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SYMMETRY IN MATHEMATICS*

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Abstract—In this paper the theme of symmetry in mathematics is explored. The main problems may be formulated as the determination of the symmetry groups of objects of various categories. As a result, it is possible to put the works of Lie, Cartan, Chevalley, Gel'fand, Harish-Chandra, Langlands and many others, in proper perspective.

1. INTRODUCTION

The difficulties in giving an exposition of this subject to a scientific but non mathematical audience are many and obvious. Nevertheless, I shall make the attempt here and hope that I will succeed in communicating at least a few of my main ideas. It is of course inevitable that I have to use some technical mathematical language at certain stages of this paper. I shall try to keep such instances to a minimum.

I am sure that everyone knows that the mathematical treatment of symmetry is based on the notions of *group* and *group of transformations*. Although most groups arise as groups of transformations, I shall follow the tradition of the mathematician that insists on keeping these two concepts well differentiated. A group is thus an abstract set G of elements

$$a, b, c, \dots,$$

which allows an associative multiplication law to be defined among its elements, with inverses and an identity element e:

$$\begin{aligned} a\,(b\,c) &= (a\,b)\,c \\ a\,a^{-1} &= a^{-1}a = e, \\ a\,e &= e\,a = a. \end{aligned}$$

If G is such a group and X is a set ("space") we say that *G acts on X* if for any $g \in G$ and any $x \in X$, we can define an element $g[x]$ (or $g\,x$) in X, to be thought of as the *point to which the point x is moved by the element g*, satisfying the following axioms:

$$(1) \quad g\,h[x] = g[h[x]], \qquad (g, h \in G,\, x \in X),$$
$$(2) \quad e[x] = x, \qquad (x \in X).$$

In order to avoid having to deal with useless elements we shall generally deal with only *effective* actions, i.e., those which satisfy the condition that any element g different from the identity actually moves some point of X:

$$g[x] = x, \qquad \forall x \in X \Rightarrow g = e.$$

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One should not imagine that the theory of groups is just a game played with the above axioms. That will be boring and will very soon end in utter triviality. What one should really do is to introduce a few examples of groups and transformation groups that "occur" in nature and use them to formulate some basic themes and goals. This is what I shall do next but, before I do that, I shall introduce the concept of *representation* of a group that links the two earlier concepts. Suppose we have in mind not arbitrary spaces but spaces of a particular *category*; this means typically that we consider spaces with some additional *structure*, and allow only transformations that preserve this additional structure (*automorphisms*). A *representation* of an abstract group G in the category $\mathbf{C}$ is then an action of G in a space X belonging to the category $\mathbf{C}$, with the following property:

$$\text{all the transformations } x \longrightarrow g[x] \text{ are automorphisms of } X.$$

The importance of this idea consists in the fact that the same group can have essentially different representations in a given category. This circumstance allows for great flexibility in applying the concept of transformation groups to physical problems.

Examples of some basic categories are the following:

The category of finite sets.

The category of vector spaces and linear transformations.

The category of manifolds with a differential or algebraic structure.

All these and many others are not only of theoretical importance but are actually essential in physical and mathematical applications.

Even though the origins of the concepts of groups and groups of transformations go back to antiquity, one may maintain without much exaggeration that it is only in the last two centuries that decisive progress has been made in understanding them and their role in physical applications. The entire history of the subject has been shaped by the efforts of mathematicians to "solve" the following two "problems":

PROBLEM I (STRUCTURE). *Fix a category* $\mathbf{C}$ *and a space* X *in it. What is the structure of the group of "all" automorphisms of* X*?*

PROBLEM II (REPRESENTATIONS). *Fix a group* G *that arises in answer to the preceding question and select another category* $\mathbf{C}'$ *(not necessarily related to* $\mathbf{C}$ *in any manner). What are "all" representations of* G *in spaces of the category* $\mathbf{C}'$*?*

It may be hard to believe that such very general goals can have much content or enough beauty and difficulty to keep an extraordinary number of mathematicians and physical scientists interested for two centuries and more. Yet this is exactly the case; indeed, the charm, power, and pervasive usefulness of these (and other closely related) problems have proved so seductive that there are people who do nothing else in their lifetime, and what is even worse, have a hard time accepting that there is any scientific life beyond these themes!

Before going on to the more substantive part of this discussion let me make two observations. The first has to do with the reasons behind the remarkable effectiveness of the two problems mentioned above. The point is that the notion of categories on which these problems are based is a fundamental concept in mathematics, and the examples of categories that I have listed above must convince anyone that it is also very flexible. It is through solving these problems specifically for various special categories of mathematical and physical importance that the theory of groups has emerged in the present century as the overarching discipline of impact and usefulness for the physical sciences. The second point is somewhat more personal and philosophical. No one who has ever been involved in the application of any aspect of mathematics in the physical sciences can fail to be impressed by the ambivalence inherent in the nature of mathematics. On the one hand, the development of its principal domains (topology, geometry, analysis, arithmetic, etc.) is almost entirely informed by an *internal* esthetic, so much so, that in the hands of its greatest practitioners it becomes essentially the pursuit of mathematical beauty. On the other hand, in its greatest applications to the physical world, it clearly reveals its extraordinary power to relate to the *external* world. It is my deep belief that one should not regard these two realities, the internal and the external, as separate; but rather, one should view them as two *complementary* facets of the *single* reality of the physical world.

2. FINITE GROUPS

Take a set of N elements, say $X = \{1, 2, \ldots, N\}$. The group in question is S_N, the group of *all* permutations of X. There are $N!$ elements in this group. It has a normal subgroup, A_N, the subset of all *even* permutations. It is traditional to take, for representations of S_N, its representations as groups of linear transformations in vector spaces or as groups of matrices with entries from either the real or complex number fields. I mention a typical example of such a representation. The group is S_3, acting on the plane π in 3-space defined by the equation:

$$\pi: x_1 + x_2 + x_3 = 0,$$

the action, being by permutation of coordinates. The images of a typical element are the vertices of a hexagon. A second typical example of a finite group is the group of invertible matrices or a subgroup of it, when the entries are chosen from a *finite field*. All finite groups are subgroups of the permutation groups S_N, but the vast array of finite groups and their realizations makes it essential to describe them in other ways.

In what contexts do finite groups arise "naturally"? Here is a list, very partial and very incomplete, of such situations.

2.1. Combinatorics

These are problems of gambling, i.e., problems associated with cards, dice, and so on. Indeed, it was their interest in such problems that led Pascal and Fermat to their first ideas in the theory of probability. For instance, it is not at all a trivial matter to determine the structure of the transformations that arise from "shuffling" decks of cards. If a finite set has an additional structure, the group of automorphisms becomes a *subgroup* of the full permutation group. For example, the points of the set may represent the vertices of a graph, so that it is natural to consider those permutations that leave the graph unchanged.

2.2. Engineering

These are already very sophisticated problems involving *Fourier Transform theory* on finite abelian groups, which lead to techniques of calculating ordinary Fourier transform (*Fast Fourier Transforms*).

2.3. Algebra

The finite groups enter algebra most strikingly as *Galois Groups*, names after E. Galois, who invented them for solving some very troubling questions of the theory of algebraic equations of his time. One takes an equation such as:

$$X^{17} - 1 = 0,$$

or

$$X^5 - X + 1 = 0.$$

The *Galois group* of the equation is then a *certain* subgroup of the group of permutations of the roots of the equation (in the field of complex numbers). For the first equation the roots are the 17^{th} roots of unity; for any integer m which is not a multiple of 17, the operation of multiplication by m on the integers, followed by reduction mod 17 gives a permutation of $\{1, 2, \ldots, 16\}$; this permutation depends only on the residue class of m modulo 17, and the set of such permutations is a subgroup of S_{16} that has 16 elements; this is the Galois group in question. The second equation leads to the full group S_5 as its Galois group. The fact that the Galois group of the equation $X^{17} - 1 = 0$ has $16 = 2^4$ elements was the crucial fact that allowed Gauss (at the age of 19!) to prove that the regular 17-gon can be constructed by ruler and compass. (Indeed, it was this incident that convinced Gauss that his career should be in mathematics and not in philology, cf. [1].) The fact that the Galois group of the second equation is S_5 is also of great historical interest. The central problem of the theory of algebraic equations, once the methods of solving the cubic and biquadratic equations were developed, was the solution of the problem of solving by radicals the general equation of the 5^{th} degree. It was this question that led to the epoch-making discoveries of Galois, in particular to his remarkable insight that for the solvability via radicals the necessary and sufficient condition is that the Galois group should be built up in a particularly simple manner from abelian groups, or that it is "*solvable*," the terminology of course being a reflexion of this historical circumstance. The group S_5 is *not* solvable, thus providing an example of an equation of the 5^{th} degree that is not solvable by radicals. (As an aside, let us note that if one treats the general equation as an object that varies as a function of its coefficients, the radicals define *algebraic functions* of the coefficients of a very special type; and one may ask whether the fifth degree equation can be solved with the help of more complicated transcendental functions, e.g., elliptic functions. This line of thought was pursued among others by Abel, Hermite, Kronecker, Klein and Jordan, and is now well understood. See the account of this question by Hiroshi Umemura in [2].)

2.4. Number theory

Here is a first example of Problem I:

What are all possible Galois groups of extensions of the field of rational numbers?

Let us consider all the fields generated by equations with rational coefficients as subfields of the fields of complex numbers. These fields form a *tower* (not straight up, though), any two steps of the tower being related by a Galois group. One may ask how *all* these fields are to be constructed. Classical number theory answered this question in a complete and convincing fashion, in the case when the Galois groups are restricted to be *abelian*. The starting point of this theory, known as *Class field theory*, is the celebrated theorem of Kronecker-Weber:

The cyclotomic fields, namely the fields generated by roots of unity, give all extension fields with abelian Galois groups.

The great quest of modern number theory is to generalize this classical achievement to include the case[1] of field extensions that have a *non abelian* Galois group. It appears that one has a better access to the *linear representations of the Galois groups* rather than the groups themselves, and there is a remarkable contemporary programme, the so-called *Langlands Programme*, that sets out with a map for this voyage. It was first conceived by Robert Langlands, in the 1960's, and it has dominated all work in this direction since then. I shall have occasion to refer to it a little later, but at this time I shall just limit myself to the statement that this programme is a great synthesis of algebraic geometry, group theory and analysis, and that it has attracted the attention and efforts of a large number of contemporary mathematicians (see the two volumes issued by the American Mathematical Society [3]). Apropos of this, I must mention a theorem of Igor Shafaraevich that asserts that any *solvable* finite group of odd order can occur as the Galois group of an extension of the rational field.

2.5. Function Theory

The mathematicians of the 19^{th} century, starting with Riemann, pursued in great depth the analogy between algebraic numbers and algebraic functions. Their work led to the realization that the algebraic notion of the Galois group and the topological notions of the *fundamental group* and *monodromy group*, in the context of compact Riemann surfaces spread over the (extended)

[1] It may be maintained that this transition from the *commutative* to the *noncommutative* is the central feature of the way that the mathematicians and physicists of the present century have sought to understand and extend the work of their predecessors of the last century.

complex plane or other compact Riemann surfaces, were more or less the same. The continued exploration of this analogy in the present century has led to some of the major advances in mathematics. As an example of this, I may mention the theory of *Schemes* of Grothendieck, that enables one to treat both the arithmetic Galois group and the topological fundamental group from a single unified point of view.

2.6. Discrete Groups

In the 19^{th} century many questions of geometry, algebra and analysis led to the problem of classification of the finite subgroups of the group of rotations of two and three dimensional Euclidean spaces. As almost every one would know, these are the symmetry groups of the regular polygons and polyhedra (Platonic solids). More generally, we can also include the translations and consider discrete subgroups Γ of the group G of motions of the Euclidean spaces, such that the quotient space G/Γ is compact, the so-called *Crystallographic groups*. Their non Euclidean analogues, associated with the geometries of Riemann and Lobachevsky, have proved to be of fundamental importance in number theory and topology. See the recently published volume, [4], for a beautiful treatment of this aspect of group theory, as well as of many others.

3. CONTINUOUS GROUPS

In the 19^{th} century, Sophus Lie started the theory of what are nowadays called the *Lie Groups*, namely, groups whose elements are described by parameters taking values in a continuum (such as the continuum of real or complex numbers), and whose law of multiplication is defined by analytic (or at least differentiable) functions of these parameters. From the very beginning these groups were viewed as symmetry groups of geometric objects such as the set of lines in $\mathbf{R}^2$, the set of complex lines in $\mathbf{C}^2$, or more generally, the set of linear spaces of a given dimension in an ambient space of another given dimension. These are all examples of *compact algebraic manifolds imbedded in Projective spaces.* Now it turns out that the group of automorphisms of such a geometric object has two remarkable properties:

(1) *It is transitive.*

(2) *It is not solvable and, in fact, is a product of simple groups.*

Here *simple* means that there are no continuous normal subgroups other than the trivial ones and that one has to exclude the case when the group has dimension 1. The transitivity property (1) shows that these spaces are *homogeneous*, and so, have many strong applications. In view of these remarks we have the following absolutely fundamental question:

What are the simple Lie groups?

It was Killing and Cartan who first studied this problem, limiting themselves to the case when the groups are defined over the *complex* numbers. Their work, carried out in the second half of the 19^{th} century revealed the astonishing fact that these groups belong to 4 infinite families (named much later the *Classical Groups* by Hermann Weyl) and 5 isolated or *Exceptional Groups.* The families of classical groups are of course the usual ones:

A_n : *The group of linear transformations in dimension $n+1$ of determinant 1.*

B_n : *The group of orthogonal transformations in dimension $2n+1$.*

C_n : *The group of invertible linear transformations in dimension $2n$ that leave a nondegenerate skew symmetric bilinear form invariant.*

D_n : *The group of orthogonal transformations in dimension $2n$.*

These were, of course, around for a long time but the exceptional groups were new and represented a major discovery. Later, when the methods and ideas of Killing and Cartan were fully understood, it became natural to examine the problem of simple groups when one worked in the category of *algebraic* manifolds, rather than the differentiable ones. Chevalley did precisely this, and he succeeded, around the mid 1950's, in showing that the above classification *remained the same over any field that was algebraically closed.* The story of simple groups did not end here; Chevalley, using his theory over the fields of prime characteristic, realized that his work led to the construction of new *finite simple groups.* It became very clear that in the realm of finite simple groups the groups of *Lie-Chevalley type* were very important, and the burning question of the theory of finite groups became the following:

What are all the finite simple groups?

The natural conjecture was that if one took the groups of Lie-Chevalley type and the series of groups A_N of even permutations of N symbols, then there can only remain a finite number of isolated, or *sporadic simple groups*, to complete the list of simple groups. In all, there are 26 such sporadic groups, the largest of which is the *Monster* with

$$2^{46} \times 3^{20} \times 5^9 \times 7^6 \times 11^2 \times 13^3 \times 17 \times 19 \times 23 \times 29 \times 31 \times 41 \times 47 \times 59 \times 71$$

elements *There are no other simple groups.* This result, which is one of the great climaxes of the theory of groups in the present century, is due to the combined efforts of several dozen mathematicians whose work is spread through several hundred articles running to several thousand journal pages; and very few (if any) non specialists know all the details of this monumental accomplishment [5,6].

I would like to mention something about the manner by which Killing and Cartan reached their classification. They used the *infinitesimal method*, the essence of which is to work with the *infinitesimal motions* corresponding to the group elements rather than the finite ones. The advantage (which is huge) was of course that the problem became a *linear* one. In technical language, one can say that Killing and Cartan classified *simple Lie algebras over the field of complex numbers.* It is natural (for the mathematician) to raise the question of the classification of *simple Lie algebras defined over an algebraically closed field of arbitrary characteristic.* This has been carried out only recently by Richard Block and Robert Lee Wilson [7,8]. It is interesting to note that this classification, conjectured earlier by Kostrikin and Shafaraevich, shows a very striking analogy to the classification due to Cartan of certain *infinite dimensional* Lie algebras.

The link between the infinitesimal and finite motions is provided by the theory of differential equations and can be formalized as the theory of the so-called *exponential map* (to which it reduces when we deal with *linear* motions). Since the exponential series involves division by $n!$ for all integers n, it is clear that it will not be well behaved in prime characteristic. It is this that makes Chevalley's work so interesting because he did not use the infinitesimal method. His classification was actually independent of the Killing-Cartan classification over the complex numbers, and was carried out by *global* methods which were completely geometric and dealt directly with the groups themselves, and in which the idea that the groups in question were the groups of *all* automorphisms of certain projective varieties (the variety of *flags*) played a fundamental role.

In addition to his classification of simple groups over any algebraically closed field, Chevalley made another fundamental discovery concerning these groups. If you recall the definition of the classical groups, you will notice that they are defined as *groups of matrices that satisfy polynomial equations with integer coefficients.* This circumstance makes it possible to consider the points of such groups even when the coefficients are chosen from an arbitrary *commutative ring with unit.* Such groups should not be dismissed as the creatures of the mathematician's fantasy; for instance, the groups of matrices with *integer* entries plays a basic role in number theory and in crystallography. What Chevalley discovered was that this fact is true even for the exceptional groups. This discovery of Chevalley, in my opinion, is one of the great signposts of the theory of simple groups. It made it possible to consider for any simple group *its points over arbitrary commutative rings with unit.* If one now takes this ring to be a finite field then one has obtained *finite groups of the Lie type.* These and their "*twisted forms*" are the so-called *groups of Lie-Chevalley type.* If on the other hand one takes the ring to be the field of p-adic numbers, then one obtains groups which play a role in arithmetic that corresponds to the role of their complex counterparts in geometry and analysis. As a simple example let me mention the fact that the group of 2×2 matrices with entries from the field of p-adic numbers may be viewed as the group of all automorphisms of a *tree* which is infinite and has $p+1$ branches at each vertex [9]. The group acts *transitively on the tree*, and one may regard the tree and the corresponding group, with considerable justification, as the analogue, over the field of p-adic numbers, of the Lobachevsky upper half plane and its group of motions (*Möbius transformations*). The theory of simple Lie groups, especially the work of Killing, Cartan, Chevalley and their successors, thus provides an extraordinary generalization and unification of many classical problems, constructions and themes, and so is destined to play a pivotal role in the formulation and solution of a whole range of problems coming from mathematics and physics.

I would like to go back at this stage to the two problems that I mentioned at the beginning, and reemphasize that the idea of studying all the symmetries of natural structures has been a profoundly rewarding one for the theory of groups and their actions. This achievement of mathematicians of the present and the last century has placed at the disposal of the physical scientists a tool of unparalleled subtlety and power, whose impact is already clear from the huge number of applications of group theory in both the technical and the conceptual parts of mathematics and the physical sciences.

4. INFINITE DIMENSIONAL GROUPS

The advances and demands of mathematics and physics are already forcing the consideration of groups whose elements depend on *infinitely many continuous parameters*. In a certain sense, their study was already begun by Cartan in the early years of this century, when he discovered the analogues of the classical groups in *infinite dimension*. As an example of such a group (or pseudogroup), I may mention the group of volume-preserving transformations or, infinitesimally, the Lie algebra of vector fields of zero divergence. The simple transitive pseudogroups of transformations were classified by Cartan in 1909 [10]. In the theory of algebraic varieties and the theory of nonlinear ordinary differential equations, one encounters the groups of automorphisms of algebraic spaces, while in particle physics and linear differential equations, we run across the groups of *gauge transformations*. To these, one should also add the so-called *loop groups* and their extensions that are so prominent in particle physics these days. All these are very natural generalizations of the great classical simple groups from one or the other point of view. Recently, however, there are some very intriguing ideas that go in a new direction which can be (very roughly) described as a theory of spaces and their symmetries in the *noncommutative* context. For instance, the theory of *Quantum groups* [11] and *Noncommutative Geometry* [12]. However, there is no general and definitive theory of these generalizations as of now. In spite of this, these ideas are generating a tremendous amount of interest. Perhaps the interplay between mathematics and high energy physics may result in revealing the definitive concepts and direction of research in the entire subject in the near future [13–15].

5. REPRESENTATIONS

The second problem I mentioned is that of taking a typical simple group and asking what its representations are in a given category. By far, the most extensively studied are the category of linear spaces and their invariants. Apart from the fundamental importance of this theme in mathematics itself, the principal applications of representation theory have been to physics. In Quantum theory, the symmetries of any physical system are described by unitary operators of the Hilbert space that lies at the background, and so, if one wants to give a description of the system that is covariant with respect to some symmetry group (Poincaré group, for example, in special relativity), then it is essential that one should know the unitary representations of that particular group. Thus, the classification of elementary particles and the organization of their phenomenology is highly dependent on the representation theory of some of the groups that we have talked about. As an example of the fundamental role of representation theory in number theory, I recall the Langlands Programme that I mentioned earlier. The central idea in this programme is that it is the manifold of infinite dimensional unitary representations of the simple groups, and their variants, that describes the representations of the Galois groups of the fields of algebraic numbers. The representation theory of the simple groups in finite dimensions was created by Cartan and Weyl in the 1920's and had a profound impact on the theory of geometric invariants. It was reexamined in modern times by Mumford, who discovered new aspects of it in fields of prime characteristic. The work of Mumford and Haboush and their successors have led to a rejuvenation of the classical theory of invariants. The representation theory of the simple finite dimensional groups in *infinite* dimensions is a beautiful structure of mathematics, erected by the efforts of Harish-Chandra and Gel'fand. As for the representation theory of the infinite dimensional groups, all one can say is that the subject is still very much in its beginning stages, and that although there are intriguing possibilities, definitive directions are still not visible [16,17].

These remarks should give the reader a glimpse of the beautiful world of symmetry and group theory. I cannot do better than end by recalling the words of Hermann Weyl, one of the greatest figures in group theory in this century, that his whole life was spent in the pursuit of the true and the beautiful, and that if he had to make a choice, he chose the beautiful. I feel we do not need to make this choice, and that the theory of groups will help us comprehend the marvelous interplay of truth and beauty that we see all around us.

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