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A certain class of starlike functions

Janusz Sokół

Department of Mathematics, Rzeszów University of Technology, ul. W. Pola 2, 35-959 Rzeszów, Poland

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ABSTRACT

This paper presents a new class of functions analytic in the open unit disc, and closely related to the class of starlike functions. Besides being an introduction to this field, it provides an interesting connections defined class with well known classes. The paper deals with several ideas and techniques used in geometric function theory. The order of starlikeness in the class of convex functions of negative order is also considered here.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\Delta = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0, f'(0) = 1$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in Δ will be denoted by $\mathcal{S}^*$. The set of all functions $f \in \mathcal{A}$ that are convex univalent in Δ by $\mathcal{K}$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is, if and only if the linear segment joining any two points of E lies entirely in E. Let the function f be analytic univalent in the unit disc Δ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$. Then f maps Δ onto a starlike domain with respect to $w_0 = 0$ if and only if [1]

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \Delta), \tag{1.1}$$

while f maps Δ onto a convex domain E if and only if [2]

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \Delta). \tag{1.2}$$

Such function f is said to be starlike in Δ with respect to $w_0 = 0$ (or briefly starlike) or, respectively, is said to be convex in Δ (or briefly convex). It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0, f'(0) \neq 0$, then f is univalent and starlike in Δ. Robertson introduced in [3], the classes $\mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha \leq 1$, which are defined by

$$\mathcal{S}^*(\alpha) := \left\{f \in \mathcal{A} : \mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ z \in \Delta\right\}, \tag{1.3}$$

E-mail addresses: jsokol@prz.edu.pl, jsokol@prz.rzeszow.pl.

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$$\mathcal{K}(\alpha) := \left\{ f \in \mathcal{A} : \mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \ z \in \Delta \right\}. \tag{1.4}$$

If $\alpha \in [0; 1)$, then a function in either of these sets is univalent, if $\alpha < 0$ it may fail to be univalent. In particular we denote $\mathcal{S}^*(0) = \mathcal{S}^*$, $\mathcal{K}(0) = \mathcal{K}$. Let $\mathcal{S}$ denote the subset of $\mathcal{A}$ which is composed of univalent functions. We say that f is subordinate to F in Δ, written as $f \prec F$, if and only, if $f(z) = F(\omega(z))$ for some holomorphic function $\omega(z)$, $\omega(0) = 0$, $|\omega(z)| < 1, z \in \Delta$. The class of starlike functions $\mathcal{S}^*$ can be defined in several ways, for example we say that f is starlike if it satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec p(z) \quad (z \in \Delta) \tag{1.5}$$

where $p(z) = (1+z)/(1-z)$. Many subclasses of $\mathcal{S}^*$ have been defined by the condition (1.5) with a convex univalent function p, given arbitrary. If we restrict considerations to the absorbing geometric shape of $p(\Delta)$, then it is proper to recall the papers [4,5], where $p(\Delta)$ is a disc. In [6,7] the set $p(\Delta)$ is an angle while in [8–10] $p(\Delta)$ is a parabolic domain. In [11–13] the set $p(\Delta)$ is an interior of hyperbola or of an elliptic domain. For the case when $p(\Delta)$ is an interior of the right loop of the Lemniscate of Bernoulli see [14,15] or when $p(\Delta)$ is a leaf-like domain see [16]. An interesting case when the function p is convex but is not univalent was considered in [17]. A function p that is not univalent and is not convex and maps unit circle onto the trisectrix of Maclaurin was considered in [18]. In the current paper we shall study a class defined by Eq. (1.5) with univalent function p which maps Δ onto a concave set. One of the results obtained applies to the order of starlikeness of the class of convex functions of negative order.

2. Preliminary results

At the beginning of this chapter we shall investigate the properties of a one-parameter family of functions used in the sequel.

Lemma 1. *Let*

$$p_b(z) = \frac{1}{1-(1+b)z+bz^2} \quad (z \in \Delta). \tag{2.1}$$

If $-1 < b < 1$, *then*

$$\mathfrak{Re}\{p_b(z)\} > \frac{1-3b}{2(1-b)^2} \quad (z \in \Delta). \tag{2.2}$$

Proof. Note that $1-(1+b)z+bz^2 = b(z-1)(z-1/b)$ so the function p_b does not have any poles in $\overline{\Delta} \setminus \{1\}$ and is analytic in Δ, thus looking for the $\min\{\mathfrak{Re}\{p_b(z)\} : |z| < 1\}$ it is sufficient to consider it on the boundary $\partial p_b(\Delta) = \{p_b(e^{i\varphi}) : \varphi \in (0, 2\pi)\}$. We have

$$\frac{1}{1+az+bz^2} = \frac{1+a\cos\varphi + b\cos 2\varphi - i(a\sin\varphi + b\sin 2\varphi)}{1+a^2+b^2+2a(1+b)\cos\varphi + 2b\cos 2\varphi}, \tag{2.3}$$

thus we can write

$$\begin{aligned}
\mathfrak{Re}\{p_b(e^{i\varphi})\} &= \frac{1-(1+b)\cos\varphi + b\cos 2\varphi}{1+(1+b)^2+b^2-2(1+b)^2\cos\varphi + 2b\cos 2\varphi} \\
&= \frac{1-b-(1+b)\cos\varphi + 2b\cos^2\varphi}{2(1+b^2)-2(1+b)^2\cos\varphi + 4b\cos^2\varphi} \\
&= \frac{(1-\cos\varphi)(1-b-2b\cos\varphi)}{(1-\cos\varphi)(2(1+b^2)-4b\cos\varphi)} \\
&= \frac{1-b-2b\cos\varphi}{2(1+b^2)-4b\cos\varphi}.
\end{aligned} \tag{2.4}$$

So we can see that $\mathfrak{Re}\{p_b(e^{i\varphi})\}$ is well defined also for $\varphi = 0$. The function

$$f(x) = \frac{1-b-2bx}{2(1+b^2)-4bx}$$

decreases for $b > -1$ so (2.4) attains its minimal value when $\cos\varphi = 1$. Substituting it in (2.4) we get (2.2). $\square$

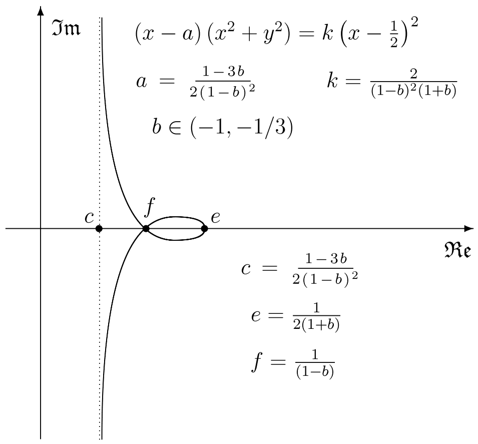

Fig. 1. $p_b(\mathrm{e}^{\mathrm{i}\varphi}),\ b\in(-1,-1/3)$.

Note that the function

$$g(b)=\frac{1-3b}{2(1-b)^2}\quad b\in(-1,1) \tag{2.5}$$

increases in $(-1,-1/3]$ from $g(-1)=1/2$ to $g(-1/3)=9/16$ and decreases in $(-1/3,1)$ to $-\infty$.

Lemma 2. *If $b\in[-1/3,1]$, then the function p_b defined in (2.1) is univalent in Δ.*

Proof. We have

$$p_b(z_1)=p_b(z_2)\Leftrightarrow (z_1-z_2)(1+b-b(z_1+z_2))=0. \tag{2.6}$$

It is easy to see that for $b=0$ we have $p_b(z_1)=p_b(z_2)\Leftrightarrow(z_1=z_2)$. When $b\neq 0$ we have

$$1+b-b(z_1+z_2)=0\Leftrightarrow z_1+z_2=\frac{1+b}{b}.$$

If $b\in[-1/3,1]\setminus\{0\}$, then $|(1+b)/b|\geq 2$ so there are no $z_1,z_2\in\Delta$ such that $1+b-b(z_1+z_2)=0$. Therefore by (2.6) the function p_b is univalent in Δ when $b\in[-1/3,1]$. $\square$

Let the function p_b be given by (2.1) and let us denote $\mathfrak{Re}\{p_b(\mathrm{e}^{\mathrm{i}\varphi})\}=x$ and $\mathfrak{Im}\{p_b(\mathrm{e}^{\mathrm{i}\varphi})\}=y$, $\varphi\in(0,2\pi)$. Then by (2.3) and (2.4) after simple calculation, we get

$$x=\frac{1-b-2b\cos\varphi}{2(1+b^2)-4b\cos\varphi},\qquad y=\frac{(1+b-2b\cos\varphi)\sin\varphi}{(1-\cos\varphi)(2(1+b^2)-4b\cos\varphi)}. \tag{2.7}$$

Therefore we can find that the image of the unit circle $|z|=1$ under the function p_b is a curve described by

$$\gamma_1:\ (x-a)(x^2+y^2)-k(x-1/2)^2=0,\quad \text{where } a=\frac{1-3b}{2(1-b)^2},\qquad k=\frac{2}{(1-b)^2(1+b)}. \tag{2.8}$$

Thus the curve γ_1 is symmetric with respect to real axis. Investigating the ordinate y in (2.7) it easy to see that for b such that the equation $1+b-2b\cos\varphi=0$ has solutions $\varphi_1,\ \varphi_2\in(0,2\pi)$ the curve γ_1 has a loop intersecting the real axis at the points $1/(1-b)$ and $1/(2+2b)$. A simple calculations shows that it is when $b\in(-1,-1/3)$, see Fig. 1. For $b\in[-1/3,1)$ the curve γ_1 has no loops and it is like a conchoid (see Fig. 2) such that

$$\frac{1-3b}{2(1-b)^2}<\mathfrak{Re}\{p_b(\mathrm{e}^{\mathrm{i}\varphi})\}\leq\frac{1}{2(1+b)}=p_b(-1)\quad \varphi\in(0,2\pi). \tag{2.9}$$

Lemma 3. *If $-1/3\leq b_1<b_2<1$, then*

$$p_{b_1}\prec p_{b_2}\quad (z\in\Delta). \tag{2.10}$$

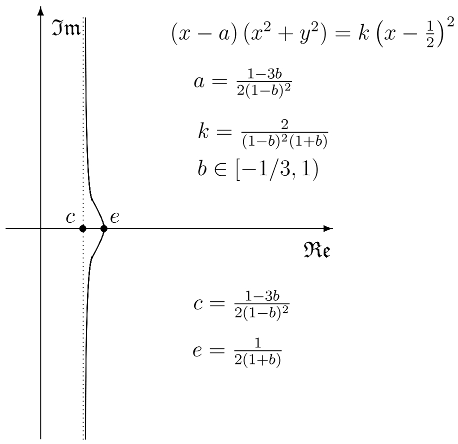

Fig. 2. $p_b(e^{i\varphi}),\ \ b \in [-1/3, 1)$.

Proof. If $-1/3 \le b_1 < b_2 < 1$, then by Lemma 2 the functions p_{b_1} and p_{b_2} are univalent in Δ so for the proof of (2.10) it suffices to show that $p_{b_1}(\Delta) \subset p_{b_2}(\Delta)$. That is $\partial p_{b_1}(\Delta) \cap \partial p_{b_2}(\Delta) = \emptyset$. Thus we need to show that the system of equations of boundary curves

$$\begin{cases}(x-a_1)(x^2+y^2)-k_1(x-1/2)^2=0\\ (x-a_2)(x^2+y^2)-k_2(x-1/2)^2=0\end{cases} \tag{2.11}$$

where $a_i = \frac{1-3b_i}{2(1-b_i)^2},\ k_i = \frac{2}{(1-b_i)^2(1+b_i)},\ i = 1, 2$, has no solutions (x, y). If the system (2.11) has a solution (x, y), then by (2.9) should to be $\frac{1-3b_2}{2(1-b_2)^2} < x \le \frac{1}{2(1+b_2)}$. We will now show that it is impossible. Comparing $x^2 + y^2$ in the Eqs. (2.11) we obtain

$$x = \frac{k_1a_2 - k_2a_1}{k_1 - k_2} = \frac{2 + 3(b_1 + b_2)}{2[1 + b_1 + b_2 - (b_1^2 + b_1b_2 + b_2^2)]}. \tag{2.12}$$

The function $a(b) = \frac{1-3b}{2(1-b)^2}$ increases, while the function $k(b) = \frac{2}{(1-b)^2(1+b)}$ decreases for $-1 < b < 1$ so $a_1 < a_2$ and $k_2 < k_1$. Thus x in (2.12) is a positive number. Moreover, because $2 + 3(b_1 + b_2) > 0$ for $-1/3 \le b_1 < b_2 < 1$, then the denominator of (2.12) is positive too. Further calculation of (2.12) shows that

$$x = \frac{1}{2(b_2+1)} + \frac{(2b_2+b_1+1)^2}{2(1+b_1)[1+b_1+b_2-(b_1^2+b_1b_2+b_2^2)]}$$

so $x > \frac{1}{2(b_2+1)}$ for $-1/3 \le b_1 < b_2 < 1$, so the system (2.11) has no solution. □

Lemma 4. *Let q be analytic in Δ with $q(0) = 1$. A function g is in the class*

$$\mathcal{B}(q) = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec q(z)\ (z \in \Delta)\right\}$$

if and only if there exists an analytic function p, $p \prec q$, such that

$$g(z) = z \exp \int_0^z \frac{p(t) - 1}{t}\, dt \quad (z \in \Delta). \tag{2.13}$$

Proof. Let $g \in \mathcal{B}(q)$ and let $p(z) := zg'(z)/g(z)$. Then $p \prec q$ and integrating this equation we obtain (2.13). If g is given by (2.13) with an analytic p, $p(0) = 0$, $p \prec q$, then differentiating logarithmically (2.13) we obtain $zg'(z)/g(z) = p(z)$ therefore $zg'(z)/g(z) \prec q(z)$ and $g \in \mathcal{B}(q)$. □

3. The class $\mathcal{SK}(\alpha)$ and its properties

Definition 1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SK}(\alpha)$, $\alpha \in (-3, 1]$, if it satisfies the condition

$$\frac{zf'(z)}{f(z)} \prec \widetilde{q}_\alpha(z) := \frac{3}{3 + (\alpha - 3)z - \alpha z^2} \quad (z \in \Delta). \tag{3.1}$$

It is easily observed that the function (2.1) with $b = -\alpha/3$ becomes the function $\widetilde{q}_\alpha$. Moreover, by Lemma 2 the function $\widetilde{q}_\alpha$ is univalent in Δ when $\alpha \in (-3, 1]$. We have made this normalization because of the reductions of formulas in the next considerations. The name of this class is from the fact that the curve $\widetilde{q}_\alpha(e^{i\varphi})$, $\varphi \in (0, 2\pi)$ is geometrically similar to a conchoid in Fig. 2. From Lemma 1 we obtain

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{9(1+\alpha)}{2(3+\alpha)^2} \quad (z \in \Delta),$$

when $f \in \mathcal{SK}(\alpha)$. Therefore we obtain the following corollary.

Corollary 1. *$\mathcal{SK}(\alpha) \subset \mathcal{S}^*(\gamma) = \{f \in \mathcal{A} : \mathfrak{Re}\{\frac{zf'(z)}{f(z)}\} > \gamma,\ z \in \Delta\}$, where $\gamma = \frac{9(1+\alpha)}{2(3+\alpha)^2}$, this means that if $f \in \mathcal{SK}(\alpha)$, then it is starlike of order γ. If $-1 \le \alpha < 1$, then f belonging to the class $\mathcal{SK}(\alpha)$ is a starlike function and so it is univalent in the unit disc Δ.*

Corollary 1 and (3.1) give that $\mathcal{SK}(0) = \mathcal{S}^*(1/2)$ and $\mathcal{SK}(1) \subset \mathcal{S}^*(9/16)$, $\mathcal{SK}(-1) \subset \mathcal{S}^*$. Moreover, by Lemma 2 the function $\widetilde{q}_\alpha, \alpha \in (-3, 1]$, is univalent in Δ so by the subordination principle the subordination (2.8) with $b = -\alpha/3$ and (3.1) give the next corollary.

Corollary 2. *A function $f \in \mathcal{A}$ belongs to the class $\mathcal{SK}(\alpha)$, $\alpha \in (-3; 1]$, if and only if for $z \in \Delta$ the quantity $zf'(z)/f(z)$ takes all its values on the right-hand side of the curve*

$$\gamma_2 : \ (x-a)(x^2+y^2) - k(x-1/2)^2 = 0, \quad \text{where } a = \frac{9(1+\alpha)}{2(3+\alpha)^2} \qquad k = \frac{54}{(3+\alpha)^2(3-\alpha)}. \tag{3.2}$$

Using Lemma 3 we directly can obtain the next corollary.

Corollary 3. *If $-3 \le \alpha_1 < \alpha_2 \le 1$, then*

$$\mathcal{S}^*(1/2) = \mathcal{SK}(0) \supset \mathcal{SK}(\alpha_1) \supset \mathcal{SK}(\alpha_2) \supset \mathcal{SK}(1) \subset \mathcal{S}^*(9/16).$$

Note that

$$\begin{aligned}\widetilde{q}_\alpha(z) &= \frac{3}{3+(\alpha-3)z-\alpha z^2} = \frac{3}{3+\alpha}\left[\frac{1}{1-z} + \frac{\alpha}{\alpha z+3}\right] \\ &= \frac{3}{3+\alpha}\sum_{n=0}^{\infty}\left[1+(-1)^n\left(\frac{\alpha}{3}\right)^{n+1}\right]z^n = 1 + \frac{(3-\alpha)^2}{3(3+\alpha)}z + \cdots.\end{aligned} \tag{3.3}$$

Let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order α defined in (1.3), and let $\mathcal{S}^*[B]$ be the subclass of $\mathcal{S}^*$ defined by

$$\mathcal{S}^*[B] = \left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1}{1+Bz}\right\}, \tag{3.4}$$

where $-1 \le B \le 1$, $B \ne 0$. Observe that for $B = 1$ the function $p(z) = \frac{1}{1+Bz}$ maps the unit disc Δ onto the half-plane $\mathfrak{Re}\,w > 1/2$, while onto the disc $\mathbf{D}(C(B), R(B))$ with the center $C(B) = 1/(1-B^2)$ and the radius $R(B) = |B|/(1-B^2)$, when $B \ne 0$. We now formulate the following theorem for these classes of functions.

Theorem 1. *If a function f belongs to the class $\mathcal{SK}(\alpha)$, $\alpha \in (-3, 1]\setminus\{0\}$, then there exists a function $g \in \mathcal{S}^*(1/2)$ and a function $h \in \mathcal{S}^*[\alpha/3]$ such that*

$$f(z) = [g(z)]^{\frac{3}{3+\alpha}}[h(z)]^{\frac{\alpha}{3+\alpha}} \quad (z \in \Delta). \tag{3.5}$$

If $\alpha = 0$, then $\mathcal{SK}(0) = \mathcal{S}^*(1/2)$.

Proof. Let $f \in \mathcal{SK}(\alpha)$. Then by Lemma 4, there exists an analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in \Delta$ such that

$$f(z) = z\exp\int_0^z \frac{\widetilde{q}_\alpha(\omega(t))-1}{t}\,dt. \tag{3.6}$$

Notice that from (3.3) we have

$$\widetilde{q}_\alpha(\omega(t)) = \frac{3}{3+\alpha}\,\frac{1}{1-\omega(t)} + \frac{3\alpha}{3+\alpha}\,\frac{1}{3+\alpha\omega(t)} \tag{3.7}$$

and hence, we can rewrite (3.6) in the form

$$
\begin{aligned}
f(z) &= z\exp\left[\int_0^z \frac{\frac{3}{3+\alpha}\left[\frac{1}{1-\omega(t)}-1\right]}{t}\,\mathrm{d}t + \int_0^z \frac{\frac{\alpha}{3+\alpha}\left[\frac{1}{1+\frac{\alpha}{3}\omega(t)}-1\right]}{t}\,\mathrm{d}t\right] \\
&= \left[z\exp\int_0^z \frac{\left[\frac{1}{1-\omega(t)}-1\right]}{t}\,\mathrm{d}t\right]^{\frac{3}{3+\alpha}} \times \left[z\exp\int_0^z \frac{\left[\frac{1}{1+\frac{\alpha}{3}\omega(t)}-1\right]}{t}\,\mathrm{d}t\right]^{\frac{\alpha}{3+\alpha}} \\
&= [g(z)]^{\frac{3}{3+\alpha}}[h(z)]^{\frac{\alpha}{3+\alpha}}.
\end{aligned}
\tag{3.8}
$$

Using the structural formulas for the classes $\mathcal{S}^*(\alpha)$ (see [19, p. 172]) and $\mathcal{S}^*[B]$ (see Lemma 4), we find that the functions g, h defined in (3.8) satisfy $g \in \mathcal{S}^*(1/2)$, and $h \in \mathcal{S}^*[\alpha/3]$ which proves Theorem 1. □

Theorem 2. *If there exist a function $g \in \mathcal{S}^*(1/2)$ and a function $h \in \mathcal{S}^*[\alpha/3]$ such that*

$$\frac{zg'(z)}{g(z)} = \frac{1}{1-\omega(z)}, \qquad \frac{zh'(z)}{h(z)} = \frac{1}{1+\alpha\omega(z)/3} \quad (z \in \Delta), \tag{3.9}$$

for certain analytic function ω with $\omega(0) = 0$, $|\omega(z)| < 1$, $z \in \Delta$, then the function

$$f(z) = [g(z)]^{\frac{3}{3+\alpha}}[h(z)]^{\frac{\alpha}{3+\alpha}}$$

belongs to the class $f \in \mathcal{SK}(\alpha)$.

Proof. The conditions (3.9) say that the functions $g \in \mathcal{S}^*(1/2)$ and $h \in \mathcal{S}^*[\alpha/3]$ are generated by (2.13) with the same function ω. Therefore, by the considerations in the previous proof and by (3.7) and (3.8) we can get

$$[g(z)]^{\frac{3}{3+\alpha}}[h(z)]^{\frac{\alpha}{3+\alpha}} = z\exp\int_0^z \frac{\widetilde{q}_\alpha(\omega(t))-1}{t}\,\mathrm{d}t.$$

This shows that $[g(z)]^{\frac{3}{3+\alpha}}[h(z)]^{\frac{\alpha}{3+\alpha}} \in \mathcal{SK}(\alpha)$ which proves Theorem 2. □

Theorem 3. *If $f \in \mathcal{SK}(\alpha)$, $\alpha \in (-3, 1]$ and $|z| = r$, $0 \le r < 1$, then*

$$\left(\frac{r}{1+r}\right)^{\frac{3}{3+\alpha}}\left(\frac{r}{1+\alpha r/3}\right)^{\frac{\alpha}{3+\alpha}} \le |f(z)| \le \left(\frac{r}{1-r}\right)^{\frac{3}{3+\alpha}}\left(\frac{r}{1-\alpha r/3}\right)^{\frac{\alpha}{3+\alpha}}. \tag{3.10}$$

Proof. Suppose that $\alpha \in (-3, 0)$. To find (3.10) let us recall, see [5, pp. 315–317], that if $h \in \mathcal{S}^*[\alpha/3]$, $\alpha \in (-3, 0]$, then for $|z| = r$, $0 \le r < 1$ we have

$$\frac{r}{1-\alpha r/3} \le |h(z)| \le \frac{r}{1+\alpha r/3}. \tag{3.11}$$

Recall also that, if $g \in \mathcal{S}^*(1/2)$, then for $|z| = r$, $0 \le r < 1$ we have

$$\frac{r}{1+r} \le |g(z)| \le \frac{r}{1-r}. \tag{3.12}$$

Moreover, from (3.11) we get

$$\left(\frac{r}{1+\alpha r/3}\right)^{\frac{\alpha}{3+\alpha}} \le |h(z)|^{\frac{\alpha}{3+\alpha}} \le \left(\frac{r}{1-\alpha r/3}\right)^{\frac{\alpha}{3+\alpha}} \tag{3.13}$$

because $\frac{\alpha}{3+\alpha} < 0$ when $\alpha \in (-3, 0)$. By Theorem 1 we have $|f(z)| = |g(z)|^{\frac{3}{3+\alpha}}|h(z)|^{\frac{\alpha}{3+\alpha}}$ with $g \in \mathcal{S}^*(1/2)$ and $g \in \mathcal{S}^*(1/2)$. Raising (3.12) to the power $\frac{3}{3+\alpha}$ and then multiplying by sides with (3.13) we get the condition (3.10). If $\alpha \in (0, 1)$, then analogously as (3.11) we can obtain

$$\frac{r}{1+\alpha r/3} \le |h(z)| \le \frac{r}{1-\alpha r/3}. \tag{3.14}$$

when $h \in \mathcal{S}^*[\alpha/3]$ and $|z| = r$. Using again the fact that $|f(z)| = |g(z)|^{\frac{3}{3+\alpha}}|h(z)|^{\frac{\alpha}{3+\alpha}}$, where both exponents are positive, and accordingly multiplying by sides (3.11) and (3.14) we obtain (3.10). If $\alpha = 0$, then $\mathcal{SK}(0) = \mathcal{S}^*(1/2)$ and (3.10) becomes (3.12). □

Theorem 4. *The function $g_n(z) = z + cz^n$ belongs to the class $\mathcal{SK}(\alpha)$, whenever*

$$|c| \le \frac{2(3-\alpha)-3}{2(3-\alpha)n-3}. \tag{3.15}$$

Proof. Let us denote

$$G(z) := \frac{zg_n'(z)}{g_n(z)} = \frac{1+ncz^{n-1}}{1+cz^{n-1}} \quad (z \in \Delta).$$

To prove that $g_n \in \mathcal{SK}(\alpha)$, it suffices to show that $G \prec \widetilde{q}_\alpha$ or equivalently that $G(\Delta) \subset \widetilde{q}_\alpha(\Delta)$ because the function $\widetilde{q}_\alpha$ is univalent. The set $\widetilde{q}_\alpha(\Delta)$ is on the right of the curve in Fig. 1 with $\max\{\mathfrak{Re}\widetilde{q}_\alpha(e^{i\varphi})\} = \frac{3}{2(3-\alpha)}$ on the real axis. The set $G(\Delta)$ is a disc with the diameter from $x_1 = \frac{1-n|c|}{1-|c|}$ to $x_2 = \frac{1+n|c|}{1+|c|}$. We have that x_i, $i = 1, 2$, satisfy $x_i > \frac{3}{2(3-\alpha)}$, thus $G(\Delta) \subset \widetilde{q}_\alpha(\Delta)$. This proves the theorem. □

Making use of the formula (2.13) with the function $p(t) = \widetilde{q}_\alpha(t)$ we obtain the other example of function of the class $\mathcal{SK}(\alpha)$:

$$\widetilde{f}_\alpha(z) = \left(\frac{z}{1+\alpha z/3}\right)^{\frac{\alpha}{3+\alpha}} \left(\frac{z}{1-z}\right)^{\frac{3}{3+\alpha}} = \frac{z}{(1+\alpha z/3)^{\frac{\alpha}{3+\alpha}}(1-z)^{\frac{3}{3+\alpha}}} \in \mathcal{SK}(\alpha). \tag{3.16}$$

4. The order of starlikeness in the class of convex functions of negative order

A function convex of order zero is starlike of order one-half [20,21]. Several different results have been made on the way to obtain the order of starlikeness of the class of convex functions of order α. MacGregor [22] proved that if $f \in \mathcal{A}$, $\alpha \in [0, 1)$, then

$$\mathfrak{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \;\Rightarrow\; \frac{zf'(z)}{f(z)} \prec \widetilde{q}(z), \tag{4.1}$$

where

$$\widetilde{q}(z) = \begin{cases} \dfrac{(1-2\alpha)z}{(1-z)[1-(1-z)^{1-2\alpha}]} & \text{if } \alpha \ne \dfrac{1}{2} \\ \dfrac{z}{(z-1)\log(1-z)} & \text{if } \alpha = \dfrac{1}{2}. \end{cases}$$

The exact value of $\min\{\mathfrak{Re}\widetilde{q}(z) : |z| = 1\}$, as conjectured in [22], one can find in [23, p. 115]. This value is the order of starlikeness of convex functions of positive order $\alpha \in [0, 1)$ and is given by

$$\delta(\alpha) = \begin{cases} \dfrac{2\alpha - 1}{2 - 2^{2(1-\alpha)}} & \text{if } \alpha \ne \dfrac{1}{2} \\ 1/\log 4 & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{4.2}$$

In the current paper we consider an improvement of the result (4.1) for functions of certain negative order of convexity.

Let us denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\Delta} \setminus E(f)$, where

$$E(f) := \{\zeta : \zeta \in \partial\Delta \text{ and } \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that

$$f'(\zeta) \ne 0 \qquad (\zeta \in \partial\Delta \setminus E(f)).$$

Lemma 5 ([23]). *Let $p \in \mathcal{Q}$ with $p(0) = a$ and let*

$$q(z) = a + a_n z^n + \cdots$$

be analytic in Δ with

$$q(z) \not\equiv a \quad \text{and} \quad n \in \mathbb{N}.$$

If q is not subordinate to p, then there exist points

$$z_0 = r_0 e^{i\theta} \in \Delta \quad \text{and} \quad \zeta \in \partial\Delta \setminus E(f),$$

and there exists a number $m \ge n$ for which

$$q(|z| < r_0) \subset p(\Delta), \qquad q(z_0) = p(\zeta) \quad \text{and} \quad z_0 q'(z_0) = m\zeta p'(\zeta).$$

Theorem 5. *Let $-3 \le \alpha \le 1$. If a function f belongs to the class $\mathcal{A}$ and*

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{3\alpha}{2(3-\alpha)} \tag{4.3}$$

for $z \in \Delta$, then $f \in \mathcal{SK}(\alpha)$.

Proof. If $\alpha = 0$, then Theorem 5 becomes well known result that a function convex of order zero is starlike of order one-half. Suppose that $\alpha \neq 0$ and that $f \notin \mathcal{SK}(\alpha)$ or equivalently

$$\frac{zf'(z)}{f(z)} \not\prec \widetilde{q}_\alpha(z),$$

then by Lemma 4 there exist $z_0 \in \Delta$ and ζ, $|\zeta| = 1$, $\zeta \neq 1$, and $m > 1$ such that

$$\frac{z_0 f'(z_0)}{f(z_0)} = \widetilde{q}_\alpha(\zeta) \quad \text{and} \quad \left[z\left(\frac{zf'(z)}{f(z)}\right)'\right]\Bigg|_{z=z_0} = m\zeta(\widetilde{q}_\alpha(\zeta))'. \tag{4.4}$$

Then, after some calculation we get

$$\begin{aligned} 1+\frac{z_0 f''(z_0)}{f'(z_0)} &= \widetilde{q}_\alpha(\zeta) + \frac{m\zeta(\widetilde{q}_\alpha(\zeta))'}{\widetilde{q}_\alpha(\zeta)} \\ &= \frac{-3}{(\zeta-1)(\alpha\zeta+3)} - m\frac{(3-\alpha)\zeta+2\alpha\zeta^2}{(\zeta-1)(\alpha\zeta+3)} \\ &= \frac{-3}{(\zeta-1)(\alpha\zeta+3)} - m\frac{(\alpha\zeta+3)(2\zeta-(\alpha+3)/\alpha)+3(\alpha+3)/\alpha}{(\zeta-1)(\alpha\zeta+3)} \\ &= \frac{-3}{(\zeta-1)(\alpha\zeta+3)}\left[1+\frac{m(\alpha+3)}{\alpha}\right] - \frac{m(2\zeta-(\alpha+3)/\alpha)}{\zeta-1}. \end{aligned} \tag{4.5}$$

If $|\zeta| = 1$, $\zeta \neq 1$, then the last expression in (4.5) takes its values on the vertical line $\mathfrak{Re}\,w = m[1+(3+\alpha)/(2\alpha)]$. Moreover, by (2.9) we have

$$\frac{9(1+\alpha)}{2(3+\alpha)^2} \le \mathfrak{Re}\{\widetilde{q}_\alpha(\zeta)\} = \mathfrak{Re}\left\{\frac{-3}{(\zeta-1)(\alpha\zeta+3)}\right\} \le \frac{3}{2(3-\alpha)}.$$

Then after some calculations and then using Lemma 4 we obtain

$$\begin{aligned} \mathfrak{Re}\left\{1+\frac{z_0 f''(z_0)}{f'(z_0)}\right\} &= \mathfrak{Re}\left\{\frac{-3(1+m(\alpha+3)/\alpha)}{(\zeta-1)(\alpha\zeta+3)}\right\} - \mathfrak{Re}\left\{\frac{m(2\zeta-(\alpha+3)/\alpha)}{\zeta-1}\right\} \\ &= \mathfrak{Re}\left\{\frac{-3(1+m(\alpha+3)/\alpha)}{(\zeta-1)(\alpha\zeta+3)}\right\} - m\left(1+\frac{\alpha+3}{2\alpha}\right) \\ &< \frac{3}{2(3-\alpha)}\left[1+\frac{m(\alpha+3)}{\alpha}\right] - m\frac{3(\alpha+1)}{2\alpha} \\ &= \frac{3}{2(3-\alpha)} + m\left[\frac{3(3+\alpha)}{2(3-\alpha)\alpha} - \frac{3(\alpha+1)}{2\alpha}\right] \\ &= \frac{3}{2(3-\alpha)} - m\frac{3(1-\alpha)}{2(3-\alpha)} \\ &< \frac{3}{2(3-\alpha)} - \frac{3(1-\alpha)}{2(3+\alpha)} = \frac{3\alpha}{2(3-\alpha)}, \end{aligned}$$

which contradicts our assumptions, hence

$$\frac{zf'(z)}{f(z)} \prec \widetilde{q}_\alpha(z),$$

and $f \in \mathcal{SK}(\alpha)$. $\square$

Corollary 4. *Let $-3 \leq \alpha < 1$. If a function f belongs to the class $\mathcal{A}$ and*

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{3\alpha}{2(3-\alpha)}$$

for $z \in \Delta$, then $f \in \mathcal{S}^*(\gamma)$, where $\gamma = \frac{9(1+\alpha)}{2(3+\alpha)^2}$.

Proof. From Corollary 1 we have that $\mathcal{SK}(\alpha) \subset \mathcal{S}^*(\gamma)$, where $\gamma = \frac{9(1+\alpha)}{2(3+\alpha)^2}$, this means that if $f \in \mathcal{SK}(\alpha)$, then it is starlike of order γ. $\square$

Thus Corollary 4 adds a relationship between the order of convexity $\frac{3\alpha}{2(3-\alpha)}$ and the order of starlikeness $\frac{9(1+\alpha)}{2(3+\alpha)^2}$. Below there are some examples of this relationship.

α	-3	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$-\frac{1}{3}$	$\frac{1}{2}$	1
$\frac{3\alpha}{2(3-\alpha)}$	$-\frac{3}{4}$	$-\frac{15}{22}$	$-\frac{3}{5}$	$-\frac{1}{2}$	$-\frac{3}{8}$	$-\frac{3}{14}$	$-\frac{3}{20}$	$-\frac{3}{26}$	0	$\frac{3}{22}$	$\frac{3}{16}$	$\frac{3}{10}$	$\frac{3}{4}$
$\frac{9(1+\alpha)}{2(3+\alpha)^2}$	$-\infty$	-27	$-\frac{9}{2}$	-1	0	$\frac{9}{25}$	$\frac{27}{64}$	$\frac{54}{121}$	$\frac{1}{2}$	$\frac{90}{169}$	$\frac{54}{100}$	$\frac{27}{49}$	$\frac{9}{16}$

We can see that if f is convex of order -3/8, then f is starlike. When f is convex of order 0, then f is starlike of order 1/2 which was proved earlier, see (4.2).

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