cong \operatorname{Hom}_A(A/\mathfrak{a}, \mathrm{H}^1_\mathfrak{a}(N/L))$. Also $\mathrm{H}^1_\mathfrak{a}(N) \cong \mathrm{H}^1_\mathfrak{a}(N/L)$, so $\operatorname{Hom}_A(A/\mathfrak{a}, \mathrm{H}^1_\mathfrak{a}(N))$ is finite. Hence by 4.5 the module $\mathrm{H}^1_\mathfrak{a}(N)$ is $\mathfrak{a}$-cofinite. Since $\mathrm{H}^i_\mathfrak{a}(N) = 0$ for $i > 1$, 3.10 implies that $N = 0 :_M x$ satisfies (ii). From the exactness of $0 \to N \to M \to xM \to 0$, we therefore get that $\operatorname{Ext}^1_A(A/\mathfrak{a}, xM)$ and $\operatorname{Ext}^2_A(A/\mathfrak{a}, xM)$ are finite. Hence from the exactness of $0 \to xM \to M \to M/xM \to 0$ we get that $\operatorname{Hom}_A(A/\mathfrak{a}, P)$ and $\operatorname{Ext}^1_A(A/\mathfrak{a}, P)$, where $P = M/xM$, are finite modules. An argument similar to that one, we used to show that $\mathrm{H}^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all i, shows that $\mathrm{H}^i_{\mathfrak{a}}(P)$ is $\mathfrak{a}$-cofinite for all i.

Consider the homomorphism $f = x1_M$, so $N = \operatorname{Ker} f$ and $P = \operatorname{Coker} f$. We have shown that $\mathrm{H}^i_{\mathfrak{a}}(\operatorname{Ker} f)$ and $\mathrm{H}^i_{\mathfrak{a}}(\operatorname{Coker} f)$ are cofinite with respect to $\mathfrak{a}$ for each i. By 4.5 the class of $\mathfrak{a}$-cofinite modules, which are modules over $\bar{A}$ annihilated by x constitute a Serre subcategory of the category of A-modules. Hence it follows from 3.2 that for all i the modules $\operatorname{Ker} \mathrm{H}^i_{\mathfrak{a}}(f)$ and $\operatorname{Coker} \mathrm{H}^i_{\mathfrak{a}}(f)$ belong to the same category. Since $x \in \mathfrak{a}$ our criterion 3.4 implies that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i. □

Corollary 7.11. *If M is a finite A-module or more generally $\mathfrak{b}$-cofinite for some ideal $\mathfrak{b} \subset \mathfrak{a}$, then $\mathrm{H}^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all i.*

Proof. If M is $\mathfrak{b}$-cofinite and $\mathfrak{a} \supset \mathfrak{b}$, then 3.10 and [4, Proposition 1] implies that (ii) is satisfied by M. □

Proposition 7.12. *Let A be a complete local ring and $\mathfrak{a} \subset A$ an ideal, such that* $\dim A/\mathfrak{a} = 1$.

Consider the conditions:

- $(*)$ *For every $\mathfrak{a}$-cofinite A-module L and each prime ideal $\mathfrak{p}$ minimal over $\mathfrak{a}$ the modules $\Gamma_{\mathfrak{p}}(L)$ and $\mathrm{H}^1_{\mathfrak{p}}(L)$ are $\mathfrak{p}$-cofinite.*
- $(**)$ *For any homomorphism $f : M \to N$ between $\mathfrak{a}$-cofinite modules* $\operatorname{Ker} f$, $\operatorname{Coker} f$ *and* $\operatorname{Im} f$ *are $\mathfrak{a}$-cofinite.*

Then $(*) \Rightarrow (*)$.

Proof. Assume that $(*)$ holds. Let $K = \operatorname{Ker} f$, $I = \operatorname{Im} f$ and $C = \operatorname{Coker} f$, where $f : M \to N$ is a homomorphism between the $\mathfrak{a}$-cofinite modules M and N. We show that $\mathrm{H}^i_{\mathfrak{p}}(K)$ is $\mathfrak{p}$-cofinite for all i and every prime ideal $\mathfrak{p}$ minimal over $\mathfrak{a}$. Then we get from 3.10 that $\operatorname{Ext}^i_A(A/\mathfrak{p}, K)$ is a finite A-module for all i and therefore we get from [4, Corollary 1] that K is $\mathfrak{a}$-cofinite and then so are I and C. Since by our assumption $(*)$, the modules $\Gamma_{\mathfrak{p}}(M)$ and $\Gamma_{\mathfrak{p}}(N)$ are $\mathfrak{p}$-cofinite, $\Gamma_{\mathfrak{p}}(K) = \operatorname{Ker} \Gamma_{\mathfrak{p}}(f)$ is $\mathfrak{p}$-cofinite, since $\mathfrak{p}$ is a one-dimensional prime in a complete local ring and therefore the category of $\mathfrak{p}$-cofinite modules is an abelian subcategory of the category of A-modules as shown by Delfino and Marley in [4]. We have the exact sequence

$$0 \to \Gamma_{\mathfrak{p}}(K) \to \Gamma_{\mathfrak{p}}(M) \to \Gamma_{\mathfrak{p}}(I) \xrightarrow{\delta} \mathrm{H}^1_{\mathfrak{p}}(K) \to \mathrm{H}^1_{\mathfrak{p}}(M) \to \mathrm{H}^1_{\mathfrak{p}}(I) \to 0.$$

Since $\Gamma_{\mathfrak{p}}(K)$ and $\Gamma_{\mathfrak{p}}(M)$ are $\mathfrak{p}$-cofinite, $\operatorname{Ker} \delta$ is $\mathfrak{p}$-cofinite, again by [4]. Also $\operatorname{Coker} \delta$ is $\mathfrak{p}$-cofinite, since it is isomorphic to a submodule of $\mathrm{H}^1_{\mathfrak{p}}(M)$, which is $\mathfrak{p}$-cofinite by assumption and artinian by 5.3. Hence by 3.3 the cokernel of the map $\operatorname{Hom}_A(A/\mathfrak{p}, \Gamma_{\mathfrak{p}}(I)) \to \operatorname{Hom}_A(A/\mathfrak{p}, \mathrm{H}^1_{\mathfrak{p}}(K))$ is finite. But $\operatorname{Hom}_A(A/\mathfrak{p}, \Gamma_{\mathfrak{p}}(I)) \cong 0 :_I \mathfrak{p} \subset 0 :_N \mathfrak{p}$, which is finite. Hence $\operatorname{Hom}_A(A/\mathfrak{p}, \mathrm{H}^1_{\mathfrak{p}}(K))$ is finite. But $\operatorname{Supp}_A \mathrm{H}^1_{\mathfrak{p}}(K) \subset \mathrm{V}(\mathfrak{m})$. Consequently $\mathrm{H}^1_{\mathfrak{p}}(K)$ is artinian and $\mathfrak{p}$-cofinite. Note that $\mathrm{H}^i_{\mathfrak{p}}(K) = 0$ for all $i > 1$, since K has support in the one-dimensional set $\mathrm{V}(\mathfrak{a})$. □

References

[1] R. Belshoff, C. Wickham, A note on local duality, Bull. London Math. Soc. 29 (1997) 25–31.
[2] M. Brodmann, R.Y. Sharp, Local Cohomology: an Algebraic Introduction with Geometric Applications, Cambridge Univ. Press, Cambridge, UK, 1998.
[3] D. Delfino, On the cofiniteness of local cohomology modules, Math. Proc. Cambridge Philos. Soc. 115 (1994) 79–84.
[4] D. Delfino, T. Marley, Cofinite modules and local cohomology, J. Pure Appl. Algebra 121 (1997) 45–52.
[5] E. Enochs, Flat covers and flat cotorsion modules, Proc. Amer. Math. Soc. 92 (1984) 179–184.
[6] A. Grothendieck, Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), North-Holland, Amsterdam, 1968.
[7] R. Hartshorne, Cohomological dimension of algebraic varieties, Ann. of Math. 88 (1968) 403–450.
[8] R. Hartshorne, Affine duality and cofiniteness, Invent. Math. 9 (1970) 145–164.
[9] C. Huneke, J. Koh, Cofiniteness and vanishing of local cohomology modules, Math. Proc. Cambridge Philos. Soc. 110 (1991) 421–429.
[10] K.-I. Kawasaki, Cofiniteness of local cohomology modules for principal ideals, Bull. London Math. Soc. 30 (1998) 241–246.
[11] J. Lescot, Séries de Poincaré et modules inertes, J. Algebra 132 (1990) 22–49.
[12] T. Marley, J. Vassilev, Cofiniteness and associated primes of local cohomology modules, J. Algebra 256 (2002) 180–193.
[13] S. McAdam, Primes Associated to an Ideal, Contemp. Math., vol. 102, Amer. Math. Soc., Providence, RI, 1989.
[14] L. Melkersson, On asymptotic stability for sets of prime ideals connected with the powers of an ideal, Math. Proc. Cambridge Philos. Soc. 107 (1990) 267–271.
[15] L. Melkersson, Some applications of a criterion for artinianness of a module, J. Pure Appl. Algebra 101 (1995) 291–303.
[16] L. Melkersson, Properties of cofinite modules and applications to local cohomology, Math. Proc. Cambridge Philos. Soc. 125 (1999) 417–423.
[17] J. Rotman, An Introduction to Homological Algebra, Academic Press, San Diego, 1979.
[18] P. Rudlof, On minimax and related modules, Canad. J. Math. 44 (1992) 154–166.
[19] P. Schenzel, Flatness and ideal-transforms of finite type, in: Commutative Algebra, Proceedings, Salvador, 1988, in: Lecture Notes in Math., vol. 1430, Springer-Verlag, Berlin, 1990, pp. 88–97.
[20] P. Schenzel, N.V. Trung, N.T. Cuong, Verallgemeinerte Cohen–Macaulay-Moduln, Math. Nachr. 85 (1978) 57–73.
[21] J. Stückrad, W. Vogel, Buchsbaum Rings and Applications, Springer-Verlag, Berlin, 1986.
[22] K.-I. Yoshida, Cofiniteness of local cohomology modules for ideals of dimension one, Nagoya Math. J. 147 (1997) 179–191.
[23] T. Zink, Endlichkeitsbedingungen für Moduln über einem Noetherschen Ring, Math. Nachr. 164 (1974) 239–252.
[24] H. Zöschinger, Minimax Moduln, J. Algebra 102 (1986) 1–32.
[25] H. Zöschinger, Über die Maximalbedingung für radikalvolle Untermoduln, Hokkaido Math. J. 17 (1988) 101–116.