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A fast algorithm for Prandtl's integro-differential equation [1]

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Abstract

Collocation and quadrature methods for singular integro-differential equations of Prandtl's type are studied in weighted Sobolev spaces. A fast algorithm basing on the quadrature method is proposed. Convergence results and error estimates are given.

Keywords: Hypersingular integral equation; Weighted Sobolev spaces; Discrete sine function

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1. Introduction

In this paper we consider collocation and discrete collocation (quadrature) methods for solving a singular integro-differential equation of Prandtl's type

$$g(x)v(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{v'(t)}{t-x}\,\mathrm{d}t + \frac{1}{\pi}\int_{-1}^{1}h(x,t)v(t)\,\mathrm{d}t = f(x), \quad -1 < x < 1, \tag{1.1}$$

where the unknown function $v(x)$ has to fulfil the additional conditions

$$v(-1) = v(1) = 0. \tag{1.2}$$

Several authors have studied this type of integro-differential equations and related numerical methods. (Among others we refer the reader to [14, Ch. 3, Section I; 13, Section 3; 19, Section 9.53]). Since,

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for a function $v \in \mathbf{L}^p(-1,1)$ possessing a generalized derivative $v' \in \mathbf{L}^p(-1,1)$, we have (see [17, Ch. II, Lemma 6.1])

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\frac{v(t)}{t-x}\,\mathrm{d}t = \int_{-1}^{1}\frac{v'(t)}{t-x}\,\mathrm{d}t - \frac{v(-1)}{1+x} - \frac{v(1)}{1-x}$$

for all $x \in (-1,1)$, Eq. (1.1) together with (1.2) can be written in the form

$$g(x)v(x) - \frac{1}{\pi}\int_{-1}^{1}\frac{v(t)}{(t-x)^2}\,\mathrm{d}t + \frac{1}{\pi}\int_{-1}^{1}h(x,t)v(t)\,\mathrm{d}t = f(x), \quad -1 < x < 1, \tag{1.3}$$

where the hypersingular integral operator has to be understood in the sense of

$$\int_{-1}^{1}\frac{v(t)}{(t-x)^2}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-1}^{1}\frac{v(t)}{t-x}\,\mathrm{d}t. \tag{1.4}$$

Galerkin and collocation methods for Eq. (1.3) in case of $g(x) \equiv 0$ were considered in weighted $\mathbf{L}^2$-spaces in [10] and in case of $h(x,t) \equiv 0$ in [15]. Recently, in [9] convergence results in weighted Sobolev norms were given for the case $g(x) \equiv h(x,t) \equiv 0$.

The aim of the present paper is to prove optimal convergence rates for collocation and quadrature methods for Eq. (1.3) in weighted Sobolev norms and to found the idea of a fast algorithm for the numerical solution of (1.3) based on the quadrature method. Moreover, also the case of weakly singular perturbation kernels $h(x,t)$ is investigated. For this, following [9, 10, 13, 15, 19, 20], we recognize that the solution of (1.1) or (1.3) (together with (1.2)) has an endpoint behavior of the form $\sqrt{1-x^2}$. Thus, it is convenient to represent $v(x)$ as the product

$$v(x) = \varphi(x)u(x) \tag{1.5}$$

of the weight function $\varphi(x) = \sqrt{1-x^2}$ and another unknown function $u(x)$.

2. Notations and preliminaries

With this agreement we write Eq. (1.3) in the form

$$(M_\Gamma + V + H)u = f, \tag{2.1}$$

where M_Γ denotes the multiplication operator

$$(M_\Gamma u)(x) = \Gamma(x)u(x), \qquad \Gamma(x) = g(x)\varphi(x), \tag{2.2}$$

H the integral operator

$$(Hu)(x) = \frac{1}{\pi}\int_{-1}^{1}h(x,t)u(t)\varphi(t)\,\mathrm{d}t, \tag{2.3}$$

and $V = -DS$ the finite part integral operator with (comp. (1.4)) the operator $D = \mathrm{d}/\mathrm{d}x$ of generalized differentiation and the Cauchy singular integral operator

$$(Su)(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{t-x}\varphi(t)\,\mathrm{d}t. \tag{2.4}$$

For real numbers a and b with $a - \mathrm{i}\,b = e^{\mathrm{i}\,\pi\alpha}$, $0 < \alpha < 1$, define $\beta = 1 - \alpha$, the Jacobi weight function

$$v^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta},$$

and the singular integral operator of Cauchy type

$$(Au)(x) = a\,v^{\alpha,\beta}(x)u(x) + \frac{b}{\pi}\int_{-1}^{1}\frac{u(t)}{t-x}\,v^{\alpha,\beta}(t)\,\mathrm{d}t. \tag{2.5}$$

For $\gamma > -1$ and $\delta > -1$, let $\mathbf{L}^2_{\gamma,\delta}$ denote the weighted space of square integrable functions on the interval $[-1,1]$ endowed with the scalar product and the norm

$$\langle u,v\rangle_{\gamma,\delta} = \frac{1}{\pi}\int_{-1}^{1} u(x)\overline{v(x)}v^{\gamma,\delta}(x)\,\mathrm{d}x \quad \text{and} \quad \|u\|_{\gamma,\delta} = \sqrt{\langle u,u\rangle_{\gamma,\delta}},$$

respectively. Moreover, let $p_n^{\gamma,\delta}$ refer to as the normalized Jacobi polynomials (with positive leading coefficient) of degree n with respect to the Jacobi weight $v^{\gamma,\delta}(x)$. For real numbers $s \geqslant 0$ define the weighted Sobolev space $\mathbf{L}^{2,s}_{\gamma,\delta}$ by (comp. [5])

$$\mathbf{L}^{2,s}_{\gamma,\delta} = \left\{ u \in \mathbf{L}^2_{\gamma,\delta} :\ \sum_{n=0}^{\infty}(1+n)^{2s}\left|\langle u, p_n^{\gamma,\delta}\rangle_{\gamma,\delta}\right|^2 < \infty \right\}$$

with the norm

$$\|u\|_{\gamma,\delta,s} = \left(\sum_{n=0}^{\infty}(1+n)^{2s}\left|\langle u, p_n^{\gamma,\delta}\rangle_{\gamma,\delta}\right|^2\right)^{1/2}.$$

In the following we summarize some results concerning the properties of weighted Sobolev spaces, of interpolation operators with respect to the zeros of the orthogonal polynomials $p_n^{\gamma,\delta}$, and the singular integral operator A defined by (2.5). By $\mathcal{L}(\mathbf{X},\mathbf{Y})$ we will denote the Banach space of all bounded linear operators between the Banach spaces $\mathbf{X}$ and $\mathbf{Y}$.

Lemma 2.1 (Berthold et al. [5, Conclusion 2.3]). *For $0 \leqslant s < t$, the space $\mathbf{L}^{2,t}_{\gamma,\delta}$ is compactly imbedded in $\mathbf{L}^{2,s}_{\gamma,\delta}$.*

Lemma 2.2 (Berezanski [3, Ch. III, Section 6.9, Theorem 6.10], Junghanns [11, Remark 1.5]). *If the operator B belongs to $\mathcal{L}(\mathbf{L}^{2,s_1}_{\alpha_1,\beta_1},\mathbf{L}^{2,s_2}_{\alpha_2,\beta_2})$ and $\mathcal{L}(\mathbf{L}^{2,t_1}_{\alpha_1,\beta_1},\mathbf{L}^{2,t_2}_{\alpha_2,\beta_2})$ then $B \in \mathcal{L}(\mathbf{L}^{2,s(\tau)}_{\alpha_1,\beta_1},\mathbf{L}^{2,t(\tau)}_{\alpha_2,\beta_2})$, where $s(\tau) = (1-\tau)s_1 + \tau t_1$ and $t(\tau) = (1-\tau)s_2 + \tau t_2$, $0 \leqslant \tau \leqslant 1$.*

Lemma 2.3 (Berthold et al. [5, pp. 196,197]). *Let $r \geqslant 0$ be an integer. Then $u \in \mathbf{L}^{2,r}_{\gamma,\delta}$ if and only if $u^{(k)}\varphi^k$ belongs to $\mathbf{L}^2_{\gamma,\delta}$ for all $k = 0,\dots,r$. Moreover, the norms $\|u\|_{\gamma,\delta,r}$ and*

$$\|u\|_{\gamma,\delta,r,\varphi} = \sum_{k=0}^{r}\|u^{(k)}\varphi^k\|_{\gamma,\delta}$$

are equivalent.

Let $x_{nk}^{\gamma,\delta}$ with $-1 < x_{nn}^{\gamma,\delta} < \cdots < x_{n1}^{\gamma,\delta}$ be the zeros of $p_n^{\gamma,\delta}$ and denote by $L_n^{\gamma,\delta}$ the Lagrange interpolation operator

$$L_n^{\gamma,\delta} f = \sum_{k=1}^{n} f(x_{nk}^{\gamma,\delta})\, l_{nk}^{\gamma,\delta}, \qquad l_{nk}^{\gamma,\delta}(x) = \prod_{j=1,j\neq k}^{n} \frac{x - x_{nj}^{\gamma,\delta}}{x_{nk}^{\gamma,\delta} - x_{nj}^{\gamma,\delta}}.$$

Lemma 2.4. *For $s > \frac{1}{2}$ we have*
(a) $\lim_{n\to\infty} \|f - L_n^{\gamma,\delta} f\|_{\gamma,\delta,s} = 0$ *for all* $f \in \mathbf{L}_{\gamma,\delta}^{2,s}$,
(b) $\|f - L_n^{\gamma,\delta} f\|_{\gamma,\delta,t} \leqslant \text{const } n^{t-s} \|f\|_{\gamma,\delta,s}$, *if* $0 \leqslant t \leqslant s$.

Proof. This lemma was proved in [5, Theorem 3.4] in case of $|\gamma| = |\delta| = \frac{1}{2}$. For the case $s \geqslant 1$, $\gamma, \delta > -1$ arbitrary, the proof is given in [7, Theorem 2.3]. The general case is considered in [16]. □

Lemma 2.5 (Prössdorf and Silbermann [19, Theorems 9.9 and 9.14, Remark 9.15]). *For the singular integral operator A defined in* (2.5) *we have the relation*

$$A p_n^{\alpha,\beta} = A p_{n+1}^{-\alpha,-\beta}, \quad n = 0, 1, 2, \ldots$$

The following corollary is an immediate consequence of the previous lemma.

Corollary 2.6 (Berthold et al. [5, Lemma 4.1]). *For all $s \geqslant 0$, the singular integral operator A belongs to $\mathcal{L}(\mathbf{L}_{\alpha,\beta}^{2,s}, \mathbf{L}_{-\alpha,-\beta}^{2,s})$. Moreover, $A : \mathbf{L}_{\alpha,\beta}^{2,s} \to \mathbf{L}_{-\alpha,-\beta}^{2,s,0}$ is a bijection, where*

$$\mathbf{L}_{\gamma,\delta}^{2,s,0} = \left\{ f \in \mathbf{L}_{\gamma,\delta}^{2,s} : \langle f, p_0^{\gamma,\delta} \rangle_{\gamma,\delta} = 0 \right\},$$

and the inverse operator is given by

$$A^{-1} = \widehat{A}, \quad (\widehat{A} f)(t) := a\, v^{-\alpha,-\beta}(t) f(t) - \frac{b}{\pi} \int_{-1}^{1} \frac{f(x)}{x - t}\, v^{-\alpha,-\beta}(x)\, \mathrm{d}x.$$

Lemma 2.7. *For all $s \geqslant 0$ and $\gamma, \delta > -1$, the operator* D *of generalized differentiation is a continuous isomorphism from $\mathbf{L}_{\gamma,\delta}^{2,s+1,0}$ onto $\mathbf{L}_{1+\gamma,1+\delta}^{2,s}$.*

Proof. First of all we remark that, in view of Lemma 2.3, the operator D is defined on all functions of $\mathbf{L}_{\gamma,\delta}^{2,s+1}$. Further, we use the relation

$$v^{\gamma,\delta}(x) p_n^{\gamma,\delta}(x) = -[n(n+\gamma+\delta+1)]^{-1/2} \frac{\mathrm{d}}{\mathrm{d}x} \left[v^{1+\gamma,1+\delta}(x) p_{n-1}^{1+\gamma,1+\delta}(x) \right], \quad n = 1, 2, \ldots,$$

which follows from [22, Eqs. (4.10.1) and (4.3.4)]. Now, with the notation $\langle .,. \rangle = \langle .,. \rangle_{0,0}$, we have

$$\|\mathrm{D}u\|^2_{1+\gamma,1+\delta,s} = \sum_{n=1}^{\infty} n^{2s} \left| \langle \mathrm{D}u, p_{n-1}^{1+\gamma,1+\delta} v^{1+\gamma,1+\delta} \rangle \right|^2$$
$$= \sum_{n=1}^{\infty} n^{2s} n(n+\gamma+\delta+1) \left| \langle u, p_n^{\gamma,\delta} v^{\gamma,\delta} \rangle \right|^2,$$

which is equivalent to $\|u\|^2_{\gamma,\delta,s+1}$ for $u \in \mathbf{L}^{2,s+1}_{\gamma,\delta,0}$. Thus, the lemma is proved. □

As a generalization of the finite part integral operator $V = -\mathrm{D}S$ we consider the operator $\mathrm{D}A$, which can be written in the form

$$(\mathrm{D}Au)(x) = a \frac{\mathrm{d}}{\mathrm{d}x}[v^{\alpha,\beta}(x)u(x)] + \frac{b}{\pi} \int_{-1}^{1} \frac{u(t)}{(t-x)^2} v^{\alpha,\beta}(t)\,\mathrm{d}t.$$

The following corollary is a consequence of Lemmas 2.5 and 2.7, Corollary 2.6, and the relation (see [22, Eq. (4.21.7)])

$$\frac{\mathrm{d}}{\mathrm{d}x} p_n^{\gamma,\delta}(x) = \sqrt{n(n+\gamma+\delta+1)}\, p_{n-1}^{1+\gamma,1+\delta}(x), \quad n = 1,2,\ldots$$

One remember that $\alpha + \beta = 1$.

Corollary 2.8. *For each $s \geqslant 0$, the finite part integral operator* DA *is a continuous isomorphism between the spaces $\mathbf{L}^{2,s+1}_{\alpha,\beta}$ and $\mathbf{L}^{2,s}_{\beta,\alpha}$. Moreover, for $u \in \mathbf{L}^{2,s+1}_{\alpha,\beta}$,*

$$\mathrm{D}Au = \sum_{n=0}^{\infty} (n+1) \langle u, p_n^{\alpha,\beta} \rangle_{\alpha,\beta}\, p_n^{\beta,\alpha}. \tag{2.6}$$

Remark 2.9 (Ervin and Stephan [9, Theorem 1]). In the special case $a=0$, $b=-1$ (i.e., $\alpha=\beta=\frac{1}{2}$), Corollary 2.8 implies $V \in \mathcal{L}(\mathbf{L}^{2,s+1}_{\varphi}, \mathbf{L}^{2,s}_{\varphi})$ and

$$Vu = -\mathrm{D}Su = \sum_{n=0}^{\infty} (n+1) \langle u, p_n^{\varphi} \rangle_{\varphi}\, p_n^{\varphi},$$

where we use the notations $\mathbf{L}^{2,s}_{\varphi} = \mathbf{L}^{2,s}_{1/2,1/2}$, $\langle .,. \rangle_{\varphi} = \langle .,. \rangle_{1/2,1/2}$, and $p_n^{\varphi} = p_n^{1/2,1/2}$.

For the kernel $h(x,t)$ of the integral operator (comp. (2.3))

$$(Hu)(x) = \frac{1}{\pi} \int_{-1}^{1} h(x,t)u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t \tag{2.7}$$

we consider three cases:

(a) $h(x,t) = h_1(x,t)$,

$$(H_1 u)(x) = \frac{1}{\pi} \int_{-1}^{1} h_1(x,t)u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t, \tag{2.8}$$

(b) $h(x,t) = h_2(x,t)\ln|x-t|$,

$$(H_2u)(x) = \frac{1}{\pi}\int_{-1}^{1} h_2(x,t)\ln|x-t|u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t\,, \tag{2.9}$$

(c) $h(x,t) = h_3(x,t)|x-t|^{-\eta}$, $0<\eta<1$,

$$(H_3u)(x) = \frac{1}{\pi}\int_{-1}^{1} h_3(x,t)|x-t|^{-\eta}u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t.$$

We assume that the functions h_j are continuous on $[-1,1]^2$, and in what follows we summarize some mapping properties of these operators.

Lemma 2.10 (Berthold et al. [5, Lemma 4.2]). *If $h_1(.,t)\in \mathbf{L}^{2,s}_{\gamma,\delta}$ uniformly w.r.t. $t\in[-1,1]$ then $H_1\in\mathcal{L}(\mathbf{L}^2_{\alpha,\beta},\mathbf{L}^{2,s}_{\gamma,\delta})$.*

The following lemma is needed to study the case (b). By $\mathbf{C}^r_\varphi$, $r\geqslant 0$ an integer, we denote the space of all r times differentiable functions $u:(-1,1)\to\mathbb{C}$ satisfying the conditions $u^{(k)}\varphi^k\in\mathbf{C}[-1,1]$ for $k=0,1,\ldots,r$. Let $\|u\|_{\mathbf{C}^r_\varphi} = \sum_{k=0}^r \|u^{(k)}\varphi^k\|_\infty$.

Lemma 2.11 (Junghanns [11, Lemma 3.5]). *Let $r\geqslant 0$ be an integer and $\Gamma\in\mathbf{C}^r_\varphi$. Then the multiplication operator M_Γ belongs to $\mathcal{L}(\mathbf{L}^{2,r}_{\gamma,\delta},\mathbf{L}^{2,r}_{\gamma,\delta})$ and $\|M_\Gamma\|_{\mathbf{L}^{2,r}_{\gamma,\delta}\to\mathbf{L}^{2,r}_{\gamma,\delta}} \leqslant$ const $\|\Gamma\|_{\mathbf{C}^r_\varphi}$.*

Corollary 2.12. *Taking into account Lemma 2 and (under the assumptions of Lemma 2.11) $M_\Gamma\in\mathcal{L}(\mathbf{L}^2_{\gamma,\delta},\mathbf{L}^2_{\gamma,\delta})$ the condition $\Gamma\in\mathbf{C}^r_\varphi$ implies $M_\Gamma\in\mathcal{L}(\mathbf{L}^{2,s}_{\gamma,\delta},\mathbf{L}^{2,s}_{\gamma,\delta})$ for $0\leqslant s\leqslant r$.*

For a given continuous function $h_2:[-1,1]^2\to\mathbb{C}$ define

$$(\widetilde{H}_2u)(x) = a\int_{-1}^{x} h_2(x,t)u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t - \frac{b}{\pi}\int_{-1}^{1} h_2(x,t)\ln|x-t|u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t.$$

By h'_x we shall denote the partial derivative $\frac{\partial h}{\partial x}$ of a function $h(x,t)$.

Lemma 2.13. *Let $h_2, h'_{2x}:[-1,1]^2\to\mathbb{C}$ possess continuous partial derivatives up to order r. Then $\widetilde{H}_2\in\mathcal{L}(\mathbf{L}^{2,s}_{\alpha,\beta},\mathbf{L}^{2,s+1}_{-\alpha,-\beta})$ for $0\leqslant s\leqslant r$.*

Proof. The proof goes on the same lines as the proofs of [11, Lemmas 3.6 and 3.7], We have

$$\mathrm{D}\widetilde{H}_2 = \widetilde{H}_2^{(1)} + T_1 + AM_\Phi\,, \tag{2.10}$$

where

$$(\widetilde{H}_2^{(1)}u)(x) = a\int_{-1}^{x} h'_{2x}(x,t)u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t - \frac{b}{\pi}\int_{-1}^{1} h'_{2x}(x,t)\,\ln|x-t|u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t,$$

$$(T_1u)(x) = \frac{b}{\pi}\int_{-1}^{1}\tilde{h}_2(x,t)u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t,$$

$$\widetilde{h}_2(x,t) = \frac{h_2(x,t) - h_2(t,t)}{t-x}, \quad \Phi(t) = h_2(t,t).$$

Since $h'_{2x}, \widetilde{h}_2 : [-1,1]^2 \to \mathbb{C}$ and $\Phi : [-1,1] \to \mathbb{C}$ are continuous functions it follows that

$$\left\|\varphi \mathrm{D}\widetilde{H}_2 u\right\|_{-\alpha,-\beta} \leqslant \left\|\mathrm{D}\widetilde{H}_2 u\right\|_{-\alpha,-\beta} \leqslant \text{const } \|u\|_{\alpha,\beta},$$

taking into account Corollary 2.6. In view of $\widetilde{H}_2 \in \mathcal{L}(\mathbf{L}^2_{\alpha,\beta}, \mathbf{L}^2_{-\alpha,-\beta})$ and Lemma 2.3 this implies $\widetilde{H}_2 \in \mathcal{L}(\mathbf{L}^2_{\alpha,\beta}, \mathbf{L}^{2,1}_{-\alpha,-\beta})$. Thus, having regard to Lemma 2, the lemma is true for $r = 0$. Assume that the assertion of the lemma is valid for $r \leqslant m$ and that the conditions of the lemma are fulfilled for $r = m+1$. Then (2.10) holds true with $\widetilde{H}_2^{(1)}$, $T_1 \in \mathcal{L}(\mathbf{L}^{2,m}_{\alpha,\beta}, \mathbf{L}^{2,m+1}_{-\alpha,-\beta})$. (For the operator T_1 we refer to Lemmas 2.10 and 2.3.) Moreover, in view of Lemma 2.11, $M_\Phi \in \mathcal{L}(\mathbf{L}^{2,m+1}_{\alpha,\beta}, \mathbf{L}^{2,m+1}_{\alpha,\beta})$. Applying Lemma 2.11 together with Corollary 2.6 we obtain, for $u \in \mathbf{L}^{2,m+1}_{\alpha,\beta}$,

$$\begin{aligned}
\left\|\widetilde{H}_2 u\right\|_{-\alpha,-\beta,m+2} &\leqslant \text{const } \left(\left\|\widetilde{H}_2 u\right\|_{-\alpha,-\beta,m+1} + \left\|\varphi^{m+2}\mathrm{D}^{m+2}\widetilde{H}_2 u\right\|_{-\alpha,-\beta}\right) \\
&\leqslant \text{const } \left(\|u\|_{\alpha,\beta,m} + \left\|\varphi^{m+1}\mathrm{D}^{m+1}(\widetilde{H}_2^{(1)} + T_1 + AM_\Phi)u\right\|_{-\alpha,-\beta}\right) \\
&\leqslant \text{const } \left(\|u\|_{\alpha,\beta,m} + \left\|(\widetilde{H}_2^{(1)} + T_1 + AM_\Phi)u\right\|_{-\alpha,-\beta,m+1}\right) \\
&\leqslant \text{const } \|u\|_{\alpha,\beta,m+1}.
\end{aligned}$$

This proves the lemma by induction. □

Since the space $\mathbf{L}^{2,s+1}_{-1/2,-1/2}$ is continuously embedded into $\mathbf{L}^{2,s+1}_{\varphi}$ (one can see this by using an equivalent norm in $\mathbf{L}^{2,s}_{\gamma,\delta}$ of the form (2.11) below), Lemma 2.13 implies the following corollary.

Corollary 2.14. *If the function $h_2(x,t)$ satisfies the conditions of Lemma* 2.13, *then the operator H_2 with $\alpha = \beta = \frac{1}{2}$* (i.e. $a = 0$, $b = -1$) *belongs to $\mathcal{L}(\mathbf{L}^{2,s}_{\varphi}, \mathbf{L}^{2,s+1}_{\varphi})$ for $0 \leqslant s \leqslant r$.*

To study the case (c) we introduce the following weighted spaces of continuous functions. Let ρ and τ be nonnegative real numbers. By $\mathbf{C}_{\rho,\tau}$ we denote the Banach space of all continuous functions $u\colon (-1,1) \to \mathbb{C}$, for which $v^{\rho,\tau}u$ is continuous on $[-1,1]$, equipped with the norm $\|u\|_{\infty,\rho,\tau} = \sup\{v^{\rho,\tau}(x)|u(x)| : x \in [-1,1]\}$. Let $\mathbb{P}_n$ be the set of algebraic polynomials of degree not greater than n. For $f \in \mathbf{C}_{\rho,\tau}$ we denote by $E_n^{\rho,\tau}(f)$ the best weighted uniform approximation of f by polynomials belonging to $\mathbb{P}_n$, i.e.,

$$E_n^{\rho,\tau}(f) = \inf\{\|f - p\|_{\infty,\rho,\tau} : p \in \mathbb{P}_n\}.$$

Let $\chi > 0$ and $q \geqslant 0$ be real numbers. The subspace

$$\mathbf{C}^{\chi,q}_{\rho,\tau} = \left\{u \in \mathbf{C}_{\rho,\tau} : \sup_{n=1,2,\ldots} \frac{n^\chi E_n^{\rho,\tau}(u)}{\ln^q(n+1)} < \infty\right\}$$

equipped with the norm

$$\|u\|_{\rho,\tau,\chi,q} = \|u\|_{\infty,\rho,\tau} + \sup_{n=1,2,\dots} \frac{n^\chi E_n^{\rho,\tau}(u)}{\ln^q(n+1)}$$

becomes a Banach space (see [12, Proposition 3.1]). We remark that $\mathbf{C}_{\rho,\tau}^{\chi,0}$ coincides with the weighted Besov space $\mathbf{B}_{\chi,\infty}^{\infty}(\varphi, v^{\rho,\tau})$ (see [8, Theorem 3.1]). The following lemma is a generalization of [18, Theorem 7], (also mentioned in [18, Remark 8]).

Lemma 2.15. *For $s > \frac{1}{2}$, the space $\mathbf{L}_{\gamma,\delta}^{2,s}$ is continuously embedded into the space $\mathbf{C}_{\tilde{\gamma},\tilde{\delta}}$, where*

$$\tilde{\gamma} = \tfrac{1}{2}\max\{0, \gamma + \tfrac{1}{2}\} \quad \text{and} \quad \tilde{\delta} = \tfrac{1}{2}\max\{0, \delta + \tfrac{1}{2}\}.$$

Proof. The proof follows the proof of [18, Theorem 7] (case $\gamma = -\frac{1}{2}$, $\delta = \frac{1}{2}$). Let $u \in \mathbf{L}_{\gamma,\delta}^{2,s}$, $s > \frac{1}{2}$, and

$$\tilde{u}(x) = v^{\tilde{\gamma},\tilde{\delta}}(x)u(x), \quad \tilde{p}_n^{\gamma,\delta}(x) = v^{\tilde{\gamma},\tilde{\delta}}(x)p_n^{\gamma,\delta}(x).$$

Moreover, define

$$\gamma_1 = \gamma - 2\tilde{\gamma} = \gamma - \max\{0, \gamma + \tfrac{1}{2}\}, \qquad \delta_1 = \delta - 2\tilde{\delta} = \delta - \max\{0, \delta + \tfrac{1}{2}\}.$$

Then

$$\langle \tilde{p}_n^{\gamma,\delta}, \tilde{p}_m^{\gamma,\delta}\rangle_{\gamma_1,\delta_1} = \langle p_n^{\gamma,\delta}, p_m^{\gamma,\delta}\rangle_{\gamma,\delta} = \delta_{mn} \qquad \text{and} \qquad \langle \tilde{u}, \tilde{p}_n^{\gamma,\delta}\rangle_{\gamma_1,\delta_1} = \langle u, p_n^{\gamma,\delta}\rangle_{\gamma,\delta}.$$

Since (see [2, Theorem 1.1])

$$\left|p_n^{\gamma,\delta}(x)\right| \left(\sqrt{1-x} + \frac{1}{n}\right)^{\gamma+1/2} \left(\sqrt{1+x} + \frac{1}{n}\right)^{\delta+1/2} \leqslant \text{const}$$

we have $\tilde{p}_n^{\gamma,\delta}(x) \leqslant \text{const}$ and, consequently,

$$\sum_{n=0}^{\infty} \left|\langle \tilde{u}, \tilde{p}_n^{\gamma,\delta}\rangle_{\gamma_1,\delta_1} \tilde{p}_n^{\gamma,\delta}(x)\right| \leqslant \text{const} \sum_{n=0}^{\infty} \left|\langle \tilde{u}, \tilde{p}_n^{\gamma,\delta}\rangle_{\gamma_1,\delta_1}\right|$$

$$\leqslant \text{const}\ \|u\|_{\gamma,\delta,s} \sqrt{\sum_{n=0}^{\infty}(n+1)^{-2s}} \leqslant \text{const}\ \|u\|_{\gamma,\delta,s}.$$

Thus, the Fourier series of the function $\tilde{u}$ with respect to the orthonormal system $\{\tilde{p}_n^{\gamma,\delta}\}_{n=0}^{\infty}$ in $\mathbf{L}_{\gamma_1,\delta_1}^2$ converges uniformly on $[-1,1]$, which implies that $\tilde{u} : [-1,1] \to \mathbb{C}$ is continuous. Moreover, the last estimate shows that $\|u\|_{\infty,\tilde{\gamma},\tilde{\delta}} \leqslant \text{const}\ \|u\|_{\gamma,\delta,s}$. □

Lemma 2.16 (Junghanns and Luther [12, Proposition 4.13]). *Let* $h(x,t) = (k(x,t) - k(t,t))/(x-t)$ *with* $k : [-1,1]^2 \to \mathbb{C}$ *continuous and* $k(.,t) \in \mathbf{C}_{0,0}^{\chi,q}$ *uniformly w.r.t.* $t \in [-1,1]$. *Then the operator* H *defined by* (2.7) *belongs to* $\mathcal{L}(\mathbf{C}_{\alpha^+,\beta^+}, \mathbf{C}_{\alpha^-,\beta^-}^{\chi,q+1})$. *Here we use the notations* $\alpha^{\pm} = \max\{0, \pm\alpha\}$ *and* $\beta^{\pm} = \max\{0, \pm\beta\}$.

For a real number $\chi > 0$ and an integer $r \geqslant 0$, let $\mathbf{C}^{r,\chi}$ denote the space of r times continuously differentiable functions on $[-1,1]$, whose rth derivative is Hölder continuous with exponent χ.

Lemma 2.17. *If* $h_3(.,t) \in \mathbf{C}^{0,1-\eta}$ *uniformly w.r.t.* $t \in [-1,1]$, *then the operator* H_3 *is a compact operator from* $\mathbf{L}_{\gamma,\delta}^{2,s}$ *into* $\mathbf{L}_{\gamma',\delta'}^{2,\eta'}$ *for* $s > \frac{1}{2}$, $\gamma = 2\alpha^+ - \frac{1}{2}$, $\delta = 2\beta^+ - \frac{1}{2}$, $\gamma' > 2\alpha^- - 1$, $\delta' > 2\beta^- - 1$, *and* $0 \leqslant \eta' < 1 - \eta$.

Proof. Since $h_3(x,t) = (k(x,t) - k(t,t))/(x-t)$, where $k(x,t) = h_3(x,t)(x-t)|x-t|^{-\eta}$ and $k(.,t)$ belongs to $\mathbf{C}^{0,1-\eta}$ uniformly w.r.t. $t \in [-1,1]$, we conclude from Lemma 2.16 that $H_3 \in \mathcal{L}(\mathbf{C}_{\alpha^+,\beta^+}, \mathbf{C}_{\alpha^-,\beta^-}^{1-\eta,1})$. From Lemma 2.15 we have the continuous embedding of $\mathbf{L}_{\gamma,\delta}^{2,s}$ into $\mathbf{C}_{\alpha^+,\beta^+}$. Let $\varepsilon > 0$ such that $\eta' + \varepsilon < 1 - \eta$. Then the space $\mathbf{C}_{\alpha^-,\beta^-}^{1-\eta,1}$ is compactly embedded into the space $\mathbf{C}_{\alpha^-,\beta^-}^{\eta'+\varepsilon,0}$ ([12, Lemma 3.2]). So it remains to show that the last space is continuously embedded into $\mathbf{L}_{\gamma',\delta'}^{2,\eta'}$. But this is a consequence of the equivalence of the norm in $\mathbf{L}_{\gamma',\delta'}^{2,\eta'}$ and the norm

$$\sqrt{\|u\|_{\gamma',\delta'} + \sum_{n=1}^{\infty} n^{2\eta'-1} \left[E_n^{\gamma',\delta'}(u)_2\right]^2} \tag{2.11}$$

as well as of

$$\|u\|_{\gamma',\delta'} \leqslant \text{const } \|u\|_{\infty,\alpha^-,\beta^-}, \qquad E_n^{\gamma',\delta'}(u)_2 \leqslant \text{const } E_n^{\alpha^-,\beta^-}(u),$$

where

$$E_n^{\gamma',\delta'}(u)_2 = \inf\{\|u - p\|_{\gamma',\delta'} : p \in \mathbb{P}_n\}.$$

Indeed, for $u \in \mathbf{C}_{\alpha^-,\beta^-}^{\eta'+\varepsilon,0}$, we can estimate

$$\begin{aligned}\|u\|_{\gamma',\delta',\eta'} &\leqslant \text{const } \sqrt{\|u\|_{\infty,\alpha^-,\beta^-}^2 + \left[\sup_{n=1,2,\ldots} n^{\eta'+\varepsilon} E_n^{\alpha^-,\beta^-}(u)\right]^2 \sum_{n=1}^{\infty} n^{-2\varepsilon-1}} \\ &\leqslant \text{const } \|u\|_{\alpha^-,\beta^-,\eta'+\varepsilon,0}.\end{aligned}$$

This proves the lemma. □

Corollary 2.18. *Let* $h_3(x,t)$ *satisfy the conditions of Lemma* 2.17 *and* $\alpha = \beta = \frac{1}{2}$. *Then the operator* $H_3 : \mathbf{L}_{\varphi}^{2,s} \to \mathbf{L}_{\varphi}^{2,\eta'}$ *is compact for* $s > \frac{1}{2}$ *and* $0 \leqslant \eta' < 1 - \eta$.

3. Collocation and quadrature methods

Instead of Eq. (2.1) we investigate the more general equation

$$Bu := (M_\Psi + DA + H_1 + \widetilde{H}_2 + H_3)u = f, \tag{3.1}$$

which we consider in the pair of spaces

$$(\mathbf{L}^{2,s+1}_{\alpha,\beta}, \mathbf{L}^{2,s}_{\beta,\alpha}). \tag{3.2}$$

The collocation method consists in looking for an approximate solution $u_n \in \mathbb{P}_{n-1}$ of Eq. (3.1) by solving the equation

$$L_n^{\beta,\alpha}(M_\Psi + DA + H_1 + \widetilde{H}_2 + H_3)u_n = L_n^{\beta,\alpha} f.$$

In view of relation (2.6) this equation is equivalent to

$$B_n u_n := [DA + L_n^{\beta,\alpha}(M_\Psi + H_1 + \widetilde{H}_2 + H_3)]u_n = L_n^{\beta,\alpha} f. \tag{3.3}$$

Since, again in view of Corollary 2.8, each solution $u_n \in \mathbf{L}^{2,s+1}_{\alpha,\beta}$ of (3.3) belongs to $\mathbb{P}_{n-1}$, we can also consider Eq. (3.3) in the pair of spaces (3.2).

For all what follows we assume:

(A0) For $f \equiv 0$ Eq. (3.1) has only the trivial solution $u \equiv 0$ in $\mathbf{L}^{2,1}_{\alpha,\beta}$.

With respect to the continuous functions $\Psi(x)$ and $h_j(x,t)$, $j = 1,2,3$, we make the following assumptions:

(A1) $\Psi \in \mathbf{C}^r_\varphi$ for some integer $r \geqslant 0$.

(A2) $h_1(.,t) \in \mathbf{L}^{2,\tilde{s}}_{\beta,\alpha}$ uniformly w.r.t. $t \in [-1,1]$.

(A3) h_2 and h'_{2x} possess continuous partial derivatives up to order $\tilde{r}$.

(A4) $h_3(.,t) \in \mathbf{C}^{0,1-\eta}$ uniformly w.r.t. $t \in [-1,1]$.

In all cases, which we will consider, the operators H_1, $\widetilde{H}_2$, and H_3 as well as M_Ψ are compact in the pair of spaces (3.2) as well as in the pair $(\mathbf{L}^{2,1}_{\alpha,\beta}, \mathbf{L}^{2}_{\beta,\alpha})$. This will be a consequence of the Lemmata 2.10, 2.13, 2.17, and Corollary 2.12 as well as Lemma 2. Then, in view of Corollary 2.8 and Lemma 2, the operator $B : \mathbf{L}^{2,t+1}_{\alpha,\beta} \to \mathbf{L}^{2,t}_{\beta,\alpha}$ is invertible for $0 \leqslant t \leqslant s$ and Eq. (3.1) possesses a unique solution $u^* \in \mathbf{L}^{2,s+1}_{\alpha,\beta}$.

Theorem 3.1. *Let* $s > \frac{1}{2}$, $\Psi \equiv 0$, $h_3 \equiv 0$, $f \in \mathbf{L}^{2,s}_{\beta,\alpha}$, *Assume* (A0), (A2) *and* (A3) *be fulfilled for* $\tilde{s} = s$ *and* $\tilde{r} \geqslant \max\{0, s-1\}$. *Then, for all sufficiently large n, Eq.* (3.3) *is uniquely solvable, and the solution* u_n^* *converges to the unique solution* u^* *of* (3.1) *in the norm of* $\mathbf{L}^{2,s+1}_{\alpha,\beta}$. *Moreover, for* $0 \leqslant t \leqslant s$,

$$\|u_n^* - u^*\|_{\alpha,\beta,t+1} \leqslant \text{const } n^{t-s} \|u^*\|_{\alpha,\beta,s+1}. \tag{3.4}$$

Proof. Since $H_1 + \widetilde{H}_2 : \mathbf{L}^{2,s+1}_{\alpha,\beta} \to \mathbf{L}^{2,s}_{\beta,\alpha}$ is compact, it follows that

$$\lim_{n\to\infty} \|B_n - B\|_{\mathbf{L}^{2,s+1}_{\alpha,\beta} \to \mathbf{L}^{2,s}_{\beta,\alpha}} = 0 \tag{3.5}$$

taking into account Lemma 2.4. With the help of Lemmata 2.10, 2.13, and 2.4 (b) as well as the continuous embedding $\mathbf{L}^{2,1}_{-\alpha,-\beta} \subset \mathbf{L}^{2,1}_{\beta,\alpha}$ we obtain

$$\|B_n - B\|_{\mathbf{L}^{2,1}_{\alpha,\beta} \to \mathbf{L}^{2}_{\beta,\alpha}} \leqslant \text{const } n^{-\tilde{t}},$$

where $\tilde{t} = \min\{s, 1\}$. This, (3.5), and Lemma 2 imply the existence and uniform boundedness of $B_n^{-1} \in \mathcal{L}(\mathbf{L}^{2,t}_{\beta,\alpha}, \mathbf{L}^{2,t+1}_{\alpha,\beta})$ for $0 \leqslant t \leqslant s$ and for all sufficiently large n. Consequently, in view of

$$u_n^* - u^* = B_n^{-1}\left[L_n^{\beta,\alpha} f - f + \left(I - L_n^{\beta,\alpha}\right)(H_1 + \tilde{H}_2)u^*\right],$$

we have with the help of Lemma 2.4 (b)

$$\|u_n^* - u^*\|_{\alpha,\beta,t+1} \leqslant \text{const } n^{t-s}\left(\|f\|_{\beta,\alpha,s} + \left\|(H_1 + \tilde{H}_2)u^*\right\|_{\beta,\alpha,s}\right),$$

which leads to (3.4). □

Theorem 3.2. *In case $\Psi \not\equiv 0$ and $\alpha = \beta = \frac{1}{2}$ (i.e., $a = 0$, $b = -1$) Theorem* 3.1 *remains true if we additionally assume that* (A1) *with $r \geqslant s$ is fulfilled.*

Proof. We remark that in this case $\mathbf{L}^{2,s}_{\alpha,\beta} = \mathbf{L}^{2,s}_{\beta,\alpha} = \mathbf{L}^{2,s}_{\varphi}$ and Corollary 2.12 applies to see the compactness of $M_\Psi : \mathbf{L}^{2,t+1}_{\varphi} \to \mathbf{L}^{2,t}_{\varphi}$ for $0 \leqslant t \leqslant s$. Moreover, since $r \geqslant 1$, $\|(M_\Psi - L_n^\varphi M_\Psi)u\|_\varphi \leqslant \text{const } n^{-1}\|u\|_{\varphi,1}$ for $u \in \mathbf{L}^{2,1}_{\varphi}$. □

Remark 3.3. In case $\Psi \not\equiv 0$, $h_3 \not\equiv 0$ and $\alpha = \beta = \frac{1}{2}$ (i.e., $a = 0$, $b = -1$) Theorem 3.1 remains true for $\frac{1}{2} < s < 1 - \eta$, if we additionally assume that $0 < \eta < \frac{1}{2}$ and (A1) with $r \geqslant s$ as well as (A4) are fulfilled.

Proof. For the proof, at first, we refer to Corollary 2.18. Furthermore, if we apply Lemma 2.17 with $\eta' = s$ and $\alpha^+ = \beta^+ = \frac{1}{2}$, it follows $\|H_3 - L_n^\varphi H_3\|_{\mathbf{L}^{2,1}_\varphi \to \mathbf{L}^2_\varphi} \leqslant \text{const } n^{-s}$. □

With the help of $Q_n^{\gamma,\delta}$ we will denote the application of the Gaussian rule with respect to the Jacobi weight $v^{\gamma,\delta}$, which means

$$\frac{1}{\pi}\int_{-1}^{1} u(t)v^{\gamma,\delta}(t)\,\mathrm{d}t \approx Q_n^{\gamma,\delta}(u) := \sum_{k=1}^{n} \lambda_{nk}^{\gamma,\delta} u(x_{nk}^{\gamma,\delta}) \tag{3.6}$$

with

$$\lambda_{nk}^{\gamma,\delta} = \frac{1}{\pi}\int_{-1}^{1} l_{nk}^{\gamma,\delta}(t)v^{\gamma,\delta}(t)\,\mathrm{d}t.$$

Now, we can approximate the operator H_1 by

$$(H_{1n}u)(x) = \sum_{k=1}^{n} \lambda_{nk}^{\alpha,\beta} h(x, x_{nk}^{\alpha,\beta}) u(x_{nk}^{\alpha,\beta}). \tag{3.7}$$

To approximate the operators $\widetilde{H}_2$ and H_3 we use product integration rules of the following kind:

$$a\int_{-1}^{x} u(t)v^{\alpha,\beta}(t)\,\mathrm{d}t - \frac{b}{\pi}\int_{-1}^{1} u(t)\ \ln|x-t|v^{\alpha,\beta}(t)\,\mathrm{d}t \approx \sum_{k=1}^{n}\widetilde{\lambda}_{nk}^{\alpha,\beta}(x)u(x_{nk}^{\alpha,\beta}), \tag{3.8}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{u(t)}{|t-x|^{\eta}}v^{\gamma,\delta}(t)\,\mathrm{d}t \approx \sum_{k=1}^{n}\omega_{nk}^{\gamma,\delta}(x)u(x_{nk}^{\gamma,\delta}), \tag{3.9}$$

where

$$\widetilde{\lambda}_{nk}^{\alpha,\beta}(x) = a\int_{-1}^{x} l_{nk}^{\alpha,\beta}(t)v^{\alpha,\beta}(t)\,\mathrm{d}t - \frac{b}{\pi}\int_{-1}^{1} l_{nk}^{\alpha,\beta}(t)\ \ln|x-t|v^{\alpha,\beta}(t)\,\mathrm{d}t,$$

$$\omega_{nk}^{\gamma,\delta}(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{l_{nk}^{\gamma,\delta}(t)}{|t-x|^{\eta}}v^{\gamma,\delta}(t)\,\mathrm{d}t.$$

Application of these quadrature rules to the operators $\widetilde{H}_2$ and H_3 leads to

$$(\widetilde{H}_{2n}u)(x) = \sum_{k=1}^{n}\widetilde{\lambda}_{nk}^{\alpha,\beta}(x)h_2(x,x_{nk}^{\alpha,\beta})u(x_{nk}^{\alpha,\beta}) \tag{3.10}$$

and

$$(H_3u)(x) = \sum_{k=1}^{n}\omega_{nk}^{\alpha,\beta}(x)h_3(x,x_{nk}^{\alpha,\beta})u(x_{nk}^{\alpha,\beta}). \tag{3.11}$$

The quadrature or discrete collocation method consists in solving the equation

$$\mathrm{D}A + L_n^{\beta,\alpha}(M_\Psi + H_{1n} + \widetilde{H}_{2n} + H_{3n})u_n = L_n^{\beta,\alpha}f. \tag{3.12}$$

The solution of this equation again belongs to $\mathbb{P}_{n-1}$. Since, for such u_n, we have

$$\begin{aligned}(H_1u_n)(x) &= Q_n^{\alpha,\beta}\left(u_nL_n^{\alpha,\beta}[h_1(x,.)]\right)\\ &= \frac{1}{\pi}\int_{-1}^{1} u_n(t)L_{nt}^{\alpha,\beta}[h(x,t)]v^{\alpha,\beta}(t)\,\mathrm{d}t =: (\widehat{H}_{1n}u_n)(x),\end{aligned} \tag{3.13}$$

the approximate Eq. (3.12) is equivalent to

$$\widetilde{B}_nu_n := \mathrm{D}A + L_n^{\beta,\alpha}(M_\Psi + \widehat{H}_{1n} + \widetilde{H}_{2n} + H_{3n})u_n = L_n^{\beta,\alpha}f. \tag{3.14}$$

The following three lemmata are generalizations of [11, Lemma 3.10], (comp. also [5, Lemma 4.4]).

Lemma 3.4. *Assume $h_1(x,.)\in\mathbf{L}_{\alpha,\beta}^{2,s}$ for some $s>\frac{1}{2}$ uniformly w.r.t. $x\in[-1,1]$. Then, for $0\leqslant t\leqslant s$ and $u\in\mathbf{L}_{\alpha,\beta}^{2}$,*

$$\left\|L_m^{\gamma,\delta}(\widehat{H}_{1n}-H_1)u\right\|_{\gamma,\delta,t} \leqslant \text{const}\ m^tn^{-s}\,\|u\|_{\alpha,\beta}.$$

Proof. Since, for a polynomial v_m of degree less than m, $\|v_m\|_{\gamma,\delta,t} \leqslant m^t \|v_m\|_{\gamma,\delta}$, we are able to estimate with the help of Schwarz' inequality

$$\begin{aligned}
\left\|L_m^{\gamma,\delta}(\widehat{H}_{1n} - H_1)u\right\|_{\gamma,\delta,t}^2 &\leqslant m^{2t} \left\|L_m^{\gamma,\delta}(\widehat{H}_{1n} - H_1)u\right\|_{\gamma,\delta}^2 \\
&= m^{2t} \sum_{j=1}^{m} \lambda_{mj}^{\gamma,\delta} \left\{ \frac{1}{\pi} \int_{-1}^{1} u(\tau)[L_{n\tau}^{\alpha,\beta} h(x_{mj}^{\gamma,\delta},\tau) - h(x_{mj}^{\gamma,\delta},\tau)] v^{\alpha,\beta}(\tau)\,\mathrm{d}\tau \right\}^2 \\
&\leqslant m^{2t} \|u\|_{\alpha,\beta}^2 \sum_{j=1}^{m} \lambda_{mj}^{\gamma,\delta} \left\|L_n^{\alpha,\beta} h(x_{mj}^{\gamma,\delta},.) - h(x_{mj}^{\gamma,\delta},.)\right\|_{\alpha,\beta}^2 \\
&\leqslant \text{const } m^{2t} n^{-2s} \|u\|_{\alpha,\beta}^2 \sum_{j=1}^{m} \lambda_{mj}^{\gamma,\delta} \left\|h(x_{mj}^{\gamma,\delta},.)\right\|_{\alpha,\beta,s}^2 \leqslant \text{const } m^{2t} n^{-2s} \|u\|_{\alpha,\beta}^2
\end{aligned}$$

taking into account Lemma 2.4 (b). □

Lemma 3.5. *Let, for some integer $q \geqslant s > \frac{1}{2}$, $h_2(x,.) \in \mathbf{C}_\varphi^q$ uniformly w.r.t. $x \in [-1,1]$. Then, for $0 \leqslant t \leqslant s$ and $u \in \mathbf{L}_{\alpha,\beta}^{2,s}$,*

$$\left\|L_m^{\gamma,\delta}(\widetilde{H}_{2n} - \widetilde{H}_2)u\right\|_{\gamma,\delta,t} \leqslant \text{const } m^t n^{-s} \|u\|_{\alpha,\beta,s}.$$

Proof. Using the Gaussian rule and Schwarz' inequality we obtain

$$\begin{aligned}
&\left\|L_m^{\gamma,\delta}(\widetilde{H}_{2n} - \widetilde{H}_2)u\right\|_{\gamma,\delta,t}^2 \\
&\leqslant m^{2t} \left\|L_m^{\gamma,\delta}(\widetilde{H}_{2n} - \widetilde{H}_2)u\right\|_{\gamma,\delta}^2 \\
&= m^{2t} \sum_{j=1}^{m} \lambda_{mj}^{\gamma,\delta} \left| a \int_{-1}^{x_{mj}^{\gamma,\delta}} \left[\sum_{k=1}^{n} h_2(x_{mj}^{\gamma,\delta}, x_{nk}^{\alpha,\beta}) u(x_{nk}^{\alpha,\beta}) l_{nk}^{\alpha,\beta}(\tau) - h_2(x_{mj}^{\gamma,\delta},\tau) u(\tau) \right] v^{\alpha,\beta}(\tau)\,\mathrm{d}\tau \right. \\
&\quad \left. - \frac{b}{\pi} \int_{-1}^{1} \left[\sum_{k=1}^{n} h_2(x_{mj}^{\gamma,\delta}, x_{nk}^{\alpha,\beta}) u(x_{nk}^{\alpha,\beta}) l_{nk}^{\alpha,\beta}(\tau) - h_2(x_{mj}^{\gamma,\delta},\tau) u(\tau) \right] \ln|x_{mj}^{\gamma,\delta} - \tau| v^{\alpha,\beta}(\tau)\,\mathrm{d}\tau \right|^2 \\
&\leqslant 2m^{2t} \sum_{j=1}^{m} \lambda_{mj}^{\gamma,\delta} \left\|(L_n^{\alpha,\beta} - I)[h_2(x_{mj}^{\gamma,\delta},.)u]\right\|_{\alpha,\beta}^2 \\
&\quad \times \left[a^2 \pi \int_{-1}^{1} v^{\alpha,\beta}(\tau)\,\mathrm{d}\tau + \frac{b^2}{\pi} \int_{-1}^{1} \ln^2 |x_{mj}^{\gamma,\delta} - \tau| v^{\alpha,\beta}(\tau)\,\mathrm{d}\tau \right].
\end{aligned}$$

With the help of Lemma 2.4 (b), Corollary 2.12, and the uniform boundedness of

$$\int_{-1}^{1} \ln^2 |x - \tau| v^{\alpha,\beta}(\tau)\,\mathrm{d}\tau, \quad x \in [-1,1],$$

the assertion of the lemma follows. □

Lemma 3.6. *Let* $0<\eta<\frac{1}{2}$ *and, for some integer* $q\geqslant s>\frac{1}{2}$, $h_3(x,.)\in \mathbf{C}^q_\varphi$, *uniformly w.r.t.* $x\in[-1,1]$. *Then, for* $0\leqslant t\leqslant s$ *and* $u\in \mathbf{L}^{2,s}_{\alpha,\beta}$,

$$\left\| L^{\gamma,\delta}_m (H_{3n}-H_3)u \right\|_{\gamma,\delta,t} \leqslant \text{const } m^t n^{-s} \|u\|_{\alpha,\beta,s}.$$

Proof. Analogous to the proof of Lemma 3.5 we find

$$\begin{aligned} &\left\| L^{\gamma,\delta}_m (H_{3n}-H_3)u \right\|^2_{\gamma,\delta,t} \\ &\quad \leqslant m^{2t} \sum_{j=1}^m \lambda^{\gamma,\delta}_{mj} \left\| (L^{\alpha,\beta}_n - I)[h_3(x^{\gamma,\delta}_{mj},.)u] \right\|^2_{\alpha,\beta} \frac{1}{\pi}\int_{-1}^1 \frac{v^{\alpha,\beta}(\tau)}{|x^{\gamma,\delta}_{mj}-\tau|^{2\eta}}\,\mathrm{d}\tau. \end{aligned}$$

The assumption on η and the fact that $\alpha>0$, $\beta>0$, guarantee the uniform boundedness of

$$\frac{1}{\pi}\int_{-1}^1 \frac{v^{\alpha,\beta}(\tau)}{|x-\tau|^{2\eta}}\,\mathrm{d}\tau, \quad x\in[-1,1].$$

Thus, the assertion follows by Lemma 2.4 (b) and Corollary 2.12. □

Theorem 3.7. *Let* $s>\frac{1}{2}$, $\Psi\equiv 0$, $h_3\equiv 0$, $f\in \mathbf{L}^{2,s}_{\beta,\alpha}$. *Assume that* (A0), (A2) *and* (A3) *are fulfilled for* $\tilde{s}=s$ *and* $\tilde{r}\geqslant \max\{0,s-1\}$. *Moreover, assume that* $h_1(x,.)\in \mathbf{L}^{2,s}_{\alpha,\beta}$ *and* $h_2(x,.)\in \mathbf{C}^q_\varphi$ *for some integer* $q\geqslant s$ *uniformly w.r.t.* $x\in[-1,1]$. *Then, for all sufficiently large n, Eq.* (3.14) *is uniquely solvable, and the solution* u_n^* *converges in the norm of the space* $\mathbf{L}^{2,t+1}_{\alpha,\beta}$, $0\leqslant t<s$, *to the unique solution* u^* *of* (3.1), *where*

$$\|u_n^*-u^*\|_{\alpha,\beta,t+1} \leqslant \text{const } n^{t-s}\|u^*\|_{\alpha,\beta,s+1}. \tag{3.15}$$

Proof. Referring to Lemmata 3.4, 3.5, and 2.4 (b) one can see that, for $H=H_1+\widetilde{H}_2$, $H_n=\widehat{H}_{1n}+\widetilde{H}_{2n}$, and $\tilde{t}=\min\{1,s\}$,

$$\begin{aligned} \left\| L^{\beta,\alpha}_n H_n - H \right\|_{\mathbf{L}^{2,1}_{\alpha,\beta}\to \mathbf{L}^2_{\beta,\alpha}} &\leqslant \text{const } \left\| L^{\beta,\alpha}_n H_n - H \right\|_{\mathbf{L}^{2,\tilde{t}}_{\alpha,\beta}\to \mathbf{L}^2_{\beta,\alpha}} \\ &\leqslant \left\| L^{\beta,\alpha}_n (H_n - H) \right\|_{\mathbf{L}^{2,\tilde{t}}_{\alpha,\beta}\to \mathbf{L}^2_{\beta,\alpha}} + \left\| L^{\beta,\alpha}_n H - H \right\|_{\mathbf{L}^{2,\tilde{t}}_{\alpha,\beta}\to \mathbf{L}^2_{\beta,\alpha}} \\ &\leqslant \text{const } n^{-\tilde{t}}. \end{aligned}$$

We remark that the operator $H:\mathbf{L}^{2,\tilde{t}}_{\alpha,\beta}\to \mathbf{L}^{2,\tilde{t}}_{\beta,\alpha}$ is continuous because of the following sequences of continuous mappings and embeddings (see Lemmata 2.10 and 2.13)

$$\mathbf{L}^{2,\tilde{t}}_{\alpha,\beta} \subset \mathbf{L}^2_{\alpha,\beta} \xrightarrow{H_1} \mathbf{L}^{2,s}_{-\alpha,-\beta} \subset \mathbf{L}^{2,s}_{\beta,\alpha} \subset \mathbf{L}^{2,\tilde{t}}_{\beta,\alpha}$$

and

$$\mathbf{L}^{2,\tilde{t}}_{\alpha,\beta} \subset \mathbf{L}^2_{\alpha,\beta} \xrightarrow{\widetilde{H}_2} \mathbf{L}^{2,1}_{-\alpha,-\beta} \subset \mathbf{L}^{2,1}_{\beta,\alpha} \subset \mathbf{L}^{2,\tilde{t}}_{\beta,\alpha}.$$

Consequently,

$$\lim_{n\to\infty} \left\| \widetilde{B}_n - B \right\|_{\mathbf{L}^{2,1}_{\alpha,\beta} \to \mathbf{L}^2_{\beta,\alpha}} = 0,$$

which implies the uniform boundedness of

$$\widetilde{B}_n^{-1} \in \mathcal{L}(\mathbf{L}^2_{\beta,\alpha}, \mathbf{L}^{2,1}_{\alpha,\beta}).$$

With the help of this result and Lemmata 3.4, 3.5, as well as Lemma 2.4 (b) we can estimate

$$\begin{aligned}
\left\| u_n^* - L_n^{\alpha,\beta} u^* \right\|_{\alpha,\beta,t+1} &\leqslant n^t \left\| u_n^* - L_n^{\alpha,\beta} u^* \right\|_{\alpha,\beta,1} \\
&\leqslant \text{const } n^t \left\| \widetilde{B}_n (u_n^* - L_n^{\alpha,\beta} u^*) \right\|_{\alpha,\beta} \\
&\leqslant \text{const } n^t \left(\left\| L_n^{\beta,\alpha} f - f \right\|_{\alpha,\beta} + \left\| (H - L_n^{\beta,\alpha} H_n) L_n^{\alpha,\beta} u^* \right\|_{\alpha,\beta} + \left\| B(u^* - L_n^{\alpha,\beta} u^*) \right\|_{\alpha,\beta} \right) \\
&\leqslant \text{const } n^{t-s} \left\| u^* \right\|_{\alpha,\beta,s+1}.
\end{aligned}$$

Thus, the estimate (3.15) is proved, if we remember $u^* \in \mathbf{L}^{2,s+1}_{\alpha,\beta}$ and Lemma 2.4 (b). □

Theorem 3.8. *In case $\Psi \not\equiv 0$ and $\alpha = \beta = \frac{1}{2}$* (i.e., $a = 0$, $b = -1$) *Theorem* 3.7 *remains true if we additionally assume that* (A1) *with $r \geqslant s$ is fulfilled.*

Proof. With $\tilde{t}$ defined in the proof of Theorem 3.7 we have

$$\| L_n^{\varphi} M_{\Psi} - M_{\Psi} \|_{\mathbf{L}^{2,1}_{\varphi} \to \mathbf{L}^2_{\varphi}} \leqslant \text{const } n^{-\tilde{t}}$$

having regard to Lemma 2.4 (b) and Corollary 2.12. The proof of the estimate (3.15) is the same as in the proof of Theorem 3.7, if we additionally take into account $L_n^{\varphi} M_{\Psi} L_n^{\varphi} u^* = L_n^{\varphi} M_{\Psi} u^*$ and again apply Corollary 2.12. □

Remark 3.9. In case $\Psi \not\equiv 0$, $h_3 \not\equiv 0$ and $\alpha = \beta = \frac{1}{2}$ (i.e., $a = 0$, $b = -1$) Theorem 3.7 remains true for $\frac{1}{2} < s < 1 - \eta$, if we additionally assume that $h_3(x, .) \in \mathbf{C}^1_{\varphi}$ uniformly w.r.t. $x \in [-1, 1]$, $0 < \eta < \frac{1}{2}$, and that (A1) with $r \geqslant s$ as well as (A4) are fulfilled.

Proof. With the help of Lemma 3.6 we have

$$\| L_n^{\varphi} (H_{3n} - H_3) \|_{\mathbf{L}^{2,1}_{\varphi} \to \mathbf{L}^2_{\varphi}} \leqslant \text{const } n^{-s}.$$

Corollary 2.18 together with Lemma 2.4 (b) gives

$$\| L_n^{\varphi} H_3 - H_3 \|_{\mathbf{L}^{2,1}_{\varphi} \to \mathbf{L}^2_{\varphi}} \leqslant \text{const } n^{-s}.$$

The proof of the estimate (3.15) is the same as in the proof of Theorem 3.7. □

4. A fast algorithm

In this section we consider the original Eq. (1.1) or, which is the same, Eq. (2.1) in the case of $\Gamma(x) = \gamma_0 = \text{const}$ and $h(x,t) = h_1(x,t) - \gamma_1 \ln|x-t|$ which means an equation of the form

$$\gamma_0 u(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\varphi(t)\,dt + \frac{1}{\pi}\int_{-1}^{1}\Big[h_1(x,t)) - \gamma_1 \ln|x-t|\Big]u(t)\varphi(t)\,dt = f(x). \tag{4.1}$$

We write this equation as

$$(\mathcal{A}+\mathcal{H})u = f, \tag{4.2}$$

where

$$\mathcal{A} = M_{\gamma_0} + V + \gamma_1 W, \qquad (Wu)(x) = -\frac{1}{\pi}\int_{-1}^{1} \ln|x-t|u(t)\varphi(t)\,dt, \tag{4.3}$$

and

$$(\mathcal{H}u)(x) = \frac{1}{\pi}\int_{-1}^{1} h_1(x,t)u(t)\varphi(t)\,dt.$$

We investigate Eq. (4.2) in the pair of spaces

$$(\mathbf{L}_\varphi^{2,s+1}, \mathbf{L}_\varphi^{2,s}) \tag{4.4}$$

for some $s > \frac{1}{2}$ and make the following assumptions:

(a0) For $f \equiv 0$ Eq. (4.2) possesses in $\mathbf{L}_\varphi^{2,1}$ only the trivial solution $u \equiv 0$. The same is assumed for the equation $\mathcal{A}u = 0$.

(a1) $h_1(.,t) \in \mathbf{L}_\varphi^{2,s+\delta}$ uniformly w.r.t. $t \in [-1,1]$ and

(a2) $h_1(x,.) \in \mathbf{L}_\varphi^{2,s+\delta}$ uniformly w.r.t. $x \in [-1,1]$ for some $\delta \geqslant 0$.

(a3) The right-hand side f of Eq. (4.2) belongs to $\mathbf{L}_\varphi^{2,s}$.

To construct a fast algorithm for the numerical solution of Eq. (4.2) (basing on the quadrature method considered in Section 3) we will follow the idea of [5, Section 6], which is based on the fundamental approach given in [1].

First of all, let us summarize some results of the previous sections. As a consequence of Lemmata 2.10 and 2 we have

(b1) The operator $\mathcal{H}$ belongs to $\mathcal{L}(\mathbf{L}_\varphi^{2}, \mathbf{L}_\varphi^{2,s+\delta})$. Especially, $\mathcal{H} : \mathbf{L}_\varphi^{2,s+1} \to \mathbf{L}_\varphi^{2,s}$ is compact.

Analogously

(b2) The operator $M_{\gamma_0} : \mathbf{L}_\varphi^{2,s+1} \to \mathbf{L}_\varphi^{2,s}$ is compact.

Corollary 2.14 gives

(b3) The operator $W : \mathbf{L}_\varphi^{2,t} \to \mathbf{L}_\varphi^{2,t+1}$ is continuous for all $t \geqslant 0$ and, consequently, compact in the pair of spaces (4.4).

Taking into account Corollary 2.8 (see also Remark 2.9) from (b1), (b2), and (b3) we have

(b4) $\mathcal{A}, \mathcal{A}+\mathcal{H} : \mathbf{L}_\varphi^{2,s+1} \to \mathbf{L}_\varphi^{2,s}$ are continuous isomorphisms.

The operator $\mathcal{A}+\mathcal{H}$ is approximated using the quadrature method (compare Section 3). Thus, at first we consider the approximate equation

$$(\mathcal{A}_n+\mathcal{H}_n)u_n = L_n^{\varphi} f, \tag{4.5}$$

where

$$\mathcal{A}_n = M_{\gamma_0} + V + \gamma_1 L_n^{\varphi} W,$$

$$\mathcal{H}_n = L_n^{\varphi}\widehat{H}_{1n}, (\widehat{H}_{1n}u)(x) = \frac{1}{\pi}\int_{-1}^{1} u(t) L_{nt}^{\varphi}[h_1(x,t)]\varphi(t)\,\mathrm{d}t.$$

Let us reformulate Theorem 3.8 for the case under consideration here.

Theorem 4.1. *Let $s>\frac{1}{2}$. Assume that* (a0)–(a3) *be fulfilled. Then, for all sufficiently large n, Eq.* (4.5) *is uniquely solvable, and for the solutions u_n^* we have the error estimate*

$$\|u_n^* - u^*\|_{\varphi,t+1} \leqslant \text{const } n^{t-s}\,\|u^*\|_{\varphi,s+1}, \tag{4.6}$$

where $0\leqslant t\leqslant s$ and $u^*\in \mathbf{L}_{\varphi}^{2,s+1}$ is the unique solution of (4.2).

We again remark that each solution u_n of (4.5) belongs to $\mathbb{P}_{n-1}$, such that

$$(\widehat{H}_{1n}u_n)(x) = (H_{1n}u_n)(x) = \sum_{k=1}^{n} \lambda_{nk}^{\varphi} h_1(x, x_{nk}^{\varphi}) u_n(x_{nk}^{\varphi}).$$

It is well known that

$$x_{nk}^{\varphi} = \cos\frac{k\pi}{n+1}, \quad \lambda_{nk}^{\varphi} = \frac{1-(x_{nk}^{\varphi})^2}{n+1} = \frac{1}{n+1}\sin^2\frac{k\pi}{n+1}, \quad k+1,\ldots,n.$$

To find a formula for the product integration weights $\widetilde{\lambda}_{nk}^{\varphi}(x)$ (comp. (3.8)) we use the following lemma.

Lemma 4.2 (Berthold et al. [4, Theorem 3.2]). *Let $T_n(x)=\cos(n\xi)$, $x=\cos\xi$, be the Tschebyscheff polynomial of degree n and of the first kind. Then, for $x\in[-1,1]$,*

$$-\frac{1}{\pi}\int_{-1}^{1} \ln|x-t|\,T_n(t)\frac{\mathrm{d}t}{\sqrt{1-t^2}} = \begin{cases} \ln 2, & n=0,\\ \dfrac{1}{n}T_n(x), & n=1,2,\ldots \end{cases}$$

Corollary 4.3. *For the operator W defined in* (4.3) *we have the relations*

$$Wp_n^{\varphi} = \frac{1}{\sqrt{2}}\begin{cases} \ln 2 - \dfrac{1}{2}T_2, & n=0,\\ \dfrac{1}{n}T_n - \dfrac{1}{n+2}T_{n+2}, & n=1,2,\ldots \end{cases}$$

Proof. Since $p_n^\varphi = \sqrt{2}U_n$, where

$$U_n(x) = \frac{\sin[(n+1)\xi]}{\sin\xi},$$

$x = \cos\xi$, is the nth Tschebyscheff polynomial of the second kind, and

$$U_n(x)(1-x^2) = \tfrac{1}{2}\,[T_n(x) - T_{n+2}(x)],$$

the assertion follows immediately from Lemma 4.2. □

As a consequence of

$$T_0(x) = U_0(x) = 1\,, \quad T_1(x) + \tfrac{1}{2}U_1(x),$$

and

$$T_n(x) = \tfrac{1}{2}\,[U_n(x) - U_{n-2}(x)]\,, \quad n = 2,3,\ldots\,,$$

from Corollary 4.3 we have

$$Wp_0^\varphi = \tfrac{1}{4}\left[(1+2\ln 2)p_0^\varphi - p_2^\varphi\right] =: \omega_{00}\,p_0^\varphi + \omega_{02}\,p_2^\varphi \tag{4.7}$$

$$Wp_1^\varphi = \tfrac{1}{4}\left[(1+\tfrac{1}{3})p_1^\varphi - \tfrac{1}{3}p_3^\varphi\right] =: \omega_{11}\,p_1^\varphi + \omega_{13}\,p_3^\varphi \tag{4.8}$$

$$\begin{aligned} Wp_n^\varphi &= \tfrac{1}{4}\left[-\frac{1}{n}p_{n-2}^\varphi + \left(\frac{1}{n} + \frac{1}{n+2}\right)p_n^\varphi - \frac{1}{n+2}p_{n+2}^\varphi\right] \\ &=: \omega_{n,n-2}\,p_{n-2}^\varphi + \omega_{nn}\,p_n^\varphi + \omega_{n,n+2}\,p_{n+2}^\varphi, \quad n = 2,3,\ldots \end{aligned} \tag{4.9}$$

Set

$$\omega_{jk} = 0 \quad \text{if} \quad |j-k| \neq 0 \;\text{ and }\; |j-k| \neq 2. \tag{4.10}$$

Now we use the representation

$$l_{nk}^\varphi(x) = \lambda_{nk}^\varphi \sum_{j=0}^{n-1} p_j^\varphi(x_{nk}^\varphi)\,p_j^\varphi(x) \tag{4.11}$$

of the fundamental Lagrange polynomials and obtain from Corollary 4.3

$$\begin{aligned} \widetilde{\lambda}_{nk}^\varphi(x) &= (Wl_{nk}^\varphi)(x) \\ &= \lambda_{nk}^\varphi\left\{U_0(x_{nk}^\varphi)\left[\ln 2 - \frac{1}{2}T_2(x)\right] + \sum_{j=1}^{n-1} U_j(x_{nk}^\varphi)\left[\frac{1}{j}T_j(x) - \frac{1}{j+2}T_{j+2}(x)\right]\right\} \end{aligned}$$

as the weights in the product integration rule

$$-\frac{1}{\pi}\int_{-1}^{1} u(t)\ln|x-t|\varphi(t)\,\mathrm{d}t \approx \sum_{k=1}^{n} \widetilde{\lambda}_{nk}^\varphi(x)\,u(x_{nk}^\varphi)\,.$$

Thus, if we seek the approximate solution of (4.5) in the form

$$u_n(x) = \sum_{k=1}^{n} \xi_{nk} l_{nk}^{\varphi}(x),$$

then, using Remark 2.9,

$$V l_{nk}^{\varphi} = \lambda_{nk}^{\varphi} \sum_{j=0}^{n-1} p_j^{\varphi}(x_{nk}^{\varphi})(j+1) p_j^{\varphi}$$

and (4.5) can be written in the form

$$(\gamma_0 \boldsymbol{I}_n + \boldsymbol{V}_n \Lambda_n + \gamma_1 \boldsymbol{W}_n + \boldsymbol{H}_n \Lambda_n)\, \xi_n = \boldsymbol{\eta}_n \tag{4.12}$$

with $\xi_n = [\xi_{nk}]_{k=1}^n$, $\eta_n = [f(x_{nj}^{\varphi})]_{j=1}^n$, and

$$\boldsymbol{I}_n = [\delta_{jk}]_{j,k=1}^n, \quad \boldsymbol{V}_n = \boldsymbol{U}_n^{\mathrm{T}} \boldsymbol{D}_n \boldsymbol{U}_n, \quad \boldsymbol{W}_n = [\widetilde{\lambda}_{nk}^{\varphi}(x_{nj}^{\varphi})]_{j,k=1}^n, \quad \boldsymbol{H}_n = [h_1(x_{nj}^{\varphi}, x_{nk}^{\varphi})]_{j,k=1}^n,$$

$$\boldsymbol{U}_n = [p_j^{\varphi}(x_{nk}^{\varphi})]_{j=0,k=1}^{n-1,n}, \quad \boldsymbol{D}_n = \mathrm{diag}[1,\dots,n], \quad \Lambda_n = \mathrm{diag}[\lambda_{n1}^{\varphi},\dots,\lambda_{nn}^{\varphi}].$$

From $\delta_{jk} = \langle p_k^{\varphi}, p_j^{\varphi} \rangle_{\varphi} = \sum_{l=1}^{n} \lambda_{nl}^{\varphi} p_k^{\varphi}(x_{nl}^{\varphi}) p_j^{\varphi}(x_{nl}^{\varphi})$ it follows that

$$\boldsymbol{I}_n = \boldsymbol{U}_n \Lambda_n \boldsymbol{U}_n^{\mathrm{T}}. \tag{4.13}$$

We will see that it is not necessary to generate the Matrix $\boldsymbol{W}_n$ in order to solve (4.12) (see Remark 4.4 below).

In what follows we assume that the vector η_n of the values of the function f at the collocation points x_{nj}^{φ}, $j = 1,\dots,n$, as well as the values $h_1(x_{nj}^{\varphi}, x_{nk}^{\varphi})$, $j,k = 1,\dots,n$, are given. Choose an integer $0 < m < n$ and write

$$u_n = \sum_{k=0}^{m-1} \alpha_k p_k^{\varphi} + \sum_{k=m}^{n-1} \alpha_k p_k^{\varphi} = \mathcal{P}_m u_n + \mathcal{Q}_m u_n,$$

where

$$\mathcal{P}_m u = \sum_{k=0}^{m-1} \langle u, p_k^{\varphi} \rangle_{\varphi} p_k^{\varphi} \quad \text{and} \quad \mathcal{Q}_m = I - \mathcal{P}_m.$$

Set $\alpha_k = \langle v_n^*, p_k^{\varphi} \rangle_{\varphi}$, $k = m,\dots,n-1$, where $v_n^* = \sum_{k=0}^{n-1} \beta_{nk}^* p_k^{\varphi}$ is the solution of

$$\mathcal{A}_n v_n = L_n^{\varphi} f. \tag{4.14}$$

In view of Theorem 4.1 (for the case of $h_1 \equiv 0$) Eq. (4.14) is uniquely solvable for all sufficiently large n, if (a0) is satisfied. For $\beta_n = [\beta_{nk}]_{k=0}^{n-1}$ we have

$$[(M_{\gamma_0} v_n)(x_{nj}^{\varphi})]_{j=1}^n = \left[\gamma_0 \sum_{k=0}^{n-1} \beta_{nk} p_k^{\varphi}(x_{nj}^{\varphi}) \right]_{j=1}^n = \gamma_0 \boldsymbol{U}_n^{\mathrm{T}} \beta_n,$$

and

$$[(Vv_n)(x_{nj}^\varphi)]_{j=1}^n = \left[\sum_{k=0}^{n-1} \beta_{nk}(k+1)p_k^\varphi(x_{nj}^\varphi)\right]_{j=1}^n = \boldsymbol{U}_n^{\mathrm{T}}\boldsymbol{D}_n\beta_n .$$

To find the Matrix $\widehat{\boldsymbol{W}}_n$ with

$$[(Wv_n)(x_{nj}^\varphi)]_{j=1}^n = \left[\sum_{k=0}^{n-1} \beta_{nk}\left(Wp_k^\varphi(x_{nj}^\varphi)\right)\right]_{j=1}^n = \boldsymbol{U}_n^{\mathrm{T}}\widehat{\boldsymbol{W}}_n\beta_n$$

define $\widetilde{\boldsymbol{W}}_n = [\omega_{jk}]_{j,k=0}^{n-1}$, where ω_{jk} is defined in (4.7)–(4.10). Then, for $\widetilde{\beta}_n = \widetilde{\boldsymbol{W}}_n\beta_n$,

$$L_n^\varphi W v_n = \sum_{k=0}^{n-1} \widetilde{\beta}_{nk}\, p_k^\varphi + \beta_{n,n-1}\omega_{n,n+2}L_n^\varphi\, p_{n+1}^\varphi$$

in view of $L_n^\varphi p_n^\varphi = 0$. Since

$$p_{n+1}^\varphi(x) = 2x\, p_n^\varphi(x) - p_{n-1}^\varphi(x)$$

the relation $L_n^\varphi p_{n+1}^\varphi = -L_n^\varphi p_{n-1}^\varphi = -p_{n-1}^\varphi$ holds true. Thus,

$$L_n^\varphi W\, v_n = \sum_{k=0}^{n-2} \widetilde{\beta}_{nk}\, p_k^\varphi + (\widetilde{\beta}_{n,n-1} - \omega_{n,n+2}\beta_{n,n-1})p_{n-1}^\varphi ,$$

which shows that $\widehat{\boldsymbol{W}}_n = [\widehat{\omega}_{jk}]_{j,k=0}^{n-1}$, where $\widehat{\omega}_{jk} = \omega_{jk}$ with the one exeption $\widehat{\omega}_{n-1,n-1} = \omega_{n-1,n-1} - \omega_{n,n+2}$. Consequently, Eq. (4.14) is equivalent to

$$\boldsymbol{U}_n^{\mathrm{T}}(\gamma_0\boldsymbol{I}_n + \boldsymbol{D}_n + \gamma_1\widehat{\boldsymbol{W}}_n)\beta_n = \eta_n$$

or, having regard to (4.13),

$$(\gamma_0\boldsymbol{I}_n + \boldsymbol{D}_n + \gamma_1\widehat{\boldsymbol{W}}_n)\beta_n = \boldsymbol{U}_n\Lambda_n\eta_n. \tag{4.15}$$

Remark 4.4. *Using these observations and* (4.13) *we see that Eq.* (4.12) *can be written in the equivalent form*

$$\left[\boldsymbol{U}_n^{\mathrm{T}}(\gamma_0\boldsymbol{I}_n + \boldsymbol{D}_n + \gamma_1\widehat{\boldsymbol{W}}_n)\boldsymbol{U}_n + \boldsymbol{H}_n\right]\Lambda_n\xi_n = \eta_n .$$

Since the transform

$$\boldsymbol{U}_n\Lambda_n = \frac{\sqrt{2}}{n+1}\left[\sin\frac{jk\pi}{n+1}\right]_{j,k=1}^n \operatorname{diag}\left[\sin\frac{k\pi}{n+1}\right]_{k=1}^n$$

can be applied to a vector with O(n ln n) computational complexity (comp., for example, [21, 23]), we can compute β_n (and so $\alpha_m, \ldots, \alpha_{n-1}$) with O($n$ ln n)-complexity taking into account the simple structure of the matrix on the left-hand side of (4.15).

Lemma 4.5 (Berthold et al. [5, Lemma 2.2]). *For $s \geqslant 0$ and $u \in \mathbf{L}_\varphi^{2,s}$ we have*

$$\|Q_m u\|_{\varphi,t} \leqslant (1+m)^{t-s}\,\|u\|_{\varphi,s}, \quad m = 0, 1, 2, \ldots$$

Lemma 4.6. *Assume* (a0), (a1), *and* (a3) *to be satisfied. Let* u^* *be the solution of* (4.2) *and let* $\mathcal{Q}_m u_n$, *be defined with the help of the solution* v_n^* *of* (4.14) (*i.e.*, $\mathcal{Q}_m u_n = \mathcal{Q}_m v_n^*$). *Then, for* $0 \leqslant t \leqslant s$,

$$\|\mathcal{Q}_m u_n - \mathcal{Q}_m u^*\|_{\varphi,t+1} \leqslant \text{const } \left(m^{t-s-\delta} + n^{t-s}\right) \|u^*\|_{\varphi,s+1} .$$

Proof. Write

$$\begin{aligned}\mathcal{Q}_m u_n - \mathcal{Q}_m u^* &= \mathcal{Q}_m(v_n^* - u^*) = \mathcal{Q}_m\left[\mathcal{A}_n^{-1} L_n^\varphi f - u^*\right] \\ &= \mathcal{Q}_m \mathcal{A}_n^{-1}\left[L_n^\varphi f - f + (W - L_n^\varphi W)u^* + \mathcal{H}u^*\right] .\end{aligned}$$

It follows that

$$\begin{aligned}&\|\mathcal{Q}_m u_n - \mathcal{Q}_m u^*\|_{\varphi,t+1} \\ &\quad \leqslant \|\mathcal{Q}_m \mathcal{A}_n^{-1}(L_n^\varphi f - f)\|_{\varphi,t+1} + \|\mathcal{Q}_m \mathcal{A}_n^{-1}(W - L_n^\varphi W)u^*\|_{\varphi,t+1} + \|\mathcal{Q}_m \mathcal{A}_n^{-1}\mathcal{H}u^*\|_{\varphi,t+1} .\end{aligned}$$

With the help of Lemma 4.5, Lemma 2.4(b), (b1), (b3), and the uniform boundedness of $\|\mathcal{A}_n^{-1}\|_{\mathbf{L}_\varphi^{2,t} \to \mathbf{L}_\varphi^{2,t+1}}$, $t \geqslant 0$, we estimate

$$\begin{aligned}\|\mathcal{Q}_m \mathcal{A}_n^{-1}(L_n^\varphi f - f)\|_{\varphi,t+1} &\leqslant \text{const } \|L_n^\varphi f - f\|_{\varphi,t} \\ &\leqslant \text{const } n^{t-s}\|f\|_{\varphi,s} \leqslant \text{const } n^{t-s}\|u^*\|_{\varphi,s+1} ,\end{aligned}$$

$$\begin{aligned}\|\mathcal{Q}_m \mathcal{A}_n^{-1}(W - L_n^\varphi W)u^*\|_{\varphi,t+1} &\leqslant \text{const } \|(W - L_n^\varphi W)u^*\|_{\varphi,t} \\ &\leqslant \text{const } n^{t-s-2}\|Wu^*\|_{\varphi,s+2} \\ &\leqslant \text{const } n^{t-s-2}\|u^*\|_{\varphi,s+1} ,\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{Q}_m \mathcal{A}_n^{-1}\mathcal{H}u^*\|_{\varphi,t+1} &\leqslant \text{const } m^{t-s-\delta}\|\mathcal{A}_n^{-1}\mathcal{H}u^*\|_{\varphi,s+1+\delta} \\ &\leqslant \text{const } m^{t-s-\delta}\|\mathcal{H}u^*\|_{\varphi,s+\delta} \leqslant \text{const } m^{t-s-\delta}\|u^*\|_{\varphi} ,\end{aligned}$$

which proves the lemma. □

The second step of our algorithm consists in setting $\mathcal{P}_m u_n = w_m^*$, where w_m^* is the solution of

$$(\mathcal{A}_m + \mathcal{H}_m) w_m = L_m^\varphi (f - \mathcal{A}_n \mathcal{Q}_m v_n^*) . \tag{4.16}$$

This equation is equivalent to (see Remark 4.4)

$$\left[\boldsymbol{U}_m^{\mathrm{T}}(\gamma_0 \boldsymbol{I}_m + \boldsymbol{D}_m + \gamma_1 \widehat{\boldsymbol{W}}_m)\boldsymbol{U}_m + \boldsymbol{H}_m\right] \Lambda_m \omega_m = \widetilde{\eta}_m , \tag{4.17}$$

where $\omega_m = [w_m(x_{mk}^\varphi)]_{k=1}^m$ and $\widetilde{\eta}_m = [f(x_{mj}^\varphi) - (\mathcal{A}_n \mathcal{Q}_m v_n^*)(x_{mj}^\varphi)]_{j=1}^m$. The matrix $\boldsymbol{U}_m$ can be generated with $\mathrm{O}(m^2)$-complexity using the three-term recurrence relation of the orthogonal polynomials $p_j^\varphi(x)$. Thus, for given $\widetilde{\eta}_m$, Eq. (4.17) can be solved with $\mathrm{O}(m^3)$-complexity. The values $f(x_{mj}^\varphi)$ are already been given if we choose m in such a way that $(n+1)/(m+1)$ is an integer, which implies $x_{mj}^{\varphi} \in \{x_{nk}^{\varphi} : k = 1,\dots,n\}$ for $j = 1,\dots,m$. So, it remains to compute $\boldsymbol{U}_n^{\mathrm{T}}(\gamma_0 \boldsymbol{I}_n + \boldsymbol{D}_n)\widetilde{\beta}_n$, where $\widetilde{\beta}_n = [0,\dots,0,\beta_{nm},\dots,\beta_{n,n-1}]^{\mathrm{T}}$. This can be done with $\mathrm{O}(n \ln n)$ operations taking into account that

$$\boldsymbol{U}_n^{\mathrm{T}} = \sqrt{2}\,\mathrm{diag}\left[\sin^{-1}\frac{k\pi}{n+1}\right]_{k=1}^{n}\left[\sin\frac{jk\pi}{n+1}\right]_{k,j=1}^{n}$$

can again be handled as fast discrete sine transform (comp. [21, 23]). The determination of the Fourier coefficients α_{nk}, $k = 0,\dots,m-1$, needs $\mathrm{O}(m \ln m)$ operations, since $[\alpha_{nk}]_{k=0}^{m-1} = \boldsymbol{U}_m \Lambda_m \omega_m$. Summarizing these considerations we have

Remark 4.7. *The computation of the Fourier coefficients of $u_n = w_m^* + \mathcal{Q}_m v_n^*$, where v_n^* and w_m^* are the solutions of* (4.14) *and* (4.16), *respectively, can be done with $\mathrm{O}(m^3 + n \ln n)$ numerical complexity.*

Lemma 4.8. *If the assumptions* (a0)–(a3) *are fulfilled and if $\frac{1}{2} < t \leqslant s$, then, for all sufficiently large m, Eq.* (4.16) *is uniquely solvable and*

$$\|w_m^* - \mathcal{P}_m u^*\|_{\varphi,t+1} \leqslant \text{const}\,\left(m^{t-s-\delta} + n^{t-s}\right)\|u^*\|_{\varphi,s+1}.$$

Proof. First of all, since $L_m^{\varphi} L_n^{\varphi} = L_m^{\varphi}$ (because of $(n+1)/(m+1)$ is assumed to be an integer) and $(M_{\gamma_0} + V)\mathcal{P}_m u^* \in \mathbb{P}_{m-1}$ we have

$$\begin{aligned}
&(\mathcal{A}_m + \mathcal{H}_m)(w_m^* - \mathcal{P}_m u^*)\\
&= L_m^{\varphi} f - L_m^{\varphi}\left(M_{\gamma_0} + V + \gamma_1 L_n^{\varphi} W\right)\mathcal{Q}_m v_n^* - \left(M_{\gamma_0} + V + \gamma_1 L_m^{\varphi} W\right)\mathcal{P}_m u^* - \mathcal{H}_m \mathcal{P}_m u^*\\
&= L_m^{\varphi}(\mathcal{A} + \mathcal{H})u^* - L_m^{\varphi}\mathcal{A}\mathcal{Q}_m v_n^* - L_m^{\varphi}\mathcal{A}\mathcal{P}_m u^* - \mathcal{H}_m \mathcal{P}_m u^*\\
&= L_m^{\varphi}\mathcal{A}(\mathcal{Q}_m u^* - \mathcal{Q}_m v_n^*) + L_m^{\varphi}(\mathcal{H} - \widehat{H}_{1m})u^* + L_m^{\varphi}\widehat{H}_{1m}\mathcal{Q}_m u^*.
\end{aligned}$$

From Lemma 2.4, (b4), and Lemma 4.6 it follows that

$$\begin{aligned}
&\|L_m^{\varphi}\mathcal{A}(\mathcal{Q}_m u^* - \mathcal{Q}_m v_n^*)\|_{\varphi,t}\\
&\quad\leqslant \text{const}\,\|\mathcal{Q}_m u^* - \mathcal{Q}_m v_n^*\|_{\varphi,t+1} \leqslant \text{const}\,\left(m^{t-s-\delta} + n^{t-s}\right)\|u^*\|_{\varphi,s+1}.
\end{aligned}$$

With the help of Lemma 3.4 we can estimate

$$\left\|L_m^{\varphi}(\mathcal{H} - \widehat{H}_{1m})u^*\right\|_{\varphi,t} \leqslant \text{const}\; m^{t-s-\delta}\,\|u^*\|_{\varphi}.$$

Moreover, by the definition of the operator $\widehat{H}_{1m}$ we see that $\widehat{H}_{1m}\mathcal{Q}_m u^* \equiv 0$. Thus, it remains to apply the uniform boundedness of $\|(\mathcal{A}_m + \mathcal{H}_m)^{-1}\|_{\mathbf{L}_{\varphi}^{2,t} \to \mathbf{L}_{\varphi}^{2,t+1}}$ for all sufficiently large m. □

Now we can summarize our results.

Theorem 4.9. *Let $s > \frac{1}{2}$,* (a0)–(a3) *be satisfied and $m, n, 0 < m < n$, be integers such that $(n+1)/(m+1)$ is an integer and $c_1 n \leqslant m^3 \leqslant c_2 n$ with some positive constants c_1 and c_2. Then,* *for all sufficiently large m, Eqs.* (4.14) *and* (4.16) *are uniquely solvable and* $\widetilde{u}_n^* = w_m^* + \mathcal{Q}_m v_n^*$ *converges in the norm of* $\mathbf{L}_\varphi^{2,t+1}$, $0 \leqslant t < s$, *to the unique solution* $u^* \in \mathbf{L}_\varphi^{2,s+1}$ *of Eq.* (4.2), *where, for* $\max\left\{\frac{1}{2}, s - \frac{1}{2}\delta\right\} < t \leqslant s$,

$$\|\widetilde{u}_n^* - u^*\|_{\varphi,t+1} \leqslant \text{const } n^{t-s} \|u^*\|_{\varphi,s+1}. \tag{4.18}$$

Moreover, the solution of (4.14) *and* (4.16) *needs* $O(n \ln n)$ *operations.*

Proof. Lemmata 4.6 and 4.8 yield

$$\|\widetilde{u}_n^* - u^*\|_{\varphi,t+1} \leqslant \text{const } \left(m^{t-s-\delta} + n^{t-s}\right) \|u^*\|_{\varphi,s+1},$$

which implies together with $t > s - \frac{1}{2}\delta$ and $m \geqslant c_1 n^{1/3}$ the estimate (4.18). Remark 4.7 together with $m^3 \leqslant c_2 n$ leads to a complexity of $O(n \ln n)$. □

At least we want to discuss, what results are possible if instead of M_{γ_0} and/or $\gamma_1 W$ operators M_Γ (see (2.2)) or H_2 (see (2.9)) occur. That means, in place of Eq. (4.1) we will consider an equation. of the form

$$\Gamma(x)u(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{u(t)}{(t-x)^2}\varphi(t)\,\mathrm{d}t + \frac{1}{\pi}\int_{-1}^{1}\left[h_1(x,t) + h_2(x,t)\ln|x-t|\right]u(t)\varphi(t)\,\mathrm{d}t = f(x). \tag{4.19}$$

We also write this equation in the form (4.2), but now with

$$\mathcal{A} = V, \qquad \mathcal{H} = M_\Gamma + H_1 + H_2$$

(H_1 and H_2 are defined in (2.8) and (2.9) with $v^{\alpha,\beta} = \varphi$). The approximating operators are defined as

$$\mathcal{A}_n = \mathcal{A} \quad \text{and} \quad \mathcal{H}_n = L_n^\varphi(M_\Gamma + \widehat{H}_{1n} + H_{2n}).$$

(see (3.13) and (3.10), $H_{2n} = \widetilde{H}_{2n}$ for case of $a = 0$, $b = -1$). We have to check if the assertions of Lemmata 4.6 and 4.8 remain true. The crusial point in the proof of Lemma 4.6 is the estimation of $\|\mathcal{Q}_m \mathcal{A}_n^{-1}\mathcal{H}u^*\|_{\varphi,t+1}$. If we suppose that h_2 and h'_{2x} possess continuous partial derivatives up to order $r \geqslant s+1$ on $[-1,1]$ and that Γ belongs to $\mathbf{C}_\varphi^r$, we can apply Corollary 2.14 and obtain

$$\|\mathcal{Q}_m \mathcal{A}^{-1} M_\Gamma u^*\|_{\varphi,t+1} \leqslant \text{const } m^{t-s-1} \|\mathcal{A}^{-1} M_\Gamma u^*\|_{\varphi,s+2} \leqslant \text{const } m^{t-s-1} \|u^*\|_{\varphi,s+1}$$

and

$$\|\mathcal{Q}\mathcal{A}^{-1} H_2 u^*\|_{\varphi,t+1} \leqslant \text{const } m^{t-s-2} \|\mathcal{A}^{-1} H_2 u^*\|_{\varphi,s+3} \leqslant \text{const } m^{t-s-2} \|u^*\|_{\varphi,s+1}.$$

The essential steps in the proof of Lemma 4.8 are the estimations of

$$\left\|L_m^\varphi(\mathcal{H} - M_\Gamma - \widehat{H}_{1m} - H_{2m})u^*\right\|_{\varphi,t} = \left\|L_m^\varphi(H_1 + H_2 - \widehat{H}_{1m} - H_{2m})u^*\right\|_{\varphi,t}$$

and

$$\|\mathcal{H}_m \mathcal{Q}_m u^*\|_{\varphi,t}.$$

If we assume that $h_2(x,.)\in \mathbf{C}_\varphi^q$ uniformly w.r.t. $t\in[-1,1]$ for some integer $q\geqslant s+1$ then, having regard to Lemma 3.5,

$$\|L_m^\varphi(H_2-H_{2m})u^*\|_{\varphi,t}\leqslant \text{const}\ m^{t-s-1}\ \|u^*\|_{\varphi,s+1}\,.$$

Furthermore,

$$\|L_m^\varphi M_\Gamma \mathcal{Q}_m u^*\|_{\varphi,t}\leqslant \text{const}\ \|\mathcal{Q}_m u^*\|_{\varphi,t}\leqslant \text{const}\ m^{t-s-1}\ \|u^*\|_{\varphi,s+1}$$

and

$$\|L_m^\varphi H_{2m}\mathcal{Q}_m u^*\|_{\varphi,t}\leqslant \text{const}\ \|\mathcal{Q}_m u^*\|_{\varphi,\max\{0,t-1\}}\leqslant \text{const}\ m^{\max\{0,t-1\}-s-1}\ \|u^*\|_{\varphi,s+1}\,.$$

The summary of these observations is that it is possible to hold true the assertions of Theorem 4.9 in case of Eq. (4.19) for $\delta'=\min\{\delta,1\}$ instead of δ.

5. Numerical Examples

In this section we apply the fast algorithm presented in Section 4 to Eq. (4.1) with $\gamma_0=1$, $\gamma_1=0$, and

(i) $h_1(x,t)=|x|+|t|$, $f(x)=2+|x|/2+2/3\pi$,

(ii) $h_1(x,t)=t\,(x^2|x|+t|t|)$,

$$f(x)=x\left[(1+4x/15\pi)\,|x|+6/\pi+((3x^2-2)/\pi\sqrt{1-x^2})\ln[(1+\sqrt{1-x^2})/(1-\sqrt{1-x^2})]\right].$$

In case (i) Eq. (4.1) possesses the solution $u(x)\equiv 1$ and in case (ii) the solution $u(x)=x|x|$. Moreover, for $\varepsilon>0$, we have

(i) $h_1(.,t)$, $h_1(x,.)$, $f\in L^2_{\varphi,3/2-\varepsilon}$,

(ii) $h_1(.,t)$, $h_1(x,.)\in L^2_{\varphi,7/2-\varepsilon}$, and $f\in L^2_{\varphi,3/2-\varepsilon}$

in the respective examples. Hence, in case (i) the assumptions of Theorem 4.9 are satisfied, for example, for $s=0.8$ and $\delta=0.6$. In case (ii) the same holds true for $s=1.5-\varepsilon$ and $\delta=2$. Therefore, in case (i) we can expect theoretically the convergence rate

$$\|u_n^*-u^*\|_{\varphi,t+1}\leqslant \text{const}\ n^{t-0.8}\|u^*\|_{\varphi,1.8},\quad 0.5<t\leqslant 0.8,$$

Table 1

Example (i)		
n	m	$\|u_n^*-u^*\|_{\varphi,1.51}$
8	2	0.123D-00
27	3	0.801D-01
64	4	0.562D-01
125	5	0.519D-01
216	6	0.386D-01
343	7	0.393D-01
399	15	0.205D-01

Table 2

Example (ii)			
n	m	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.51}$	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.85}$
8	2	0.552D-01	0.107D-00
27	3	0.186D-01	0.520D-01
64	4	0.838D-02	0.283D-01
125	5	0.447D-02	0.212D-01
216	6	0.226D-02	0.125D-01
343	7	0.168D-02	0.112D-01
399	15	0.141D-02	0.102D-01

Table 3

Example (i)	
n	$\|\|u_n^* - P_n u^*\|\|_{\varphi,1.51}$
8	0.980D-03
27	0.199D-03
64	0.184D-04
125	0.980D-05
216	0.165D-05
343	0.131D-05
399	0.972D-06

Table 4

Example (ii)		
n	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.51}$	$\|\|u_n^* - \mathcal{P}_n u^*\|\|_{\varphi,1.85}$
8	0.522D-01	0.103D-00
27	0.178D-01	0.511D-01
64	0.695D-02	0.271D-01
125	0.436D-02	0.211D-01
216	0.211D-02	0.124D-01
343	0.164D-02	0.112D-01
399	0.141D-02	0.102D-01

and in case (ii), for $s < 1.5$,

$$||u_n^* - u^*||_{\varphi,t+1} \leqslant \text{const } n^{t-s}||u^*||_{\varphi,s+1}, \quad 0.5 < t \leqslant s.$$

Tables 1 and 2 show the actual values of the error in the examples considered.

Finally, in Tables 3 and 4 one can see the results obtained by means of the quadrature method. Of course, Example (ii) is more convenient than Example (i) for applying the fast algorithm, since already for small m in comparision with n the errors for the quadrature method and the fast algorithm are essentially the same. The reason for this is that in Example (ii) the kernel $h_1(x,t)$ is really smoother than the right-hand side $f(x)$.

References

[1] B.A. Amosov, On the approximate solution of elliptic pseudodifferential equations on a smooth closed curve, *Zeitschr. Anal. Anw.* **9** (1990) 545–563 (in Russian).

[2] V.M. Badkov, Convergence in mean and almost everywhere of Fourier series in polynomials orthogonal on a segment, *Mat. Sb.* **95** (137) (1974) 229–262 (in Russian).

[3] J.M. Berezanski, Expansion with respect to eigenfunctions of selfadjoint operators (in Russian), (Naukova Dumka, Kiev 1965).

[4] D. Berthold, W. Hoppe and B. Silbermann, The numerical solution of the generalized airfoil equation, *J. Integral Equations Appl.* **4** (2) (1992) 309–336.

[5] D. Berthold, W. Hoppe and B. Silbermann, A fast algorithm for solving the generalized airfoil equation, *J. Comput. Appl. Math.* **43** (1992) 185–219.

[6] D. Berthold and P. Junghanns, New error bounds for the quadrature method for the solution of Cauchy singular integral equations, *SIAM J. Numer. Anal.* **30** (1993) 1351–1372.

[7] M.R. Capobianco and G. Mastroianni, Uniform boundedness of Lagrange operator in weighted Sobolev-type spaces, *Math. Nachr.*, to appear.

[8] Z. Ditzian and V. Totik, Remarks on Besov spaces and best polynomial approximation, *Proc. Amer. Math. Soc.* **104** (1988) 1059–1066.

[9] V.J. Ervin and E.P. Stephan, Collocation with Chebyshev polynomials for a hypersingular integral equation on an interval, *J. Comput. Appl. Math.* **43** (1992) 221–229.

[10] M.A. Golberg, The convergence of several algorithms for solving integral equations with finite-part integrals, *J. Integral Equations* **5** (1983) 329–340.

[11] P. Junghanns, Product integration for the generalized airfoil equation, in: E. Schock, Ed., Beiträge zur Angewandten Analysis und Informatik (Shaker, Aachen, 1994) 171–188.

[12] P. Junghanns and U. Luther, Cauchy singular integral equations in spaces of continuous functions and methods for their numerical solution. *J. Comput. Appl. Math.* **77** (1997) 201–237.

[13] P. Junghanns and B. Silbermann, The numerical treatment of singular integral equations by means of polynomial approximations. I, preprint P-Math-35/86, AdW der DDR, Karl-Weierstraß-Institut für Mathematik, Berlin, 1986.

[14] A.I. Kalandiya, *Mathematical Methods of Two-Dimensional Elasticity*, (Mir, Moscow, 1975).

[15] A.C. Kaya and F. Erdogan, On the solution of integral equations with strongly singular kernels, *Quart. Appl. Math.* **XLV** (1987) 105–122.

[16] G. Mastroianni and M. G. Russo, Lagrange interpolation in weighted Besov spaces, to appear.

[17] S.G. Mikhlin and S. Prössdorf, *Singular Integral Operators* (Akademie Verlag, Berlin, 1986).

[18] G. Monegato and S. Prössdorf, Uniform convergence estimates for a collocation and discrete collocation method for the generalized airfoil equation, in: A.G. Agarval, Ed., *Contribution to Numerical Mathematics* (World Scientific, Singapore, 1993) 285–299. See also the errata corrige in the Internal Report no. 14, 1993, Dipartimento di Matematica, Politecnico di Torino.

[19] S. Prössdorf and B. Silbermann, *Numerical Analysis for Integral and Related Operator Equations* (Akademie Verlag, Berlin, 1991).

[20] E.P. Stephan and W.L. Wendland, A hypersingular boundary integral method for two dimensional screen and crack problems, *Arch. Rational Mech. Anal.* **112** (1990) 363–390.

[21] G. Steidl, Fast radix-p discrete cosine transform, *AAECC* **3** (1992) 39–46.

[22] G. Szegö, *Orthogonal Polynomials* (AMS Providence, Rhode Island, 1939).

[23] M. Tasche, Fast algorithms for discrete Chebyshev–Vandermonde transforms and applications, *Numer. Algorithm.* **5** (1993), 453–464.