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Application of the variational iteration method to inverse heat source problems

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ABSTRACT

This paper investigates the inverse problem of determining a heat source in the parabolic heat equation using the usual conditions. The numerical solution is developed by using the variational iteration method. This method is based on the use of Lagrange multipliers for the identification of optimal values of parameters in a functional. Using this method a rapid convergent sequence can be obtained which tends to the exact solution of the problem. Furthermore, the variational iteration method does not require the discretization of the problem. Thus the variational iteration method is suitable for finding the approximation of the solution without discretization of the problem. Two numerical examples are presented to illustrate the strength of the method.

1. Introduction

In the process of transportation, diffusion and conduction of natural materials, the following heat equation is induced:

$$u_t - a^2 \Delta u = f(x, t; u), \quad (x, t) \in \Omega \times (0, T), \tag{1.1}$$

where u represents the state variable, a is the diffusion coefficient, Ω is a bounded domain in $\mathbb{R}^d$ and f denotes physical laws, which means a source term here. There are many researches on such an inverse problem of determining the source term from 1970s, since the characteristic of source in the practical problem is always unknown. However, the solution of this problem is unstable, hence the inverse problem is ill-posed. The major difficulty in establishing any numerical algorithm for approximating the solution is the ill-posedness of the problem and the ill-conditioning of the resultant discretized matrix. The inverse problem of determining an unknown heat source function in the heat conduction equation has been considered in many papers [1–8]. For $f = f(u)$, the inverse source problem with additional data was studied by Cannon, Duchateau and Fatullayev [1,2]. In [3,4], the source was sought as a function of both space and time but is additive or separable. However, in all the other studies the source has been sought as a function of space or time only [5–8].

In this paper, the heat source is taken to be a function of both space and time but is separable, and the overdetermination is the transient temperature measured at time T. This measurement ensures that the inverse problem has a unique solution, but this solution is unstable, hence the problem is ill-posed. The inverse problem is formulated in Section 2. Several numerical methods have been proposed for the inverse source problem [5–8]. In this work, we extent the use of variational iteration method (VIM) to this inverse source problem. The VIM was proposed originally by He [9–16]. This method is based on the use of Lagrange multipliers for the identification of optimal values of parameters in a functional. This method gives rapidly convergent successive approximations of the exact solution if such a solution exists. Furthermore, VIM does not

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require discretization of the problem. Thus the variational iteration method is suitable for finding the approximation of the solution without discretization of the problem. It was successfully applied to two-point boundary value problems, partial differential equations, evolution equations and other fields [9–30].

The rest of the paper is organized as follows. In Section 2, we formulate the problem mathematically. The VIM is introduced and applied to the inverse source problem in Section 3. The numerical examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Formulation of the inverse problem

Consider the one-dimensional problem in which the source $f(x,t;u)=\varphi(t)f(x)$:

$$u_t-a^2u_{xx}=\varphi(t)f(x),\quad (x,t)\in(0,1)\times(0,T), \tag{2.1}$$

subject to the initial and boundary conditions

$$u(x,0)=v_0(x),\quad 0\le x\le 1, \tag{2.2}$$

$$u(0,t)=g_0(t),\quad u(1,t)=g_1(t),\quad 0\le t\le T, \tag{2.3}$$

and the overspecified condition

$$u(x,T)=h(x),\quad 0\le x\le 1, \tag{2.4}$$

where $v_0(x),\ g_0(t),\ g_1(t)$ are given functions satisfying the compatibility conditions

$$v_0(0)=g_0(0),\qquad v_0(1)=g_1(0) \tag{2.5}$$

and the functions $u(x,t)$ and $f(x)$ are unknown.

Although sufficient conditions for the solvability of the problem are provided [5], problem (2.1)–(2.5) is still ill-posed since small errors, inherently present in any practical measurement, give rise to unbounded and highly oscillatory solutions. We will change (2.1) to an equation which is easy to handle using VIM.

From (2.1) and (2.4), one obtains

$$u_{xx}(x,T)=h''(x)=\frac{1}{a^2}(u_t(x,T)-\varphi(T)f(x)). \tag{2.6}$$

Hence,

$$f(x)=\frac{u_t(x,T)-a^2h''(x)}{\varphi(T)}. \tag{2.7}$$

Substituting (2.7) into (2.1) yields

$$u_{xx}(x,t)-\frac{u_t(x,t)}{a^2}+\frac{\varphi(t)u_t(x,T)}{a^2\varphi(T)}-\frac{\varphi(t)h''(x)}{\varphi(T)}=0. \tag{2.8}$$

3. Analysis and application of He's variational iteration method

Consider the differential equation

$$Lu+Nu=g(x), \tag{3.1}$$

where L and N are linear and nonlinear operators, respectively, and $g(x)$ is the source inhomogeneous term. In [9–15], the VIM was introduced by He where a correct functional for (3.1) can be written as

$$u_{n+1}(x)=u_n(x)+\int_0^x\lambda[Lu_n(t)+N\tilde{u}_n(t)-g(t)]\mathrm{d}t, \tag{3.2}$$

where λ is a general Lagrangian multiplier [10], which can be identified optimally via variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n=0$. By this method, it is firstly required to determine the Lagrangian multiplier λ that will be identified optimally. The successive approximations $u_{n+1}(x)$, $n\ge 0$, of the solution $u(x)$ will be readily obtained upon using the determined Lagrangian multiplier and any selective function $u_0(x)$. Consequently, the solution is given by

$$u(x)=\lim_{n\to\infty}u_n(x).$$

For variational iteration method, the key element is the identification of the Lagrangian multiplier. For linear problems, their exact solutions can be obtained by only one iteration step due to the fact that the Lagrangian multiplier can be identified exactly. For nonlinear problems, the Lagrangian multiplier is difficult to be identified exactly. To overcome the difficulty, we apply restricted variations to nonlinear terms. Due to the approximate identification of the Lagrangian multiplier, the approximate solutions converge to their exact solutions relatively slowly. It should be specially pointed out that the more accurate the identification of the multiplier, the faster the approximations converge to their exact solutions.

For (2.8), according to the VIM, we consider its correct functional in x-direction in the following form:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x \lambda \left[u_{n_{\xi\xi}}(\xi,t) - \frac{1}{a^2}\tilde{u}_{n_t}(\xi,t) + \frac{\varphi(t)\tilde{u}_{n_t}(\xi,T)}{a^2\varphi(T)} - \frac{\varphi(t)h''(\xi)}{\varphi(T)} \right] \mathrm{d}\xi, \tag{3.3}$$

where λ is the general Lagrangian multiplier, which can be identified optimally via variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$.

To find the optimal value of λ, we have

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^x \lambda \left[u_{n_{\xi\xi}}(\xi,t) - \frac{1}{a^2}\tilde{u}_{n_t}(\xi,t) + \frac{\varphi(t)\tilde{u}_{n_t}(\xi,T)}{a^2\varphi(T)} - \frac{\varphi(t)h''(\xi)}{\varphi(T)} \right] \mathrm{d}\xi = 0, \tag{3.4}$$

that results

$$\delta u_{n+1}(x,t) = \delta u_n(x,t)(1-\lambda'(x)) + \delta u_n'(x,t)\lambda(x) - \int_0^x \delta u_n(\xi,t)\lambda''(\xi)\mathrm{d}\xi = 0. \tag{3.5}$$

We, therefore, have the following stationary conditions:

$$\lambda''(\xi) = 0,$$
$$1 - \lambda'(x) = 0, \qquad \lambda(x) = 0.$$

The Lagrangian multiplier can be identified in the form

$$\lambda(\xi) = \xi - x,$$

Therefore, we have the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (\xi - x) \left[u_{n_{\xi\xi}}(\xi,t) - \frac{1}{a^2}u_{n_t}(\xi,t) + \frac{\varphi(t)u_{n_t}(\xi,T)}{a^2\varphi(T)} - \frac{\varphi(t)h''(\xi)}{\varphi(T)} \right] \mathrm{d}\xi. \tag{3.6}$$

Now taking $u_0(x,t) = u(0,t) + \alpha u(1,t) = g_0(t) + \alpha g_1(t)$ as an initial value, where α is the unknown parameter to be further determined, according to (3.6), we can obtain the n-order approximate solution $u_n(x,t)$ of (2.1). Incorporating the initial condition $u_0(x,0) = v_0(x)$ into $u_n(x,t)$, the unknown parameter α can be obtained. Therefore, the n-order approximation $u_n(x,t)$ is obtained.

Hence, from (2.7), the n-order approximation to $f(x)$ is

$$f_n(x) = \frac{u_{n_t}(x,T) - a^2h''(x)}{\varphi(T)}. \tag{3.7}$$

4. Numerical examples

In this section, we present and discuss the numerical results by employing VIM for two test examples. For these examples, we have taken $a = 1$ and $T = 1$. The results demonstrate that the present method is remarkably effective.

Example 4.1. With the input data

$$u(x,0) = v_0(x) = \mathrm{e}^{2x},$$
$$u(0,t) = g_0(t) = \mathrm{e}^{t}, \qquad u(1,t) = g_1(t) = \mathrm{e}^{2+t},$$
$$u(x,1) = h(x) = \mathrm{e}^{1+2x}, \qquad \varphi(t) = \mathrm{e}^{t},$$

the inverse problem (2.1)–(2.5) has the unique solution given by

$$u(x,t) = \mathrm{e}^{2x+t},$$
$$f(x) = 3\mathrm{e}^{2x}.$$

Beginning with

$$u_0(x,t) = g_0(t) + \alpha g_1(t) = \mathrm{e}^{t} + \alpha\mathrm{e}^{2+t},$$

where α is the unknown parameter to be further determined, according to (3.6), one can obtain the first-order approximation $u_1(x,t)$.

Incorporating the initial condition $u_0(x,0) = v_0(x) = \mathrm{e}^{2x}$ of Example 4.1 into $u_1(x,t)$, the unknown parameter α can be obtained. Therefore, the first-order approximation $u_1(x,t)$ is obtained and $u_1(x,t) = \mathrm{e}^{t+2x}$, which is the exact solution of Example 4.1. From (3.7), we have $f_1(x) = 3\mathrm{e}^{2x}$, which is equal to the exact $f(x)$ of Example 4.1.

Example 4.2. With the input data

$$u(x,0) = v_0(x) = \sin(x),$$
$$u(0,t) = g_0(t) = 0, \qquad u(1,t) = g_1(t) = \sin(1)e^t,$$
$$u(x,1) = h(x) = e\sin(x), \quad \varphi(t) = e^t,$$

the inverse problem (2.1)–(2.5) has the unique solution given by

$$u(x,t) = \sin(x)e^t,$$
$$f(x) = -2\sin(x).$$

Beginning with

$$u_0(x,t) = g_0(t) + \alpha g_1(t) = \alpha\sin(1)e^t,$$

where α is the unknown parameter to be further determined, according to (3.6), one can obtain the first-order approximation $u_1(x,t)$.

Incorporating the initial condition $u_0(x,0) = v_0(x) = \sin(x)$ of Example 4.2 into $u_1(x,t)$, the unknown parameter α can be obtained. Therefore, the first-order approximation $u_1(x,t)$ is obtained and $u_1(x,t) = \sin(x)e^t$, which is the exact solution of Example 4.2. From (3.7), we have $f_1(x) = -\sin(x)$, which is equal to the exact $f(x)$ of Example 4.2.

Remark. From the above two numerical examples, it can be seen that the exact solutions are obtained by using one iteration step only.

5. Conclusion

In this paper, He's variational iteration method was employed successfully for solving the inverse heat source problem. This method solves the problem without any discretization of the variables. Therefore, it is not affected by rounding the errors in the computational process. The numerical results show that the VIM is an accurate and reliable numerical technique for the solution of the inverse time-dependent heat source problem.

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