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Regular rapidly decreasing nonlinear generalized functions. Application to microlocal regularity

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Abstract

We present new types of regularity for nonlinear generalized functions, based on the notion of regular growth with respect to the regularizing parameter of the Colombeau simplified model. This generalizes the notion of $\mathcal{G}^\infty$-regularity introduced by M. Oberguggenberger. A key point is that these regularities can be characterized, for compactly supported generalized functions, by a property of their Fourier transform. This opens the door to microanalysis of singularities of generalized functions, with respect to these regularities. We present a complete study of this topic, including properties of the Fourier transform (exchange and regularity theorems) and relationship with classical theory, via suitable results of embeddings.

Keywords: Colombeau generalized functions; Rapidly decreasing generalized functions; Fourier transform; Microlocal regularity

1. Introduction

The various theories of nonlinear generalized functions are suitable frameworks to set and solve differential or integral problems with irregular operators or data. Even for linear problems, these theories are efficient to overcome some limitations of the distributional framework. We follow in this paper the theory introduced by Colombeau [1,2,10,19]. To be short, a special Colombeau type algebra is a factor space $\mathcal{G} = \mathcal{X}/\mathcal{N}$ of moderate modulo negligible nets. The moderateness (respectively the negligibility) of nets is defined by their asymptotic behavior when a real parameter ε tends to 0.

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A local and microlocal analysis of singularities of nonlinear generalized functions has been developed during the last decade, based on the notion of $\mathcal{G}^\infty$-regularity [20]. A generalized function is $\mathcal{G}^\infty$-regular if it has uniform growth bounds, with respect to the regularization parameter ε, for all derivatives. In fact, this notion appears to be the exact generalization of the C^∞-regularity for distributions, in the sense given by the result of [20] which asserts that $\mathcal{G}^\infty \cap \mathcal{D}'$ is equal to C^∞.

In this paper we include $\mathcal{G}^\infty$ and $\mathcal{G}$ in a new framework of spaces of *$\mathcal{R}$-regular nonlinear generalized functions*, in which the growth bounds are defined with the help of spaces $\mathcal{R}$ of sequences satisfying natural conditions of stability. One main property of those spaces is that the elements with compact support can be characterized by a "$\mathcal{R}$-property" of their Fourier transform. (Those Fourier transforms belong to some regular subspaces of spaces of *rapidly decreasing generalized functions* [8,22].) Thus, the parallel is complete with the C^∞-regularity of compactly supported distributions. Moreover, from this characterization, we deduce that the microlocal behavior of a generalized function with respect to a given $\mathcal{R}$-regularity is completely similar to the one of a distribution with respect to the C^∞-regularity. In particular, we can handle the $\mathcal{R}$-wavefront of an element of $\mathcal{G}$ as the C^∞ one of a distribution. Finally, the $\mathcal{G}^\infty$-regularity for an element of $\mathcal{G}$ appears as a remarkable particular case. With this new notion of $\mathcal{R}$-regularity, we enlarge the possibilities for the study of the propagation of singularities through differential and pseudo differential operators, and expect to be able to study nonlinear situations. A first attempt (in a slightly different framework) as began in [6].

We want to emphasize here that this type of spaces also showed its efficiency in problems of Schwarz kernel-type theorem. More precisely, we proved in [3] that some nets of linear maps (parametrized by $\varepsilon \in (0,1]$), satisfying some growth conditions similar to those introduced for $\mathcal{R}$-regular spaces, give rise to linear maps between spaces of generalized functions. Moreover, those maps can be represented by generalized integral kernel on some special $\mathcal{R}$-regular subspaces of $\mathcal{G}(\Omega)$ in which the growth bounds are at most sublinear with respect to the order of derivation. These results are due to the fact that the convolution admits an unity in these $\mathcal{R}$-regular spaces, whereas this is not true in $\mathcal{G}(\Omega)$. Similar notions are also used in [25] for another type of kernel problem in $\mathcal{G}(\Omega)$. Finally, this kind of result has been extended in [5] to kernel problems in spaces of tempered generalized functions.

The paper is organized as follows. In Section 2, we introduce the spaces of $\mathcal{R}$-regular generalized functions and we precise some classical results about the embedding of $\mathcal{D}'$ into these spaces. Section 3 is devoted to the study of the space $\mathcal{G}_\mathcal{S}$ of rapidly decreasing generalized functions. In particular, we show that $\mathcal{O}'_C$, the space of rapidly decreasing distributions, is embedded in $\mathcal{G}_\mathcal{S}$. Thus, $\mathcal{G}_\mathcal{S}$ plays for $\mathcal{O}'_C$ the role that $\mathcal{G}$ plays for $\mathcal{D}'$. Section 4 contains the material related to Fourier transform of elements of $\mathcal{G}_\mathcal{S}$ and especially an exchange theorem which is, in the context of $\mathcal{R}$-regularity, an analogon and a generalization of the classical exchange theorem between $\mathcal{O}'_C$ and $\mathcal{O}_M$. Section 5 gives the above mentioned characterization by Fourier transform of compactly supported $\mathcal{R}$-regular generalized functions whereas, in Section 6, we present the $\mathcal{R}$-local and $\mathcal{R}$-microlocal analysis of generalized functions.

2. The sheaf of Colombeau simplified algebras and related subsheaves

2.1. Sheaves of regular generalized functions

Definition 1. We say that a subspace $\mathcal{R}$ of $\mathbb{R}_+^\mathbb{N}$ is *regular* if $\mathcal{R}$ is non-empty and

(i) $\mathcal{R}$ is "overstable" by translation and by maximum

$$\forall N \in \mathcal{R},\ \forall (k,k') \in \mathbb{N}^2,\ \exists N' \in \mathcal{R},\ \forall n \in \mathbb{N},\quad N(n+k)+k' \leqslant N'(n), \tag{1}$$

$$\forall N_1 \in \mathcal{R},\ \forall N_2 \in \mathcal{R},\ \exists N \in \mathcal{R},\ \forall n \in \mathbb{N},\quad \max\bigl(N_1(n), N_2(n)\bigr) \leqslant N(n). \tag{2}$$

(ii) For all N_1 and N_2 in $\mathcal{R}$, there exists $N \in \mathcal{R}$ such that

$$\forall (l_1,l_2) \in \mathbb{N}^2,\quad N_1(l_1)+N_2(l_2) \leqslant N(l_1+l_2). \tag{3}$$

Example 1. (i) The set $\mathcal{B}$ of bounded sequences and the set $\mathcal{A}$ of affine sequences are regular subsets of $\mathbb{R}_+^{\mathbb{N}}$, which is itself regular.

(ii) The set $\mathcal{L}_{\mathrm{og}} = \{N \in \mathbb{R}_+^{\mathbb{N}} \mid \exists b \in \mathbb{R}_+,\ N: n \mapsto \ln n + b\}$ is not regular ((3) is not satisfied), whereas $\mathcal{L}_{\mathrm{og}}^1 = \{N \in \mathbb{R}_+^{\mathbb{N}} \mid \exists (a,b) \in \mathbb{R}_+^2,\ N: n \mapsto a\ln n + b\}$ is regular. ((3) comes, for example, from $\ln x + \ln y \leqslant 2\ln(x+y)$, for $x>0$ and $y>0$.)

Let Ω be an open subset of $\mathbb{R}^d$ ($d \in \mathbb{N}$) and consider the algebra $\mathrm{C}^\infty(\Omega)$ of complex valued smooth functions, endowed with its usual topology. This topology can be described by the family of seminorms $(p_{K,l})_{K \Subset \Omega, l \in \mathbb{N}}$ defined by $p_{K,l}(f) = \sup_{x\in K,\, |\alpha|\leqslant l} |\partial^\alpha f(x)|$. For any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, we set

$$\begin{aligned}&\mathcal{X}^{\mathcal{R}}(\Omega)\\&\quad= \bigl\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \bigm| \forall K \Subset \Omega,\ \exists N \in \mathcal{R},\ \forall l \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}\bigl(\varepsilon^{-N(l)}\bigr) \text{ as } \varepsilon \to 0\bigr\},\\&\mathcal{N}^{\mathcal{R}}(\Omega)\\&\quad= \bigl\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \bigm| \forall K \Subset \Omega,\ \forall m \in \mathcal{R},\ \forall l \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}\bigl(\varepsilon^{m(l)}\bigr) \text{ as } \varepsilon \to 0\bigr\}.\end{aligned}$$

Proposition 1.

(i) *For any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the functor $\Omega \to \mathcal{X}^{\mathcal{R}}(\Omega)$ defines a sheaf of differential algebras over the ring*

$$\mathcal{X}(\mathbb{C}) = \bigl\{(r_\varepsilon)_\varepsilon \in \mathbb{C}^{(0,1]} \bigm| \exists q \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}\bigl(\varepsilon^{-q}\bigr) \textit{ as } \varepsilon \to 0\bigr\}.$$

(ii) *The set $\mathcal{N}^{\mathcal{R}}(\Omega)$ is equal to Colombeau's ideal*

$$\begin{aligned}&\mathcal{N}(\Omega)\\&\quad= \bigl\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \bigm| \forall K \Subset \Omega,\ \forall l \in \mathbb{N},\ \forall m \in \mathbb{N},\ p_{K,l}(f_\varepsilon) = \mathrm{O}\bigl(\varepsilon^{m}\bigr) \textit{ as } \varepsilon \to 0\bigr\}.\end{aligned}$$

Thus, the functor $\mathcal{N}^{\mathcal{R}}: \Omega \to \mathcal{N}^{\mathcal{R}}(\Omega)$ defines a sheaf of ideals of the sheaf $\mathcal{X}^{\mathcal{R}}(\cdot)$.

(iii) *For any regular subspaces $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathbb{R}_+^{\mathbb{N}}$, with $\mathcal{R}_1 \subset \mathcal{R}_2$, the sheaf $\mathcal{X}^{\mathcal{R}_1}(\Omega)$ is a subsheaf of the sheaf $\mathcal{X}^{\mathcal{R}_2}(\Omega)$.*

Proof. We give the main ideas. (a) *Algebraical properties.* First, for a given Ω, properties (1) and (2) imply that $\mathcal{X}^{\mathcal{R}}(\Omega)$ is stable by multiplication by elements of $\mathcal{X}_M(\mathbb{R})$ and by sum. Then property (3), combined with the Leibniz rule, renders $\mathcal{X}^{\mathcal{R}}(\Omega)$ stable by product. Finally, for the equality $\mathcal{N}^{\mathcal{R}}(\Omega) = \mathcal{N}(\Omega)$, take first $(f_\varepsilon) \in \mathcal{N}^{\mathcal{R}}(\Omega)$. For any $K \Subset \Omega$, $l \in \mathbb{N}$ and $m \in \mathbb{N}$, choose $N \in \mathcal{R}$. According to (1), there exists $N' \in \mathcal{R}$ such that $N + m \leqslant N'$. Thus, $p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{N'(l)}) = \mathrm{O}(\varepsilon^m)$ and $(f_\varepsilon) \in \mathcal{N}(\Omega)$. Conversely, given $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$ and $N \in \mathcal{R}$, we have $p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{N(l)})$ since this estimates holds for all $m \in \mathbb{N}$.

(b) *Sheaf properties*. The proof follows the same lines as in the case of Colombeau simplified algebras (see [10, Theorem 1.2.4]). First, the definition of restriction (by mean of restriction of representatives) is straightforward. For the sheaf properties, we have to replace Colombeau's usual estimates by $\mathcal{X}^{\mathcal{R}}$-estimates. In each place where these estimates appear, we have only to consider a finite number of terms by compactness properties. Thus, the stability by maximum of $\mathcal{R}$ (property (2)) induces the result. Finally, point (iii) of the proposition follows directly from the obvious inclusion $\mathcal{X}^{\mathcal{R}_1}(\Omega) \subset \mathcal{X}^{\mathcal{R}_2}(\Omega)$. $\square$

The sheaf $\mathcal{G}^{\mathcal{R}}(\cdot) = \mathcal{X}^{\mathcal{R}}(\cdot)/\mathcal{N}^{\mathcal{R}}(\cdot) = \mathcal{X}^{\mathcal{R}}(\cdot)/\mathcal{N}(\cdot)$ turns to be a sheaf of differentiable algebras on the ring $\mathcal{X}_M(\mathbb{C})/\mathcal{N}(\mathbb{C})$ with

$$\mathcal{N}(\mathbb{K}) = \left\{(r_\varepsilon) \in \mathbb{K}^{(0,1]} \;\middle|\; \forall q \in \mathbb{N},\ |r_\varepsilon| = \mathrm{O}\left(\varepsilon^q\right)\right\}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}.$$

Definition 2. For any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the sheaf of algebras

$$\mathcal{G}^{\mathcal{R}}(\cdot) = \mathcal{X}^{\mathcal{R}}(\cdot)/\mathcal{N}^{\mathcal{R}}(\cdot)$$

is called the sheaf of $\mathcal{R}$-regular algebras of (nonlinear) generalized functions.

Example 2. Taking $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$, we recover the sheaf of *Colombeau simplified* or *special algebras*.

Notation 1. In the sequel, we shall write $\mathcal{G}(\Omega)$ (respectively $\mathcal{X}_M(\Omega)$) instead of $\mathcal{G}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$ (respectively $\mathcal{X}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$). For $(f_\varepsilon)_\varepsilon$ in $\mathcal{X}_M(\Omega)$ or $\mathcal{X}^{\mathcal{R}}(\Omega)$, $[(f_\varepsilon)_\varepsilon]$ will be its class in $\mathcal{G}(\Omega)$ or in $\mathcal{G}^{\mathcal{R}}(\Omega)$, since these classes are obtained modulo the same ideal. (We consider $\mathcal{G}^{\mathcal{R}}(\Omega)$ as a subspace of $\mathcal{G}(\Omega)$.)

Example 3. Taking $\mathcal{R} = \mathcal{B}$, introduced in Example 1, we obtain the sheaf of $\mathcal{G}^{\infty}$-generalized functions [20].

Example 4. Take a in $[0, +\infty]$ and set

$$\mathcal{R}_0 = \left\{N \in \mathbb{R}_+^{\mathbb{N}} \;\middle|\; \lim_{l \to +\infty}\left(N(l)/l\right) = 0\right\},$$

$$\mathcal{R}_a = \left\{N \in \mathbb{R}_+^{\mathbb{N}} \;\middle|\; \limsup_{l \to +\infty}\left(N(l)/l\right) < a\right\}, \quad \text{for } a > 0.$$

For any a in $[0, +\infty]$, $\mathcal{R}_a$ is a regular subset of $\mathbb{R}_+^{\mathbb{N}}$. The corresponding sheaves $\mathcal{G}^{\mathcal{R}_a}(\cdot)$ are the sheaves of algebras of generalized functions with slow growth introduced in [3] and mentioned in the introduction. Note that, for a in $(0, +\infty]$, a sequence N is in $\mathcal{R}_a$ iff there exists $(a', b) \in (\mathbb{R}^+)^2$ with $a' < a$ such that $N(l) \leqslant a'l + b$. The growth of the sequence N is at most linear.

For any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the notion of support of a section $f \in \mathcal{G}^{\mathcal{R}}(\Omega)$ makes sense since $\mathcal{G}^{\mathcal{R}}(\cdot)$ is a sheaf. The following definition will be sufficient for this paper.

Definition 3. The support of a generalized function $f \in \mathcal{G}^{\mathcal{R}}(\Omega)$ is the complement in Ω of the largest open subset of Ω where f is null.

Notation 2. We denote by $\mathcal{G}_C^{\mathcal{R}}(\Omega)$ the subset of $\mathcal{G}^{\mathcal{R}}(\Omega)$ of elements with compact support.

Lemma 2. *Every $f \in \mathcal{G}_C^{\mathcal{R}}$ has a representative $(f_\varepsilon)_\varepsilon$, such that each f_ε has the same compact support.*

We shall not prove this result here, since Lemma 10 below gives the main ideas of the proof.

2.2. Some embeddings

As we need in the sequel some results related to the embeddings of classical spaces into the spaces of nonlinear generalized functions, we recall and precise here such constructions. For any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ and any Ω open subset of $\mathbb{R}^d$, $\mathrm{C}^\infty(\Omega)$ is embedded into $\mathcal{G}^{\mathcal{R}}(\Omega)$ by the canonical embedding

$$\sigma:\quad \mathrm{C}^\infty(\Omega) \to \mathcal{G}^{\mathcal{R}}(\Omega), \quad f \to \big[(f_\varepsilon)_\varepsilon\big] \quad \text{with } f_\varepsilon = f \text{ for all } \varepsilon \in (0,1].$$

For the embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$, we follow the ideas of [19]. Consider $\rho \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\int \rho(x)\,\mathrm{d}x = 1, \qquad \int x^m \rho(x)\,\mathrm{d}x = 0 \quad \text{for all } m \in \mathbb{N}^d \setminus \{0\}.$$

We now choose $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $\overline{B(0,1)}$ and $\chi \equiv 0$ on $\mathbb{R}^d \setminus B(0,2)$. Define

$$\forall \varepsilon \in (0,1],\ \forall x \in \mathbb{R}^d, \quad \theta_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right)\chi\big(|\ln\varepsilon| x\big).$$

Finally, consider $(\kappa_\varepsilon)_\varepsilon \in (\mathcal{D}(\mathbb{R}^d))^{(0,1]}$ such that

$$\forall \varepsilon \in (0,1), \quad 0 \leqslant \kappa_\varepsilon \leqslant 1, \quad \kappa_\varepsilon \equiv 1 \text{ on } \big\{x \in \Omega \mid d\big(x, \mathbb{R}^d \setminus \Omega\big) \geqslant \varepsilon \text{ and } d(x,0) \leqslant 1/\varepsilon\big\}.$$

With these ingredients, the map

$$\iota:\quad \mathcal{D}'(\Omega) \to \mathcal{G}(\Omega), \quad T \mapsto (\kappa_\varepsilon T * \theta_\varepsilon)_\varepsilon + \mathcal{N}(\Omega) \tag{4}$$

is an embedding of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ such that $\iota|_{\mathrm{C}^\infty(\Omega)} = \sigma$. The proof is mainly based on the following property of $(\theta_\varepsilon)_\varepsilon$:

$$\int \theta_\varepsilon(x)\,\mathrm{d}x = 1 + \mathrm{O}\big(\varepsilon^k\big) \quad \text{as } \varepsilon \to 0,\ \forall m \in \mathbb{N}^d \setminus \{0\},$$
$$\int x^m \theta_\varepsilon(x)\,\mathrm{d}x = \mathrm{O}\big(\varepsilon^k\big) \quad \text{as } \varepsilon \to 0. \tag{5}$$

Set

$$\mathcal{R}_1 = \big\{N \in \mathbb{R}_+^{\mathbb{N}} \,\big|\, \exists b \in \mathbb{R}_+,\ \forall l \in \mathbb{R},\ N(l) \leqslant l + b\big\}. \tag{6}$$

One can verify that the set $\mathcal{R}$ is regular and we set $\mathcal{G}^{(1)}(\cdot) = \mathcal{G}^{\mathcal{R}_1}(\cdot)$. By refining the classical proof (see [4]), we can show the proposition.

Proposition 3. *The image of $\mathcal{D}'(\Omega)$ by the embedding, defined by* (4)*, is included in $\mathcal{G}^{(1)}(\Omega)$.*

These results are summarized in the following commutative diagram (all arrows are embeddings):

$$\begin{tikzcd} \mathrm{C}^\infty(\Omega) \arrow[r] \arrow[d, "\sigma"] & \mathcal{D}'(\Omega) \arrow[d, "\iota"] & \\ \mathcal{G}^\infty(\Omega) \arrow[r] & \mathcal{G}^{(1)}(\Omega) \arrow[r] & \mathcal{G}(\Omega). \end{tikzcd} \tag{7}$$

3. Rapidly decreasing generalized functions

3.1. Definition and first properties

Spaces of rapidly decreasing generalized functions have been introduced in the literature [8,22,23], notably in view of the definition of the Fourier transform in convenient spaces of nonlinear generalized functions. We give here a more complete description of this type of space in the framework of $\mathcal{R}$-regular spaces.

Definition 4. We say that a subspace $\mathcal{R}'$ of the space $\mathbb{R}_+^{\mathbb{N}^2}$ of maps from $\mathbb{N}^2$ to $\mathbb{R}_+$ is *regular* if

(i) $\mathcal{R}'$ is "overstable" by translation and by maximum

$$\forall N \in \mathcal{R}',\ \forall (k,k',k'') \in \mathbb{N}^3,\ \exists N' \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2,$$
$$N(q+k, l+k') + k'' \leqslant N'(q,l), \tag{8}$$
$$\forall N_1 \in \mathcal{R}',\ \forall N_2 \in \mathcal{R}',\ \exists N \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2,$$
$$\max\big(N_1(q,l), N_2(q,l)\big) \leqslant N(q,l). \tag{9}$$

(ii) For any N_1 and N_2 in $\mathcal{R}'$, there exists $N \in \mathcal{R}'$ such that

$$\forall (q_1,q_2,l_1,l_2) \in \mathbb{N}^4, \quad N_1(q_1,l_1) + N_2(q_2,l_2) \leqslant N(q_1+q_2, l_1+l_2). \tag{10}$$

Example 5.

(i) The set $\mathcal{B}'$ of bounded maps from $\mathbb{N}^2$ to $\mathbb{R}_+$ is a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$.
(ii) The set $\mathbb{R}_+^{\mathbb{N}^2}$ of all maps from $\mathbb{N}^2$ to $\mathbb{R}_+$ is a regular set.

We consider Ω an open subset of $\mathbb{R}^d$ and the space $\mathcal{S}(\Omega)$ of rapidly decreasing functions defined on Ω, endowed with the family of seminorms $\mathcal{Q}(\Omega) = (\mu_{q,l})_{(q,l)\in\mathbb{N}^2}$ defined by

$$\mu_{q,l}(f) = \sup_{x\in\Omega,\ |\alpha|\leqslant l} \big(1+|x|\big)^q \big|\partial^\alpha f(x)\big|.$$

Let $\mathcal{R}'$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$ and set

$$\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in \mathcal{S}(\Omega)^{(0,1]} \bigm| \exists N \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2,\ \mu_{q,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-N(q,l)}\big) \text{ as } \varepsilon \to 0\big\},$$
$$\mathcal{N}_{\mathcal{S}}^{\mathcal{R}'}(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in \mathcal{S}(\Omega)^{(0,1]} \bigm| \forall m \in \mathcal{R}',\ \forall (q,l) \in \mathbb{N}^2,\ \mu_{q,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{m(q,l)}\big) \text{ as } \varepsilon \to 0\big\}.$$

Using the same techniques as in the proof of Proposition 1, we have $\mathcal{N}_{\mathcal{S}}^{\mathcal{R}'}(\Omega) = \mathcal{N}_{\mathcal{S}}(\Omega)$, with

$$\mathcal{N}_{\mathcal{S}}(\Omega) = \big\{(f_\varepsilon)_\varepsilon \in \mathcal{S}(\Omega)^{(0,1]} \bigm| \forall (q,l) \in \mathbb{N}^2,\ \forall m \in \mathbb{N},\ \mu_{q,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{m}\big) \text{ as } \varepsilon \to 0\big\}.$$

Thus, the functor $\mathcal{N}_{\mathcal{S}}: \Omega \to \mathcal{N}_{\mathcal{S}}(\Omega)$ defines a presheaf of ideals of the presheaf $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\cdot)$. We also have:

Proposition 4.

(i) *For any regular subspace $\mathcal{R}'$ of $\mathbb{R}_+^{\mathbb{N}^2}$, the functor $\Omega \to \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$ defines a presheaf (it allows restrictions) of differential algebras over the ring $\mathcal{X}(\mathbb{C})$.*

(ii) *For any regular subspaces $\mathcal{R}'_1$ and $\mathcal{R}'_2$ of $\mathbb{R}_+^{\mathbb{N}^2}$, with $\mathcal{R}'_1 \subset \mathcal{R}'_2$, the presheaf $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'_1}(\Omega)$ is a subpresheaf of the presheaf $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}'_2}(\Omega)$.*

Definition 5. The presheaf $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\cdot) = \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\cdot)/\mathcal{N}_{\mathcal{S}}(\cdot)$ is called the presheaf of $\mathcal{R}'$-*regular rapidly decreasing generalized functions*.

As for the case of $\mathcal{G}^{\mathcal{R}}(\cdot)$, the presheaf $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\cdot)$ is a presheaf of differential algebras and a sheaf of modules over the factor ring $\overline{\mathbb{C}} = \mathcal{X}(\mathbb{C})/\mathcal{N}(\mathbb{C})$.

Example 6. Taking $\mathcal{R}' = \mathbb{R}_+^{\mathbb{N}^2}$, we obtain the presheaf of algebras of rapidly decreasing generalized functions [8,22,23].

Notation 3. In the sequel, we shall note $\mathcal{G}_{\mathcal{S}}(\Omega)$ (respectively $\mathcal{X}_{\mathcal{S}}(\Omega)$) instead of $\mathcal{G}_{\mathcal{S}}^{\mathbb{R}_+^{\mathbb{N}^2}}(\Omega)$ (respectively $\mathcal{X}_{\mathcal{S}}^{\mathbb{R}_+^{\mathbb{N}^2}}(\Omega)$). For all regular subset $\mathcal{R}'$ and $(f_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$, $[(f_\varepsilon)_\varepsilon]_{\mathcal{S}}$ denotes its class in $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$.

Example 7. Taking $\mathcal{R}' = \mathcal{B}'$, we obtain the presheaf of $\mathcal{G}_{\mathcal{S}}^{\infty}$ generalized functions or of *regular rapidly decreasing generalized functions*.

Set

$$\mathcal{N}_{\mathcal{S}_*}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\middle|\; \forall m \in \mathbb{N},\ \forall q \in \mathbb{N},\ \mu_{q,0}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^m\right) \text{ as } \varepsilon \to 0 \right\}. \tag{11}$$

We have the exact counterpart of Theorems 1.2.25 and 1.2.27 of [10] (the proof is similar):

Lemma 5. *If the open set Ω is a box, i.e. the product of d open intervals of $\mathbb{R}$ (bounded or not) then $\mathcal{N}_{\mathcal{S}}(\Omega)$ is equal to $\mathcal{N}_{\mathcal{S}_*}(\Omega) \cap \mathcal{X}_{\mathcal{S}}(\Omega)$.*

3.2. Embeddings

The embedding of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$ is done by the canonical injective map

$$\sigma_{\mathcal{S}}: \quad \mathcal{S}\left(\mathbb{R}^d\right) \to \mathcal{G}_{\mathcal{S}}\left(\mathbb{R}^d\right), \quad f \mapsto \left[(f_\varepsilon)_\varepsilon\right]_{\mathcal{S}} \quad \text{with } f_\varepsilon = f \text{ for all } \varepsilon \in (0,1].$$

In fact, the image of $\sigma_{\mathcal{S}}$ is included in $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\mathbb{R}^d)$ for any regular subset of $\mathcal{R}' \subset \mathbb{R}_+^{\mathbb{N}^2}$. For the embedding of $\mathcal{O}'_C(\mathbb{R}^d)$ into $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$, we consider $\rho \in \mathcal{S}(\mathbb{R}^d)$ which satisfies

$$\int \rho(x)\,\mathrm{d}x = 1, \qquad \int x^m \rho(x)\,\mathrm{d}x = 0 \quad \text{for all } m \in \mathbb{N}^d \setminus \{0\}. \tag{12}$$

Set

$$\forall \varepsilon \in (0,1],\ \forall x \in \mathbb{R}^d, \quad \rho_\varepsilon(x) = \frac{1}{\varepsilon^d}\rho\left(\frac{x}{\varepsilon}\right). \tag{13}$$

Theorem 6. *The map*

$$\iota_{\mathcal{S}}\colon \quad \mathcal{O}'_C(\mathbb{R}^d) \to \mathcal{G}_{\mathcal{S}}(\mathbb{R}^d), \quad u \mapsto \left[(u * \rho_\varepsilon)_\varepsilon\right]_{\mathcal{S}} \tag{14}$$

is a linear embedding which commutes with partial derivatives.

Proof. Take $u \in \mathcal{O}'_C(\mathbb{R}^d)$. As $\rho_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$, $u_\varepsilon * \rho_\varepsilon$ is in $\mathcal{S}(\mathbb{R}^d)$ for all $\varepsilon \in (0,1]$. Consider $q \in \mathbb{N}$. The structure of elements of $\mathcal{O}'_C(\mathbb{R}^d)$ [24] shows the existence of a finite family $(f_j)_{1 \leqslant j \leqslant l(q)}$ of continuous functions such that $(1+|x|)^q f_j$ is bounded (for $1 \leqslant j \leqslant l(q)$), and $(\alpha_j)_{1 \leqslant j \leqslant l(q)} \in (\mathbb{N}^d)^{l(q)}$ such that $u = \sum_{j=1}^{l(q)} \partial^{\alpha_j} f_j$. In order to simplify notations, we shall suppose that this family is reduced to one element f, that is $u = \partial^\alpha f$. Take now $\beta \in \mathbb{N}^d$. We have

$$\begin{aligned}\forall x \in \mathbb{R}^d, \quad \partial^\beta(u * \rho_\varepsilon)(x) &= \partial^\beta\left(\partial^\alpha f * \rho_\varepsilon\right)(x) \\ &= \left(f * \partial^{\alpha+\beta}(\rho_\varepsilon)\right)(x) = \int f(x-y)\partial^{\alpha+\beta}(\rho_\varepsilon)(y)\,\mathrm{d}y \\ &= \varepsilon^{-|\alpha|-|\beta|}\int f(x-\varepsilon v)\partial^{\alpha+\beta}\rho(v)\,\mathrm{d}v.\end{aligned}$$

On one hand, there exists a constant $C_1 > 0$ such that

$$\forall (x,v) \in \mathbb{R}^{2d}, \quad \left|f(x-\varepsilon v)\right| \leqslant C_1\left(1+|x-\varepsilon v|\right)^{-q}.$$

On the other hand, as ρ is rapidly decreasing, there exists $C_2 > 0$ such that $\partial^{\alpha+\beta}\rho(v) \leqslant C_2 \times (1+|v|)^{-q-d-1}$. These estimates imply the existence of a constant C_3 such that

$$\forall x \in \mathbb{R}^d, \quad \left|\partial^\beta(u * \rho_\varepsilon)(x)\right| \leqslant C_3\varepsilon^{-|\alpha|-|\beta|}\int \left(\left(1+|x-\varepsilon v|\right)\left(1+|v|\right)\right)^{-q}\left(1+|v|\right)^{-d-1}\mathrm{d}v.$$

We have $(1+|x-\varepsilon v|) \geqslant (1+||x|-\varepsilon|v||)$. A short study of the family of functions $\phi_{|x|,\varepsilon}: t \mapsto (1+||x|-\varepsilon t|)(1+t)$ for positive t shows that $\phi_{|x|,\varepsilon}(t) \geqslant 1+|x|$. Consequently

$$\begin{aligned}\forall x \in \mathbb{R}^d, \quad \left|\partial^\beta(u * \rho_\varepsilon)(x)\right| &\leqslant C_3\varepsilon^{-|\alpha|-|\beta|}\left(1+|x|\right)^{-q}\int\left(1+|v|\right)^{-d-1}\mathrm{d}v \\ &\leqslant C_4\varepsilon^{-|\alpha|-|\beta|}\left(1+|x|\right)^{-q} \quad (C_4 \text{ positive constant}).\end{aligned}$$

It follows that $\mu_{q,l}(u * \rho_\varepsilon) = \mathrm{O}(\varepsilon^{-N(q,l)})$ as $\varepsilon \to 0$ with $N(q,l) = |\alpha| + l$. (α may depends on q) This shows that $(u * \rho_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}_{\mathcal{S}}(\mathbb{R}^d)$. Finally, it is clear that $(u * \rho_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{S}}(\mathbb{R}^d)$ implies that $u_\varepsilon * \rho_\varepsilon \to 0$ in $\mathcal{S}'$, as $\varepsilon \to 0$. As $u_\varepsilon * \rho_\varepsilon \to u$ as $\varepsilon \to 0$, u is therefore null. □

Theorem 7. *We have* $\iota_{\mathcal{S}|\mathcal{S}(\mathbb{R}^d)} = \sigma_{\mathcal{S}}$.

The *proof* is close to the one which asserts that diagram (7) is commutative [4].

Consider

$$\mathcal{R}'_1 = \left\{N' \in \mathbb{R}_+^{\mathbb{N}^2} \;\middle|\; \exists N \in \mathcal{R}_1,\ N' = 1 \otimes N\right\}, \tag{15}$$

where $\mathcal{R}_1$ is defined by (6). (This amounts to: $N \in \mathcal{R}'_1$ iff there exists $b \in \mathbb{R}_+$ such that $N(q,l) \leqslant l + b$.) The set $\mathcal{R}'_1$ is clearly regular and we note

$$\mathcal{G}_{\mathcal{S}}^{(1)}(\mathbb{R}^d) = \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'_1}(\mathbb{R}^d)/\mathcal{N}_{\mathcal{S}}(\mathbb{R}^d). \tag{16}$$

Proposition 8. *The image of $\mathcal{O}'_M(\mathbb{R}^d)$ by $\iota_{\mathcal{S}}$ is included in $\mathcal{G}_{\mathcal{S}}^{(1)}(\mathbb{R}^d)$.*

Proof. Let u be in $\mathcal{O}'_M(\mathbb{R}^d)$. According to the characterization of elements of $\mathcal{O}'_M(\mathbb{R}^d)$ [11], there exists a finite family $(f_j)_{1\leqslant j\leqslant l}$ of rapidly decreasing continuous functions and $(\alpha_j)_{1\leqslant j\leqslant l} \in (\mathbb{N}^d)^l$ such that $u = \sum_{j=1}^{l} \partial^{\alpha_j} f_j$. For sake of simplicity, we shall suppose that $u = \partial^\alpha f$, with f as above. For $\beta \in \mathbb{N}^d$, the same estimates as in proof of Theorem 6 lead to the following property

$$\forall q \in \mathbb{N},\ \exists C_q > 0,\ \forall x \in \mathbb{R}^d, \quad \big(1+|x|\big)^q \big|\partial^\beta (u * \rho_\varepsilon)(x)\big| \leqslant C_q \varepsilon^{-|\alpha|-|\beta|},$$

since, in the present case, f is rapidly decreasing. (The only difference is here that f and α do not depend on the chosen integer q.) Then $\mu_{q,l}(u * \rho_\varepsilon) \leqslant C_q\, \varepsilon^{-l-|\alpha|}$. Our claim follows, with $N'(q,l) = l + |\alpha|$, where $|\alpha|$ only depends on u. □

We can summarize Theorems 6, 7 and Proposition 8 in the following commutative diagram in which all arrows are embeddings (compare with diagram (7)):

$$\begin{tikzcd}
\mathcal{S}(\mathbb{R}^d) \arrow[r] \arrow[dr, "\sigma_{\mathcal{S}}"] & \mathcal{O}'_M(\mathbb{R}^d) \arrow[r] \arrow[d, "\iota_{\mathcal{S}}"] & \mathcal{O}'_C(\mathbb{R}^d) \arrow[d, "\iota_{\mathcal{S}}"] \\
& \mathcal{G}_{\mathcal{S}}^{(1)}(\mathbb{R}^d) \arrow[r] & \mathcal{G}_{\mathcal{S}}(\mathbb{R}^d).
\end{tikzcd} \tag{17}$$

Remark 1. In order to embed $\mathcal{S}'(\mathbb{R}^d)$ into an algebra playing the role of $\mathcal{G}(\mathbb{R}^d)$ for $\mathcal{D}'(\mathbb{R}^d)$, a space $\mathcal{G}_\tau(\mathbb{R}^d)$ of tempered generalized functions is often introduced (see [1,10]). This space $\mathcal{G}_\tau(\mathbb{R}^d)$ does not fit in the general scheme of construction of Colombeau type algebras, since the growth estimates for $\mathcal{G}_\tau(\mathbb{R}^d)$ are not based on the natural topology of the space $\mathcal{O}_M(\mathbb{R}^d)$, which replaces $\mathrm{C}^\infty(\mathbb{R}^d)$ in this case. Although it is possible to construct a space $\mathcal{G}_\tau(\mathbb{R}^d)$ based on the topology of $\mathcal{O}_M(\mathbb{R}^d)$, we do not need it in the sequel. Nevertheless, with the notations (12) and (13), one can verify that the map

$$\iota_{\mathcal{S}'}\colon \quad \mathcal{S}'\big(\mathbb{R}^d\big) \to \mathcal{G}_{\mathcal{S}}\big(\mathbb{R}^d\big), \quad u \mapsto \big[\big((u * \rho_\varepsilon)\hat{\rho}_\varepsilon\big)_\varepsilon\big]_{\mathcal{S}}$$

is a linear embedding.

4. Fourier transform and exchange theorem

4.1. Fourier transform in $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$

The Fourier transform $\mathcal{F}$ is a continuous linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. According to [7, Proposition 3.2], $\mathcal{F}$ has a canonical extension $\mathcal{F}_{\mathcal{S}}$ from $\mathcal{G}_{\mathcal{S}}$ to $\mathcal{G}_{\mathcal{S}}$ defined by

$$\mathcal{G}_{\mathcal{S}}\big(\mathbb{R}^d\big) \to \mathcal{G}_{\mathcal{S}}\big(\mathbb{R}^d\big), \quad u \mapsto \hat{u} = \left[\left(x \mapsto \int \mathrm{e}^{-ix\xi} u_\varepsilon(\xi)\, \mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{S}}, \tag{18}$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}(\mathbb{R}^d)$ is any representative of u.

The proof of this result uses mainly the continuity of $\mathcal{F}$. More precisely, any linear continuous map is continuously moderate in the sense of [7] and, therefore, admits such a canonical extension.

Definition 6. The map $\mathcal{F}_{\mathcal{S}}$ defined by (18) is called the Fourier transform in $\mathcal{G}_{\mathcal{S}}$.

In the same way, we can define $\mathcal{F}_{\mathcal{S}}^{-1}$ by

$$\mathcal{G}_{\mathcal{S}}\big(\mathbb{R}^d\big) \to \mathcal{G}_{\mathcal{S}}\big(\mathbb{R}^d\big), \quad u \mapsto \left[\left(x \mapsto (2\pi)^{-d} \int \mathrm{e}^{ix\xi} u_\varepsilon(\xi)\,\mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{S}}, \tag{19}$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}(\mathbb{R}^d)$ is any representative of u.

Theorem 9. $\mathcal{F}_{\mathcal{S}} : \mathcal{G}_{\mathcal{S}}(\mathbb{R}^d) \to \mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$ *is a one to one linear map, whose inverse is* $\mathcal{F}_{\mathcal{S}}^{-1}$.

Proof. Let u be in $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}(\mathbb{R}^d)$ be one of its representative. As representative of $\mathcal{F}(\mathcal{F}^{-1}(u))$, we can choose $(\tilde{u}_\varepsilon)_\varepsilon$ defined by

$$\forall \varepsilon \in (0,1],\ \forall x \in \mathbb{R}^d, \quad \tilde{u}_\varepsilon(x) = (2\pi)^{-d} \int \mathrm{e}^{ix\xi} \hat{u}_\varepsilon(\xi)\,\mathrm{d}\xi.$$

Since the Fourier transform is an isomorphism in $\mathcal{S}(\mathbb{R}^d)$, we get $\tilde{u}_\varepsilon = u_\varepsilon$, for all $\varepsilon \in (0,1]$, and $\mathcal{F}_{\mathcal{S}}(\mathcal{F}_{\mathcal{S}}^{-1}(u)) = [(u_\varepsilon(x))_\varepsilon]_{\mathcal{S}} = u$. □

4.2. Regular subpresheaves of $\mathcal{G}_{\mathcal{S}}(\cdot)$

We introduce here some regular subpresheaves of $\mathcal{G}_{\mathcal{S}}(\cdot)$ needed for our further microlocal analysis.

Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$ and set

$$\mathcal{R}_u = \big\{N' \in \mathbb{R}_+^{\mathbb{N}^2} \;\big|\; \exists N \in \mathcal{R},\ N' = 1 \otimes N\big\}; \qquad \mathcal{R}_\partial = \big\{N' \in \mathbb{R}_+^{\mathbb{N}^2} \;\big|\; \exists N \in \mathcal{R},\ N = N \otimes 1\big\}.$$

In other words, $N' \in \mathcal{R}_u$ (respectively $\mathcal{R}_\partial$) iff there exists $N \in \mathcal{R}$ such that $N'(q,l) = N(l)$ (respectively $N'(q,l) = N(q)$) or, equivalently, iff N only depends (in a $\mathcal{R}$-regular way) of l (respectively q).

Notation 4. We shall write, with a slight abuse, $\mathcal{R}_u = \{1\} \otimes \mathcal{R}$, $\mathcal{R}_\partial = \mathcal{R} \otimes \{1\}$.

Obviously, $\mathcal{R}_u$ (respectively $\mathcal{R}_\partial$) is a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$.

Example 8. Take $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$. We set: $\mathcal{G}_{\mathcal{S}}^u(\cdot) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}(\cdot)$ (respectively $\mathcal{G}_{\mathcal{S}}^\partial(\cdot) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\cdot)$). In this case, we have $\mathcal{R}_u = \{1\} \otimes \mathbb{R}_+^{\mathbb{N}}$ (respectively $\mathcal{R}_\partial = \mathbb{R}_+^{\mathbb{N}} \otimes \{1\}$).

The elements of $\mathcal{G}_{\mathcal{S}}^u(\Omega)$ (Ω open subset of $\mathbb{R}^d$) have uniform growth bounds with respect to the regularization parameter ε for all factors $(1 + |x|)^q$. For $\mathcal{G}_{\mathcal{S}}^\partial(\Omega)$, those bounds are uniform for all derivatives. For $\mathcal{G}_{\mathcal{S}}^\infty(\Omega)$, introduced in Example 7, the uniformity is global, in some sense stronger than the $\mathcal{G}^\infty$-regularity considered for the algebra $\mathcal{G}$. (In this last case, the uniformity is not required with respect to the compact sets.) We have the obvious embeddings for any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$:

$$\begin{tikzcd}
& \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\Omega) \arrow[r] & \mathcal{G}_{\mathcal{S}}^{\partial}(\Omega) \arrow[dr] & \\
\mathcal{G}_{\mathcal{S}}^{\infty}(\Omega) \arrow[ur] \arrow[dr] & & & \mathcal{G}_{\mathcal{S}}(\Omega)\,. \\
& \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}(\Omega) \arrow[r] & \mathcal{G}_{\mathcal{S}}^{u}(\Omega) \arrow[ur] &
\end{tikzcd} \tag{20}$$

Example 9. The algebra $\mathcal{G}_{\mathcal{S}}^{(1)}(\mathbb{R}^d) = \mathcal{X}_{\mathcal{S}}^{\mathcal{R}'_1}(\mathbb{R}^d)/\mathcal{N}_{\mathcal{S}}(\mathbb{R}^d)$ introduced in relation (16) for the embedding of $\mathcal{O}'_M(\mathbb{R}^d)$ into $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$ (Proposition 8) can be written as $\mathcal{G}_{\mathcal{S}}^{(\mathcal{R}_1)_u}$, with

$$\mathcal{R}_1 = \bigl\{N \in \mathbb{R}_+^{\mathbb{N}} \bigm| \exists b \in \mathbb{R}_+,\ \forall l \in \mathbb{R}\ N(l) \leqslant l + b\bigr\}.$$

As a first illustration of the properties of these spaces, we can show the existence of a canonical embedding of algebras of compactly supported generalized functions into particular spaces of rapidly decreasing generalized functions.

Lemma 10. *Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$ and u be in $\mathcal{G}_C^{\mathcal{R}}(\Omega)$ (Ω open subset of $\mathbb{R}^d$), with $(u_\varepsilon)_\varepsilon$ a representative of u. Let κ be in $\mathcal{D}(\Omega)$, with $0 \leqslant \kappa \leqslant 1$ and $\kappa \equiv 1$ on a neighborhood of* $\operatorname{supp} u$. *Then $(\kappa u_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}(\mathbb{R}^d)$ and $[(\kappa u_\varepsilon)_\varepsilon]_{\mathcal{S}}$ only depends on u and κ.*

Proof. We first show that $(\kappa u_\varepsilon)_\varepsilon$ is in $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}(\mathbb{R}^d)$ and then the independence with respect to the representation.

(a) There exists a compact set $K \subset \Omega$ such that, for all $\varepsilon \in (0,1]$, $\operatorname{supp} \kappa u_\varepsilon \subset K$. It follows that κu_ε is compactly supported and therefore rapidly decreasing. Moreover

$$\forall (q,l) \in \mathbb{N}^2,\ \forall \varepsilon \in (0,1], \quad \mu_{q,l}(\kappa u_\varepsilon) \leqslant \sup_{x \in K} \bigl(1 + |x|\bigr)^q p_{K,l}(\kappa u_\varepsilon) \leqslant C_{K,q}\, p_{K,l}(\kappa u_\varepsilon),$$
$$C_{K,q} > 0.$$

Thus, $(\kappa u_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}(\mathbb{R}^d)$. Indeed, by using the Leibniz rule for estimating $p_{K,l}(\kappa u_\varepsilon)$, we can find a constant $C_{K,q,\kappa} > 0$ such that

$$\forall (q,l) \in \mathbb{N}^2,\ \forall \varepsilon \in (0,1] \quad \mu_{q,l}(\kappa u_\varepsilon) \leqslant C_{K,q,\kappa}\, p_{K,l}(u_\varepsilon). \tag{21}$$

(b) Let $(\tilde{u}_\varepsilon)_\varepsilon$ be another representative of u and $\tilde{\kappa}$ be in $\mathcal{D}(\Omega)$, with $0 \leqslant \tilde{\kappa} \leqslant 1$ and $\tilde{\kappa} = 1$ on a neighborhood of $\operatorname{supp} \tilde{u}$. Let L be a compact set such that $\operatorname{supp} \kappa u_\varepsilon \cup \operatorname{supp} \tilde{\kappa} \tilde{u}_\varepsilon \subset L \subset \Omega$. According to the previous estimate, we have

$$\begin{aligned}
&\forall (q,l) \in \mathbb{N}^2,\ \forall \varepsilon \in (0,1],\\
&\quad \mu_{q,l}(\kappa u_\varepsilon - \tilde{\kappa} \tilde{u}_\varepsilon) \leqslant \mu_{q,l}\bigl((\kappa - \tilde{\kappa}) u_\varepsilon\bigr) + \mu_{q,l}\bigl(\tilde{\kappa}(u_\varepsilon - \tilde{u}_\varepsilon)\bigr)\\
&\qquad\qquad\qquad\qquad \leqslant C_{L,q}\, p_{L,l}\bigl((\kappa - \tilde{\kappa}) u_\varepsilon\bigr) + C_{L,q}\, p_{L,l}\bigl(\tilde{\kappa}(u_\varepsilon - \tilde{u}_\varepsilon)\bigr).
\end{aligned}$$

As $\kappa = \tilde{\kappa}$ on a closed neighborhood V of $\operatorname{supp} u$, it follows that $p_{V,l}((\kappa - \tilde{\kappa}) u_\varepsilon) = 0$. Moreover, for all $m \in \mathbb{N}$, $p_{L \setminus V,l}((\kappa - \tilde{\kappa}) u_\varepsilon) = \mathrm{O}(\varepsilon^m)$ as $\varepsilon \to 0$, since $(L \setminus V) \cap \operatorname{supp} u = \emptyset$. Then $p_{L,l}((\kappa - \tilde{\kappa}) u_\varepsilon) = \mathrm{O}(\varepsilon^m)$ as $\varepsilon \to 0$. As $[(u_\varepsilon)_\varepsilon] = [(\tilde{u}_\varepsilon)_\varepsilon]$, we have $p_{L,l}(\tilde{\kappa}(u_\varepsilon - \tilde{u}_\varepsilon)) = \mathrm{O}(\varepsilon^m)$ as $\varepsilon \to 0$. Then $\mu_{q,l}(\kappa u_\varepsilon - \tilde{\kappa} \tilde{u}_\varepsilon) = \mathrm{O}(\varepsilon^m)$ and $[(\kappa u_\varepsilon)_\varepsilon]_{\mathcal{S}} = [(\tilde{\kappa} \tilde{u})_\varepsilon]_{\mathcal{S}}$. $\square$

From Lemma 10, we deduce easily the following proposition.

Proposition 11. *With the notations of Lemma* 10*, the map*

$$\iota_{C,S}\colon\quad \mathcal{G}_C^{\mathcal{R}}(\Omega) \to \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\big(\mathbb{R}^d\big),\quad u \mapsto \big[(\kappa u_\varepsilon)_\varepsilon\big]_{\mathcal{S}}$$

is a linear embedding.

From the embedding $\iota_{C,S}$, one can then verify that the Fourier transform of a compactly supported generalized functions $u \in \mathcal{G}^{\mathcal{R}}(\Omega)$, which can be straightforwardly considered as an element of $\mathcal{G}_C(\mathbb{R}^d)$, is defined by one of the following equalities

$$\mathcal{F}(u) = \mathcal{F}\big(\iota_{C,S}(u)\big) = \left[\left(x \mapsto (2\pi)^{-d}\int\limits_W \mathrm{e}^{ix\xi} u_\varepsilon(\xi)\,\mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{S}},$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}(\mathbb{R}^d)$ is any representative of u and W any relatively compact neighborhood of $\operatorname{supp} u$.

4.3. Exchange and regularity theorems

Theorem 12 *(Exchange theorem). For any regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, we have*

$$\mathcal{F}\big(\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\big(\mathbb{R}^d\big)\big) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\big(\mathbb{R}^d\big),\qquad \mathcal{F}\big(\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}\big(\mathbb{R}^d\big)\big) = \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}\big(\mathbb{R}^d\big). \tag{22}$$

The *proof* is based on the following refinement of a classical result [17] (the proof is left to the reader).

Lemma 13. *For all $u \in \mathcal{S}(\mathbb{R}^d)$ and $(q,l) \in \mathbb{N}^2$, there exists a constant $C_{q,l} > 0$ such that*

$$\mu_{q,l}(\hat{u}) \leqslant C_{q,l}\,\mu_{l+d+1,q}(u). \tag{23}$$

Indeed, let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$. (a) Take $u \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}(\mathbb{R}^d)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}_u}(\mathbb{R}^d)$ a representative of u. There exists a sequence $N \in \mathcal{R}$ such that $\mu_{r,q}(u_\varepsilon) = \mathrm{O}(\varepsilon^{-N(q)})$ as $\varepsilon \to 0$, for all $r \in \mathbb{N}$. Lemma 13 implies that $\mu_{q,l}(\hat{u}_\varepsilon) = \mathrm{O}(\varepsilon^{-N(q)})$ as $\varepsilon \to 0$, for all $l \in \mathbb{N}$. Thus, $\mathcal{F}(u) \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\mathbb{R}^d)$.

(b) Conversely, take $u \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\mathbb{R}^d)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}}^{\mathcal{R}_\partial}(\mathbb{R}^d)$ a representative of u. There exists a sequence $N \in \mathcal{R}$ such that $\mu_{r,m}(u_\varepsilon) = \mathrm{O}(\varepsilon^{-N(r)})$ as $\varepsilon \to 0$, for all $r \in \mathbb{N}$. According to the stability of regular sets, there exists a sequence $N' \in \mathcal{R}$ such that

$$\forall l \in \mathbb{N},\quad N(l+d+1) \leqslant N'(l).$$

Lemma 13 implies that $\mu_{q,l}(\hat{u}_\varepsilon) = \mathrm{O}(\varepsilon^{-N'(q)})$ as $\varepsilon \to 0$, for all $l \in \mathbb{N}$. Thus, $\mathcal{F}(u) \in \mathcal{G}_{\mathcal{S}}^{\mathcal{R}_u}(\mathbb{R}^d)$.

So, we proved the inclusions of the sets in the left-hand side of relations (22), into the sets of the right-hand side. The equalities follow directly from a similar study with the inverse Fourier transform.

Example 10. Take $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$. We get $\mathcal{F}(\mathcal{G}_{\mathcal{S}}^{\partial}(\mathbb{R}^d)) = \mathcal{G}_{\mathcal{S}}^{u}(\mathbb{R}^d)$ and $\mathcal{F}(\mathcal{G}_{\mathcal{S}}^{u}(\mathbb{R}^d)) = \mathcal{G}_{\mathcal{S}}^{\partial}(\mathbb{R}^d)$, result which is closely related to the classical exchange theorem between $\mathcal{O}_M(\mathbb{R}^d)$ and $\mathcal{O}_C'(\mathbb{R}^d)$.

Indeed, take $u \in \mathcal{O}'_C(\mathbb{R}^d)$ and consider $(u_\varepsilon)_\varepsilon = (u * \rho_\varepsilon)_\varepsilon$ which is a representative of its image by the embedding $\iota_{\mathcal{S}}$. Its Fourier image $\mathcal{F}(\iota_{\mathcal{S}}(u)) = [(\hat{u}\hat{\rho}_\varepsilon)_\varepsilon]_{\mathcal{S}}$ belongs to $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$, with $\hat{u} \in \mathcal{O}_M(\mathbb{R}^d)$ and $\hat{\rho}_\varepsilon \in \mathcal{S}(\mathbb{R}^d)$. As $\lim_{\varepsilon\to 0} \hat{\rho}_\varepsilon = 1$, we get $\lim_{\varepsilon\to 0}(\hat{u}\hat{\rho}_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^d)$. (For those limits, we consider $\mathcal{O}_M(\mathbb{R}^d)$ equipped with its usual topology; see [16,24].) This shows the consistency of our result with the classical one. The generalized function $\mathcal{F}(\iota_{\mathcal{S}}(u))$ belongs to a space of rapidly decreasing generalized functions, but the limit of its representatives when $\varepsilon \to 0$ is in a space of functions of moderate growth.

Corollary 14 *(Regularity theorem). We have $\mathcal{F}(\mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d)) = \mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d)$.*

Proof. Apply Theorem 12 with $\mathcal{R} = \mathcal{B}$, the set of bounded sequences, for which $\mathcal{B}_u = \mathcal{B}_\partial$. □

We can now complete diagram (20) in the case of $\Omega = \mathbb{R}^d$:

$$\begin{tikzcd}
 & \mathcal{G}^{\mathcal{R}_\partial}_{\mathcal{S}}(\mathbb{R}^d) \arrow[r] \arrow[d, <->, "\mathcal{F}"] & \mathcal{G}^{\partial}_{\mathcal{S}}(\mathbb{R}^d) \arrow[dr] \arrow[d, <->, "\mathcal{F}"] & \\
\mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d) \arrow[ur] \arrow[dr] & & & \mathcal{G}_{\mathcal{S}}(\mathbb{R}^d) . \\
 & \mathcal{G}^{\mathcal{R}_u}_{\mathcal{S}}(\mathbb{R}^d) \arrow[r] & \mathcal{G}^{u}_{\mathcal{S}}(\mathbb{R}^d) \arrow[ur] &
\end{tikzcd} \tag{24}$$

An interesting consequence of Corollary 14 is the following property, also proved in [8], which is the equivalent for rapidly decreasing generalized functions of the result mentioned in the introduction for the $\mathcal{G}^\infty$-regularity: $\mathcal{D}'(\Omega) \cap \mathcal{G}^\infty(\Omega) = \mathrm{C}^\infty(\Omega)$ [20].

Proposition 15. *We have $\mathcal{O}'_C(\mathbb{R}^d) \cap \mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$.*

Proof. We follow here the ideas of [19] for the proof of the above mentioned result about $\mathcal{G}^\infty(\mathbb{R}^d)$. Let u be in $\mathcal{O}'_C(\mathbb{R}^d)$ and set $(u_\varepsilon)_\varepsilon = (u * \rho_\varepsilon)_\varepsilon$. By assumption $[(u * \rho_\varepsilon)_\varepsilon]_{\mathcal{S}}$ is in $\mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d)$. According to Corollary 14, $\mathcal{F}_S([(u * \rho_\varepsilon)_\varepsilon]_{\mathcal{S}})$ is also in $\mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d)$. It follows that there exists $N \in \mathbb{N}$ such that

$$\forall q \in \mathbb{N},\ \exists C_q > 0, \quad \sup_{\xi\in\mathbb{R}^d} \big(1 + |\xi|\big)^q \big|\hat{u}(\xi)\hat{\rho}_\varepsilon(\xi)\big| \leqslant C_q \varepsilon^{-N}, \quad \text{for } \varepsilon \text{ small enough.}$$

By choice of ρ, $\hat{\rho}_\varepsilon$ is an element of $\mathcal{D}(\mathbb{R}^d)$. Moreover, a straightforward calculation shows that $\hat{\rho}_\varepsilon(\xi) = \hat{\rho}(\varepsilon\xi)$, for all $\xi \in \mathbb{R}^d$, with $\hat{\rho}$ equal to 1 on a neighborhood of 0. It follows that, for all $q \in \mathbb{N}$, we have

$$\begin{aligned}
\forall \xi \in \mathbb{R}^d, \quad \big(1 + |\xi|\big)^q \big|\hat{u}(\xi)\big| &\leqslant \big(1 + |\xi|\big)^q \big|\hat{u}(\xi)\big|\big(\big|1 - \hat{\rho}(\varepsilon\xi)\big| + \big|\hat{\rho}(\varepsilon\xi)\big|\big) \\
&\leqslant \big(1 + |\xi|\big)^q \big|\hat{u}(\xi)\big|\big|1 - \hat{\rho}(\varepsilon\xi)\big| + C_q \varepsilon^{-N}.
\end{aligned}$$

Since $1 - \hat{\rho}(\varepsilon\xi) = \hat{\rho}(0) - \hat{\rho}(\varepsilon\xi) = -\varepsilon\xi \int_0^1 \hat{\rho}'(\varepsilon\xi t)\,\mathrm{d}t$, with $\hat{\rho}'$ bounded, there exists a constant $C > 0$ such that

$$\forall \xi \in \mathbb{R}^d, \quad \big(1 + |\xi|\big)^q \big|\hat{u}(\xi)\big| \leqslant C\big(1 + |\xi|\big)^q \big|\hat{u}(\xi)\big|\varepsilon|\xi| + C_q \varepsilon^{-N}.$$

As $\hat{u}$ is in $\mathcal{O}_M(\mathbb{R}^d)$, there exist $m \in \mathbb{N}$ and a constant $C_1 > 0$ such that $\sup_{\xi\in\mathbb{R}^d}(1 + |\xi|)^{-m+1}|\hat{u}(\xi)| \leqslant C_1$. Therefore, by setting $C_2 = \max(CC_1, C_q)$, we get

$$\forall \xi \in \mathbb{R}^d, \quad \left(1+|\xi|\right)^q \left|\hat{u}(\xi)\right| \leqslant C_2\left(\left(1+|\xi|\right)^{q+m-1}\varepsilon|\xi| + \varepsilon^{-N}\right)$$
$$\leqslant C_2\left(\left(1+|\xi|\right)^{q+m}\varepsilon + \varepsilon^{-N}\right).$$

By minimizing the function $f_\xi : \varepsilon \mapsto (1+|\xi|)^{q+m}\varepsilon + \varepsilon^{-N}$, we get the existence of a constant $C_3 > 0$ such that

$$\forall \xi \in \mathbb{R}^d, \quad \left(1+|\xi|\right)^q \left|\hat{u}(\xi)\right| \leqslant C_3\left(\left(1+|\xi|\right)^{N(q+m)/(N+1)}\right), \quad \text{and}$$
$$\forall \xi \in \mathbb{R}^d, \quad \left|\hat{u}(\xi)\right| \leqslant C_3\left(\left(1+|\xi|\right)^{-q/(N+1)+mN/(N+1)}\right),$$

for all $q \in \mathbb{N}$. (m only depends on u.) Treating the derivatives in the same way, we obtain the same type of estimates. Therefore $\hat{u}$ and its derivatives are rapidly decreasing. This shows that $\mathcal{O}'_C(\Omega) \cap \mathcal{G}^\infty_\mathcal{S}(\Omega) \subset \mathcal{S}(\Omega)$. As the other inclusion is obvious, our claim is proved. □

5. Global regularity of compactly supported generalized functions

5.1. C^∞-regularity for compactly supported distributions

In order to render easier the comparison between the distributional case and the generalized case, we are going to recall the classical theorem and complete it by some equivalent statements.

Theorem 16. *For u in $\mathcal{E}'(\mathbb{R}^d)$, the following equivalences hold*:

$$\begin{aligned} &\text{(i)} \quad u \in \mathrm{C}^\infty\left(\mathbb{R}^d\right) &&\Leftrightarrow \quad \text{(ii)} \quad \mathcal{F}(u) \in \mathcal{S}\left(\mathbb{R}^d\right)\\ & &&\Leftrightarrow \quad \text{(iii)} \quad \mathcal{F}(u) \in \mathcal{S}_*\left(\mathbb{R}^d\right)\\ & &&\Leftrightarrow \quad \text{(iv)} \quad \mathcal{F}(u) \in \mathcal{O}'_M\left(\mathbb{R}^d\right)\\ & &&\Leftrightarrow \quad \text{(v)} \quad \mathcal{F}(u) \in \mathcal{O}'_C\left(\mathbb{R}^d\right).\end{aligned}$$

Proof. The equivalence (i) ⇔ (ii) is the classical result. The trivial inclusion $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}_*(\mathbb{R}^d)$ shows (ii) ⇒ (iii). Then, the structure of elements of $\mathcal{O}'_M(\mathbb{R}^d)$ [21] shows that $\mathcal{S}_*(\Omega)$ is canonically embedded in $\mathcal{O}'_M(\mathbb{R}^d)$: this shows (iii) ⇒ (iv). As $\mathcal{O}'_M(\mathbb{R}^d) \subset \mathcal{O}'_C(\mathbb{R}^d)$, (iv) ⇒ (v) is obvious. For (v) ⇒ (i), note that $\mathcal{F}(u)$ belongs to $\mathcal{O}_M(\mathbb{R}^d)$ and better to $\mathcal{O}_C(\mathbb{R}^d)$ since u is in $\mathcal{E}'(\mathbb{R}^d)$. (This last assertion is a refinement of the classical previous one.) Then, if (v) holds, $\mathcal{F}(u)$ is in $\mathcal{O}_C(\mathbb{R}^d) \cap \mathcal{O}'_C(\mathbb{R}^d)$ which is equal to $\mathcal{S}(\mathbb{R}^d)$ [21]. Then (ii) holds. □

Theorem 16 shows, at least, that there is no need to consider spaces of functions with all the derivatives rapidly decreasing to characterize elements of $\mathcal{E}'(\mathbb{R}^d)$ which are C^∞. In fact, we can only consider functions rapidly decreasing, with no other hypothesis on the derivatives. A similar situation holds for generalized functions, justifying the introduction of rough generalized functions in the following subsection.

5.2. Rough rapidly decreasing generalized functions

5.2.1. Definitions

Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^\mathbb{N}$ and Ω an open subset or $\mathbb{R}^d$. Set

$$\mathcal{S}_*(\Omega) = \left\{ f \in \mathrm{C}^\infty(\Omega) \mid \forall q \in \mathbb{N},\ \mu_{q,0}(f) < +\infty \right\},$$
$$\mathcal{X}^\mathcal{R}_{\mathcal{S}_*}(\Omega) = \left\{ (f_\varepsilon)_\varepsilon \in \mathcal{S}_*(\Omega)^{(0,1]} \;\middle|\; \exists N \in \mathcal{R},\ \forall q \in \mathbb{N},\ \mu_{q,0}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-N(q)}\right) \text{ as } \varepsilon \to 0 \right\},$$

$$\mathcal{N}_{\mathcal{S}_*}(\Omega)=\left\{(f_\varepsilon)_\varepsilon\in\mathcal{S}_*(\Omega)^{(0,1]}\;\middle|\;\forall N\in\mathbb{R}_+^{\mathbb{N}},\ \forall q\in\mathbb{N},\ \mu_{q,0}(f_\varepsilon)=\mathrm{O}\big(\varepsilon^{N(q)}\big)\text{ as }\varepsilon\to 0\right\}. \tag{25}$$

One can show that $\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$ is a subalgebra of $\mathcal{S}_*(\Omega)^{(0,1]}$ and $\mathcal{N}_{\mathcal{S}_*}(\Omega)$ an ideal of $\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$. (The proof is similar to that of Proposition 1.)

Definition 7. The space $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)=\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)/\mathcal{N}_{\mathcal{S}_*}(\Omega)$ is called the algebra of $\mathcal{R}$-*regular* rough rapidly decreasing generalized functions.

Example 11. Taking $\mathcal{R}=\mathbb{R}_+^{\mathbb{N}}$, we obtain the space $\mathcal{G}_{\mathcal{S}_*}(\Omega)$ of *rough rapidly decreasing generalized functions*.

Example 12. Taking $\mathcal{R}=\mathcal{B}$, the set of bounded sequences, we obtain the space $\mathcal{G}_{\mathcal{S}_*}^{\infty}(\Omega)$, of *regular rough rapidly decreasing generalized functions*.

Lemma 5 implies immediately the following proposition.

Proposition 17. *If the open set Ω is a box and $\mathcal{R}'$ a regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$, then $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}'}(\Omega)$ is included in $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}'_0}(\Omega)$, where $\mathcal{R}'_0$ is the regular subset of $\mathbb{R}_+^{\mathbb{N}}$ defined by*

$$\mathcal{R}'_0=\left\{N(\cdot,0),\ N\in\mathcal{R}'\right\}.$$

Example 13. If Ω is a box, for all $\mathcal{R}\subset\mathbb{R}_+^{\mathbb{N}}$, $\mathcal{G}_{\mathcal{S}}^{\mathcal{R}_\partial}(\Omega)$ is included in $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$.

Indeed, $\mathcal{R}_\partial=\mathcal{R}\otimes\{1\}$, which implies that $(\mathcal{R}_\partial)_0=\mathcal{R}$. Let us mention two other applications of Proposition 17.

Corollary 18. *If the open set Ω is a box, then*

(i) *$\mathcal{G}_{\mathcal{S}}(\Omega)$, obtained for $\mathcal{R}'=\mathbb{R}_+^{\mathbb{N}^2}$, is included in $\mathcal{G}_{\mathcal{S}_*}(\Omega)$.*
(ii) *$\mathcal{G}_{\mathcal{S}}^{\infty}(\Omega)$, obtained for $\mathcal{R}'=\mathcal{B}'$, is included in $\mathcal{G}_{\mathcal{S}_*}^{\infty}(\Omega)$.*

Indeed, (i) (respectively (ii)) holds, since $(\mathbb{R}_+^{\mathbb{N}^2})_0=\mathbb{R}_+^{\mathbb{N}}$ (respectively $(\mathcal{B}')_0=\mathcal{B}'$). Note that the proof of Proposition 15 shows also that $\mathcal{G}_{\mathcal{S}_*}^{\infty}(\mathbb{R}^d)\cap\mathcal{O}'_C(\mathbb{R}^d)=\mathcal{S}_*(\mathbb{R}^d)$.

We turn to the question of embeddings. First, the structure of elements of $\mathcal{O}'_C(\mathbb{R}^d)$ [16,21] shows that $\mathcal{S}_*(\mathbb{R}^d)$ is canonically embedded in $\mathcal{O}'_C(\mathbb{R}^d)$. The embedding of $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ is done by the canonical injective map

$$\sigma_{\mathcal{S}_*}:\quad \mathcal{S}_*\big(\mathbb{R}^d\big)\to\mathcal{G}_{\mathcal{S}_*}\big(\mathbb{R}^d\big),\quad f\mapsto(f_\varepsilon)_\varepsilon+\mathcal{N}_{\mathcal{S}_*}\big(\mathbb{R}^d\big)\quad\text{with } f_\varepsilon=f\text{ for }\varepsilon\in(0,1].$$

Finally, a simplification of the proofs of Theorems 6, 7 and Proposition 8 leads to the following theorem, where $(\rho_\varepsilon)_\varepsilon$ is defined by (12) and (13).

Theorem 19.

(i) *The map*

$$\iota_{\mathcal{S}_*}:\quad \mathcal{O}'_C\big(\mathbb{R}^d\big)\to\mathcal{G}_{\mathcal{S}_*}\big(\mathbb{R}^d\big),\quad u\mapsto(u*\rho_\varepsilon)_\varepsilon+\mathcal{N}_{\mathcal{S}_*}\big(\mathbb{R}^d\big)$$

is a linear embedding which commutes with partial derivatives.

(ii) *We have*: $\iota_{\mathcal{S}_*|\mathcal{S}_*(\mathbb{R}^d)} = \sigma_{\mathcal{S}_*}$.
(iii) *We have*: $\iota_{\mathcal{S}_*}(\mathcal{O}'_M(\mathbb{R}^d)) \subset \mathcal{G}^{\infty}_{\mathcal{S}_*}(\mathbb{R}^d)$.

Remark 2. Theorems 6, 7 and 19 combined together show that all the arrows are injective and all the diagrams commutative in the following schemes:

$$\begin{tikzcd}
\mathcal{S}(\mathbb{R}^d) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{S}_*(\mathbb{R}^d) \arrow[dl] \arrow[dd] \\
& \mathcal{O}'_C(\mathbb{R}^d) \arrow[dl] \arrow[dr] & \\
\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d) \arrow[rr] & & \mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{S}(\mathbb{R}^d) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathcal{S}_*(\mathbb{R}^d) \arrow[dl] \arrow[dd] \\
& \mathcal{O}'_M(\mathbb{R}^d) \arrow[dl] \arrow[dr] & \\
\mathcal{G}^u_{\mathcal{S}}(\mathbb{R}^d) \arrow[rr] & & \mathcal{G}^{\infty}_{\mathcal{S}_*}(\mathbb{R}^d)
\end{tikzcd}$$

5.2.2. Fourier transform in $\mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$

We need in the sequel to define a Fourier transform (or an inverse Fourier transform) in $\mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\mathbb{R}^d)$. This is done in the following way. Set, for any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$,

$$\begin{aligned}
\mathcal{X}_{\mathcal{B}}(\Omega) &= \left\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\middle|\; \exists N \in \mathbb{R}_+^{\mathbb{N}},\ \forall l \in \mathbb{N},\ \mu_{0,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-N(l)}\big) \text{ as } \varepsilon \to 0\right\},\\
\mathcal{X}^{\mathcal{R}}_{\mathcal{B}}(\Omega) &= \left\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\middle|\; \exists N \in \mathcal{R},\ \forall l \in \mathbb{N},\ \mu_{0,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{-N(l)}\big) \text{ as } \varepsilon \to 0\right\},\\
\mathcal{N}_{\mathcal{B}}(\Omega) &= \left\{(f_\varepsilon)_\varepsilon \in \mathrm{C}^\infty(\Omega)^{(0,1]} \;\middle|\; \forall N \in \mathbb{R}_+^{\mathbb{N}},\ \forall l \in \mathbb{N},\ \mu_{0,l}(f_\varepsilon) = \mathrm{O}\big(\varepsilon^{N(l)}\big) \text{ as } \varepsilon \to 0\right\}.
\end{aligned}$$

According to the general scheme of construction of Colombeau type algebras, $\mathcal{G}_{\mathcal{B}}(\Omega) = \mathcal{X}_{\mathcal{B}}(\Omega)/\mathcal{N}_{\mathcal{B}}(\Omega)$ is an algebra, called the *algebra of bounded generalized functions*. Moreover, $\mathcal{X}^{\mathcal{R}}_{\mathcal{B}}(\Omega)$ is a subalgebra of $\mathcal{X}_{\mathcal{B}}(\Omega)$. (The proof is similar to that of Proposition 1.) The space $\mathcal{G}^{\mathcal{R}}_{\mathcal{B}}(\Omega) = \mathcal{X}^{\mathcal{R}}_{\mathcal{B}}(\Omega)/\mathcal{N}_{\mathcal{B}}(\Omega)$ is called the space of *$\mathcal{R}$-regular bounded generalized functions*.

Notation 5. We shall note $[(f_\varepsilon)_\varepsilon]_{\mathcal{B}}$ the class of $(f_\varepsilon)_\varepsilon$ in $\mathcal{G}^{\mathcal{R}}_{\mathcal{B}}(\Omega)$.

Remark 3. One can verify that $\mathcal{G}_C(\Omega)$ (respectively $\mathcal{G}^{\mathcal{R}}_C(\Omega)$) is embedded into $\mathcal{G}_{\mathcal{B}}(\Omega)$ (respectively $\mathcal{G}^{\mathcal{R}}_{\mathcal{B}}(\Omega)$).

Proposition 20.

(i) *For all $u \in \mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$ a representative of u, the expression*

$$\hat{u}: \quad \left[\hat{u}_\varepsilon = \left(\xi \mapsto \int \mathrm{e}^{-ix\xi} u_\varepsilon(x)\,\mathrm{d}x\right)_\varepsilon\right]_{\mathcal{B}} \tag{26}$$

defines an element of $\mathcal{G}_{\mathcal{B}}(\Omega)$ depending only on u.
(ii) *For any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}_{\mathcal{S}}(\Omega)$, we have $(\hat{u}_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}_{\mathcal{B}}(\Omega)$.*

Proof. *Assertion* (i). Take $u \in \mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ and $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$ a representative of u. Then Lemma 13 (applied with $q = 0$) implies that

$$\forall l \in \mathbb{N},\ \exists C_l > 0,\ \forall \varepsilon \in (0,1], \quad \mu_{0,l}(\hat{u}_\varepsilon) \leqslant C_\alpha \mu_{l+d+1,0}(u_\varepsilon). \tag{27}$$

This estimate shows that $(\hat{u}_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{B}}(\mathbb{R}^d)$. Indeed, if $(u_\varepsilon)_\varepsilon$ is in $\mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$, there exists a sequence $N \in \mathcal{R}$ such that $\mu_{l,0}(u_\varepsilon) = \mathrm{O}(\varepsilon^{-N(l)})$ as $\varepsilon \to 0$ and setting $N_1 : l \mapsto N(l+d+1)$, we get that $\mu_{0,l}(\hat{u}_\varepsilon) = (\varepsilon^{-N_1(l)})$ as $\varepsilon \to 0$. According to the overstability by translation of the subset $\mathcal{R}$, $(\hat{u}_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}_{\mathcal{B}}(\mathbb{R}^d)$. Similar arguments show that, if $(\eta_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{S}_*}(\mathbb{R}^d)$, then $(\hat{\eta}_\varepsilon)_\varepsilon \in \mathcal{N}_{\mathcal{B}}(\Omega)$. Therefore, relation (26) defines an element of $\mathcal{G}_{\mathcal{B}}(\mathbb{R}^d)$, depending only on u.

Assertion (ii). The estimate (27) implies that the regularity of the sequences in the definition of moderate elements transfers by Fourier transform from the space index q in the $\mathcal{S}_*$-type spaces to the derivative index l in the Colombeau type space (here of bounded functions), showing our claim. □

We define the *Fourier transform* $\mathcal{F}_*$ on $\mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ by the formula

$$\mathcal{F}_*:\quad \mathcal{G}_{\mathcal{S}_*}\big(\mathbb{R}^d\big) \to \mathcal{G}_B\big(\mathbb{R}^d\big), \quad u \mapsto \left[\left(x \mapsto \int \mathrm{e}^{-ix\xi} u_\varepsilon(\xi)\,\mathrm{d}\xi\right)_\varepsilon\right]_{\mathcal{B}},$$

where $(u_\varepsilon)_\varepsilon \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$ is any representative of u. (The inverse Fourier on $\mathcal{G}_{\mathcal{S}_*}(\mathbb{R}^d)$ is defined analogously.)

The assertion (ii) of Proposition 20 implies:

Proposition 21 *(Small exchange theorem). We have $\mathcal{F}(\mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}) \subset \mathcal{G}^{\mathcal{R}}_B(\mathbb{R}^d)$.*

5.3. $\mathcal{G}^{\mathcal{R}}$-regularity for compactly supported generalized functions

We have now all the elements to formulate and prove the following fundamental theorem.

Theorem 22. *Let $\mathcal{R}$ be regular subspace of $\mathbb{R}_+^{\mathbb{N}}$. For u in $\mathcal{G}_C(\mathbb{R}^d)$, the following equivalences hold*:

$$\begin{aligned}\text{(i)}\quad u \in \mathcal{G}^{\mathcal{R}}\big(\mathbb{R}^d\big) \quad &\Leftrightarrow \quad \text{(ii)}\quad \mathcal{F}(u) \in \mathcal{G}^{\mathcal{R}_\partial}_{\mathcal{S}}\big(\mathbb{R}^d\big)\\ &\Leftrightarrow \quad \text{(iii)}\quad \mathcal{F}(u) \in \mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}\big(\mathbb{R}^d\big).\end{aligned}$$

Proof. (i) ⇒ (ii). As u is in $\mathcal{G}_C(\mathbb{R}^d) \cap \mathcal{G}^{\mathcal{R}}(\mathbb{R}^d) = \mathcal{G}^{\mathcal{R}}_C(\mathbb{R}^d)$, u is in $\mathcal{G}^{\mathcal{R}_u}_{\mathcal{S}}(\mathbb{R}^d)$ according to Proposition 11. Then, applying Theorem 12, $\mathcal{F}(u)$ is in $\mathcal{G}^{\mathcal{R}_\partial}_{\mathcal{S}}(\mathbb{R}^d)$.

(ii) ⇒ (iii). We have $\mathcal{G}^{\mathcal{R}_\partial}_{\mathcal{S}}(\mathbb{R}^d) \subset \mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\mathbb{R}^d)$, according to Example 13.

(iii) ⇒ (i). Let u be in $\mathcal{G}_C(\mathbb{R}^d)$, $(u_\varepsilon)_\varepsilon$ be a representative of u and K a compact set such that $\operatorname{supp} u_\varepsilon \subset K$, for all ε in $(0,1]$. We have $\mathcal{F}_{\mathcal{S}}(u) = [(\hat{u}_\varepsilon)_\varepsilon]_{\mathcal{G}_{\mathcal{S}}}$ where ˆ denotes the classical Fourier transform in $\mathcal{S}$. By assumption $\mathcal{F}_{\mathcal{S}}(u)$ is in $\mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\mathbb{R}^d)$ and we can consider its inverse Fourier transform $\mathcal{F}_*^{-1}$, with $\mathcal{F}_*^{-1}(\mathcal{F}_{\mathcal{S}}(u))$ in $\mathcal{G}^{\mathcal{R}}_B(\mathbb{R}^d)$ and

$$\mathcal{F}_*^{-1}\big(\mathcal{F}_{\mathcal{S}}(u)\big) = \big[\big(\mathcal{F}^{-1}(\hat{u}_\varepsilon)\big)_\varepsilon\big]_{\mathcal{B}}.$$

Using the classical isomorphism theorem in $\mathcal{S}$, we have $\mathcal{F}^{-1}(\hat{u}_\varepsilon) = u_\varepsilon$ for all ε in $(0,1]$. Then

$$\mathcal{F}_*^{-1}\big(\mathcal{F}_{\mathcal{S}}(u)\big) = \big[(u_\varepsilon)_\varepsilon\big]_{\mathcal{B}}.$$

Since all the u_ε have their support included in the same compact set, we obviously have $[(u_\varepsilon)_\varepsilon]_{\mathcal{B}} = \iota_{C,B}(u)$, where $\iota_{C,B}$ is the canonical embedding of $\mathcal{G}_C(\mathbb{R}^d)$ in $\mathcal{G}_B(\mathbb{R}^d)$. Therefore, $u \in \mathcal{G}^{\mathcal{R}}_B(\mathbb{R}^d) \cap \mathcal{G}_C(\mathbb{R}^d) = \mathcal{G}^{\mathcal{R}}(\mathbb{R}^d) \cap \mathcal{G}_C(\mathbb{R}^d)$. □

Example 14. The case $\mathcal{R}=\mathcal{B}$ in Theorem 22 gives a characterization of the global $\mathcal{G}^\infty$-regularity of compactly supported generalized functions.

Moreover, we can refine Theorem 22 in this particular case and prove:

Theorem 23. *For u in $\mathcal{G}_C(\mathbb{R}^d)$, the following statements are equivalent*:

$$\begin{aligned}
&\text{(i)}\quad u\in\mathcal{G}^\infty(\mathbb{R}^d) \quad\Leftrightarrow\quad \text{(ii)}\quad \mathcal{F}(u)\in\mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d)\\
&\qquad\qquad\qquad\qquad\Leftrightarrow\quad \text{(iii)}\quad \mathcal{F}(u)\in\mathcal{G}^u_{\mathcal{S}}(\mathbb{R}^d)\\
&\qquad\qquad\qquad\qquad\Leftrightarrow\quad \text{(iv)}\quad \mathcal{F}(u)\in\mathcal{G}^\infty_{\mathcal{S}_*}(\mathbb{R}^d).
\end{aligned}$$

Indeed, (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (i) follow directly from Theorem 22 applied with $\mathcal{R}=\mathcal{B}$, since $\mathcal{B}_\partial=\mathcal{B}'$, the set of bounded elements of $\mathbb{R}_+^{\mathbb{N}^2}$. For (ii) $\Rightarrow$ (iii), we have $\mathcal{G}^\infty_{\mathcal{S}}(\mathbb{R}^d)\subset\mathcal{G}^u_{\mathcal{S}}(\mathbb{R}^d)$. For (iii) $\Rightarrow$ (iv), we remark that $\mathcal{G}^u_{\mathcal{S}}(\mathbb{R}^d)$ is obtained with $\mathcal{R}'=\{1\}\otimes\mathbb{R}_+^{\mathbb{N}}$ as regular subset of $\mathbb{R}_+^{\mathbb{N}^2}$. This implies that $(\mathcal{R}')_0=\mathcal{B}'$, with the notations of Proposition 17.

6. Local and microlocal $\mathcal{R}$-regularity

We follow here the presentation of [15] and show that, with the previously introduced material, the $\mathcal{G}^{\mathcal{R}}$-wavefront of a generalized function is defined exactly like the C^∞-wavefront of a distribution. First, as $\mathcal{G}^{\mathcal{R}}$ is a subsheaf of $\mathcal{G}$, the following definition makes sense.

Definition 8. Let u be in $\mathcal{G}(\Omega)$. The singular $\mathcal{G}^{\mathcal{R}}$-support of u is the set

$$\operatorname{sing\,supp}_{\mathcal{R}} u=\Omega\setminus\{x\in\Omega\mid \exists V\in\mathcal{V}_x,\ u\in\mathcal{G}^{\mathcal{R}}(V)\}.$$

Proposition 24. *$\mathcal{G}_{\mathcal{S}_*}:\Omega\to\mathcal{G}_{\mathcal{S}_*}(\Omega)$ is a presheaf. It allows restrictions.*

The *proof* is similar to the part (b) of the one of Proposition 4.

Notation 6. For $(x,\xi)\in\Omega\times\mathbb{R}^d\setminus\{0\}$ (Ω open subset of $\mathbb{R}^d$), we shall denote by:

(i) $\mathcal{V}_x$ (respectively $\mathcal{V}_\xi^\Gamma$), the set of all open neighborhoods (respectively open convex conic neighborhoods) of x (respectively ξ),
(ii) $\mathcal{D}_x(\Omega)$, the set of elements $\mathcal{D}(\Omega)$ nonvanishing at x.

For $\Gamma\in\mathcal{V}_\xi^\Gamma$, we say that $\hat{u}\in\mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\Gamma)$ if $u|_\Gamma\in\mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\Gamma)$. Let us fix a regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ and set, for $u\in\mathcal{G}_C(\mathbb{R}^d)$,

$$O^{\mathcal{R}}(u)=\{\xi\in\mathbb{R}^d\setminus\{0\}\mid \exists\Gamma\in\mathcal{V}_\xi^\Gamma\ \hat{u}\in\mathcal{G}^{\mathcal{R}}_{\mathcal{S}_*}(\Gamma)\},\qquad \Sigma^{\mathcal{R}}(u)=(\mathbb{R}^d\setminus\{0\})\setminus O^{\mathcal{R}}(u).$$

Lemma 25. *For $u\in\mathcal{G}_C(\mathbb{R}^d)$ and $\varphi\in D(\mathbb{R}^d)$, $O^{\mathcal{R}}(u)\subset O^{\mathcal{R}}(\varphi u)$ (or, equivalently, $\Sigma^{\mathcal{R}}(\varphi u)\subset\Sigma^{\mathcal{R}}(u)$).*

Proof. Let $(u_\varepsilon)_\varepsilon\in\mathcal{X}(\mathbb{R}^d)$ be a representative of u with $\operatorname{supp}u_\varepsilon$ included in the same compact set, for all ε in $(0,1]$. We have

$$\widehat{\varphi u}_\varepsilon(y)=\hat{\varphi}*\hat{u}_\varepsilon(y)=\int\hat{\varphi}(\eta)\hat{u}_\varepsilon(y-\eta)\,\mathrm{d}\eta.$$

Let ξ be in $O^{\mathcal{R}}(u)$ and $\Gamma \in \mathcal{V}_{\xi}^{\Gamma}$ such that $\hat{u} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$. There exists an open conic neighborhood $\Gamma_1 \subset \Gamma$ of ξ and a real number $c \in (0,1)$ such that, for all (y,η) with $y \in \Gamma_1$ and $|\eta| \leqslant c|y|$, $y-\eta \in \Gamma$. Then

$$\begin{aligned}\widehat{\varphi u}_{\varepsilon}(y) &= \int\limits_{|\eta|\leqslant c|y|} \hat{\varphi}(\eta)\hat{u}_{\varepsilon}(y-\eta)\,\mathrm{d}\eta + \int\limits_{|\eta|>c|y|} \hat{\varphi}(\eta)\hat{u}_{\varepsilon}(y-\eta)\,\mathrm{d}\eta \\ &= \underbrace{\int\limits_{|\eta|\leqslant c|y|} \hat{\varphi}(\eta)\hat{u}_{\varepsilon}(y-\eta)\,\mathrm{d}\eta}_{v_{1,\varepsilon}(y)} + \underbrace{\int\limits_{|y-\eta|>c|y|} \hat{\varphi}(y-\eta)\hat{u}_{\varepsilon}(\eta)\,\mathrm{d}\eta}_{v_{2,\varepsilon}(y)}.\end{aligned}$$

In order to estimate $v_{1,\varepsilon}$, let us remark that $\hat{u} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$. There exists a sequence $N \in \mathcal{R}$ such that, for all $q \in \mathbb{N}$, there exists a constant $C_1 > 0$ with

$$\forall (y,\eta) \in \Gamma_1 \times \mathbb{R}^d \text{ with } |\eta| \leqslant c|y|, \quad \left|\hat{u}_{\varepsilon}(y-\eta)\right| \leqslant C_1 \varepsilon^{-N(q)}\left(1+|y-\eta|\right)^{-q},$$

for ε small enough.

As, for $|\eta| \leqslant c|y|$, we have $|y-\eta| \geqslant ||y|-|\eta|| \geqslant |y|(1-c)$, it follows that

$$\forall (y,\eta) \in \Gamma_1 \times \mathbb{R}^d \text{ with } |\eta| \leqslant c|y|, \quad \left|\hat{u}_{\varepsilon}(y-\eta)\right| \leqslant C_1 \varepsilon^{-N(q)}\left(1+|y|(1-c)\right)^{-q}.$$

Since $\hat{\varphi}$ is rapidly decreasing, we get the existence of a constant $C_2 > 0$ such that

$$\forall \eta \in \mathbb{R}^d, \quad \hat{\varphi}(\eta) \leqslant C_2\left(1+|\eta|\right)^{-d-1}.$$

Replacing in the definition of $|v_{1,\varepsilon}(y)|$, we get the existence of a constant $C_3 > 0$ such that

$$\forall y \in \Gamma_1, \quad \left(1+|y|\right)^q \left|v_{1,\varepsilon}(y)\right| \leqslant C_3 \varepsilon^{-N(q)} \int \left(\frac{1+|y|}{(1+|y|(1-c))}\right)^q \frac{1}{(1+|\eta|)^{d+1}}\,\mathrm{d}\eta.$$

The function $t \mapsto (1+t)/(1+t(1-c))$ is bounded on $\mathbb{R}_+$. It follows that the integral in the previous inequality converges, we, finally, get a constant $C_4 > 0$ such that

$$\forall y \in \Gamma_1 \quad \left|v_{1,\varepsilon}(y)\right| \leqslant C_4 \varepsilon^{-N(q)}\left(1+|y|\right)^{-q}. \tag{28}$$

For $v_{2,\varepsilon}$, note that $(u_{\varepsilon})_{\varepsilon} \in \mathcal{X}_{\mathcal{S}_*}(\mathbb{R}^d)$. Therefore, there exist $M > 0$ and $C_5 > 0$ such that $|\hat{u}_{\varepsilon}(\eta)| \leqslant C_5 \varepsilon^{-M}(1+|\eta|)^{-d-1}$ for ε small enough. As $\hat{\varphi} \in S(\mathbb{R}^d)$, there exists $C_6 > 0$ such that

$$\begin{aligned}&\forall (y,\eta) \in \Gamma_1 \times \mathbb{R}^d \text{ with } |y-\eta| \geqslant c|y|, \\ &\quad \left|\hat{\varphi}(y-\eta)\right| \leqslant C_6\left(1+|y-\eta|\right)^{-q} \leqslant C_6\left(1+c|y|\right)^{-q}.\end{aligned}$$

Then $|\hat{\varphi}(y-\eta)| = \mathrm{O}((1+|y|)^{-q})$ as $y \to +\infty$. Thus, there exists a constant $C_7 > 0$ such that

$$\forall y \in \Gamma_1, \quad \left|v_{2,\varepsilon}(y)\right| \leqslant C_7 \varepsilon^{-M}\left(1+|y|\right)^{-q}, \quad \text{for } \varepsilon \text{ small enough.} \tag{29}$$

From (28) and (29), we get that, for all $q \in \mathbb{N}$, there exists a constant $C > 0$ (depending on q) such that

$$\forall y \in \Gamma_1, \quad \left|\widehat{\varphi u}_{\varepsilon}(y)\right| \leqslant C\varepsilon^{-(N(q)+M)}\left(1+|y|\right)^{-q}.$$

Since $\mathcal{R}$ is overstable by translation, there exists a sequence $N'(\cdot) \in \mathcal{R}$ such that $N(\cdot) + M \leqslant N'(\cdot)$ and $\mu_{q,0}(\widehat{\varphi u}_{\varepsilon}) = \mathrm{O}(\varepsilon^{-N'(q)})$ as $\varepsilon \to 0$. Finally, $\widehat{\varphi u} = [(\widehat{\varphi u}_{\varepsilon})_{\varepsilon}]_{\mathcal{G}_{\mathcal{S}_*}} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma_1)$ and $\xi \in O^{\mathcal{R}}(\varphi u)$. $\square$

Definition 9. An element $u \in \mathcal{G}(\Omega)$ is said to be $\mathcal{R}$ microregular on $(x, \xi) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$ if there exist $\varphi \in \mathcal{D}_x(\Omega)$ and $\Gamma \in \mathcal{V}_\xi^\Gamma$, such that $\widehat{\varphi u} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$.

We set, for $u \in \mathcal{G}(\Omega)$ and $x \in \Omega$,

$$O_x^{\mathcal{R}}(u) = \bigcup_{\varphi \in \mathcal{D}_x} O^{\mathcal{R}}(\varphi u) = \bigl\{\xi \in \bigl(\mathbb{R}^d \setminus \{0\}\bigr) \bigm| u \text{ is microregular on } (x, \xi)\bigr\},$$

$$\Sigma_x^{\mathcal{R}}(u) = \bigcap_{\varphi \in \mathcal{D}_x} \Sigma^{\mathcal{R}}(\varphi u) = \bigl(\mathbb{R}^d \setminus \{0\}\bigr) \setminus O_x^{\mathcal{R}}(u).$$

Definition 10. For $u \in \mathcal{G}(\Omega)$ the set

$$WF_{\mathcal{R}}(u) = \bigl\{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \bigm| \xi \in \Sigma_x^{\mathcal{R}}(u)\bigr\}$$

is called the $\mathcal{R}$-wavefront of u.

Proposition 26. *For $u \in \mathcal{G}(\Omega)$, the projection on the first component of $WF_{\mathcal{R}}(u)$ is equal to* $\operatorname{sing\,supp}_{\mathcal{R}} u$.

The *proof* of this proposition follows the same lines as the one for the C^∞-wavefront of a distribution. First, for $u \in \mathcal{G}(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, φu, which is a priori in $\mathcal{G}_C(\Omega)$, can be straightforwardly considered as an element of $\mathcal{G}_C(\mathbb{R}^d)$. As $\mathcal{G}_C(\mathbb{R}^d)$ is included in $\mathcal{G}_{\mathcal{S}}(\mathbb{R}^d)$ (see Proposition 11), the Fourier transform of φu can be defined. (In the distributional case, that is, $u \in \mathcal{D}'(\Omega)$, φu is identified to an element of $\mathcal{E}'(\mathbb{R}^d)$.) From this, we can follow the arguments of [15, p. 253] for the C^∞-wavefront, which use mainly the compactness of the sphere S^{d-1} and Lemma 25, which holds in both cases; see [15, Lemma 8.1.1] for the distributional case.

Example 15. Taking $\mathcal{R} = \mathcal{B}$, the set of bounded sequences, we recover the $\mathcal{G}^\infty$-wavefront, which has here a definition independent of representatives.

Example 16. Taking $\mathcal{R} = \mathcal{R}_1$, we get a wavefront "containing" the distributional microlocal singularities of a generalized function, since $\mathcal{D}'(\cdot)$ is embedded in $\mathcal{G}^{(1)}(\cdot)$. This allows the study of distributional type singularities for a generalized functions.

In [18], it is shown that the analog of this lemma holds for the analytic singularities of a generalized function, giving rise to the corresponding wavefront set and the projection property of Proposition 26. Our future aim is to apply this theory to the propagation of singularities through integral generalized operators [3]. We also refer the reader to [9,12–14,19] and the literature therein for other presentations of the $\mathcal{G}^\infty$-wavefront (which is a particular case of $\mathcal{R}$-wavefront).

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