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New characterizations of spheres, cylinders and W-curves[☆]

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ABSTRACT

We study hypersurfaces (curves, resp.) of Euclidean space of arbitrary dimension such that the chord joining any two points on the hypersurface (curve, resp.) meets it at the same angle.

1. Introduction

It is well-known that a circle is characterized as a closed plane curve such that the chord joining any two points on it meets the curve at the same angle at the two points (cf. [6, pp. 160–162]). From the viewpoint of differential geometry, this characteristic property of circles can be stated as follows:

Theorem 1. *Let $X = X(s)$ be a unit speed closed curve in Euclidean plane $\mathbb{E}^2$ and $T(s) = X'(s)$ be its unit tangent vector field. Then $X = X(s)$ is a circle if and only if it satisfies the following condition:*
(C) : $\langle X(t) - X(s), T(t) - T(s)\rangle = 0$ *holds identically.*

Generalizing above, Chen and the authors proved the following [3]:

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Theorem 2. *A unit speed curve $X = X(s)$ in Euclidean m-space $\mathbb{E}^m$ $(m \geqslant 2)$ is a W-curve if and only if it satisfies the Condition (C).*

Here, $X = X(s)$ is called a W-curve if its Frenet curvatures are constant along X. On a hypersurface of Euclidean space, a unit normal vector field G is naturally defined on M. Such G is also called the Gauss map of M. For a hypersphere M of Euclidean space, the chord joining any two points on it meets the sphere at the same angle at the two points, that is, the sphere satisfies the Condition:

(D): $\langle y - x, G(x) + G(y)\rangle = 0$ holds identically.

From the previous theorems, it is natural to ask the following question:

"What are hypersurfaces of Euclidean space which satisfy the Condition (D)?"

In differential geometry, the shape operator is the most natural tool to observe the extrinsic shape of submanifolds. Among those, the isoparametric hypersurface is one of nice hypersurface of Euclidean space, which have constant principal curvatures.

Boas [1,2] studied the hypersurfaces of Euclidean space which satisfy the Condition (D) and later, Wegner [8] gave a differential geometric proof for such hypersurfaces.

In this article, we provide a much easier and elementary characterization of isoparametric hypersurfaces of Euclidean space and characterize the W-curves by using the similar techniques.

In Section 2, we study hypersurfaces of Euclidean space $\mathbb{E}^m$ which satisfy the Condition (D). For this we have the following:

Theorem A. *For a hypersurface M in $\mathbb{E}^m$, the following are equivalent*:

(i) *M satisfies the Condition (D).*
(ii) *For an $m \times m$ matrix A and a vector $b \in \mathbb{E}^m$ we have $G(x) = Ax + b$.*
(iii) *M is a isoparametric hypersurface.*
(iv) *M is an open part of one of the following hypersurfaces*:

$$\mathbb{E}^{m-1}, S^{m-1}(r), S^{p-1}(r) \times \mathbb{E}^{m-p}.$$

In Section 3 we give a characterization of W-curves as follows:

Theorem B. *For a unit speed curve $X(s)$ in $\mathbb{E}^m$, the following five statements are equivalent*:

(i) *$X(s)$ satisfies the Condition (C).*
(ii) *For an $m \times m$ matrix A and a vector $b \in \mathbb{E}^m$ we have $X'(s) = AX(s) + b$.*
(iii) *For each k $(k = 1, 2, \ldots, m)$, $|X^{(k)}(s)|$ is constant.*
(iv) *$X(s)$ is a W-curve.*
(v) *$X(s)$ can be written as one of the following*:

$$\begin{aligned} X(s) &= (a_1 \cos c_1 s, a_1 \sin c_1 s, \ldots, a_n \cos c_n s, a_n \sin c_n s, 0, \ldots, 0), \\ X(s) &= (a_1 \cos c_1 s, a_1 \sin c_1 s, \ldots, a_n \cos c_n s, a_n \sin c_n s, bs, 0, \ldots, 0) \end{aligned} \tag{1.1}$$

for distinct nonzero numbers $c_1, \ldots, c_n$ and a nonzero number b.

Throughout this article, we assume that all objects are smooth and connected unless otherwise mentioned.

2. Proof of Theorem A

Let M be a hypersurface in $\mathbb{E}^m$ which satisfies the Condition (D). Without loss of generality, we may assume that M is not contained in any hyperplane, that is, M is full in $\mathbb{E}^m$. Then on M, there exist points $y_0, y_1, \ldots, y_m$ such that the set $\{y_j - y_0 | j = 1, 2, \ldots, m\}$ spans $\mathbb{E}^m$.

From the Condition (D) we have

$$\langle G(x), y_0\rangle = \langle G(x), x\rangle - \langle G(y_0), y_0\rangle + \langle G(y_0), x\rangle, \tag{2.1}$$

$$\langle G(x), y_j\rangle = \langle G(x), x\rangle - \langle G(y_j), y_j\rangle + \langle G(y_j), x\rangle, \quad j = 1, 2, \ldots, m. \tag{2.2}$$

By subtracting (2.1) from (2.2), we obtain

$$\langle G(x), A_j\rangle = \langle B_j, x\rangle + c_j, \quad j = 1, 2, \ldots, m, \tag{2.3}$$

where we put

$$A_j = y_j - y_0, B_j = G(y_j) - G(y_0), \quad c_j = \langle G(y_0), y_0\rangle - \langle G(y_j), y_j\rangle$$

for $j = 1, 2, \ldots, m$.

Lemma 2.1. *For an $m \times m$ matrix A and a vector $b \in \mathbb{E}^m$ we have $G(x) = Ax + b$.*

Proof. Let A denote the matrix defined by $A^t = [B_1, B_2, \ldots, B_m][A_1, A_2, \ldots, A_m]^{-1}$, where $[B_1, B_2, \ldots, B_m]$ denotes the matrix with column vectors $B_1, B_2, \ldots, B_m$. If we let $b = \sum b_j A_j$, where b_j is defined by

$$(b_1, b_2, \ldots, b_m)^t = (C_{jk})^{-1}(c_1, c_2, \ldots, c_m)^t, C_{jk} = \langle A_j, A_k\rangle,$$

then we have $G(x) = Ax + b$. □

By differentiating G covariantly with respect to a tangent vector X to M, it follows from Lemma 2.1 that

$$AX = -S(X), \quad X \in T_xM, \tag{2.4}$$

where S denotes the shape operator. Choose an orthonormal frame $E_1, \ldots, E_{m-1}$ such that $E_1, \ldots, E_{m-1}$ are eigenvectors of S associated with eigenvalues $\mu_1, \ldots, \mu_{m-1}$. Then from (2.4), for all $x \in M$ we have

$$AE_j(x) = -\mu_j(x)E_j(x), \quad j = 1, 2, \ldots, m-1. \tag{2.5}$$

Since A is a constant matrix and the set of eigenvalues of a matrix is discrete, the principal curvatures $\mu_1, \ldots, \mu_{m-1}$ are all constant, that is, M is an isoparametric hypersurface. Hence it follows from a well-known theorem (cf. [5,7]) that M is an open part of either a sphere $S^{m-1}(r)$ or a generalized cylinder $S^{p-1}(r) \times \mathbb{E}^{m-p}$.

Here, we give an elementary proof. Since S is self-adjoint and $\{X \in T_xM | x \in M\}$ spans $\mathbb{E}^m$, (2.4) shows that A is symmetric. For the function $f : \mathbb{E}^m \to \mathbb{R}$ defined by $f(x) = \langle Ax + b, Ax + b\rangle$, it follows from Lemma 2.1 that $M \subset f^{-1}(1)$ because $G(x)$ is a unit vector field. This shows that the gradient vector $\nabla f(x) = 2A(Ax + b)$ is proportional to $G(x)$. Hence for some function $\lambda(x)$ we have

$$A(Ax + b) = \lambda(x)(Ax + b), \quad x \in M. \tag{2.6}$$

Since the set of eigenvalues of a matrix is discrete, $\lambda(x)$ must be a constant. It follows from (2.6) that $V = \{Ax + b | x \in M\}$ is contained in an eigenspace of A corresponding to eigenvalue λ.

From the assumption that M is not contained in any hyperplane, as in the proof of Lemma 2.1 we see that

$$\operatorname{Im} A = \operatorname{Span}\{AA_j | j = 1, 2, \ldots, m\} \subset V. \tag{2.7}$$

It follows from (2.6) that

$$(A^2 - \lambda A)x = -Ab + \lambda b, \quad x \in M. \tag{2.8}$$

Hence we have

$$(A^2 - \lambda A)(y_j - y_0) = 0, \quad j = 1, 2, \ldots, m, \tag{2.9}$$

which shows

$$A^2 - \lambda A = 0. \tag{2.10}$$

Suppose $\lambda = 0$. Then (2.10) shows that $A^2 = 0$. Since A is symmetric, A must vanish. Together with Lemma 2.1, this shows that M is an open part of an hyperplane $\mathbb{E}^{m-1}$. This contradiction implies that $\lambda \neq 0$. Together with (2.10) and (2.8) shows that $b = \frac{1}{\lambda}Ab \in \mathrm{Im}A$. Hence (2.7) implies $V = \mathrm{Im}\,A$.

Let us denote $\lambda = \pm\frac{1}{r}$ with $r > 0$. Then we have $A|_V = \pm\frac{1}{r}I$.

Case 1. Suppose that V is of dimension m. Then we have $A = \pm\frac{1}{r}I$. Therefore we see that $G(x) = \pm\frac{1}{r}x + b$, which shows that M is a hypersphere of radius r.

Case 2. Suppose that V is of dimension p with $2 \leqslant p \leqslant m-1$. Then the orthogonal complement $V^{\perp}$ of V is of dimension $(m-p)$ and it is contained in the tangent space T_xM for all $x \in M$. Around a fixed $x_0 \in M$, choose an orthonormal frame $E_1(x), \ldots, E_{m-1}(x)$ such that $E_1, \ldots, E_{m-p}$ are all constant vectors in $V^{\perp}$. Then we see that $\{E_{m-p+1}(x), \ldots, E_{m-1}(x), G(x)\}$ generates V. For the distribution T spanned by $\{E_{m-p+1}(x), \ldots, E_{m-1}(x)\}$, it is obvious that T is integrable and its integral submanifold M_1 through x_0 is nothing but the intersection $M_1 = M \cap (x_0 + V)$. Thus M is decomposed as $M = M_1 \times \mathbb{E}^{m-p}$, where $\mathbb{E}^{m-p} = V^{\perp}$.

Note that M_1 is a hypersurface in $V = \mathbb{E}^p$. Its Gauss map $G_1(x)$ in $\mathbb{E}^p$ satisfies

$$G_1(x) = G(x), \quad x \in M_1. \tag{2.11}$$

This shows that

$$G_1(x) = A_1x + b, \quad x \in M_1, \tag{2.12}$$

where A_1 denotes the $p \times p$ matrix $A|_V$. In fact, $A_1 = \pm\frac{1}{r}I$. Hence it follows from Case 1 that M_1 is a $(p-1)$-dimensional sphere $S^{p-1}(r)$, which shows that M is an open part of a generalized cylinder $S^{p-1}(r) \times \mathbb{E}^{m-p}$.

Case 3. Suppose that V is 1-dimensional. Then $G(x)$ is constant. Thus M is an open part of a hyperplane. The remaining part of the proof of Theorem A is straightforward.

3. Proof of theorem B

Let $X(s)$ be a unit speed curve in $\mathbb{E}^m$ which satisfies the Condition (C). Without loss of generality, we may assume that $X(s)$ lies fully in $\mathbb{E}^m$. Then on X, there exist points $X(t_0), X(t_1), \ldots, X(t_m)$ such that the set $\{X(t_j) - X(t_0) | j = 1, 2, \ldots, m\}$ spans $\mathbb{E}^m$. From the Condition (C) we have

$$\langle T(s), X(t_0)\rangle = \langle T(s), X(s)\rangle + \langle T(t_0), X(t_0)\rangle - \langle T(t_0), X(s)\rangle \tag{3.1}$$

and for $j = 1, 2, \ldots, m$ we also have

$$\left\langle T(s), X(t_j)\right\rangle = \langle T(s), X(s)\rangle + \left\langle T(t_j), X(t_j)\right\rangle - \left\langle T(t_j), X(s)\right\rangle. \tag{3.2}$$

Hence we obtain

$$\left\langle T(s), A_j\right\rangle = \left\langle B_j, X(s)\right\rangle + c_j, \quad j = 1, 2, \ldots, m, \tag{3.3}$$

where we denote for $j = 1, 2, \ldots, m$

$$A_j = X(t_j) - X(t_0), B_j = T(t_0) - T(t_j), \quad c_j = \left\langle T(t_j), X(t_j)\right\rangle - \langle T(t_0), X(t_0)\rangle.$$

As in Section 2, we get

Lemma 3.1. *For an $m \times m$ matrix A and a vector $b \in \mathbb{E}^m$ we have $X'(s) = AX(s) + b$.*

Together with the Condition (C), Lemma 3.1 shows that

$$\langle A(X(s) - X(t)), X(s) - X(t)\rangle = 0. \tag{3.4}$$

Hence for the symmetric matrix $B = A^t + A$, we also have

$$\langle B(X(s) - X(t)), X(s) - X(t)\rangle = 0. \tag{3.5}$$

This implies that

$$\langle BX(s), X(t_0)\rangle = \frac{1}{2}\left\{\langle BX(s), X(s)\rangle + \langle BX(t_0), X(t_0)\rangle\right\} \tag{3.6}$$

and for $j = 1, 2, \ldots, m$

$$\left\langle BX(s), X(t_j)\right\rangle = \frac{1}{2}\left\{\langle BX(s), X(s)\rangle + \left\langle BX(t_j), X(t_j)\right\rangle\right\}. \tag{3.7}$$

By subtracting (3.6) from (3.7), we get

$$\left\langle BX(s), A_j\right\rangle = d_j, \tag{3.8}$$

where we denote by A_j, d_j for $j = 1, 2, \ldots, m$

$$A_j = X(t_j) - X(t_0), \quad d_j = \frac{1}{2}\{\left\langle BX(t_j), X(t_j)\right\rangle - \langle BX(t_0), X(t_0)\rangle\}.$$

Since $\{A_j\}$ is a basis for $\mathbb{E}^m$, (3.8) shows that $BX(s)$ is a constant vector. In particular, $B(A_j) = BX(t_j) - BX(t_0) = 0$ for $j = 1, 2, \ldots, m$. This shows that B must vanish, that is, A is skew symmetric.

From $X'(s) = AX(s) + b$, we have

$$AX^{(k)}(s) = X^{(k+1)}(s), \quad k = 1, 2, \ldots, m.$$

Since A is skew symmetric, we see that for $k = 1, 2, \ldots, m$

$$\left\langle X^{(k)}(s), X^{(k)}(s)\right\rangle' = 2\left\langle AX^{(k)}(s), X^{(k)}(s)\right\rangle = 0.$$

Thus for each $k = 1, 2, \ldots, m$, $|X^{(k)}(s)|$ is constant.

The remaining part of the proof of Theorem B follows from [3,4] and a straightforward calculation.

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