# Analytic Multivalued Functions in Banach Algebras and Uniform Algebras* 

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## Introduction

In this paper many problems left unsolved in my book [6] are now solved using subharmonic techniques or the theory of several complex variables.

In the first section which concerns properties of the spectrum in Banach algebras I give some applications of the subharmonicity of $\lambda \rightarrow \log \delta_{n}(f(\lambda))$ and $\lambda \rightarrow \log c(f(\lambda))$, where $\lambda \rightarrow f(\lambda)$ is an analytic function from $\mathbb{C}$ into a Banach algebra and $\delta_{n}, c_{n}$ are, respectively, the $n$th diameter and the capacity of the spectrum of $f(\lambda)$. One important result is the general theorem of perturbation by inessential elements, which is a generalization, to any Banach algebra, of a result of I. C. Gohberg. It implies in particular the general conjecture of Perczyński for Banach algebras with involutions, which has been conjectured for a long time.

In the second section concerning uniform algebras and problems of analytic strucure, I give a simple proof of the subharmonicity of $\lambda \rightarrow \log \delta_{n}\left(K_{g}(\lambda)\right)$ and $\lambda \rightarrow \log c\left(K_{g}(\lambda)\right)$, where $\lambda \rightarrow K_{g}(\lambda)$ denotes the "fiber function." I conjectured this result several years ago and it was solved independently by Stodkowski [41] and Senichkin [39]. From this I can simplify and even generalize very strongly results of E. Bishop, R. Basener, B. Aupetit and J. Wermer, and N. Sibony, about analytic structure. I also give the very important proof by $Z$. Shodkowski on subharmonicity of $\lambda \rightarrow \log \rho\left(K_{g}(\lambda)\right)$, which avoids Rossi's local maximum principle. At the end of this part I also show that subharmonic methods can be used to prove results of Seidel-Frostman and Tsuji which concern cluster sets theory.

The third section is the most important because it includes and generalizes the two previous ones. It introduces a general theory of analytic multivalued functions which originates from my conjecture on the scarcity of operators with countable spectrum (Conjecture 3) and from some ideas of $Z$. Shodkowski. I give rather easy proofs of the fact that $\lambda \rightarrow \operatorname{Sp} f(\lambda)$ and $\lambda \rightarrow K_{g}(\lambda)$ are analytic multivalued functions. Then using several results of K .

[^0]Oka, K. Nishino and H. Yamaguchi, and theorems about domains of holomorphy, conjecture 3 can be proved. And this last conjecture implies the general conjecture of Pelczyński and all the problems of analytic structure in the case of finite or countable fibers.

To conclude I would like to thank John Wermer and Zbigniew Srodkowski for the many discussions we had on these problems. Of course many of their ideas have been used to improve this work.

## 1. Subharmonicity of the Spectrum in Banach Algebras

In [46, 47], E. Vesentini proved the subharmonicity of $\lambda \rightarrow \log \rho(f(\lambda))$, where $\lambda \rightarrow f(\lambda)$ is an analytic function from a domain $D$ of $\mathbb{C}$ into a Banach algebra $A$ and where $\rho(x)$ denotes the spectral radius of $x$ in $A$. From this, I proved the subharmonicity of $\lambda \rightarrow \log \delta(f(\lambda))$, where $\delta(x)$ denotes the diameter of the spectrum of $x$, and a lot of results (see Chaps. 1, 2 and 3 of [6]), in particular the following important scarcity theorem.

Theorem $1.1[4 ; 6, \mathrm{p} .66]$. Let $\lambda \rightarrow f(\lambda)$ be an analytic function from $a$ domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$ then:

- either the set of $\lambda$ for which $\operatorname{Sp} f(\lambda)$ is finite is of outer capacity zero,
- or there exists an integer $n \geqslant 1$ such that $\# \operatorname{Sp} f(\lambda)=n$, for every $\lambda$ in $D$, except on a closed discrete countable set $E$ of $D$. In this case the points of the spectrum vary holomorphically if $\lambda$ is outside of $E$.

This theorem has numerous applications (see [6]). For $K$ compact in $\mathbb{C}$ denote by $\delta_{n}(K)$ the $n t h$ diameter, i.e., $\operatorname{Max}\left(\prod_{i<j}\left|\lambda_{i}-\lambda_{j}\right|\right)^{2 /(n+1) n}$ for $\lambda_{1}, \ldots, \lambda_{n+1}$ in $K$. In [6, pp. 15, 68], I conjectured that the functions $\lambda \rightarrow \log \delta_{n}(f(\lambda))$ and $\lambda \rightarrow \log c(f(\lambda))$ are subharmonic, where $\delta_{n}(x)$ denotes $\delta_{n}(\mathrm{Sp} x)$ and $c(x)$ denotes the capacity of $\mathrm{Sp} x$ or equivalently the transfinite diameter $\lim _{n \rightarrow \infty} \delta_{n}(x)$. Clearly it is enough to prove the subharmonicity of the first one and this will give via Cartan's theorem (see the next remark) a very simple proof of Theorem 1.1. Z. Slodkowski [41] suggested the use of a classical theorem of M. Schechter of which I gave a simplified version in [6, p. 140].

Theorem 1.2 (M. Schechter [37]). Let $X_{1}, \ldots, X_{n}$ be complex Banach spaces and $Y$ the completion of $X_{1} \otimes \cdots \otimes X_{n}$ for some tensor norm and let $A_{i}$ be linear bounded operators on $X_{i}$, for $i=1, \ldots, n$. We define $T_{1}, \ldots, T_{n} \in \mathscr{L}(Y)$ by

$$
T_{i}=I_{1} \otimes \cdots \otimes I_{i-1} \otimes A_{i} \otimes I_{i+1} \otimes \cdots \otimes I_{n}
$$

where $I_{j}$ is the identity operator on $X_{j}$. The $T_{i}$ commute and for every polynomial $P\left(z_{1}, \ldots, z_{n}\right)$ of $n$ complex variables we have

$$
\mathrm{Sp}_{\mathscr{\mathscr { C }}(Y)} P\left(T_{1}, \ldots, T_{n}\right)=P\left(\mathrm{Sp}_{\mathscr{L}_{\left(X_{1}\right)}} A_{1}, \ldots, \mathrm{Sp}_{\mathscr{Q}_{\left(X_{n}\right)}} A_{n}\right) .
$$

We admit the proof of this theorem which is not too complicated (it uses the joint spectrum and elementary complex function theory) to obtain the following:

TheORem 1.3. Let $\lambda \rightarrow f(\lambda)$ be an analytic function from a domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$ and $P\left(z_{1}, \ldots, z_{n}\right)$ a polynomial of $n$ complex variables; then $\lambda \rightarrow \log \operatorname{Max}\left|p\left(\alpha_{1}, \ldots, \alpha_{n}\right)\right|$, for $\alpha_{i} \in \operatorname{Sp} f(\lambda)$, is subharmonic.

Proof. Considering the right representation $x \rightarrow T_{x}$ from $A$ into $\mathscr{L}(A)$ defined by $T_{x}: y \rightarrow x y$, it is easy to verify that $\mathrm{Sp}_{A} x=\mathrm{Sp}_{\notin(A)} T_{x}$ (see [34, Theorem 1.6.9, p. 32]). If $\lambda \rightarrow f(\lambda)$ is an analytic function from $D \subset \mathbb{C}$ into $A$ then $\lambda \rightarrow F(\lambda)=T_{f(\lambda)}$ is an analytic function from $D$ into $\mathscr{L}(A)$. Let $Y$ be the projective tensor product of $n$ copies equal to $A$ and consider

$$
F_{i}(\lambda)=I_{1} \otimes \cdots \otimes I_{i-1} \otimes F(\lambda) \otimes I_{i+1} \otimes \cdots \otimes I_{n} \in \mathscr{L}(Y)
$$

It is easy to verify that $\lambda \rightarrow F_{i}(\lambda)$ is an analytic function from $D$ into $\mathscr{L}(Y)$. By Schechter's theorem, $\operatorname{Max}\left|p\left(\alpha_{1}, \ldots, \alpha_{n}\right)\right|$, for $\alpha_{i} \in \operatorname{Sp} f(\lambda), i=1, \ldots, n$, is equal to $\rho\left(p\left(F_{1}(\lambda), \ldots, F_{n}(\lambda)\right)\right)$, but by Vesentini's theorem $\lambda \rightarrow$ $\log \rho\left(p\left(F_{1}(\lambda), \ldots, F_{n}(\lambda)\right)\right)$ is subharmonic because $\lambda \rightarrow p\left(F_{1}(\lambda), \ldots, F_{n}(\lambda)\right)$ is analytic from $D$ into $\mathscr{L}(Y)$.

Corollary 1.4. If $\lambda \rightarrow f(\lambda)$ is an analytic function from a domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$ then $\lambda \rightarrow \log \delta_{n}(f(\lambda))$ is subharmonic, for $n \geqslant 1$.

Proof. It is enough to use the previous theorem with $p\left(z_{1}, \ldots, z_{n}\right)=$ $\prod_{1 \leqslant i<j \leqslant n+1}\left(z_{i}-z_{j}\right)$.

Remark. Theorem 1.1 comes more easily from Corollary 1.4. If the set of $\lambda$, for which $\operatorname{Sp} f(\lambda)$ is finite, is of positive outer capacity then, using the same argument that I used, at the beginning of the proof given in $[6, \mathrm{p} .66]$, we conclude that there exists an integer $n \geqslant 1$ such that $\# \operatorname{Sp} f(\lambda) \leqslant n$ on a set $E_{n}$ with $c^{+}\left(E_{n}\right)>0$. Then $\log \delta_{n}(f(\lambda))=-\infty$ on $E_{n}$ and Cartan's theorem says that $\log \delta_{n}(f(\lambda)) \equiv-\infty$, i.e., $\neq \operatorname{Sp} f(\lambda) \leqslant n$ for every $\lambda \in D$. The end of proof is done in the same way. The important point is that Corollary 1.4 avoids localizing the problem to the case $n=1$ and gives directly the global result.

Corollary 1.5. If $\lambda \rightarrow f(\lambda)$ is an analytic function from a domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$ then $\lambda \rightarrow \log c(f(\lambda))$ is subharmonic.
Proof. It is enough to remark that $\log c(f(\lambda))$ is the decreasing limit of the sequence of $\log \delta_{n}(f(\lambda))$, when $n \rightarrow \infty$, and to use part 3 of Theorem 1 , Appendix II, in [6].

This result gives a local characterization of algebras which are quasialgebraic in the sense of Halmos ([19] or [6, p. 19])-an operator $x$ being quasi-algebraic iff $c(x)=0$ or equivalently if $\lim _{n \rightarrow \infty} \operatorname{Inf}_{p \in \mathcal{P}_{n}}\|p(x)\|^{1 / n}=0$, where $\mathscr{F}_{n}$ is the set of polynomials of degree $n$ with leading coefficient 1 .

Corollary 1.6. Let $A$ be a complex Banach algebra and $H$ a real linear subspace, such that $A=H+i H$, containing an absorbing subset $U$ such that each element $x$ of $U$ is quasi-algebraic; then every element of $A$ is quasi-algebraic.

Proof. It is similar to the proof of Theorem 2, p. 71 of [6].
Obviously this theorem can be applied for a real algebra with its complexified algebra, also for a complex algebra with an involution $x \rightarrow x^{*}$ and $H=\left\{x \mid x=x^{*}\right\}$ the set of self-adjoint elements.

From Corollary 1.6 comes in particular that the spectrum function $x \rightarrow \mathrm{Sp} x$ is continuous on $A$, by Newburgh's theorem, if all the elements of $U$ are quasi-algebraic.

Corollary 1.5 also gives the following theorem of pseudo-continuity:

Corollary 1.7. Let $\lambda \rightarrow f(\lambda)$ be an analytic function from a domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$ and $E$ be a subset of $D$, non-thin at $\lambda_{0} \in D$. If $f(\lambda)$ is quasi-algebraic for every $\lambda$ in $E$, then $f\left(\lambda_{0}\right)$ is quasialgebraic.

This can be applied, in particular, for $E$ being a Jordan arc and $\lambda_{0}$ one of its ends. Corollary 2, p. 34 of [6], derives, from Corollary 1.4, by using the same idea.
For a given Banach algebra $A$, let $Q$ denote the set of quasi-algebraic elements. Using the same ideas as those of pp. 95-96 of [6] it is possible to prove the equivalence of the stability of $Q$ by addition with the stability of $Q$ by multiplication. And as in Theorem 6 of [6], with one of these conditions, we conclude that either $Q$ is in the center of $A$ or $Q$ contains a maximal nonzero two-sided ideal contained in $Q$.

If $k h(\operatorname{soc} A)$ denotes the intersection of all the primitive ideals of $A$ containing the socle of $A$ (for definition see, for instance, $[6$, p. 79]) then by the following Theorem 1.9, we have $c(a+x)=c(x)$, for every $x$ in $A$, if $a$ is in $k h(\operatorname{soc} A)+C I$. Conversely, what can be said about the set
$E=\{a \mid c(a+x)=c(x)$ for every $x \in A\}$ (This problem is settled in [44])? By using Corollary 1.5 one can prove that $E$ is a Lie ideal of $A$. The stability of $E$ by addition and scalar multiplication is obvious. Let us now consider $a \in E, x$ and $y$ in $A$ arbitrary, and $\lambda \in \mathbb{C}$. We have

$$
\begin{aligned}
& c\left(e^{\lambda y} a e^{-\lambda y}-a+\lambda x\right) \\
& \quad=c\left(a+e^{-\lambda y}(\lambda x-a) e^{\lambda y}\right)=c\left(e^{-\lambda y}(\lambda x-a) e^{\lambda y}\right) \\
& \quad=c(\lambda x-a)=c(\lambda x)=c\left(\lambda[y, a]+\left(\lambda^{2} / 2\right)[y,[y, a]]+\cdots+\lambda x\right)
\end{aligned}
$$

consequently $c(x)=c(f(\lambda)+x)$, where $f(\lambda)=[y, a]+(\lambda / 2)[y,[y, a]]+\cdots$, if $\lambda \neq 0$. But by Corollary 1.5, $\lambda \rightarrow c(f(\lambda)+x)$ is subharmonic; hence $c(f(0)+x)=\lim _{\lambda \rightarrow 0} c(f(\lambda)+x)=c(x)$, for every $x \in A$ (see [6, Appendix II, Corollary 1]), and this says that $[y, a] \in E$.

The following conjecture about $C^{*}$-algebras, named the Pelczynski conjecture in Poland (probably because it originates from the work of A. Pelczyński and Z. Semadeni [33]), has been studied for a long time.

Conjecture 1. Let $A$ be a $C^{*}$-algebra; suppose that every self-adjoint element has a countable spectrum; then every element of $A$ has a countable spectrum and $A$ has a particular algebraic structure.

This conjecture has been given also with equivalent conditions on scattered algebras by H. E. Jensen [25], and recently solved by purely $C^{*}$ algebras techniques (which cannot be extended to other situations) by T . Huruya $[23]$ in the following form:

Theorem (T. Huruya). In a separable $C^{*}$-algebra $A$ the following conditions are equivalent:
$1^{\circ}$ Every self-adjoint element has a countable spectrum.
$2^{\circ}$ Every element of $A$ has a countable spectrum.
$3^{\circ}$ A admits a composition sequence $\left(I_{\alpha}\right)_{0 \leqslant \alpha \leqslant \alpha_{0}}$ of two-sided closed ideals indexed by first-class and second-class ordinals such that $I_{a+1} / I_{\alpha}$ is a dual $C^{*}$-algebra.
$4^{\circ}$ The enveloping von Neumann's algebra of $A$ is a sum of type $I$ factors.

During the Spectral Theory Semester held at the Banach Center of Warsaw (from September 1977 to December 1977) I lectured on the use of subharmonicity in functional analysis and precisely on this conjecture, showing how subharmonic techiques would be a good tool to prove this problem and even the more general one which follows:

Conjecture 2. Let $A$ be a complex Banach algebra with an involution $x \rightarrow x^{*}$ and let $H$ be the set of self-adjoint elements. Suppose that $H$ contains
a non-void open subset $U$ such that $\mathrm{Sp} h$ is countable for every $h$ in $U$; then $\operatorname{Sp} x$ is countable for every $x$ in $A$.

I succeeded in proving it partially (see [6, pp. 86-87]), supposing $\operatorname{Sp} h$ with a finite number of limit points, for every $h$ self-adjoint.

For the general situation I explained that the proof would probably be similar, with an argument of condensation of singularities. Recently E . Kirchberg proved Conjecture 2 with $U=H$, but his proof is lengthy, complicated and artificially constructed [26].

I shall now give a proof of this conjecture, obtained simultaneously with that of E. Kirchberg, using a nice theorem of perturbation of Gohberg's type, whose proof is purely subharmonic. With this I also give a theorem of algebraic structure, completely similar to the result of T. Huruya, in the case of separable Banach algebras.

In fact, as we shall see in Section 3, all these results will be generalized by a scarcity theorem of countable type, similar to Theorem 1.1 (conjectured also since 1977, see [4] or [6, p. 68]). Obviously these new and very elegant results will be excessively more complicated to prove. For this we must use theorems concerning several complex variables.

Let $A$ be a complex Banach algebra; we shall say that an element is inessential if it is in $k h(\operatorname{soc} A)$. For $x$ in $A$ we denote by $(\mathrm{Sp} x)^{(\alpha)}$ the $\alpha-$ topological derivative of $\mathrm{Sp} x$ for every ordinal $\alpha$. Also we denote by $D(x)$ the set of $\lambda$ in the spectrum of $x$ which are not isolated spectral values with the associated idempotent in the socle of $A$. It is easy to see that $D(x+\alpha)=$ $\alpha+D(x)$, for $\alpha \in \mathbb{C}$.

Lemma 1.8. If $p$ is an idempotent of $k h(\operatorname{soc} A)$ then $p$ is in $\operatorname{soc} A$.
Proof. If $p$ is in $k h(\operatorname{soc} A)$ then $p^{\prime} \in \operatorname{Rad}(A / \overline{\operatorname{soc} A})$, where $p^{\prime}$ denotes the class of $p$ in the quotient algebra $A / \operatorname{soc} A$. Then $\rho\left(p^{\prime}\right)=\lim _{n \rightarrow \infty}\left\|p^{\prime n}\right\|^{1 / n}=0$. But $p^{\prime n}=p^{\prime}$ for every $n$, so $p^{\prime}=0$, i.e., $\mathrm{p} \in \operatorname{soc} A$. It is easy to see that $p(\overline{\operatorname{soc}} A) p$ is a closed subalgebra of $A$; hence a Banach algebra with identity $p$, in which $p(\operatorname{soc} A) p$ is a dense two-sided ideal. Consequently we have $p(\operatorname{soc} A) p=p(\operatorname{soc} A) p$ which contains $p$, so $p \in p(\operatorname{soc} A) p \subset \operatorname{soc} A$.

Theorem 1.9 (of perturbation by inessential elements). Let $A$ be a complex Banach algebra, then for every $x$ in $A$ and $y$ in $k h(\operatorname{soc} A)$ we have:
$1^{\circ} \quad D(x) \subset \mathrm{Sp}(x+y)$.
$2^{\circ} \mathrm{Sp} \dot{x} \subset D(x) \cup\{0\}$ and $D(x) \subset \sigma(\bar{x})$, where $\bar{x}$ denotes the class of $x$ in $A / k h(\operatorname{soc} A)$ and $\sigma$ the full spectrum, i.e., the union of the spectrum with its holes.
$3^{\circ}$ In particular $\sigma(x)^{(\alpha)}=\sigma(\bar{x})^{(\alpha)}$ for every ordinal $\alpha>0$.
Proof. $1^{\circ}$ Suppose that $\alpha \notin \operatorname{Sp}(x+y)$ with $\alpha \in \operatorname{Sp} x$. From the relation

$$
\begin{equation*}
\alpha-x=(\alpha-(x+y))\left[1+(\alpha-(x+y))^{-1} y\right] \tag{1}
\end{equation*}
$$

we obtain that $1+(\alpha-(x+y))^{-1} y$ is non-invertible. Let us consider $f(\lambda)=(\lambda-(x+y))^{-1} y$ for $|\lambda-\alpha|<r$, with $r$ small enough that $\lambda \notin \operatorname{Sp}(x+y)$ in this case. By the Ruston characterization of Riesz operators ([43, Lemma 5.2] or [6, Theorem 2, p. 83]) we know that -1 is isolated in $\mathrm{Sp} f(\lambda)$. We choose an open disk $D$ centered at $\alpha$ such that $D \cap \operatorname{Sp} f(\alpha)=$ $\{-1\}$ and $\operatorname{Sp}(\alpha) \cap \partial D=\varnothing$. Because $\operatorname{Sp} f(\lambda)$ has at most 0 as a limit point there exists $r_{1} \leqslant r$ such that $|\lambda-\alpha|<r_{1}$ implies \# $(\operatorname{Sp} f(\lambda) \cap D)<\infty$. By the localized scarcity theorem (Theorems 1 and 2, pp. 66-67 in [6]) or by the help of Corollary 1.4 we may affirm that $\operatorname{Sp} f(\lambda) \cap D=\left\{\alpha_{1}(\lambda), \ldots, \alpha_{n}(\lambda)\right\}$ for $|\lambda-\alpha|<r_{1}$, outside of a closed discrete set $E$, for some integer $n \geqslant 1$ and where the functions $\alpha_{1}, \ldots, \alpha_{n}$ are continuous on the disk and holomorphic outside of $E$. Because $E$ is discrete in $\left\{\lambda\left||\lambda-\alpha| \leqslant r_{1}\right\}\right.$ its limit points are on the boundary of this disk, so there exists $r_{2} \leqslant r_{1}$ such that for $|\lambda-\alpha|<r_{2}$ the functions $\alpha_{1}, \ldots, \alpha_{n}$ have at most a singularity at $\alpha$ and are holomorphic for $0<|\lambda-\alpha|<r_{2}$. We take $\psi(\lambda)=\left(1+\alpha_{1}(\lambda)\right) \ldots\left(1+\alpha_{n}(\lambda)\right)$ for $0<|\lambda-\alpha|<r_{2}$ and $\psi(\alpha)=0$. This function is continuous for $|\lambda-\alpha|<r_{2}$ and holomorphic outside of is zeroes so, by Radós extension theorem (see [50, Chap. $1^{\circ}$ ] or [6, Appendix II, p. 173]), it is holomorphic everywhere. Consequently there exists $r_{3} \leqslant r_{2}$ such that $0<|\lambda-\alpha|<r_{3}$ implies $1+(\lambda-(x+y))^{-1} y$ invertible, so by (1), $\lambda-x$ is invertible, which means that $\alpha$ is an isolated spectral value. We now show that the idempotent associated to $\alpha$ is in the socle of $A$. By Lemma 1.8 it is enough to prove that it is in $k h(\operatorname{soc} A)$. Let us consider a circle $\Gamma$ of center $\alpha$, of radius strictly less than $r_{3}$ and let $\mu$ on $\Gamma$. Be relation (1) we have

$$
\begin{equation*}
p=\int_{\Gamma}(x-\mu)^{-1} d \mu=\int_{\Gamma}[1+f(\mu)]^{-1}(\mu-(x+y))^{-1} d \mu \tag{2}
\end{equation*}
$$

If we put $[1+f(\mu)]^{-1}=1+g(\mu)$ then $f(\mu)+g(\mu)+f(\mu) g(\mu)=0$ and $f(\mu) \in k h(\operatorname{soc} A)$, because $y$ is in $k h(\operatorname{soc} A)$, which is a closed two-sided ideal of $A$. Then $g(\mu) \in k h(\operatorname{soc} A)$, but in the relation

$$
\begin{equation*}
p=\int_{\Gamma}(\mu-(x+y))^{-1} d \mu+\int_{\Gamma} g(\mu)(\mu-(x+y))^{-1} d \mu \tag{3}
\end{equation*}
$$

the first right term is 0 because $\lambda \rightarrow(\lambda-(x+y))^{-1}$ is analytic for $|\lambda-\alpha|<r_{3}$ and the second right term is in $k h(\operatorname{soc} A)$.
$2^{\circ}$ Suppose $\alpha \notin D(x)$ with $\alpha \neq 0$ and $\alpha \in \operatorname{Sp} x$; then $\alpha$ is an isolated spectral value whose idempotent $p$ is in soc $A$. In the quotient algebra $A / k h(\operatorname{soc} A)$ we have $\operatorname{Sp} \bar{x}=\operatorname{Sp} \overline{x-\alpha p} \subset \operatorname{Sp}(x-\alpha p)$, but by the holomorphic functional calculus and the fact that $\alpha \neq 0$ we have $\alpha \notin \operatorname{Sp}(x-\alpha p)$ so $\alpha \notin \operatorname{Sp} \bar{x}$. So the first inclusion is proved. By $1^{\circ}$ we know
that $D(x) \subset \operatorname{Sp}(x+y)$ for every $y \in k h(\operatorname{soc} A)$, so $D(x) \subset \bigcap \operatorname{Sp}(x+y)$, for $y \in k h(\operatorname{soc} A)$. By Harte's theorem ([20] or [6, p. 6]) we obtain the second inclusion.
$3^{\circ}$ If we write $\operatorname{Sp} x=D(x) \cup E_{1} \cup E_{2}$, where $E_{1}$ is discrete and contained in the polynomially convex hull $D(x)^{\wedge}$ of $D(x)$, and where $E_{2}$ is discrete, disjoint form $D(x)^{\wedge}$, we get easily that $\sigma(x)=D(x)^{\wedge} \cup E_{2}$, so $\sigma(x)^{\prime}=\left(D(X)^{\wedge}\right)^{\prime} \subset \sigma(\bar{x})^{\prime}$, by relation $2^{\circ}$. But obviously $\sigma(\bar{x}) \subset \sigma(x)$ implies $\sigma(\bar{x})^{\prime} \subset \sigma(x)^{\prime}$, so we have equality. By transfinite induction it is easy to conclude that $\sigma(x)^{(\alpha)}=\sigma(\bar{x})^{(\alpha)}$, considering $\alpha$ as a limit ordinal or a nonlimit ordinal.

Instead of considering all $k h(\operatorname{soc} A)$ it is enough to consider a closed twosided ideal I included in $k h(\operatorname{soc} A)$ and we obtain the same result with $D(x)$ slightly modified (the idempotent associated with an isolated spectral value is in $I$.

When $A=\mathscr{L}(H)$, where $H$ is a Hilbert space, we have $k h(\operatorname{soc} A)=\mathscr{L} \mathscr{C}(H)$, and the quotient algebra $A / k h(\operatorname{soc} A)$ is then the classical Calkin algebra. This is the reason why generally we shall give the name of first Calkin algebra associated to $A$ to the algebra $A / k h(\operatorname{soc} A)$. This perturbation theorem and the ideas contained in the proof obviously generalize the results of I. C. Gohberg on the holomorphic variation of spectral values of analytic families of compact operators in $\mathscr{L C} \mathscr{C}(H)$ and on the perturbation by compact operators (see [16, pp. 20-24]).

This also gives a new proof of a result of D. S. G. Stirling [44, Theorem 4 and Corollary 5].

Corollary 1.10. Let a be a complex Banach algebra; then for every $x$ in $A$ we have $c(x)=c(\bar{x})$, where $\bar{x}$ is the class of $x$ in the associated Calkin algebra. Consequently we have $c(x)=c(x+y)$, for every $y$ in $k h(\operatorname{soc} A)$.

Proof. Because two compacts of same outer boundary have the same capacity and using property $2^{\circ}$ in Theorem 1.9 , we deduce that $c(D(x))=c(\bar{x})$. But $D(x)=\operatorname{Sp} x \backslash G$, where $G$ is a countable borelian set, so by Theorem III. 18 p. 63 of [45] we have $c(D(x))=c(x)$.

We may begin now the proof of Conjecture 2. First we need a lemma and some terminology.

Lemma 1.11. Let $A$ be a complex Banach algebra and $\left(I_{n}\right)_{n>1}$ an increasing sequence of closed two-sided ideals of A. Taking $I=\bigcup_{n>i} I_{n}$ and
denoting by $\phi_{n}: A \rightarrow A_{n}=A / I_{n}$ and $\phi: A \rightarrow A_{\omega}=A / I$ the corresponding canonical morphisms, we have

$$
\sigma(\phi(x))=\bigcap_{n \geqslant 1} \sigma\left(\phi_{n}(x)\right),
$$

where $\sigma$ denotes the full spectrum.
Proof. By Harte's theorem mentioned just before, $\bigcap_{y \in I} \operatorname{Sp}(x+y) \subset$ $\sigma(\phi(x))$. Let $U$ be an open set containing this intersection; by a compacity argument we conclude that there exists $y_{1}, \ldots, y_{k} \in I$ such that $\mathrm{Sp}\left(x+y_{1}\right) \cap \cdots \cap \mathrm{Sp}\left(x+y_{k}\right) \subset U$. Let us show now that there exists open sets $V_{1}, \ldots, V_{k}$ such that $\operatorname{Sp}\left(x+y_{i}\right) \subset V_{i}$, for $i=1, \ldots, k$, and $V_{1} \cap \ldots$ $\cap V_{k} \subset U$. If this is not true, there exists a sequence $\left(z_{n}\right)$ such that $z_{n} \notin U$ and dist $\left(z_{n}, \operatorname{Sp}\left(x+y_{i}\right)\right)<1 / n$, for $i=1, \ldots, k$. This sequence is bounded and contains a converging subsequence, so we may suppose that $z_{n}$ converges to $z$, when $n$ goes to infinity. We have $z \notin U$ and $\operatorname{dist}\left(z, \operatorname{Sp}\left(x+y_{i}\right)\right) \leqslant 1 / n$, for every $i=1, \ldots, k$ and for every $n$; then $z \in \bigcap_{i=1}^{k} \operatorname{Sp}\left(x+y_{i}\right) \subset U$, a contradiction. By upper semi-continuity of the spectrum there exists $u_{1}, \ldots, u_{k} \in U_{n \geqslant 1} I_{n}$ such that $\operatorname{Sp}\left(x+u_{i}\right) \subset V_{i}$, for $i=1, \ldots, k$; hence $\operatorname{Sp}\left(x+u_{1}\right) \cap \cdots \cap \operatorname{Sp}\left(x+u_{k}\right) \subset U$. Let $m$ be the smallest integer for which $u_{1}, \ldots, u_{k} \in I_{m}$; then $u_{1}, \ldots, u_{k} \in I_{n}$ if $n \geqslant m$, so $\operatorname{Sp} \phi_{n}(x) \subset \operatorname{Sp}\left(x+u_{1}\right) \cap \cdots \cap$ $\operatorname{Sp}\left(x+u_{k}\right) \subset U$. But this is true for every open set $U$ containing $\bigcap_{y \in I} \operatorname{Sp}(x+y)$ so $\operatorname{Sp} \phi_{n}(x) \subset \cap \operatorname{Sp}(x+y) \subset \sigma(\phi(x))$ and $\bigcap_{n \geqslant 1} \sigma\left(\phi_{n}(x)\right) \subset$ $\sigma(\phi(x))$. The converse inclusion is easy if we observe that $\sigma(\phi(x)) \subset \sigma\left(\phi_{n}(x)\right)$, for $n \geqslant 1$, because of the canonical morphism from $A_{n}$ onto $A_{\omega}$.

Let $A$ be an arbitrary complex Banach algebra. We take $A_{0}=A / \operatorname{Rad} A, A_{1}=A_{0} / k h\left(\operatorname{soc} A_{0}\right)$, the first Calkin algebra, and inductively we define $A_{n}=A_{n-1} / k h\left(\operatorname{soc} A_{n-1}\right)$. The corresponding morphisms of $A$ onto $A_{0}, A_{1}, \ldots, A_{n}, \ldots$, are denoted by $\phi_{0}, \phi_{1}, \ldots, \phi_{n}, \ldots$, and their kernels by $I_{0}=\operatorname{Rad} A, I_{1}=k h(\operatorname{soc} A), I_{2}, \ldots, I_{n}=\operatorname{Ker} \phi_{n}, \ldots$. Then we can define $A_{\omega}$ with $I_{\omega}=\bigcup_{n \geqslant 1} I_{n}$, and $\phi_{\omega}$. For every ordinal $\alpha$ of the first class and the second class it is then possible by transfinite induction to define $A_{\alpha}$ and $\phi_{\alpha}$ in the following way:
-- if $\alpha$ is not a limit ordinal, $A_{\alpha}=A_{\alpha-1} / k h\left(\operatorname{soc} A_{\alpha-1}\right)$ and $\phi_{\alpha}=\pi_{\alpha-1} \circ \phi_{\alpha-1}$, where $\pi_{\alpha-1}$ is the canonical morphism from $A_{\alpha-1}$ onto $A_{\alpha}$,

- if $\alpha$ is a limit ordinal we take $I_{\alpha}=\overline{\bigcup_{B<\alpha} I_{\beta}}, A_{\alpha}=A / I_{\alpha}$ and $\phi_{\alpha}$ the corresponding canonical morphism.

By definition we shall say that $A_{\alpha}$ is the $\alpha$-Calkin algebra associated to $A$. It is easy to verify that it is semi-simple.

In the following, $\alpha \in \mathscr{F}$ will mean that $\alpha$ is an ordinal of the first class or the second class (see [60, Chap. 15]).

Lemma 1.12. Let $A$ be a complex Banach algebra, $\left(A_{\alpha}\right)_{a \in \mathscr{F}}$ and $\left(\phi_{a}\right)_{\alpha \in \mathscr{F}}$ the corresponding Calkin algebras and morphisms. For every $\alpha \in \mathscr{F}$ and every $x$ in $A$ we have $\sigma(x)^{(\alpha)} \subset \sigma\left(\phi_{\alpha}(x)\right)$. Consequently if $\sigma\left(\phi_{\alpha}(x)\right)$ is finite for some $\alpha \in \mathscr{F}$ and some $x$ in $A$, then $\operatorname{Sp} x$ is countable.

Proof. By Theorem 1.9 we have $\sigma(x)^{\prime}=\sigma\left(\phi_{1}(x)\right)^{\prime}$ so the inclusion $\sigma(x)^{\prime} \subset \sigma\left(\phi_{1}(x)\right)$. We prove the result by transfinite induction supposing this result true for $\beta<\alpha$. If $\alpha$ is not a limit ordinal we have

$$
\sigma(x)^{(\alpha)}=\left(\sigma(x)^{(\alpha-1)}\right)^{\prime} \subset \sigma\left(\phi_{\alpha-1}(x)\right)^{\prime}
$$

But applying Theorem 1.9 to $\pi_{\alpha-1}: A_{\alpha-1} \rightarrow A_{\alpha}$ we get $\sigma\left(\phi_{\alpha-1}(x)\right)^{\prime}=$ $\sigma\left(\phi_{a}(x)\right)^{\prime}$; hence $\sigma(x)^{(\alpha)} \subset \sigma\left(\phi_{\alpha}(x)\right)^{\prime} \subset \sigma\left(\phi_{\alpha}(x)\right)$. If $\alpha$ is a limit ordinal

$$
\sigma(x)^{(\alpha)}=\bigcap_{\beta<\alpha} \sigma(x)^{(\beta)} \subset \bigcap_{B<\alpha} \sigma\left(\phi_{\beta}(x)\right)=\sigma\left(\phi_{\alpha}(x)\right)
$$

by Lemma 1.11. If now $\sigma\left(\phi_{\alpha}(x)\right)$ is finite then $\sigma(x)^{(\alpha)}$ is finite so $\sigma(x)$ is countable and $\mathrm{Sp} x$ also.

In a real vector space $H$ we shall say that a set $F$ is absorbing if for every $a \in F$ and for every $x \in H$ there exists $s>0$ such that $a+\lambda(x-a) \in F$ for $-s \leqslant \lambda \leqslant s$. For example, an open set of $H$ is absorbing. For $x$ in $A$ and $r>0, B(x, r)$ will denote $\{x\|\|x-a\|<r\}$ and $\bar{B}(x, r)$ will denote $\{x \mid\|x-a\| \leqslant r\}$.

Lemma 1.13. Let $A$ be a complex semi-simple Banach algebra. Suppose that $H$ is a real subspace of $A$ such that $A=H+i H$ and that $H$ contains a closed absorbing set $F$ such that $\mathrm{Sp} x$ is countable for every $x$ in $F$. If $\mathrm{Sp} x$ is not identical to $\{0\}$ on $F$, then there exists $x$ in $F$ such that $\operatorname{Sp} x \neq \operatorname{Sp} \phi_{1}(x)$, where $\phi_{1}$ is the canonical morphism from $A$ onto the first Calkin algebra.

Proof. We suppose on the contrary that $\mathrm{Sp} x=\operatorname{Sp} \phi_{1}(x)$ for every $x$ in $F$. If $\operatorname{Sp} x$ is finite for every $x$ in $F$, then by Theorem 1.1 and the argument used, for example, in the proofs of Theorems 1 and $2, \mathrm{pp} .70-71$, of [6] we conclude that $A / \operatorname{Rad} A$ is finite dimensional, and so $A_{1}=\{0\}$; that is to say, with the hypothesis given in the beginning, that $\operatorname{Sp} x=\{0\}$ for every $x$ in $F$, which is a contradiction. Let $x_{0}$ be in $F$ such that $\operatorname{Sp} x_{0}$ has at least one limit point. Because $\operatorname{Sp} x=\operatorname{Sp} \phi_{1}(x)$, by Theorem 1.9 of perturbation, every isolated spectral value of $\operatorname{Sp} x_{0}$ has its associated idempotent which is not in $\operatorname{soc} A$. Let $\alpha_{0}, \alpha_{1}$ be isolated in $\operatorname{Sp} x_{0}$ and be different. We choose two open disjoint disks $D_{0}, D_{1}$ such that

$$
\begin{aligned}
\operatorname{Sp} x_{0} \cap D_{0} & =\left\{\alpha_{0}\right\}, & & \operatorname{Sp} x_{0} \cap D_{1}=\left\{\alpha_{1}\right\} \\
\operatorname{Sp} x_{0} \cap \partial D_{0} & =\varnothing, & & \operatorname{Sp} x_{0} \cap \partial D_{1}=\varnothing
\end{aligned}
$$

For $\left\|x-x_{0}\right\|<r$, with $r$ small enough, $\operatorname{Sp} x \cap \partial D_{i}=\varnothing$, for $i=0,1$ (it is upper semi-continuity of the spectrum).

We now show that there exists $x_{1} \in F \cap \bar{B}\left(x_{0}, r / 2\right)$ such that $S p x_{1} \cap D_{i}$ be infinite, for $i=0,1$. We suppose the contrary and put $E=F \cap \bar{B}\left(x_{0}, r / 2\right)$. We have

$$
\begin{aligned}
E & =\bigcup_{n=1}^{\infty}\left\{x \in E \mid \#\left(\operatorname{Sp} x \cap D_{0}\right) \leqslant n\right\} \\
& \cup \bigcup_{n=1}^{\infty}\left\{x \in E \mid \#\left(\operatorname{Sp} x \cap D_{1}\right) \leqslant n\right\} .
\end{aligned}
$$

Because the spectrum is countable on $E$, the spectrum function is continuous on $E$, by Newburgh's theorem (see [6, p. 8]), so any of the previous sets in the two unions is closed. By using Baire's argument to $E$, we conclude that there exists some integer $m$ such that, for instance, \# $\left(\mathrm{Sp} x \cap D_{0}\right) \leqslant m$, for $x \in E \cap B$, where $B$ is some open ball. But $E \cap B$ is an absorbing subset of $H$ (being the non-void intersection of two absorbing sets), then by the scarcity theorem (Theorem 1.1) applied to $p A p=p H p+i p H p$ [6, Theorem 2, p. 71], with $p$ the idempotent associated to $x_{0}$ and the disk $D_{0}$, we obtain that $p A p / \operatorname{Rad}(p A p)$ is finite dimensional. But $p A p$ is semisimple so $p A p$ is finite dimensional which implies that $p \in \operatorname{soc} A[6$, Lemma 4, p. 81] which is a contradiction. Now we know that there exists $x_{1} \in F$ such that $\left\|x_{1}-x_{0}\right\| \leqslant r / 2$ having at least four distinct and isolated points $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}$ in its spectrum such that $\alpha_{00}, \alpha_{01} \in \operatorname{Sp} x_{0} \cap D_{0}$ and $\alpha_{10}, \alpha_{11} \in \operatorname{Sp} x_{0} \cap D_{1}$. To these points we may associate four open and disjoint disks $D_{00}, D_{01}, D_{10}, D_{11}$ such that $\alpha_{i j}$ is the center of $D_{i j}, D_{00}$ and $D_{01}$ are included in $D_{0}$ and $D_{10}$ and $D_{11}$ are included in $D_{1}$. By induction, with a similar argument such as the one used previously, we can construct a sequence $\left(x_{n}\right)$ such that:

$$
1^{\circ} \quad\left\|x_{n+1}-x_{n}\right\| \leqslant r / 2^{n+1}
$$

$2^{\circ} \operatorname{Sp} x_{n}$ contains $2^{n}$ isolated and distinct points $\alpha_{i_{1} \ldots i_{n}}$ where $i_{k}$ takes the values 0,1 and $k$ goes from 1 to $n$.
$3^{\circ}$ each $\alpha_{i_{1} \ldots i_{n}}$ is the center of an open disk $D_{i_{1} \ldots i_{n}}$, in such a way that the $D_{i_{1}, \ldots, i_{n}}$, are all disjoint.

Then $\left(x_{n}\right)$ is a Cauchy sequence converging to $x \in F$ (because $F$ is closed). To obtain a contradiction we shall prove that the spectrum of $x$ is not countable.

Let $I=\left(i_{1}, i_{2}, \ldots, i_{n}, \ldots,\right)$ be an arbitrary sequence of 0 's and l's. A subsequence of $\alpha_{i_{1}}, \alpha_{i_{1} i_{2}}, \alpha_{i_{1} i_{2} i_{3}}, \ldots$, converges to $\alpha_{1}$ which is in Sp $x$. If $I \neq J$ then for some index $k$ we have $i_{k} \neq j_{k}$ with $i_{l}=j_{l}$ for $1 \leqslant l \leqslant k$. We have $\alpha_{I} \in D_{i_{1} i_{2} \cdots i_{k}}$ and $\alpha_{J} \in D_{i_{1} i_{2} \cdots i_{k-1} j_{k}}$ and these two disks are disjoint, so
$a_{I} \neq \alpha_{j}$. But the set of sequences $I$ has the cardinality of the set of the reals and the application $I \rightarrow a_{I}$ is injective, so $\operatorname{Sp} x$ is uncountable.

The end of this argument is exactly the same one used by R. Basener in [8] to extend the Bishop's analytic structure theorem in the countable case.

Theorem 1.14. Let A be a complex separable Banach algebra. Suppose that $H$ is a closed real subspace of $A$ such that $A=H+i H$ and that $H$ contains a closed absorbing set $F$ such that $\mathrm{Sp} x$ is countable for every $x$ in $F$. Then there exists $\alpha_{0} \in \mathscr{F}$ such that $A_{\alpha_{0}}=\{0\}$ which implies that $\operatorname{Sp} x$ is countable for every $x$ in $A$. There also exists an ordinal composition sequence $\left(I_{a}\right)_{\alpha<\alpha_{0}}$ of closed two-sided ideals of $A$ such that $I_{0}=\operatorname{Rad} A, I_{a_{0}}=A$ and $I_{\alpha+1} / I_{\alpha}$ is modular annihilator for every $\alpha<\alpha_{0}$.

Proof. By Lemma 1.12 it is sufficient to prove that for every $x$ in $A$ there exists $\alpha \in \mathscr{F}$ such that $\sigma\left(\phi_{a}(x)\right)$ is finite. We put $\sigma_{a}(x)=\sigma\left(\phi_{\alpha}(x)\right)$, for $\alpha \in \mathscr{F}$ and let $F_{\alpha}$ be the set $\left\{x \in F \mid \sigma_{\alpha}(x)=\sigma_{\beta}(x)\right.$ for every $\left.\beta \geqslant \alpha, \beta \in \mathscr{F}\right\}$. Because $\mathrm{Sp} x$ is countable on $F$, the $\sigma_{\alpha}(x)$ are countable on $F$; hence, by Newburgh's theorem, $F_{\alpha}$ is closed in $F$. For $x$ fixed, $\left(\sigma_{\alpha}(x)\right)_{\alpha<0}$ is decreasing so it must stabilize for some ordinal (see [28, p. 146]) so $F=\bigcup_{\alpha \in \mathcal{G}} F_{\alpha}$ where the ordinal family $F_{\alpha}$ is increasing. Using the fact that $A$ is separable, $F$ as a topological space has a countable basis, so by Theorem 3, p. 146 of [28], the previous union is countable. By Baire's argument applied to $F$, which is complete because it is closed in $A$, some of the $F_{\alpha}$ contain a nonvoid open set of $F$, so a non-void absorbing set in $H$. Let $\alpha_{0}$ be the smallest ordinal in $\mathscr{F}$ such that $F_{\alpha_{0}}$ contains a non-void absorbing set in $H$. Taking the closure $E$, relatively to $H$, of this absorbing set contained in $F_{\alpha_{0}}$ and applying Lemma 1.13 to the algebra $A_{\alpha_{0}}$, we conclude that $\operatorname{Sp} \phi_{\alpha_{0}}(x)=\{0\}$ for every $x$ in $E$, and consequently for every $x$ in $A$; hence $A_{\alpha_{0}}=\{0\}$. By Lemma 1.12, $\operatorname{Sp} x$ is countable for every $x$ in $A$. Taking the $I_{\alpha}$ as defined after Lemma 1.11, it is evident that $I_{0}=\operatorname{Rad} A, I_{\alpha_{0}}=A$. Obviously $I_{\alpha+1} / I_{\alpha}$ can be identified with $k h\left(\operatorname{soc} A_{\alpha}\right)$. By Ruston's theorem [6, p. 83], every element of $k h\left(\operatorname{soc} A_{\alpha}\right)$ has its spectrum with at most 0 as limit point. By Barnes characterization of modulator algebras (see [ $6, \mathrm{p}, 82$ ) and the fact that $\operatorname{Rad} k h\left(\operatorname{soc} A_{\alpha}\right)=k h\left(\operatorname{soc} A_{\alpha}\right) \cap \operatorname{Rad} A_{\alpha}=\{0\}$ (see [12, Corollary 20, p. 126]) we have that $I_{\alpha+1} / I_{\alpha}$ is modular annihilator.

If $A$ is a $C^{*}$-algebra every closed two-sided ideal $J$ is self-adjoint and $A / J$ is also a $C^{*}$-algebra. From Theorem 4.5 of [43] we deduce that $k h(\operatorname{soc} A)=\overline{\operatorname{soc} A}$. By Corollary 10, p. 163 of [12], $k h(\operatorname{soc} A)$ is a semisimple annihilator $C^{*}$-algebra. But it is well known that for $C^{*}$-algebras the notions of annihilator algebras, dual algebras and compact algebras are equivalent (see [3, pp. 16-17; 12, p. 171; 34, Corollary 4.10.26, p. 272]). So Theorem 1.14 gives in this situation the theorem of T. Huruya.

By using Corollary 3.18 of Section 3, which is a scarcity theorem for countable spectrum, it is possible, as we shall see later on, to prove directly that $\operatorname{Sp} x$ is countable on all $A$ if it is countable on the closed absorbing set $F$ (moreover the hypothesis " $F$ closed" is not necessary). But to obtain the structure theorem concerning $A$ with the composition sequence $\left(I_{\alpha}\right)_{\alpha \leqslant \alpha_{0}}$, we need a proof similar to the one we have just given.

Let $A$ be a complex Banach algebra; we shall say that the mapping $x \rightarrow x^{*}$ from $A$ onto $A$ is a generalized involution if:
$1^{\circ}(x+y)^{*}=x^{*}+y^{*}$, for every $x, y$ in $A$.
$2^{\circ}\left(x^{*}\right)^{*}=x$, for every $x$ in $A$.
$3^{\circ}(\lambda x)^{*}=\bar{\lambda} x^{*}$ for every $x$ in $A$ and $\lambda$ in $\mathbb{C}$.
$4^{\circ}(x y)^{*}=y^{*} x^{*}$ for every $x, y$ in $A$ or $(x y)^{*}=x^{*} y^{*}$ for every $x, y$ in $A$.

Of course a standard involution is a generalized involution. If $A$ is a real algebra, then $x+i y \rightarrow x-i y$ is a generalized continuous involution of the complexified algebra $A_{C}$.

Now we suppose that the algebras are not separable to obtain Conjecture 2.

Theorem 1.15. Let $A$ be a complex Banach algebra with a generalized continuous involution and let $H$ be the set of self-adjoint elements. Suppose that $H$ contains a closed and absorbing subset $F$ such that $\operatorname{Sp} x$ is countable for every $x$ in $F$; then $\mathrm{Sp} x$ is countable for every $x$ in $A$.

Proof. Because the generalized involution is continuous, $H$ is closed. Let $x=h+i k \in A$ with $h, k \in H$ and $h_{0} \in F$. We consider the closed subalgebra $B$ generated by $h, k, h_{0}$. It is evident that $x \in B$ and that $y \in B$ imply $y^{*} \in B$. We have $B=(H \cap B)+i(H \cap B)$ and $F \cap B$ is absorbing and closed in $H \cap B . B$ being separable because it has a finite number of generators we conclude from Theorem 1.14 that $\mathrm{Sp}_{B} x$ is countable, but $\mathrm{Sp}_{A} x \subset \mathrm{Sp}_{B} x$, and it is finished.

Corollary 1.16. Let $A$ be a complex Banach algebra with a continuous involution. If the spectrum is countable on a closed absorbing subset of the set of self-adjoint elements then every element of $A$ has a countable spectrum.

If the involution is not continuous, we know by B. Johnson's theorem (see, for example, $[6, \mathrm{p} .111]$ ) that in $A / \operatorname{Rad} A$ the corresponding involution is continuous. The image of $F$ is still absorbing but not closed, so we are unable to use Theorem 1.15. By this elementary method we are obliged to reinforce the hypothesis in the following way (but the topological conditions, closed or open, can be omitted by the help of Section 3).

Corollary 1.17. Let $A$ be a complex Banach algebra with an involution. If the spectrum is countable on an open non-void subset of the set of self-adjoint elements then every element of A has a countable spectrum.

Proof. By the canonical morphism $A \rightarrow A / \operatorname{Rad} A$, which is open, we can suppose that the spectrum of $\bar{x}$ (the same as the spectrum of $x$ ) is countable on $U^{\prime}=B^{\prime} \cap H^{\prime}$, where $B^{\prime}$ is an open ball of $A / \operatorname{Rad} A$ and $H^{\prime}$ the closed set of self-adjoint elements of $A / \operatorname{Rad} A$. Taking the trace on $H^{\prime}$ of a closed ball included in $B^{\prime}$ and using Theorem 1.15 we deduce that $\mathrm{Sp} \bar{x}$ is countable for every $x$, so is $\operatorname{Sp} x$.

Corollary 1.18. Let a be a real Banach algebra containing a closed absorbing subset on which the spectrum is countable. Then every element of the complexified algebra $A_{\mathrm{C}}$ has a countable spectrum.

Obviously all these results generalize or give a new proof of several theorems settled in pp. 79-87 of [6]. In particular the results I obtained for modular annihilator algebras.

If $A$ satisfies the hypotheses of Theorem 1.14 and has an identity we are able to obtain more results: more precisely that there exists a smallest ordinal $\beta \in \mathscr{F}$ such that $A_{\beta}$ is a sum of matrix algebras, i.e., that there exist $n_{1}, n_{2}, \ldots, n_{k} \geqslant 1$ such that $A_{\beta} \cong M_{n_{1}}(\mathbb{C}) \oplus \cdots \oplus M_{n_{k}}(\mathbb{C})$, in which case we have $A_{\beta+1}=\{0\}$. If $\alpha_{0}$ is not a limit ordinal we have $A_{\alpha_{0}-1}=k h\left(\operatorname{soc} A_{\alpha_{0}-1}\right)$; hence $A_{\alpha_{0}-1}$ is a modular annihilator algebra with idenity, and then of finite dimension (see [6, Theorem 2 and Lemma 5, pp. 83-84]), and Wedder-burn-Artin's theorem gives the desired conclusion. If $\alpha_{0}$ is a limit ordinal, then for $\beta<\alpha_{0}$ we have the identity not in $I_{\beta}$ for the same said reasons, because on the contrary $A_{\beta}$ is of finite dimension and $A_{\beta+1}=\{0\}$ with $\beta+1<\alpha_{0}$. The open ball centered at the identity of radius 1 is disjoint of $I_{\beta}$, for $\beta<\alpha_{0}$, then $I_{\alpha_{0}}=\overline{\bigcup_{B<\alpha_{0}} I_{\beta}}$ is also disjoint, which is a contradiction because $1 \in I_{\alpha_{0}}$.

The previous results can be related to an old problem of Lindenstrauss: for a Banach space $X$, is it possible for every bounded linear operator on $X$ to have a countable spectrum? If $X$ is separable, Theorem 1.14 and the last remark say that $\mathscr{L}(X)$ contains a primitive ideal $I$ such that $\mathscr{L}(X) / I$ is isomorphic to $M_{n}(\mathbb{C})$, for some $n \geqslant 1$. But is it even possible that $\mathscr{C}(X)=\mathscr{L} \mathscr{C}(X)+\mathbb{C} \Gamma$ ? Probably all these questions will be solved only by using geometrical properties of Banach spaces.

## 2. Subharmonicity of Fibers for Commutative Banach Algebras

In [49] (see also [50, pp. 132-140]), J. Wermer has proved the following result, which is the analog of E. Vesentini's theorem, as we shall see in Section 3.

Theorem 2.1 (J. Wermer). Let $A$ be a commutative Banach algebra. Denote by the set of its characters, $X$ its Shilov boundary, $f, g$ two elements of $A$. Then $\lambda \rightarrow \log \operatorname{Max}_{x \in f^{1}(\lambda)}|\chi(g)|$ is subharmonic on $\hat{f}(\mathscr{M}) \backslash \hat{f}(X)$, where $\bar{f}^{1}(\lambda)$ denotes the set of $\chi \in \mathscr{M}$ for which $\chi(f)=\lambda$ and $\hat{f}$ denotes the Gelfand's transform.
$\bar{f}^{1}(\lambda)$ is named the fiber over $\lambda$. The proof of the previous theorem comes mainly from Rossi's local principle of maximum which is a difficult theorem. This result seems also to have been obtained by V. N. Senichkin $[38,39]$. We shall see further on an elementary proof due to Z. Słodkowski [41].

Let $K_{g}(\lambda)$ denote the set of $\chi(g)$ for $\chi \in \bar{f}^{1}(\lambda)$. In [7], with J. Wermer, I proved that $\lambda \rightarrow \log \delta\left(K_{g}(\lambda)\right)$ is subharmonic on $\hat{f}(\mathcal{N}) \backslash \hat{f}(X)$, where $\delta$ is the diameter. Hence by using the ideas of [4] we succeeded in getting following generalization of Bishop's analytic structure theorem.

ThEOREM 2.2 (Bishop-Aupetit-Wermer). Let $A$ be a commutative Banach algebra. Denote by the set of its characters, $X$ its Shilov boundary, $f$ one element of $A$. Suppose that $\hat{f}(N) \backslash \hat{f}(X)$ is non-void and let $W$ be a component of this set. Suppose now that $W$ contains a set $G$ such that:
$1^{\circ}$ the outer capacity of $G$ is positive,
$2^{\circ}$ the fibers $\bar{f}^{1}(\lambda)$ are finite on $G$.
Then there exists an integer $n$ such that $\# \bar{f}^{1}(\lambda) \leqslant n$ for every $\lambda \in W$ and $\bar{f}^{1}(W)$ has the structure of a complex analytic manifold of dimension 1 on which the elements of $A$ are analytic.

The classical theorem of Bishop contains the strongest hypothesis that $G$ has a positive planar measure. Its classical proof is arduous (see [11] or [50, Chap. 11], where it is already simplified). This result is fundamental in the problem of polynomial approximation on $C^{1}$ or rectifiable arcs in $\mathbb{C}^{n}$ (see [ $5,14,50]$ for more details, and also $[2,24]$ in the rectifiable case).

In [4, 7], I conjectured that $\lambda \rightarrow \log \delta_{n}\left(K_{g}(\lambda)\right)$ and $\lambda \rightarrow \log c\left(K_{g}(\lambda)\right)$ are subharmonic. D. Kumagai [27] gave a partial answer to this when $A$ satisfies the condition

$$
\partial^{1}(A \hat{\otimes} A)=\left(\partial^{0} A \times \partial^{1} A\right) \cup\left(\partial^{1} A \times \partial^{0} A\right)
$$

where $A \widehat{\otimes} A$ is the projective tensor product, $\partial^{0} A$ the ordinary Shilov boundary and $\partial^{1} A$ the generalized Shilov boundary of order 1 (see later for definition).

We shall now prove these conjectures using the ideas of Section 1 in connection with Schechter's theorem. V.N. Senichkin [39] and Z. Stodkowski [41] obtained similar proofs also, but a little more complicated.

Lemma 2.3. Let A be a commutative Banach algebra, be its set of characters, $X$ be its Shilov boundary and $f_{1}, \ldots, f_{n}, g$ be $n+1$ elements of $A$. Denote by $F=\left(f_{1}, \ldots, f_{n}\right) \in A^{n} \quad$ and $\quad$ by $\quad \bar{F}^{1}(\lambda)=\left\{\chi \in M \mid \chi\left(f_{1}\right)=\right.$ $\left.\lambda, \ldots, \chi\left(f_{n}\right)=\lambda\right\}$. Then $\lambda \rightarrow \log _{\operatorname{Max}}^{\chi \in F^{\prime}(\lambda)}|\chi(g)|$ is subharmonic on $\bigcup_{i=1}^{n}\left(\hat{f}_{i}(N) \hat{f}_{i}(X)\right)$.

Proof. First we note that $\bar{F}^{1}(\lambda)=\left\{\chi \in \mid \chi\left(f_{1}\right)=\lambda, \chi\left(f_{2}-f_{1}\right)=0\right.$,..., $\chi\left(f_{n}-f_{1}\right)=0$. We denote by $I$ the closed ideal of $A$ generated by $f_{2}-f_{1}, \ldots, f_{n}-f_{1}$. Then $\left\{\chi \in \mathbb{M} \mid \chi\left(f_{2}-f_{1}\right)=\cdots=\chi\left(f_{n}-f_{1}\right)=0\right\}$ can be identified with the set of characters of the commutative Banach algebra $A / I$ (see [34, Theorem 2.6 .6 p. 79]). If $h$ is the class of $h$ in $A / I$, for $h \in A$ then $F^{-1}(\lambda)=f_{1}^{-1}(\lambda)$ and $\log \operatorname{Max}_{x \in F^{1}(\lambda)}|\chi(g)|=\log \operatorname{Max}_{x \in f_{1}^{\prime}(\lambda)}|\chi(\bar{g})|$ and this function is subharmonic on $\hat{f}_{1}(\mathcal{K}) \backslash \hat{V_{1}}(X)$ by Theorem 2.1 applied to $A / I$. But we can use the same argument with $f_{2}, \ldots, f_{n}$, then we get the lemma.

Theorem 2.4. Let $A, \mathcal{M}, X$ be as before and $f, g \in A$. Then for every polynomial $p\left(z_{1}, \ldots, z_{n}\right)$ of $n$ complex variables we have

$$
\lambda \rightarrow \log \operatorname{Max}_{x_{1}, \ldots, x_{n} \in F^{\prime}(\lambda)}\left|p\left(\chi_{1}(g), \ldots, \chi_{n}(g)\right)\right|
$$

subharmonic on $\hat{f}(\mathcal{M}) \backslash \hat{f}(X)$.
Proof. We can suppose that $A$ has a unit. Let $B$ denote the projective tensorial product of $k$ copies equal to $A$. It is a commutative Banach algebra whose set of characters $\boldsymbol{M}(B)$ can be identified with $\times \cdots \times \sim$ ( $k$ copies) and whose Shilov boundary $S(B)$ can be identified by the same homeomorphism to $X \times \cdots \times X$ ( $k$ copies) (see [12, Proposition 19, p. 236]). On $B$ let $\chi=\left(\chi_{1}, \ldots, \chi_{n}\right)$ be the character defined by

$$
\chi\left(\sum a_{i_{1}} \otimes a_{i_{2}} \otimes \cdots \otimes a_{i_{n}}\right)=\sum \chi_{1}\left(a_{i_{1}}\right) \cdots \chi_{n}\left(a_{i_{n}}\right) .
$$

Then $\chi_{1}, \ldots, \chi_{n} \in \bar{f}^{1}(\lambda)$ is equivalent to saying that $\chi\left(f_{1}\right)=\cdots=\chi\left(f_{n}\right)=\lambda$, for $f_{1}=f \otimes 1 \otimes \cdots \otimes 1, f_{2}=1 \otimes f \otimes \cdots \otimes 1, \ldots, f_{n}=1 \otimes 1 \otimes \cdots \otimes f$. We even have

$$
p\left(\chi_{1}(g), \ldots, \chi_{n}(g)\right)=\chi\left(p\left(g_{1}, \ldots, g_{n}\right)\right)
$$

where $g_{1}=g \otimes 1 \otimes \cdots \otimes 1, g_{2}=1 \otimes g \otimes 1 \cdots \otimes 1, \ldots, g_{n}=1 \otimes 1 \otimes \cdots \otimes g$, then $\phi(\lambda)=\operatorname{Max}_{x_{1}, \ldots, \chi_{n} \in \mathcal{F}^{1}(\mathcal{\lambda})}\left|p\left(\chi_{1}(g), \ldots, \chi_{n}(g)\right)\right|=\operatorname{Max}_{x \in F_{(\lambda)}}\left|\chi\left(p\left(g_{1}, \ldots, g_{n}\right)\right)\right|$, where $F=\left(f_{1}, \ldots, f_{n}\right) \in B^{n}, \quad p\left(g_{1}, \ldots, g_{n}\right) \in B$. We apply Lemma 2.3 to conclude that $\lambda \rightarrow \log \phi(\lambda)$ is subharmonic on $\bigcup_{i=1}^{n}\left(\hat{f}_{i}(\mathscr{M}(B)) \backslash f_{i}(S(B))\right)$ but $\hat{f}_{i}(\mathscr{M}(B))=\hat{f}(\mathscr{M})$ and $\hat{f}_{i}(S(B))=\hat{f}(X)$, and this proves the theorem.

Corollary 2.5. With the same hypotheses as in Theorem 2.4, we conclude that $\lambda \rightarrow \log \delta_{n}\left(K_{g}(\lambda)\right)$ is subharmonic on $\hat{f}(\mathcal{M}) \backslash \hat{f}(X)$.

Proof. We apply the previous theorem to $p\left(z_{1}, \ldots, z_{n}\right)=$ $\prod_{1<i<j \leqslant n+1}\left(z_{i}-z_{j}\right)$.

Remark. Using this theorem and Cartan's theorem on polar sets we obtain a much simpler proof of Theorem 2.2. It is enough to remark that there exists an integer $n$ and $E \subset W$ such that $c^{+}(E)>0$ and $\# \bar{f}^{1}(\lambda) \leqslant n$ on $E$, and then $\log \delta_{n}\left(K_{g}(\lambda)\right)=-\infty$ on $E$, so $\log \delta_{n}\left(K_{g}(\lambda)\right)=-\infty$ on $W$ and consequently $\# \bar{f}^{1}(\lambda) \leqslant n$ on $W$. The proof finishes as in [7]. To show that the elements of $A$ are analytic on $\bar{f}^{1}(W)$ we can proceed more simply by using a nice characterization of holomorphic functions, with the help of subharmonic functions, which generalizes a result of Hartogs (see [29, Lemma 3, pp. 59-60]).

Lemma 2.6. Let $\phi$ be a bounded function on a domain $D$ of $\mathbb{C}$. Then $\phi$ or $\bar{\phi}$ is holomorphic on $D$ if and only if $\lambda \rightarrow \log |\phi(\lambda)-\alpha|$ is subharmonic on $D$, for every $\alpha$ rather great in C. Particularly $\phi$ is holomorphic on $D$ if and only if $\lambda \rightarrow \log |\phi(\lambda)-\alpha \lambda-\beta|$ is subharmonic on $D$, for every $\alpha, \beta$ rather great in $\mathbb{C}$.

For the proof see [6, pp. 174-175] or [52, Lemma 2].

Corollary 2.7. With the same hypotheses as in Theorem 2.4 we conclude that $\lambda \rightarrow \log c\left(K_{g}(\lambda)\right)$ is subharmonic on $f(\mathscr{M}) \backslash \hat{f}(X)$.

Proof. As in Section 1, we remark that $\log c\left(K_{g}(\lambda)\right)$ is the decreasing limit of $\log \delta_{n}\left(K_{g}(\lambda)\right)$, when $n \rightarrow \infty$, and we use $3^{\circ}$ of Theorem 1 of [6, Appendix II].

In particular this implies that $\lambda \rightarrow \bar{f}^{1}(\lambda)$ is continuous on $W$ if $K_{g}(\lambda)$ is of capacity zero on a subset with positive outer capacity.

As in [7] we can obtain the following generalization of a result of Basener [8]:

Theorem 2.8. Let $A, \mathcal{H}, X f, g, W$ be as in Theorem 2.4. Suppose that $W$ contains a set $G$ such that:
$1^{\circ} G$ has positive outer capacity,
$2^{\circ}$ the fibers $\bar{f}^{1}(\lambda)$ are countable on $G$.
Then contains a non-void open set with the analytic structure of a complex analytic manifold of dimension 1 (it is even a polydisk) on which all the elements of $A$ are analytic.

In [8], R. Basener supposes that $G=W$ and he concludes that $\bar{f}^{1}(W)$ contains a dense open set with an analytic structure. If we suppose only that $G$ has a positive planar measure the same proof, as remarked by B. Cole, shows the existence of an analytic polydisk in $\bar{f}^{1}(W)$, but to obtain Theorem 2.8 it is necessary to use a subharmonic argument.

For $G=W$ it is also possible to give a purely topological proof (see [10]).
In Section 3 we shall show that it is even possible to globalize Theorem 2.8, when $A$ is separable, in the following form:

Theorem 2.9. Let $A, X, f, g, W, G$ be as in the previous theorem, with A separable. Then $\bar{f}^{1}(W)$ contains a dense open set with the analytic structure of a complex analytic manifold of dimension 1 on which the elements of $A$ are analytic.

Theorem 2.4 and Corollaries 2.5 and 2.7 have also been obtained by Z . Slodkowski [41], in a different and a little more complicated way. His article contains an interesting and elementary proof of Theorem 2.1 which uses the following folkloric lemma.

Lemma 2.10. Let $A, \mathcal{A}, X$ be as in Theorem 2.1. Suppose that $f \in A$ and $\lambda_{0} \notin \hat{f}(X)$. There exists $\varepsilon>0$ such that for every bounded linear form $l_{0} \in A^{*}$ satisfying $l_{0}\left(\left(f-\lambda_{0}\right) A\right)=0$ there exists en analytic function $\lambda \rightarrow l_{\lambda}$ from $B\left(\lambda_{0}, \varepsilon\right)$ into $A^{*}$ having the two following properties:

$$
\begin{array}{ll}
1^{\circ} & l_{\lambda}((f-\lambda) A)=0, \\
2^{\circ} & \left\|l_{\lambda}\right\| \leqslant 2\left\|l_{0}\right\|, \text { for }\left|\lambda-\lambda_{0}\right|<\varepsilon .
\end{array}
$$

Proof. Representing $A$ by Gelfand transform in $\mathscr{C}(N)$ it is enough to suppose that $A$ is a function algebra with unit and that $\lambda_{0}=0$. By Corollary 3.3.7, p. 137 of [34], because $0 \notin \hat{f}(X)$, there exists $r>0$ such that

$$
\begin{equation*}
\|f g\| \geqslant(1 / r)\|g\|, \quad \text { for every } g \in A \tag{1}
\end{equation*}
$$

In particular $f A$ is closed. Consider now the application from $A \times A^{*}$ to $A^{*}$ defined by

$$
(f \cdot l)(g)=l(f g) .
$$

Let us show now that for every $l \in A^{*}$ there exists $l^{\prime} \in A^{*}$ such that

$$
\begin{equation*}
l=f \cdot l^{\prime} \quad \text { and } \quad\left\|l^{\prime}\right\| \leqslant r\|l\| . \tag{2}
\end{equation*}
$$

We consider the linear form $A$ from $f A$ onto $\mathbb{C}$ defined by $\Lambda(f g)=l(g)$. By (1) it is precisely defined and its norm is bounded by $r\|l\|$. By the Hahn-Banach theorem there exists $l^{\prime} \in A^{*}$ which extends $A$ with the same norm and (2) is proved.

First we begin with $l_{0}$ and we apply (2). There exists $l_{1}$ satisfying $l_{0}=f \cdot l_{1}$ and $\left\|l_{1}\right\| \leqslant r\left\|l_{0}\right\|$. Inductively we construct a sequence $\left(l_{n}\right)$ of elements of $A^{*}$ such that

$$
\begin{equation*}
l_{n-1}=f \cdot l_{n} \text { and }\left\|l_{n}\right\| \leqslant r^{n}\left\|l_{0}\right\|, \quad \text { for } n \geqslant 1 \tag{3}
\end{equation*}
$$

Taking $\varepsilon=\frac{1}{2} r$, in $B(0, \varepsilon)$ the series $l_{\lambda}=\sum_{n=0}^{\infty} \lambda^{n} l_{n}$ is absolutely convergent to $l_{\lambda} \in A^{*}$ and $\lambda \rightarrow l_{\lambda}$ is analytic in this disk, because $\left\|\lambda^{n} l_{n}\right\| \leqslant\left\|l_{0}\right\| / 2^{n}$ we have $\left\|l_{\lambda}\right\| \leqslant 2\left\|l_{0}\right\|$. As $l_{0}(f g)=0$, for every $g \in A, f \cdot l_{0}=0$. Let us show now that $(f-\lambda) \cdot l_{\lambda}=0$, i.e., $\left.l_{\lambda}(f-\lambda) A\right)=0$. This comes immediately from the fact that

$$
(f-\lambda) \cdot l_{\lambda}=f \cdot l_{0}+\lambda\left(f \cdot l_{1}-l_{0}\right)+\lambda^{2}\left(f \cdot l_{2}-l_{1}\right)+\cdots=0
$$

Proof of Theorem 2.1 by Z. Slodkowski's Method. Let $W$ be a component of $\hat{f}(\mathbb{M}) \backslash \hat{f}(X)$. As in the proof of J. Wermer it is easy to show that $\lambda \rightarrow K_{g}(\lambda)$ is u.s.c. on $W$, so we have only to prove that $\phi(\lambda)=\log \rho\left(K_{g}(\lambda)\right)$ satisfies the mean inequality, i.e., for every $\lambda_{0} \in W$ there exists $r_{1}>0$ such that for $0<r<r_{1}$ we have

$$
\begin{equation*}
\phi\left(\lambda_{0}\right) \leqslant \frac{1}{2 \pi} \int_{0}^{2 \pi} \phi\left(\lambda_{0}+r e^{i \theta}\right) d \theta \tag{4}
\end{equation*}
$$

We fix $\lambda_{0}$; by a compacity argument there exists $\chi_{0} \in \bar{f}^{1}\left(\lambda_{0}\right)$ such that $\phi\left(\lambda_{0}\right)=\log \left|\chi_{0}(g)\right|$. This character $\chi_{0}$ is a bounded linear functional of norm 1 ; we put $l_{\lambda_{0}}=e^{i \theta} \chi_{0} \in A^{*}$, with $\theta$ such that $l_{\lambda_{0}}(g)=\phi\left(\lambda_{0}\right)$. The condition $\chi_{0} \in \bar{f}^{1}\left(\lambda_{0}\right)$ implies $\left.l_{\lambda_{0}}\left(f-\lambda_{0}\right) A\right)=0$, then by Lemma 2.10 there exists an analytic function $\lambda \rightarrow l_{\lambda}$ from $B\left(\lambda_{0}, \varepsilon\right)$, for some $\varepsilon>0$, into $A^{*}$, such that $l_{\lambda}((f-\lambda) A)=0$. For every $n \geqslant 1, \lambda \rightarrow l_{\lambda}\left(g^{n}\right)$ is holomorphic on $B\left(\lambda_{0}, \varepsilon\right)$, then $\lambda \rightarrow(1 / n) \log \left|l_{\lambda}\left(g^{n}\right)\right|$ is subharmonic. Then $\psi(\lambda)=\varlimsup_{n \rightarrow \infty}(1 / n) \log$ $\left|l_{\lambda}\left(g^{n}\right)\right|$ is perhaps not subharmonic but satisfies the mean inequality on $\left|\lambda-\lambda_{0}\right|<\varepsilon$. Because $l_{\lambda_{0}}=e^{i \theta} \chi_{0}$ and $\chi_{0}\left(g^{n}\right)=\chi_{0}(g)^{n}$ it is obvious that $\phi\left(\lambda_{0}\right)=\psi\left(\lambda_{0}\right)$. We have only to show that $\psi(\lambda) \leqslant \phi(\lambda)$ for $\left|\lambda-\lambda_{0}\right|<\varepsilon$ because in this case we shall have

$$
\phi\left(\lambda_{0}\right)=\psi\left(\lambda_{0}\right) \leqslant \frac{1}{2 \pi} \int_{0}^{2 \pi} \psi\left(\lambda_{0}+r e^{i \theta}\right) d \theta \leqslant \frac{1}{2 \pi} \int_{0}^{2 \pi} \phi\left(\lambda_{0}+r e^{i \theta}\right) d \theta
$$

and so (4) will be satisfied.

Let $I_{\lambda}$ be the closed ideal generated by $f-\lambda, l_{\lambda}$ is null on $I_{\lambda}$ and $\left\|l_{\lambda}\right\| \leqslant 2$; then $l_{\lambda}\left(g^{n}\right)\left|\leqslant 2\left\|\mid \bar{g}^{n}\right\|_{\lambda}\right.$, where $\bar{g}$ is the class of $g$ in $A / I_{\lambda}$ and $| \mid \|_{\lambda}$ the norm in this quotient algebra. Applying the spectral value formula we have

$$
\psi(\lambda) \leqslant \log \rho_{\lambda}(\bar{g}) .
$$

But the set of characters of $A / I_{\lambda}$ can be identified with the set of characters of $A$ which annihilates $I_{\lambda}$, so there exists $\chi$ a character of $A$ such that $\rho_{\lambda}(\bar{g})=|\chi(g)|$ and $\chi((f-\lambda) A)=0$. Consequently $\chi \in \bar{f}^{1}(\lambda)$ and

$$
\psi(\lambda) \leqslant \log |\chi(g)| \leqslant \phi(\lambda) .
$$

Jakóbczak [24] obtained extensions of Bishop's and Basener's analytic structure theorems but they are even weaker than the results of [7]. His paper is interesting because it clarifies Alexander's result concerning polynomial approximation on rectifiable curves. More important is the recent paper of V. N. Senichkin [39] in which he proves Corollaries 2.5 and 2.7 by a different method. He gives also interesting applications to uniform algebras.

Now we give generalizations of $n$-dimensional analytic structure theorems obtained by R. Basener [9] and used by N. Sibony [40] to get several applications.
First let us give some notations. Let $A$ be a uniform algebra, $\mathcal{M}$ be its set of characters and we denote now by $\partial^{0} A$ its classical Shilov boundary. Let $F=\left(f_{1}, \ldots, f_{n}\right) \in A^{n}$ and $V(F)=\left\{\chi \in \mathbb{M} \mid \chi\left(f_{1}\right)=\cdots=\chi\left(f_{n}\right)=0\right\}$. It is well known that $V(F)$ is $A$-convex or equivalently that

$$
A(V(F))=\{g \in \mathscr{C}(V(F)) \mid \exists f \in A \text { such that } \chi(f)=\chi(g), \forall \chi \in V(F)\}
$$

satisfies $\boldsymbol{M}(A(V(F)))=V(F)$.
By definition the $n$-generalized Shilov boundary is

$$
\partial^{n} A=\overline{\bigcup_{F} \partial^{0}(A(V(F)))}, \quad \text { for all } \quad F \in A^{n} .
$$

It can be characterized by some principle of maximum (see [40, pp. 143-144]). As in the case $n=1$ it can be proved (see [9, Lemma 2]) that:

Lemma 2.11. Suppose we have $n \geqslant 1, X \supset \partial^{n-1} \partial^{n-1} A$ and $F \in A^{n}$. Let us denote by $W$ a component of $\mathbb{C}^{n} \backslash F(X)$. Then we have either $F(\mathcal{A}) \cap W=\varnothing$ or $F(\mathcal{N}) \supset W$.

We need also the following:

Lemma 2.12. Suppose we have $g \in A, F=\left(f_{1}, \ldots, f_{n}\right) \in A^{n}, X \supset \partial^{n-1} A$ and $p$ a polynomial of $m$ complex variables. Then

$$
\lambda=\left(\lambda_{1} \ldots, \lambda_{n}\right) \rightarrow \phi(\lambda)=\log \operatorname{Max}_{\chi_{1}, \ldots, \chi_{m} \in F(\lambda)}\left|p\left(\chi_{1}(g), \ldots, \chi_{m}(g)\right)\right|,
$$

where $\bar{F}^{1}(\lambda)=\left\{\chi \mid \chi\left(f_{1}\right)=\lambda_{1}, \ldots, \chi\left(f_{n}\right)=\lambda_{n}\right\}$, is pluri-subharmonic on $F(\mathscr{M}) \backslash F(X)$.

Proof. We must show that $\lambda \rightarrow \phi(\lambda)$ is subharmonic on each complex line $D$ restricted to $F(\mathscr{M}) \backslash F(X)$. By a linear transformation of variables, which changes $F$, we may suppose that $D=\{(z, 0, \ldots, 0) \mid z \in \mathbb{C}\}$.

$$
V=F^{-1}(D)=\left\{\chi \mid \chi\left(f_{2}\right)=\cdots=\chi\left(f_{n}\right)=0\right\}=V\left(f_{2}, \ldots, f_{n}\right)
$$

is an $A$-convex subvariety of $\mathscr{M}$, of dimension $n-1$, so $\mathscr{M}(A))=V$. Let $f^{\prime}$ be the restriction of $f_{1}$ on $V$, i.e., $\chi\left(f^{\prime}\right)=\chi\left(f_{1}\right)$ for every $\chi$ in $V$. It is clear that $\chi \in f^{\prime-1}(z)$ and $\chi \in \mathscr{M}(A(V))$ is equivalent to $\chi\left(f^{\prime}\right)=z$ and $\chi \in V$, which is equivalent to $\chi\left(f_{1}\right)=z$ and $\chi \in V$, so to $\chi \in \bar{F}^{1}(z, 0, \ldots, 0)$. If we apply Theorem 2.4 to the algebra $A(V)$, with $f^{\prime}$, we conclude that the restriction of $\lambda \rightarrow \phi(\lambda)$ is subharmonic on $\hat{f}^{\prime}(V) \backslash \hat{f}^{\prime}\left(\partial^{0}(A(V))\right)$ but $\partial^{0} A(V) \subset \partial^{n-1}(A) \subset X$, by definition of $(n-1)$-generalized Shilov boundary and hypothesis, so the restriction of $\lambda \rightarrow \phi(\lambda)$ is subharmonic on $D \backslash F(X)$.

ThEOREM 2.13 (Generalization of several dimension analytic structure theorem of Basener). Suppose $F \in A^{n}$ and let $W$ be a component of $F(\mathscr{M}) \backslash F\left(\partial^{n-1} A\right)$. Suppose that $W$ contains a subset $G$ such that:
$1^{\circ} G$ is not pluri-polar, i.e., there is no pluri-subharmonic function $\phi$ on $\mathbb{C}^{n}$ such that $G \subset\left\{\lambda \in \mathbb{C}^{n} \mid \phi(\lambda)=-\infty\right\}$,
$2^{\circ}$ the fibers $\bar{F}^{1}(\lambda)$ are finite on $G$.

Then there exists an integer $n$ such that
(a) $W=\bigcup_{k=1}^{n} W_{k}$,
(b) $\bigcup_{k=1}^{n-1} W_{k}$ is a proper analytic subvariety of $W$,
(c) $\mathscr{S}=\left(\bar{F}^{1}(W), F, W\right)$ is an analytic cover, then $\bar{F}^{1}(W)$ has the structure of an analytic complex manifold of dimension $n$ on which all the elements of $A$ are analytic.

Proof. It is enough to use Cartan's theorem, Lemma 2.12 and Corollary 2.5 to conclude that $\# \bar{F}^{1}(\lambda) \leqslant n$ for every $\lambda \in W$. The rest of the
proof-which is the easiest part-is done as in Basener's proof. To prove the analyticity one can also use Lemma 2.6 .

For the definition of an analytic cover see [17, p. 101]. Originally R. Basener made the strongest hypothesis that $m_{2 n}(G)>0$, where $m_{2 n}$ is Lebesgue measure in $\mathbb{R}^{2 n}$, and this condition implies that $G$ is not pluripolar, as we explain now.

For $x=\left(x_{1}, \ldots, x_{k}\right) \in \mathbb{R}^{k},\|x\|=\left(\sum_{i=1}^{k}\left|x_{i}\right|^{2}\right)^{1 / 2}$ denotes the Euclidean norm and we put

$$
\begin{aligned}
K_{\alpha}(x) & =-\|x\|^{-\alpha} \quad \text { for } \quad \alpha>0, \\
K_{0}(x) & =\log \|x\| .
\end{aligned}
$$

Let $E$ be a compact of $\mathbb{R}^{k}$; we define the $\alpha$-potential carried by $E$ as the maximum of $\iint K_{\alpha}(x-y) d \mu(x) d \mu(y)$ for every probability measure $\mu$ concentrated on $E$ and the a-capacity of $E$, denoted by $c_{\alpha}(E)$, by $\left(-V_{\alpha}\right)^{-1 / \alpha}$ if $\alpha>0$ and $e^{-V_{0}}$ otherwise. Now we can extend this notion of $\alpha$-capacity by defining $\alpha$-outer capacity and $\alpha$-inner capacity of an arbitrary set. For $G$ arbitrary in $\mathbb{R}^{k}$ we have
$c_{\alpha}^{-}(G)=\operatorname{Sup} c_{\alpha}(E)$ for $E$ compact, $E \subset G$ and $c_{\alpha}^{+}(G)=\operatorname{Inf} c_{\alpha}^{-}(U)$ for $U$ open, $G \subset U$.

These capacities are positive and invariant by isometries. For the case $\alpha=k-2$, the bounded analytic sets $G$, consequently all bounded borelian sets, are capacitable, so $c^{+}(G)=c^{-}(G)$ (see [21, p. 273]). As in the case $n=2$ it is possible to prove an analog of Cartan's theorem, mainly that $c_{k-2}^{+}(G)=0$ if and only if there exists $\phi$ subharmonic on $\mathbb{R}^{k}, \phi \not \equiv-\infty$, such that $G \subset\left\{x \in \mathbb{R}^{k}, \phi(x)=-\infty\right\}$. If $G$ is also a $G_{\delta}$-set then, by a result of J. Deny, there exists $\phi$ subharmonic on $\mathbb{R}^{k}$ such that $G=\left\{x \in \mathbb{R}^{k} \mid \phi(x)=-\infty\right\}$.
$\mathbb{C}^{n}$ can be identified with $\mathbb{R}^{2 n}$, then we can define polar sets in $\mathbb{C}^{n}$ as sets of $2 n-2$ outer capacity zero or equivalently as sets included in $\left\{x \in \mathbb{R}^{2 n} \mid \phi(n)=-\infty\right\}$ for some subharmonic function $\phi$ on $\mathbb{R}^{2 n}$.

Particularly polar sets have $2 n$-Lebesgue measure zero. So $m_{2 n}(G)>0$ implies $G$ non-polar and $G$ non-pluri-polar, because a function plurisubharmonic on $\mathbb{C}^{n}$ is subharmonic on $\mathbb{R}^{2 n}$.

For $E \subset \mathbb{R}^{k}$ we denote by $\delta(E)$ the diameter of $E$, which can be infinite, and for $\alpha \geqslant 0$ we define

$$
\begin{array}{rlrl}
\delta^{\alpha}(E) & =\delta(E)^{\alpha} & \text { for } \quad \alpha>0 \\
\delta^{0}(E) & =1 & \text { if } & E \text { is non-void } \\
& =0 & \text { if } & E \text { is void. }
\end{array}
$$

Taking $H_{\epsilon}^{\alpha}(E)=\operatorname{Inf} \sum \delta^{\alpha}\left(E_{n}\right)$ for all the coverings $E \subset \bigcup_{n=1}^{\infty} E_{n}$, with $\delta\left(E_{n}\right)<\varepsilon$ and $H^{\alpha}(E)=\lim _{\epsilon \mid 0} H_{\epsilon}^{\alpha}(E), H^{\alpha}(E)$ is called the $\alpha$-Hausdorff measure of $E$. By Theorem 5.13, pp. 225-226 of [21] we have $c_{\alpha}(E)>0$ if $H^{\beta}(E)>0$ for some $\beta>\alpha$. Consequently in Theorem 2.13 we can replace condition $1^{\circ}$ by the strongest condition $H^{\beta}(G)>0$ for some $\beta>2 n-2$. But the condition $H^{2 n-2}(G)>0$ is not sufficient.

Let us now show how some subharmonic methods can be used to get theorems about cluster sets. Probably by such methods it is possible to get more, for example, results concerning the Iversen-Gross theorem, the Weiss theorem, etc. (See [15]).

Let $\Delta$ be the unit disk. By "inner function $z \rightarrow f(z)$ on $\Delta$ " we mean a bounded analytic function on $\Delta$ such that $\lim _{r \rightarrow 1}\left|f\left(r e^{i \theta}\right)\right|=1$ a.e. Seidel and Frostman have proved the following result. (See [31, Chap. 3, p. 37]).

Theorem 2.14. If $f$ is inner, then either $f$ is a finite Blaschke product or every value in $\Delta$-except perhaps for a closed set of capacity zero-is taken by $f$ infinitely often in $\Delta$.

In the case of $H^{\infty}(\Delta), \Delta$ can be identified topologically to an open subset of the set $M$ of characters of $H^{\infty}(\Delta)$. Then the previous theorem is strangely similar to Theorem 2.2 , with $\bar{f}^{1}(\lambda)$ replaced by $\bar{f}^{1}(\lambda) \cap \Delta$. After several discussions with J. Wermer I tried to get a general theorem englobing these two results. The main difficulty comes from the fact that $\left\{\lambda \mid \bar{f}^{1}(\lambda) \cap \Delta==\varnothing\right\}$ is of capacity 0 . This suggested to me the following subharmonic proof of the Seidel-Frostman theorem:

Proof. For $f$ inner and $\alpha \in \Delta$ take $g_{\alpha}(z)=(f(z)-\alpha) /(1-\bar{\alpha} f(z))$. This function is also inner. For $|z|<1$ fixed, the function $\alpha \rightarrow \log \left|g_{\alpha}(z)\right|=$ $\log |(f(z)-\alpha) /(1-\alpha \overline{f(z)})|$ is subharmonic on $\Delta$, because it is the difference of a subharmonic and a harmonic function. Let us state $J(r, \alpha)=(1 / 2 \pi)$ $\int_{-n}^{\pi} \log \left|g_{\alpha}\left(r e^{i \theta}\right)\right| d \theta \leqslant 0$, for $r<1$. Because $\alpha \rightarrow \log \left|g_{\alpha}(z)\right|$ is subharmonic, $\alpha \rightarrow J(r, \alpha)$ satisfies the mean inequality for every $r<1$. Then also $J(\alpha)=\lim _{r \rightarrow 1} J(r, \alpha)$ satisfies also the mean inequality. This implies that there exists a subharmonic function $\phi$ on $\Delta$ such that $J(\alpha)=\phi(\alpha)$, for $\alpha \in \Delta$ except perhaps on a set of outer capacity zero. If we succeed in proving that $J(\alpha)=0$ a.e. it will be finished because $\phi(\alpha)=0$ a.e. implies $\phi \equiv 0$ on $\Delta$ (see [48, p. 64]) and this gives $J(\alpha)=0$, except on a set of capacity zero. But $J(\alpha)=0$ means $\lim _{r \rightarrow 1} J(r, \alpha)=0$, that is to say, $g_{a}(z)$ is a Blaschke product $[22$, p. 176], and then $\alpha$ is taken by $f$. The proof of $J(\alpha)=0$ a.e. is similar to the proof given in $[22$, p. 176]. For $|\alpha| \neq|f(0)|$ we have $J(r, \alpha) \geqslant$ $\log \left|g_{\alpha}(0)\right|>-\infty$ by Jensen's inequality. Take $\rho \neq|f(0)|$ then we have

$$
\begin{aligned}
\frac{1}{2 \pi} \int_{-\pi}^{\pi} J\left(\rho e^{i \theta}\right) d \theta= & \lim _{r \rightarrow 1} \frac{1}{2 \pi} \int_{-\pi}^{\pi} J\left(r, p e^{i \theta}\right) d \theta \\
= & \lim _{r \rightarrow 1} \frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{1}{2 \pi r} \int_{|z|=r} \\
& \times \log \left|\frac{f(z)-\rho e^{i \theta}}{1-\rho e^{-i \theta} f(z)}\right||d z| d \theta \\
= & \lim _{r \rightarrow 1} \frac{1}{2 \pi r} \int_{|z|=r} I_{\rho}(z)|d z|
\end{aligned}
$$

where

$$
I_{\rho}(z)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \log \left|\frac{f(z)-\rho e^{i \theta}}{1-\rho e^{-l \theta} f(z)}\right| d \theta
$$

But

$$
I_{\rho}(z)=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \log \left|f(z)-\rho e^{i \theta}\right| d \theta-\frac{1}{2 \pi} \int_{-\pi}^{\pi} \log \left|e^{i \theta}-\rho f(z)\right| d \theta
$$

and the last integral is 0 by Jensen's inequality. Then we get

$$
I_{\rho}(z)=\operatorname{Max}(\log \rho, \log |f(z)|)
$$

But because $f$ is inner, $\lim _{r \rightarrow 1} \log \left|f\left(r e^{i t}\right)\right|=0$ a.e. in $t$. Consequently $I_{\rho}\left(r e^{i t}\right)$ goes to 0 , when $r$ goes to 1 , a.e. in $t$. But $I_{\rho}\left(r e^{t t}\right)$ is bounded, then Lebesgue's theorem says that $(1 / 2 \pi) \int_{-\pi}^{\pi} J\left(p e^{i \theta}\right) d \theta=0$, and $J(\alpha) \leqslant 0$ gives $J(\alpha)=0$ a.e.

In the case of $H^{\infty}\left(\Delta^{n}\right)$, with $n>1$, this method shows that we have almost the same result mainly: if $f$ is inner on $\Delta^{n}$ then there exists $E$ of capacity 0 in $\Delta$ such that every $\alpha$ not in $E$ is taken by $f$. If $f$ is a good inner function it comes from Theorem 5.3 .2 of $[36, \mathrm{p} .115]$, if $f$ is only inner there exists $\alpha \in \Delta$ such that $(g-\alpha) /(1-\overline{a g})$ is good and then we consider the conformal mapping $z \rightarrow(z+\alpha) /(1+\bar{\alpha} z)$ of $\Delta$ onto $\Delta$, which transforms a set of capacity 0 in a set of capacity 0 .

A similar subharmonic proof can be used to obtain the following Tsuji's theorem [45, pp. 329-331].

Theorem 2.15. Let $G(\lambda, \mu)$ be an integral function of the variables $\lambda, \mu$ and $y(\lambda)$ the analytic function defined by $G(\lambda, y(\lambda))=0$. Suppose that $y(\lambda)$ is not algebroid, i.e., we do not have $a_{0}(\lambda)+a_{1}(\lambda) y(\lambda)+\cdots+a_{n}(\lambda) y(\lambda)^{n}=0$ for some integral functions $a_{0}(\lambda) \cdots a_{n}(\lambda)$; then for every $\lambda \in \mathbb{C}$ the set $Z(\lambda)=\{\mu \mid G(\lambda, \mu)=0\}$ is infinite, except on a set of capacity zero.

It is interesting to note that it is also possible to give a purely spectral proof of this result in the following ways.

Proof. Take $E=\{\lambda \mid G(\lambda, 0)=0\}$ which is closed and discrete and $U=\mathbb{C} \backslash E$. For $\lambda \in U$ we have $Z(\lambda)$ not containing 0 . This means that $\alpha_{0}(\lambda) \neq 0$ on $U$ if we have $G(\lambda, \mu)=\sum_{n=0}^{\infty} \alpha_{n}(\lambda) \mu^{n}$. Dividing by $\alpha_{0}(\lambda)$, we can suppose $\alpha_{0}(\lambda)=1$ on $U$. Suppose that $K(\lambda)=\{(1 / \mu) \mid \mu \in Z(\lambda)\}$ and for $\varepsilon \neq 0$ consider $a_{\epsilon}(\lambda)$ the bounded operator defined on $l^{2}$ by the matrix

$$
a_{\epsilon}(\lambda)=\left[\begin{array}{cccc}
-\alpha_{1}(\lambda) & 0 & 0 & \cdots \\
-\frac{\alpha_{2}(\lambda)}{\varepsilon} & 0 & 0 & \cdots \\
-\frac{\alpha_{3}(\lambda)}{\varepsilon^{2}} & 0 & 0 & \cdots \\
\vdots & & &
\end{array}\right]
$$

$u$ the shift operator and $m_{\epsilon}(\lambda)=a_{\epsilon}(\lambda)+\varepsilon u$ which has the matrix

$$
m_{\epsilon}(\lambda)=\left[\begin{array}{ccccccc}
-\alpha_{1}(\lambda) & \varepsilon & 0 & 0 & 0 & 0 & \vdots \\
-\frac{\alpha_{2}(\lambda)}{\varepsilon} & 0 & \varepsilon & 0 & 0 & 0 & \vdots \\
-\frac{\alpha_{3}(\lambda)}{\varepsilon^{2}} & 0 & 0 & \varepsilon & 0 & 0 & \vdots \\
\vdots & \cdots & & \cdots & \cdots & \vdots
\end{array}\right]
$$

We have $m_{\epsilon}(\lambda) \in \mathscr{L}\left(l^{2}\right)$ and $a_{\epsilon}(\lambda) \in \mathscr{L} \mathscr{C}\left(l^{\star}\right)$ by using Gutzmer's formula (see [35, p. 228]):

$$
\sum\left|a_{n}\right|^{2} R^{2 n}=\frac{1}{2 \pi} \int_{0}^{2 \pi}\left|f\left(R e^{i \theta}\right)\right|^{2} d \theta
$$

Because $\operatorname{Sp} \varepsilon u=\varepsilon \bar{\lambda}$ and $m_{\epsilon}(\lambda)-\varepsilon u$ is compact we have

$$
\text { Sp } m_{\epsilon}(\lambda)=\varepsilon \bar{\Delta} \cup\left\{\text { proper values of } m_{\epsilon}(\lambda)\right\}
$$

(for this, see [18, Problem 143, p. 92]).
Now it is easy to see that $K(\lambda)$ is the set of non-zero proper values of $m_{\epsilon}(\lambda)$. Then $\operatorname{Sp} m_{\epsilon}(\lambda)=K(\lambda) \cup \varepsilon \bar{A}$, so $K(\lambda) \cup\{0\}=\bigcap_{n=1}^{\infty} \operatorname{Sp} m_{1 / n}(\lambda)$. By using Vesentini's theorem we obtain immediately that $\lambda \rightarrow \log \operatorname{Max}_{\mu \in K(\lambda)}|, \mu|$ is subharmonic and we can now make a proof similar to that of the finite spectrum scarcity theorem.

## 3. A General Theory of Analytic Multivalued Functions

The strange similarity of the results obtained in [4, 7] and in Sections 1 and 2 of this paper, concerning the spectrum, the fibers and some theorems of Tsuji about cluster sets, suggests that there exists a general theory englobing all this results. The first step in this direction has been obtained in [51].
But the real progress originates from studies on the general conjecture of Pekczyniski. In my lectures of 1977 in the Banach Center of Warsaw I saw that this conjecture could be proved very easily if we could prove the following scatcity theorem for countable spectrum, which is an adequate analog of the scarcity theorem for finite spectrum (Theorem 1.1).

Conjecture 3. Let $\lambda \rightarrow f(\lambda)$ be an analytic function from a domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$ then:
-either the set of $\lambda$ for which $\operatorname{Sp} f(\lambda)$ is countable is of outer capacity zero,
—or $\operatorname{Sp} f(\lambda)$ is countable for every $\lambda$ in $D$.
At that time I was unable to prove this conjecture, but the proof of Conjecture 2 mentioned in Section 2, that I obtained, from Conjecture 3, was the following: let $x=h+i k$ be in $A$ with $h, k$ in $H$; we take $h_{0}$ in $U$ and we consider the analytic function $\lambda \rightarrow f(\lambda)=h_{0}+\lambda\left(h-h_{0}\right)$; for $-r \leqslant \lambda \leqslant r$, with $r$ small enough, we have $\operatorname{Sp} f(\lambda)$ countable, and $[-r, r]$ is not of capacity zero, then $\operatorname{Sp} f(\lambda)$ is countable for every $\lambda$ in $\mathbb{C}$, taking $\lambda=1$ we conclude that $\mathrm{Sp} h$ is countable for every $h$ in $H$; taking now $\lambda \rightarrow g(\lambda)=h+\lambda k$, we know that $\mathrm{Sp} g(\lambda)$ is countable for every $\lambda$ real, so $\mathrm{Sp} g(\lambda)$ is countable for every $\lambda$ in $\mathbb{C}$ on in particular for $\lambda=i$.

In fact I obtained partial answers to Conjecture 3, in particular if $\operatorname{Sp} f(\lambda)$ has a finite number of limit points on a set of positive capacity (see [ 6 , pp. 86-87]). To prove this I introduced the following method. If $\alpha_{0}$ is isolated in $\operatorname{Sp} f(\lambda)$ we say that $\alpha_{0}$ is of "first kind" if for $s>0$ given, such that $\operatorname{Sp} f\left(\lambda_{0}\right) \cap B\left(\lambda_{0}, s\right)=\left\{\alpha_{0}\right\}$, there exists $r>0$ such that $\left|\lambda-\lambda_{0}\right|<r$ implies \# $\left(\operatorname{Sp} f(\lambda) \cap B\left(\lambda_{0}, s\right)\right)<\infty$. By Theorem 2, p. 67, in [6], it says that $\operatorname{Sp} f(\lambda) \cap B\left(\lambda_{0}, s\right)$ has at most $n$ points which vary holomorphically outside of a closed discrete set. Then I define $D \operatorname{Sp} f(\lambda)$ by $\operatorname{Sp} f(\lambda)$ minus the set of first kind isolated points of $\operatorname{Sp} f(\lambda)$. And then the problem is to prove that $\lambda \rightarrow D \operatorname{Sp} f(\lambda)$ is "spectral" in some sense (in fact $D \operatorname{Sp} f(\lambda)$ is "near" the spectrum of the class of $f(\lambda)$ in $A / k h(\operatorname{soc} A)$, by Theorem 1.9). At this point it is necessary to introduce a good definition of "spectral" or "analytic multivalued functions" in such a ways that $\lambda \rightarrow \operatorname{Sp} f(\lambda)$ is of type and such that if $\lambda \rightarrow K(\lambda)$ is analytic multivalued then $\lambda \rightarrow D K(\lambda)$ (where $D K(\lambda)$ is defined in a similar way) is also analytic multivalued.

Suppose that $\lambda \rightarrow K(\lambda)$ associates to $\lambda$ a compact subset $K(\lambda)$ of $\mathbb{C}$ in such a way that this multivalued function is upper semi-continuous. The graph of $\lambda \rightarrow D K(\lambda)$ is then obtained from the graph of $\lambda \rightarrow K(\lambda)$ by removing analytic varieties of complex dimension 1. After several discussions with me, Z. Slodkowski [42] had the idea to introduce the following definition:

Definition. A function $\lambda \rightarrow K(\lambda)$ from a domain $D$ of $\mathbb{C}$ into the set of non-void compact subsets of $\mathbb{C}$ is said to be analytic multivalued if it is upper semi-continuous and if $\Omega=\{(\lambda, z) \mid \lambda \in D, z \notin K(\lambda)\} \subset \mathbb{C}^{2}$ is a domain of holomorphy (or equivalently pseudoconvex).

Upper semi-continuity implies that $\Omega$ is open. This definition does not seem very easy and not very tractable. It is more natural for persons working in the field of several complex variables and, in fact, it had been introduced by K. Oka [32] in 1934, obviously without connection to spectral theory, to prove an extension of a result of F. Hartogs we mention now. If $h$ is holomorphic on $D$ it is easy to see that $\lambda \rightarrow\{h(\lambda)\}$ is analytic multivalued; conversely F . Hartogs proved the following: if $|h(\lambda)|<R$, for $\lambda \in D$ and if there exists $f$ holomorphic on $U \backslash \Gamma$ which is singular at every point of $\Gamma$, where $U=D \times\{z| | z \mid<R\}$ and $\Gamma=\{(\lambda, z) \in U \mid h(\lambda)=z\}$, then $h$ is holomorphic on $D$. In other words it says that an analytic multivalued function for which $K(\lambda)$ has always one point is in fact a holomorphic function, so it justifies the definition. For a classical proof of the theorem of F. Hartogs see [29, pp. 56-61]; for a more abstract proof using Rudin's theorem on maximum modulus algebras see [52]. See also the remark following Theorem 3.6.

When K. Oka [32], and later T. Nishino [30] and H. Yamaguchi [54], studied analytic multivalued functions (they were using the name "ensemble pseudoconcave" which means that the graph is the complement of a pseudoconvex set or equivalently a domain of holomorphy) they practically only used analytic multivalued functions of algebroid type, i.e.,

$$
K(\lambda)=\left\{\mu \mid a_{0}(\lambda)+\mu a_{1}(\lambda)+\cdots+\mu^{n} a_{n}(\lambda)=0\right\}
$$

where the $a_{i}$ are entire functions. In fact, Z. Slodkowski [42] obtained very good characterizations of analytic multivalued functions, the best one being the following: $\lambda \rightarrow K(\lambda)$ is analytic multivalued if and only if $(\lambda, z) \rightarrow$ $-\log \operatorname{dist}(z, K(\lambda))$ is pluri-subharmonic on $\Omega=\{(\lambda, z) \mid \lambda \in D, z \notin K(\lambda)\}$. From this it is possible to prove that $\lambda \rightarrow \operatorname{Sp} f(\lambda)$ is analytic multivalued if $f$ is an analytic function from a domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$. In this case $-\log \operatorname{dist}(z, K(\lambda))=\log \rho\left((z-f(\lambda))^{-1}\right)$ and we apply functional calculus to obtain that $\lambda \rightarrow(\alpha \lambda+\beta-f(\lambda))^{-1}$ is analytic on the complex line $\lambda \rightarrow \alpha \lambda+\beta$ restricted to $\Omega$, and Vesentini's theorem to obtain subharmonicity of $\lambda \rightarrow-\log \operatorname{dist}(\alpha \lambda+\beta, K(\lambda))$. In the case of commutative

Banach algebras, with the notations of Section 2, it is also possible to prove that $\lambda \rightarrow K_{s}(\lambda)$ is analytic multivalued on $\hat{f}(\mathcal{M}) \backslash \hat{f}(X)$, but the proof is much more complicated. In fact the difficulty is to prove that

$$
(\lambda, z) \rightarrow \log \operatorname{Max}_{\chi \in f^{-1}(\lambda)} \frac{1}{|z-\chi(g)|}
$$

is pluri-subharmonic. If $K_{z}(\lambda)$ does not separate the plane it can be proved easily, using Runge's approximation theorem (see the proof of Lemma 7 in [7]). In the general situation the proof obtained by Z. Slodkowski is much more complicated, because it uses Rossi's local maximum modulus principle.

Later on we shall give more elementary proofs of the fact that $\lambda \rightarrow \operatorname{Sp} f(\lambda)$ and $\lambda \rightarrow K_{g}(\lambda)$ are analytic multivalued. The first proof avoids Slodkowski's characterization and, in the case of fibers, the proof avoids Rossi's local maximum modulus principle.

Let $\Omega$ be a domain in $\mathbb{C}^{2}$. A real valued function $\phi$ defined on $\Omega$ is called a vertical function for $\Omega$ if for each $a$ such that the complex line $\{(a, z) \mid z \in \mathrm{C}\}$ meets $\Omega$ we have

$$
\lim _{z \rightarrow z_{0}} \phi\left(a, z_{0}\right)=+\infty,
$$

whenever $\left(a, z_{0}\right)$ is in the boundary of $\Omega$. It is now possible to give the following characterization of domains of holomorphy, which implies the interesting part of Slodkowski's theorem, namely, $(\lambda, z) \rightarrow-\log \operatorname{dist}(z, K(\lambda))$ pluri-subharmonic implies $\lambda \rightarrow K(\lambda)$ analytic multivalued.

Theorem 3.1. Let $\Omega$ be a domain in $\mathbb{C}^{2}$. We assume there exists a vertical function $\phi$ for $\Omega$ which is pluri-subharmonic on $\Omega$. Then $\Omega$ is a domain of holomorphy.

Proof. The argument begins as the proof of Theorem 2.6.7, p. 46, in the book of L. Hörmander [55]. Suppose $\Omega$ is not a domain of holomorphy. Denoting by $d(u)$ the distance from $u$ in $\Omega$ to $\partial \Omega$, then, by Oka-NorguetBremermann's theorem, we obtain that $-\log d$ is not pluri-subharmonic on $\Omega$. Hence there exists a complex line $L$ such that $-\log d$ is not subharmonic on $L \cap \Omega$. Then there exists a disk $D$ in $L \cap \Omega$ such that $-\log d$ violates the mean-value property on $D$. It is possible to write $D=\left\{u \in \mathbb{C}^{2} \mid u=u_{0}+t \omega\right.$, $|t| \leqslant r\}$, with $u_{0}$ in $\Omega, \omega=\left(\omega_{1}, \omega_{2}\right)$ a constant vector for which we can suppose without loss of generality that $\omega_{1} \neq 0$. Because the mean-value property is not true on $D$, there exists a polynomial $p(t)$ and $t_{1}$ such that $\left|t_{1}\right|<r$ verifying:
(1) $-\log d\left(u_{0}+t \omega\right) \leqslant \operatorname{Re} p(t) \quad$ for $|t|=r$
and
(2) $-\log d\left(u_{0}+t_{1} \omega\right)-\operatorname{Re} p\left(t_{1}\right)>0 \quad$ and maximum on $\{t||t| \leqslant r\}$.

Then $d\left(u_{0}+t_{1} \omega\right)<\left|e^{-p\left(t_{1}\right)}\right|$, so $d\left(u_{0}+t_{1} \omega\right)=\rho_{1} e^{i \alpha} e^{-p\left(t_{1}\right)}$, for some $0<\rho_{1}<1$ and $\alpha$ real. We now define the analytic disk $D_{\rho}$ by

$$
D_{\rho}=\left\{u\left|u=u_{0}+t \omega+\lambda e^{i a} e^{-p(t)} a,|t| \leqslant r\right\}\right.
$$

where $a$ is a unit vector pointing at $u_{0}+t_{1} \omega$ to the nearest point of $\partial \Omega$. With the definition of $\rho_{1}$ it is obvious that the interior of $D_{\rho_{1}}$ meets $\partial \Omega$ and using (1) and (2), $D_{\rho} \subset \Omega$ for $\rho<\rho_{1}$, with $\partial D_{\rho_{1}} \subset \Omega$. We introduce $u_{\rho}(t)=u_{0}+t \omega+\rho e^{i a} e^{-p(t)} a=\left(u_{\rho}^{1}(t), u_{\rho}^{2}(t)\right)$. Then $D_{\rho_{1}}$ is the set of $\left(u_{\rho_{1}}^{1}(t)\right.$, $\left.u_{\rho_{1}}^{2}(t)\right)$, for $|t| \leqslant r$, and $D_{\rho_{1}}$ meets $\partial \Omega$ at $\left(u_{\rho_{1}}^{1}\left(t_{1}\right), u_{\rho_{1}}^{1}\left(t_{1}\right)\right)$. The function $u_{\rho_{1}}^{1}(t)$ takes the value $u_{\rho_{1}}^{1}\left(t_{1}\right)$ on the open set $\{t||t|<r\}$; hence, by the implicit theorem, for every $\rho$ near $\rho_{0}$ there exists $t_{\rho}$ such that $u_{\rho}^{1}\left(t_{\rho}\right)=u_{\rho_{1}}^{1}\left(t_{1}\right)$, with $t_{\rho}$ going to $t_{1}$ when $\rho$ goes to $\rho_{1}$. Then $\left(u_{\rho}^{1}\left(t_{\rho}\right), u_{\rho}^{2}\left(t_{\rho}\right)\right)=\left(u_{\rho_{1}}^{1}\left(t_{1}\right), u_{\rho}^{2}\left(t_{\rho}\right)\right)$ is on the vertical line defined by $u_{\rho_{1}}^{1}\left(t_{1}\right)$, and also in $D_{\rho}$, consequently in $\Omega$ if $\rho<\rho_{1}$. But $\left(u_{\rho}^{1}\left(t_{\rho}\right), u_{\rho}^{2}\left(t_{\rho}\right)\right)$ goes to $\left(u_{\rho_{1}}^{1}\left(t_{1}\right), u_{\rho_{1}}^{2}\left(t_{1}\right)\right)$ which is in $\partial \Omega$, when $\rho$ goes to $\rho_{1}$. Hence $\phi\left(u_{\rho}\left(t_{\rho}\right)\right) \rightarrow+\infty$, because $\phi$ is a vertical function for $\Omega$. But the restriction of $\phi$ to $D_{\rho}$ is subharmonic so:

$$
\begin{equation*}
\phi\left(u_{\rho}\left(t_{\rho}\right)\right) \leqslant \operatorname{Max}_{u \in \partial D_{\rho}} \phi(u) \tag{3}
\end{equation*}
$$

for $\rho<\rho_{0}$. Because $\partial D_{\rho} \subset \Omega$, for $\rho \leqslant \rho_{1}$, there exists a compact $K$ in $\Omega$ such that $\partial D_{\rho} \subset K$, for $0 \leqslant \rho \leqslant \rho_{0}$. But $\phi$ attains its maximum $M$ on $K$, and this is a contraction with (3), because $\phi\left(u_{\rho}\left(i_{\rho}\right)\right)$ goes to infinity.

Obviously this theorem is a generalization of the theorem which says that $\Omega$ is a domain of holomorphy if and only if there exists $\phi$ pluri-subharmonic on $\Omega$ such that $\phi(u) \rightarrow+\infty$ when $u \rightarrow \partial \Omega$ (this last result being directly equivalent to Oka-Norguet-Bremermann theorem). Using this proof and some arguments of [52], J. Wermer remarked that it is then possible to prove the local maximum modulus principle for $X$ contained in $\mathbb{C}^{n}$.

We now intend to give elementary proofs of the fact that $\lambda \rightarrow \operatorname{Sp} f(\lambda)$ and $\lambda \rightarrow K_{g}(\lambda)$ are analytic multivalued functions. This is really the most important point in Section 3 of this paper, which says precisely that we now have in functional analysis very important and new examples of analytic multivalued functions.

Theorem 3.2. Let $\lambda \rightarrow f(\lambda)$ be an analytic function from a domain $D$ of $\mathbb{C}$ into a complex Banach algebra $A$; then $\lambda \rightarrow \operatorname{Sp} f(\lambda)$ is an analytic multivalued function.

Proof. Obviously $\Omega=\{(\lambda, z) \mid \lambda \in D, z \notin \operatorname{Sp} f(\lambda)\}$ is the set of $(\lambda, z)$, for which $\lambda \in D$ and $f(\lambda)-z$ is invertible. Let us take

$$
\phi(\lambda, z)=\left\|(f(\lambda)-z)^{-1}\right\|-\log \operatorname{dist}(\lambda, \partial D)
$$

The last term is subharmonic in $\lambda$ because every domain of $\mathbb{C}$ is a domain of holomorphy of $\mathbb{C}$. If we restrict $(f(\lambda)-z)^{-1}$ to the trace on $\Omega$ of a complex line $L=\{(\lambda, \alpha \lambda+\beta) \mid \lambda \in D\}$ then by using Cauchy's integral formula

$$
\left(f\left(\lambda_{0}\right)-\left(\alpha \lambda_{0}+\beta\right)\right)^{-1}=\frac{1}{2 \pi i} \int_{\Gamma} \frac{(f(\lambda)-(\alpha \lambda+\beta))^{-1}}{\lambda-\lambda_{0}} d \lambda,
$$

with subadditivity of the norm and continuity of $(\lambda, z) \rightarrow\left\|(f(\lambda)-z)^{-1}\right\|$ on $\Omega$, we conclude that the first term is pluri-subharmonic and then $\phi(\lambda, z)$ is pluri-subharmonic. Let us prove that $\phi(\lambda, z)$ goes to infinity when $(\lambda, z)$ goes to the boundary of $\Omega$. Suppose on the contrary that there exists $\left(\lambda_{0}, z_{0}\right)$ in $\partial \Omega,\left(\lambda_{n}\right) \rightarrow \lambda_{0},\left(z_{n}\right) \rightarrow z_{0}, M>0$, such that $\left(\lambda_{n}, z_{n}\right) \in \Omega$ and $\phi\left(\lambda_{n}, z_{n}\right) \leqslant M$. Because $\operatorname{dist}\left(\lambda_{n}, \partial D\right) \geqslant e^{-M}$, first we conclude that $\lambda_{0} \in D, f\left(\lambda_{0}\right)-z_{0}$ is not invertible. But

$$
\begin{aligned}
f\left(\lambda_{0}\right)-z_{0} & =f\left(\lambda_{n}\right)-z_{n}+z_{n}-z_{0}+f\left(\lambda_{0}\right)-f\left(\lambda_{n}\right) \\
& =f\left(\lambda_{n}\right)-z_{n}+u\left(\lambda_{n}, z_{n}\right) \\
& =\left[f\left(\lambda_{n}\right)-z_{n}\right]\left[1+\left(f\left(\lambda_{n}\right)-z_{n}\right)^{-1} u\left(\lambda_{n}, z_{n}\right)\right] .
\end{aligned}
$$

When $\left(\lambda_{n}\right) \rightarrow \lambda_{0}$ and $\left(z_{n}\right) \rightarrow z_{0}$ we have $\| u\left(\lambda_{n}, z_{n} \| \rightarrow 0\right.$ and

$$
\left\|\left(f\left(\lambda_{n}\right)-z_{n}\right)^{-1} u\left(\lambda_{n}, z_{n}\right)\right\| \leqslant M^{\prime}\left\|u\left(\lambda_{n}, z_{n}\right)\right\| \rightarrow 0 \quad \text { for some } \quad M^{\prime}>0 ;
$$

hence $f\left(\lambda_{0}\right)-z_{0}$ is invertible as the product of two invertible elements, which is a contradiction. $\phi$ is a pluri-subharmonic function on $\Omega$ which goes to infinity of the boundary, so $\Omega$ is a domain of holomorphy.

In the case of the fiber function $\lambda \rightarrow K_{g}(\lambda)$, the solution is more complicated.

Let $A$ be a commutative Banach algebra and $f$ be given. For $x$ in $A$ we denote by $\bar{x}^{\lambda}$ the class of $x$ in $A_{\lambda}=A /(f-\lambda) A$.

Lemma 3.3. Let $A$ be a commutative Banach algebra and $f$ be an element of $A$. Suppose that $z \rightarrow x(z)$ is an analytic function from

$$
B\left(z_{0}, s\right)=\left\{z \| z-z_{0} \mid<s\right\}
$$

into $A$ such that $\overline{x(z)}{ }^{\lambda}$ is invertible in $A_{\lambda}$ for $(\lambda, z)$ in the polydisk $B\left(\lambda_{0}, r\right) \times$ $B\left(z_{0}, s\right)$. Then there exist $r_{1}, s_{1}$ such that $0<r_{1} \leqslant r$ and $0<s_{1} \leqslant s$ and there
exist two analytic functions $(\lambda, z) \rightarrow u(\lambda, z)$ and $(\lambda, z) \rightarrow v(\lambda, z)$ from the polydisk $B\left(\lambda_{0}, r_{1}\right) \times B\left(z_{0}, s_{1}\right)$ suxh that

$$
x(z) u(\lambda, z)=1+(f-\lambda) v(\lambda, z)
$$

for every $(\lambda, z)$ in this polydisk.
Proof. We denote $x\left(z_{0}\right)$ by $x$. This element $x$ is invertible in $A_{\lambda_{0}}$, then there exists $u_{0}, v_{0}$ in $A$ such that $x u_{0}=1+\left(f-\lambda_{0}\right) v_{0}$. Then $x(z) u_{0}=$ $(x(z)-x) u_{0}+1+(f-\lambda) v_{0}+\left(\lambda-\lambda_{0}\right) v_{0}$. For $r_{1}, s_{1}$ small enough and $\left|\lambda-\lambda_{0}\right|<r_{1},\left|z-z_{0}\right|<s_{1}$ we have $\left\|\left(\lambda-\lambda_{0}\right) v_{0}+(x(z)-x) u_{0}\right\|<1$, then $1+\left(\lambda-\lambda_{0}\right) v_{0}+(x(z)-x) u_{0}$ invertible. Consequently:

$$
\begin{aligned}
x(z) u_{0}= & {\left[1+(f-\lambda) v_{0}\left(1+\left(\lambda-\lambda_{0}\right) v_{0}+(x(z)-x) u_{0}\right)^{-1}\right] } \\
& \times\left[1+\left(\lambda-\lambda_{0}\right) v_{0}+(x(z)-x) u_{0}\right]
\end{aligned}
$$

Taking

$$
u(\lambda, z)=u_{0}\left[1+\left(\lambda-\lambda_{0}\right) v_{0}+(x(z)-x) u_{0}\right]^{-1}
$$

and

$$
v(\lambda, z)=v_{0}\left[1+\left(\lambda-\lambda_{0}\right) v_{0}+(x(z)-x) u_{0}\right]^{-1}
$$

which are obviously analytic on the polydisk $B\left(\lambda_{0}, r_{1}\right) \times B\left(z_{0}, s_{1}\right)$, we obtain the result.

Theorem 3.4. Let $A, \mathscr{M}, X$ be as in Theorem 2.2. Suppose that $f$ and $g$ are in $A$ and define $K_{g}(\lambda)$ by $\left\{\chi(g) \mid \chi \in \vec{f}^{1}(\lambda), \lambda \in W\right\}$. Let us take $\Omega=\left\{(\lambda, z) \mid z \notin K_{g}(\lambda), \lambda \in W\right\}$. Then:
$1^{\circ} \quad(\lambda, z) \rightarrow-\log \operatorname{dist}\left(z, K_{g}(\lambda)\right)$ is pluri-subharmonic on $\Omega$.
$2^{\circ} \Omega$ is a domain of holomorphy, i.e., $\lambda \rightarrow K_{g}(\lambda)$ is analytic multivalued on $W$.

Proof. $1^{\circ}$ If $(\lambda, z) \in \Omega$ then $z \neq \chi(g)$, for $\chi \in \bar{f}^{1}(\lambda)$. But $\bar{f}^{1}(\lambda)$ can be interpreted as the set of characters of $A_{\lambda}$, consequently $z-\bar{g}^{-\lambda}$, the class of $z-g$ in $A_{\lambda}$, is invertible in $A_{\lambda}$. Then

$$
\begin{aligned}
\phi(\lambda, z) & =-\log \operatorname{dist}\left(z, K_{g}(\lambda)\right)=\log \operatorname{Max}_{x \in f^{\prime}(\lambda)} \frac{1}{|z-\chi(g)|} \\
& =\log \rho_{\lambda}\left(\left(z-\bar{g}^{\lambda}\right)^{-1}\right)
\end{aligned}
$$

where $\rho_{\lambda}$ denote the spectral radius in $A_{\lambda}$. We fix $\left(\lambda_{0}, z_{0}\right)$ in $\Omega$ and we
choose $r, s>0$ such that $\left|\lambda-\lambda_{0}\right|<r$ and $\left|z-z_{0}\right|<s$ implies $(\lambda, z) \in \Omega$. By a compacity argument there exists $\chi_{0} \in \bar{f}^{1}\left(\lambda_{0}\right)$ such that

$$
\begin{equation*}
\phi\left(\lambda_{0}, z_{0}\right)=\log \frac{1}{\left|z_{0}-\chi_{0}(g)\right|}=\log \left|\chi_{0}\left(\left(z_{0}-\bar{g}^{\lambda_{0}}\right)^{-1}\right)\right| . \tag{4}
\end{equation*}
$$

This character is a linear functional of norm 1 on $A$. By Lemma 2.10 , there exists an analytic function $\lambda \rightarrow l_{\lambda}$ from $B\left(\lambda_{0}, \varepsilon\right)$ into $A^{*}$, such that $l_{\lambda_{0}}=\chi_{0}, l_{\lambda}((f-\lambda) A)=0$ and $\left\|l_{\lambda}\right\| \leqslant 2\left\|\chi_{0}\right\|$, for $\left|\lambda-\lambda_{0}\right|<\varepsilon$, where $\varepsilon$ is small enough. If we decrease $r$ it is even possible to suppose that $\varepsilon=r$. For $\left|\lambda-\lambda_{0}\right|<r$ and $\left|z-z_{0}\right|<s, z-\bar{g}^{\lambda}$ is invertible in $A_{\lambda}$, then, by Lemma 3.3, there exist $r_{1}, s_{1}$ such that $0<r_{1} \leqslant r, 0<s_{1} \leqslant s$, and an analytic function $(\lambda, z) \rightarrow u(\lambda, z)$ from $B\left(\lambda_{0}, r_{1}\right) \times B\left(z_{0}, s_{1}\right)$ into $A$ such that

$$
\begin{equation*}
(z-g) u(\lambda, z)-1 \in(f-\lambda) A, \tag{5}
\end{equation*}
$$

for $\left|\lambda-\lambda_{0}\right|<r_{1},\left|z-z_{0}\right|<s_{1}$. For every integer $n \geqslant 1,(\lambda, z) \rightarrow l_{\lambda}\left(u(\lambda, z)^{n}\right)$ is analytic on this polydisk, consequently $(\lambda, z) \rightarrow(1 / n) \log \left|l_{\lambda}\left(u(\lambda, z)^{n}\right)\right|$ is pluri-subharmonic on this polydisk. Perhaps $\psi(\lambda, z)=\lim _{n \rightarrow \infty}(1 / n)$ $\log \left|l_{\lambda}\left(u(\lambda, z)^{n}\right)\right|$ is not pluri-subharmonic, but it satisfies the mean value inequality on every complex line restricted to the polydisk. We have $\chi_{0}\left(u\left(\lambda_{0}, z_{0}\right)^{n}\right)=\chi_{0}\left(u\left(\lambda_{0}, z_{0}\right)\right)^{n}$, and by (5) we obtain

$$
\begin{equation*}
\psi\left(\lambda_{0}, z_{0}\right)=\phi\left(\lambda_{0}, z_{0}\right) . \tag{6}
\end{equation*}
$$

If we succeed in proving that $\psi(\lambda, z) \leqslant \phi(\lambda, z)$ on the polydisk the proof of $1^{\circ}$ will be finished. In fact if $(\lambda, h(\lambda))$ is a parametrization of a complex line through ( $\lambda_{0}, z_{0}$ ), with $h\left(\lambda_{0}\right)=z_{0}$, we have

$$
\begin{aligned}
\phi\left(\lambda_{0}, h\left(\lambda_{0}\right)\right) & =\psi\left(\lambda_{0}, h\left(\lambda_{0}\right)\right) \leqslant \frac{1}{2 \pi} \int_{0}^{2 \pi} \psi\left(\lambda_{0}+\rho \varepsilon^{i \theta}, h\left(\lambda_{0}+\rho e^{i \theta}\right)\right) d \theta \\
& \leqslant \frac{1}{2 \pi} \int_{0}^{2 \pi} \phi\left(\lambda_{0}+\rho e^{i \theta}, h\left(\lambda_{0}+\rho e^{i \theta}\right)\right) d \theta,
\end{aligned}
$$

if $\rho$ is small enough. But it is easy to prove that $\phi$ is upper semi-continuous, so it is locally pluri-subharmonic, hence pluri-subharmonic. Now we prove that $\psi(\lambda, z) \leqslant \phi(\lambda, z)$ on the polydisk. We have

$$
\left|l_{\lambda}\left(u(\lambda, z)^{n}\right)\right| \leqslant 2\left\|\overline{u(\lambda, z)^{n^{\lambda}}}\right\| \|_{\lambda},
$$

where $\|\mid\|_{\lambda}$ denotes the norm in $A_{\lambda}$. But in $A_{\lambda}$, by relation (5),

$$
\overline{u(\lambda, z)^{n}}=\left(z-\bar{g}^{\lambda}\right)^{-n},
$$

$$
\varlimsup_{n \rightarrow \infty} \frac{1}{n} \log \left|l_{\lambda}\left(u(\lambda, z)^{n}\right)\right| \leqslant \log \rho_{\lambda}\left(\left(z-\bar{g}^{\lambda}\right)^{-1}\right)=\phi(\lambda, z)
$$

$2^{\circ}$ To prove that $\Omega$ is a domain of holomorphy we use Theorem 3.1 and the fact that $(\lambda, z) \rightarrow-\log \operatorname{dist}(z, K(\lambda))$ is a vertical function for $\Omega$, pluri-subharmonic on $\Omega$, by $1^{\circ}$.

Remark. This proof is not completely satisfactory because it uses Theorem 3.1. If it were possible to prove elementarily that there exist two analytic functions $(\lambda, z) \rightarrow u(\lambda, z)$ and $(\lambda, z)+v(\lambda, z)$, from $\Omega$ into $A$ such that

$$
(z-g) u(\lambda, z)=1+(f-\lambda) v(\lambda, z)
$$

for $(\lambda, z) \in \Omega$, it would be easy to prove that $\Omega$ is a domain of holomorphy. The proof, in fact, would be similar to the proof of Theorem 3.2. We take $\theta(\lambda, z)=\|u(\lambda, z)\|+\|v(\lambda, z)\|-\log \operatorname{dist}(\lambda, \partial D)$. It is obvious that $\theta$ is plurisubharmonic on $\Omega$. Let us prove now that $\theta(\lambda, z) \rightarrow+\infty$, when $(\lambda, z)$ goes to the boundary of $\Omega$. Suppose, on the contrary, that there exists $\left(\lambda_{0}, z_{0}\right) \in \partial \Omega$, $\left(\lambda_{n}\right) \rightarrow \lambda,\left(z_{n}\right) \rightarrow z_{0}, M>0$, such that $\left(\lambda_{n}, z_{n}\right) \in \Omega$ and $\theta\left(\lambda_{n}, z_{n}\right) \leqslant M$. We have $\lambda_{0} \in D$ and

$$
\begin{aligned}
\left(z_{0}-g\right) u\left(\lambda_{n}, z_{n}\right)= & u\left(\lambda_{n}, z_{n}\right)\left(z_{n}-g+z_{0}-z_{n}\right) \\
= & 1+\left(f-\lambda_{n}\right) v\left(\lambda_{n}, z_{n}\right)+u\left(\lambda_{n}, z_{n}\right)\left(z_{0}-z_{n}\right) \\
= & 1+\left(f-\lambda_{0}\right) v\left(\lambda_{n}, z_{n}\right)+\left(\lambda_{0}-\lambda_{n}\right) v\left(\lambda_{n}, z_{n}\right) \\
& +u\left(\lambda_{n}, z_{n}\right)\left(z_{0}-z_{n}\right) \\
= & 1+\left(\lambda_{0}-\lambda_{n}\right) v\left(\lambda_{n}, z_{n}\right)+u\left(\lambda_{n}, z_{n}\right)\left(z_{0}-z_{n}\right) \\
& +\left(f-\lambda_{0}\right) v\left(\lambda_{n}, z_{n}\right) .
\end{aligned}
$$

But $\left\|\left(\lambda_{0}-\lambda_{n}\right) v\left(\lambda_{n}, z_{n}\right)+u\left(\lambda_{n}, z_{n}\right)\left(z_{0}-z_{n}\right)\right\| \leqslant C\left(\left|\lambda_{0}-\lambda_{n}\right|+\left|z_{0}-z_{n}\right|\right)$, where $C \leqslant M+2 \log \operatorname{dist}\left(\lambda_{0}, \partial D\right)$, for $n$ large enough. Consequently $1+\left(\lambda_{0}-\lambda_{n}\right)$ $v\left(\lambda_{n}, z_{n}\right)+u\left(\lambda_{n}, z_{n}\right)\left(z_{0}-z_{n}\right)$ has an inverse $y_{n}$ in $A$ and we have

$$
\left(z_{0}-g\right) u\left(\lambda_{n}, z_{n}\right) y_{n}=1+\left(f-\lambda_{0}\right) v\left(\lambda_{n}, z_{n}\right) y_{n} .
$$

But this indicates that $z_{0}-\bar{g}^{\lambda_{0}}$ is invertible in $A_{\lambda_{0}}$, which is a contradiction, so $\Omega$ is a domain of holomorphy.

I have not been able to prove easily the result I mentioned at the beginning of this remark, but neverheless this result is true. I shall now mention quickly how it is possible to prove it using a complicated method.

Theorem 3.5. Let $A, \mathscr{M}, X, W$ be as in Theorem 2.2. Suppose that $f$ and $g$ are in $A$ and that $\Omega=\left\{(\lambda, z) \mid z \notin K_{g}(\lambda), \lambda \in W\right\}$. Then there exist two analytic functions $(\lambda, z) \rightarrow u(\lambda, z)$ and $(\lambda, z) \rightarrow v(\lambda, z)$, from $\Omega$ into $A$ such that

$$
(z-g) u(\lambda, z)+(\lambda-f) v(\lambda, z)=1
$$

for $(\lambda, z)$ in $\Omega$.
Sketch of proof. It is enough to prove that there exists a $2 \times 2$ invertible matrix $M$, where the coefficients are analytic functions from $\Omega$ into $A$, such that

$$
\begin{equation*}
M\binom{z-g}{\lambda-f}=\binom{1}{0} \tag{7}
\end{equation*}
$$

By Lemma 3.3, this property is true locally on $\Omega$. With $x(z)=z-g$, on the polydisk $B\left(\lambda_{0}, r_{1}\right) \times B\left(z_{0}, s_{1}\right)$ we consider

$$
M\left(\lambda_{0}, z_{0}, r_{1}, s_{1}\right)=\left(\begin{array}{cc}
u(\lambda, z), & v(\lambda, z) \\
\lambda-f, & g-z
\end{array}\right),
$$

such that

$$
M\left(\lambda_{0}, z_{0}, r_{1}, s_{1}\right)^{-1}=\left(\begin{array}{cc}
z-g, & v(\lambda, z) \\
\lambda-f, & -u(\lambda, z)
\end{array}\right)
$$

which verifies $M\left(\lambda_{0}, z_{0}, r_{1}, s_{1}\right)(a)=b$, where $a$ is the column vector

$$
\binom{z-g}{\lambda-f} \text { and } b \text { the column vector }\binom{1}{0} .
$$

To extend this result globally on $\Omega$ we follow now the method given by H. Cartan in [56] (which is essentially the basic paper on the foundations of coherent sheaves). We suppose that $\Delta_{1}, \Delta_{2}$ are two compact polycylinders with the same components for all the variables, except one, and for which we have a local solution of (7). Then

$$
\begin{array}{lll}
M_{1}(a)=b & \text { on } & \Delta_{1} \\
M_{2}(a)=b & \text { on } & \Delta_{2}
\end{array}
$$

and

$$
M_{1} M_{2}^{-1}(b)=b \quad \text { on } \quad \Delta_{1} \cap \Delta_{2}
$$

Every invertible $2 \times 2$ matrix $S$, with coefficients analytic on $\Delta_{1} \cap \Delta_{2}$,
leaving $b$ invariant, can be written in the from $S_{1}^{-1} S_{2}$, where $S_{1}$ is invertible, leaving $b$ invariant, with coefficients analytic on $\Delta_{1}$ and where $S_{2}$ is invertible, leaving $b$ invariant, with coefficients analytic on $\Delta_{2}$. If we have

$$
S=\left(\begin{array}{cc}
1 & \alpha \\
0 & \beta
\end{array}\right), \quad S_{1}^{-1}=\left(\begin{array}{cc}
1 & \alpha_{1} \\
0 & \beta_{1}
\end{array}\right), \quad S_{2}^{-1}=\left(\begin{array}{cc}
1 & \alpha_{2} \\
0 & \beta_{2}
\end{array}\right)
$$

we must have $\beta_{1}=\beta \beta_{2}, \alpha_{1}=\alpha_{2}+\alpha \beta_{2}$. But it is possible to solve the multiplicative and the additive Cousin's problems on $\Delta_{1} \cap A_{2}$. The proof is more complicated than in the case of analytic functions with values in $\mathbb{C}$ but adapting the proof given in [17, pp. 192-201], it is possible to prove that for the invertible analytic function $\beta$ from $\Omega$ into $A$ we can find $\beta_{1}, \beta_{2}$ invertible, with values in $A$, the first one being analytic on $\Delta_{1}$ and the second one on $\Delta_{2}$, such that $\beta_{1}=\beta \beta_{2}$. The function $\beta_{2}$ being chosen we use the analog of the additive problem to find $\alpha_{1}$ and $\alpha_{2}$ respectively analytic on $\Delta_{1}$ and $\Delta_{2}$.
Now we have $M_{1} M_{2}^{-1}=S_{1}^{-1} S_{2}$ on $\Delta_{1} \cap A_{2}$, with $S_{1}(b)=b$ and $S_{2}(b)=b$, so $S_{1} M_{1}=S_{2} M_{2}$ on $\Delta_{1} \cap \Delta_{2}$. We define an analytic matrix $M_{3}$ on $\Delta_{1} \cup \Delta_{2}$ by

$$
\begin{aligned}
M_{3} & =S_{1} M_{1} & & \text { on }
\end{aligned} \Delta_{1} .
$$

It is invertible and $M_{3}(a)=b$. So we have solved (7) on $\Delta_{1} \cup \Delta_{2}$. By Theorem 3.4, $\Omega$ is a domain of holomorphy, so there exists a sequence of compact polycylinders, having the property of the beginning, exhausting $\Omega$. Consequently, by successive steps we can solve (7) on $\Omega$. It is also possible to give a solution of (7) by using $\bar{\partial}$-theory of $C^{\infty}$-functions taking values in $A$.

Now I intend to give a certain number of theorems concerning analytic multivalued functions, but without complete proofs. For further details the reader will have to consult the given references. Later on I intend to publish a book giving a systematic introduction to this new theory.

Theorem 3.6. Let $D$ be a domain of $\mathbb{C}$. We suppose that for every integer $n \geqslant 1$ there exists an analytic multivalued function $\lambda \rightarrow K_{n}(\lambda)$ defined on $D$.

10 If for every $\lambda \in D$ the set $U_{n>1} K_{n}(\lambda)$ is relatively compact then $\lambda \rightarrow \overline{\bigcup_{n \geqslant 1} K_{n}(\lambda)}$ is analytic multivalued on $D$.

$$
2^{\circ} \text { If for every } \lambda \in D \text { we have } K_{1}(\lambda) \supset K_{2}(\lambda) \supset \cdots \supset K_{n}(\lambda) \supset
$$ $K_{n+1}(\lambda) \supset \cdots$, then $\lambda \rightarrow \bigcap_{n \geqslant 1} K_{n}(\lambda)$ is analytic multivalued on $D$.

Sketch of proof. $1^{\circ}$ This is true because the interior of the intersection of a family of pseudoconvex domains is pseudoconvex (see [48, pp. 94-95]). $2^{\circ}$

This result comes immediately from the Behnke-Stein theorem, which says that an increasing sequence of domains of holomorphy is a domain of holomorphy (see [48, pp. 146-148]).

Theorem 3.7. Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$; then:
$1^{\circ} \quad \lambda \rightarrow \log \rho(K(\lambda))$, where $\rho(K(\lambda))=\operatorname{Max}|z|$, for $z \in K(\lambda)$, is subharmonic on $D$.
$2^{\circ} \quad \lambda \rightarrow \log \delta_{n}(K(\lambda))$, where $\delta_{n}$ denotes the $n$th diameter of $K(\lambda)$ for $n \geqslant 1$, is subharmonic on $D$.
$3^{\circ} \lambda \rightarrow \log c(K(\lambda))$, where $c$ denotes the capacity of $K(\lambda)$, is subharmonic on $D$.

Sketch of proof. Part $3^{\circ}$ comes immediaely from part $2^{\circ}$. There are several methods to prove $1^{\circ}$ and $2^{\circ}$. I will merely indicate the simplest one given by H. Yamaguchi [54] (in [30], T. Nishino had obtained a proof of subharmonicity of $\lambda \rightarrow \log \delta_{1}(K(\lambda))$, similar to the proof I obtained in [4]). If $\Omega$ is a domain of holomorphy it is the union of an increasing sequence of $\Omega_{k}=\left\{(\lambda, z) \in \Omega \mid \phi_{k}(\lambda, z)<0\right\}$, where the $\phi_{k}$ are $C^{\infty}$ and pluri-subharmonic functions, and such that for every point in $\partial \Omega_{k}$ there exists an analytic variety, going through this point and non-singular at this point, which is not locally in $\Omega_{k}$. It says that

$$
K(\lambda)=\bigcap_{k>1} K_{k}(\lambda)
$$

where the graph of $K_{k}(\lambda)$ is the complement of $\Omega_{k}$, and that for each $z_{0} \in \partial K_{k}\left(\lambda_{0}\right)$ there exists an analytic function $h_{k}$ such that $h_{k}\left(\lambda_{0}\right)=z_{0}$ and $h_{k}(\lambda) \in K_{k}(\lambda)$ in a neighbourhood of $z_{0}$ (see [48, pp. 183-174] and the theorem of Levi-Krzoska [48, pp. 157-160]). By using these analytic selections it is not difficult to prove that $\lambda \rightarrow \log \rho\left(K_{k}(\lambda)\right)$ and $\lambda \rightarrow \log \delta_{n}\left(K_{k}(\lambda)\right)$ are subharmonic and taking the limits, with some technical details we obtain the result.

Remarks. Of course E. Vesentini's theorem for spectrum and J. Wermer's theorem for fibers, mentioned respectively in Sections 1 and 2, are particular cases of $1^{\circ}$ in the previous theorem. Part $2^{\circ}$ also implies directly easy proofs of the scarcity theorem for finite spectrum and of the theorem of E. Bishop for finite fibers.

If $\lambda \rightarrow\{h(\lambda)\}$ is an analytic multivalued function, where $h$ is a function from $D$ into $\mathbb{C}$; then it is easy to see that $\lambda \rightarrow\{h(\lambda)-\alpha \lambda-\beta\}$ is also analytic multivalued. By $1^{\circ}, \lambda \rightarrow \log |h(\lambda)-\alpha \lambda-\beta|$ is subharmonic, so, by Lemma $2.6, h$ is holomorphic. It is exactly the theorem of F. Hartogs.

Using part $2^{\circ}$ of Theorem 3.7 and the same argument as in the remark just after Corollary 1.4, we obtain immediately the following:

THEOREM 3.8. Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$; then:

- either the set of $\lambda$ for which $K(\lambda)$ is finite is of outer capacity zero,
- or there exists an integer $n \geqslant 1$ such that $\# K(\lambda)=n$, for every $\lambda$ in $D$, except on a closed discrete countable set $E$ of $D$. In this case the points of $K(\lambda)$ vary holomorphically if $\lambda$ is outside of $E$.

THEOREM 3.9 (Functional calculus for analytic multivalued functions). Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$ and let $u(\lambda, z)$ be holomorphic in a neighbourhood of the graph $G=\{(\lambda, z) \mid \lambda \in D, \quad z \in K(\lambda)\}$. Then $\lambda \rightarrow\{u(\lambda, z) \mid z \in K(\lambda)\}$ is analytic multivalued on $D$.

Sketch of proof. $1^{\circ}$ First we suppose that $u(\lambda, z)$ is rational, without singularities on $G$. For instance,

$$
u(\lambda, z)=p(\lambda, z) / q(\lambda, z)
$$

where $p, q$ are two polynomials and $q$ does not vanish on $G$. We consider $F(\lambda, \mu)=\{z \in \mathbb{C} \mid p(\lambda, z)-\mu q(\lambda, z)=0\}$. Because $q$ does not vanish on $G$, we have $F(\lambda, \mu) \cap K(\lambda) \neq \varnothing$ if and only if $\mu \in\{u(\lambda, z) \mid z \in K(\lambda)\}$. Let $\Omega^{\prime}$ be the set of $(\lambda, \mu)$ such that $\lambda \in D, \mu \in \mathbb{C}, \mu \notin\{u(\lambda, z) \mid z \in K(\lambda)\}$. We have to prove that this set is a domain of holomorphy. Of course $\Omega^{\prime}=\{(\lambda, \mu) \mid \lambda \in D$, $\mu \in \mathbb{C},\{\lambda\} \times F(\lambda, \mu) \subset \Omega\}$, where $\Omega$ is the domain of holomorphy associated to the analytic multivalued function $\lambda \rightarrow K(\lambda)$. We know that $(\lambda, z) \rightarrow \phi(\lambda, z)=-\log \operatorname{dist}((\lambda, z), \partial \Omega)$ is pluri-subharmonic on $\Omega$, so it is not difficult to prove that $(\lambda, \mu) \rightarrow \psi(\lambda, \mu)=\operatorname{Max}_{z \in F(\lambda, \mu)} \phi(\lambda, z)$ is plurisubharmonic on $\Omega^{\prime}$ and goes to infinity when $(\lambda, \mu)$ goes to the boundary of $\Omega^{\prime} .2^{\circ}$ We now suppose that $u(\lambda, z)$ is holomorphic in a neighbourhood of $G$. We introduce $L(\lambda)=\{u(\lambda, z) \mid z \in K(\lambda)\}$. It is easy to prove that $(\lambda, \mu) \rightarrow-\log \operatorname{dist}(\mu, L(\lambda))$ is upper semi-continuous on $\Omega^{\prime}$. We fix $\lambda_{0} \in D$. By upper semi-continuity, there exists $r>0$ and an open set $U$ in $\mathbb{C}$ such that $K(\lambda) \subset U$, for $\left|\lambda-\lambda_{0}\right|<r$ and such that $\bar{B}\left(\lambda_{0}, r\right) \times \bar{U}$ is included in the open set of $\mathbb{C}^{2}$ where $u$ is holomorphic. This product of rationally convex sets is rationally convex in $\mathbb{C}^{2}$; hence every function holomorphic in a neighbourhood of $\bar{B}\left(\lambda_{0}, r\right) \times \bar{U}$ is a uniform limit on this set of rational functions without singularities on this set (it is the generalization for $\mathbb{C}^{n}$ of Runge's theorem-see, for instance, the paper of K. Oka [57]). Hence $u(\lambda, z)$ is approximated on $\bar{B}\left(\lambda_{0}, r\right) \times \bar{U}$ by rational functions $u_{n}(\lambda, z)$ without singularities on this set. The sequence $\phi_{n}(\lambda, \mu)=-\log \operatorname{dist}\left(\mu, L_{n}(\lambda)\right)$, where
$L_{n}(\lambda)=\left\{u_{n}(\lambda, z) \mid z \in K(\lambda)\right\}$ is uniformly bounded on $\bar{B}\left(\lambda_{0}, r\right)$, for a fixed $\mu$, and converges pointwise on $\bar{B}\left(\lambda_{0}, r\right)$ to $-\log \operatorname{dist}(\mu, L(\lambda))$. If we prove that the $\phi_{n}$ are locally pluri-subharmonic then $(\lambda, \mu) \rightarrow-\log \operatorname{dist}(\mu, L(\lambda))$ satisfies the mean inequality on each complex line, and it is upper semi-continuous, so it is pluri-subharmonic. Using Theorem 3.1 the proof will be finished. The functions $\phi_{n}$ are locally subharmonic, for $n$ large enough, because $\phi_{n}(\lambda, \mu)=\log \rho\left(M_{n}(\lambda)\right)$, where

$$
M_{n}(\lambda)=\left\{\left.\frac{1}{\mu-u_{n}(\lambda, z)} \right\rvert\, z \in K(\lambda)\right\}
$$

and we use part $1^{\circ}$, and Theorem 3.7, part $1^{\circ}$. For more details see [42].
Corollary 3.10. Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$. We suppose that $\mu \notin K\left(\lambda_{0}\right)$ then there exists $r>0$ such that $\mu \notin K(\lambda)$ for $\left|\lambda-\lambda_{0}\right|<r$ and $\lambda \rightarrow-\log \operatorname{dist}(\mu, K(\lambda))$ is subharmonic on $B\left(\lambda_{0}, r\right)$.

Proof. This result comes immediately from part $1^{\circ}$ of the previous theorem and part $1^{\circ}$ of Theorem 3.7.

In fact this result, with Theorem 3.1, says that $\Omega$ is a domain of holomorphy if and only if $(\lambda, z) \rightarrow \log \operatorname{dist}(z, K(\lambda))$ is pluri-subharmonic on $\Omega$. We now give two corollaries which are extensions of two results of J. D. Newburgh, for spectrum.

The following result is obviously a generalization of Theorem 1.2.5, p. 17, in [6].

Theorem 3.11 (Holomorphic variation of isolated points). Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$. We suppose that $\alpha_{0}$ is isolated in $K\left(\lambda_{0}\right)$ and that $s>0$ is small enough such that $K\left(\lambda_{0}\right) \cap B\left(\alpha_{0}, s\right)=\left\{\alpha_{0}\right\}$. Then there exists $r>0$ such that:
$1^{\circ}$ either the set of $\lambda$ verifying $\left|\lambda-\lambda_{0}\right|<r$ for which $K(\lambda) \cap B\left(\alpha_{0}, s\right)$ has one point is of capacity zero,
$2^{\circ}$ or $K(\lambda) \cap B\left(\alpha_{0}, s\right)=\{h(\lambda)\}$, for every $\lambda$ such that $\left|\lambda-\lambda_{0}\right|<r$, where $h$ is holomorhic on this disk.

Sketch of proof. By Theorem 3.9 we localize the problem in a neighbourhood of $\lambda_{0}$. To get $1^{\circ}$ we use part $2^{\circ}$ of Theorem 3.7 and to get $2^{\circ}$ we use the last remark given after Theorem 3.7.

Using Corollary 3.10, and translating the proofs of Theorem 1.2.3., p. 12, and Theorem 1.4.1, p. 32, in [6], we then obtain:

TheOrem 3.12 (Principle of maximum). Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$. We suppose that there exists $\lambda_{0} \in D$ such that $K(\lambda) \subset K\left(\lambda_{0}\right)$, for every $\lambda \in D$. Then $\partial K\left(\lambda_{0}\right) \subset \partial K(\lambda)$, for every $\lambda \in D$. In particular, if $\hat{K}(\lambda) \subset \hat{K}\left(\lambda_{0}\right)$, for every $\lambda$ in $D$ and some $\lambda_{0}$ in $D$ then $\hat{K}(\lambda)=\hat{K}\left(\lambda_{0}\right)$, for every $\lambda$ in $D$ (here $\hat{K}(\lambda)$ denotes the polynomially convex hull of $K(\lambda)$ ).

Theorem 3.13 (Pseudo-continuity). Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$. We suppose that $\lambda_{0} \in D$ and that $E$ is a subset of $D$, non-thin at $\lambda_{0}$-for example, $E$ is a Jordan arc ending at $\lambda_{0}$ or a connected open subset with $\lambda_{0}$ in its boundary. Then there exists a sequence $\left(\lambda_{n}\right)$ of points of $E$, converging to $\lambda_{0}$ such that $\lim _{n \rightarrow \infty} \Delta\left(\hat{K}\left(\lambda_{n}\right), \hat{K}\left(\lambda_{0}\right)\right)=0$, where $\Delta$ denotes the Hausdorff's distance.

The analog propositions with only $K(\lambda)$ are not true. See the corresponding examples, with spectrum, given in $|6, \mathrm{pp} .13,34-41|$.

Theorem 3.14 (Desintegration of analytic multivalued functions). Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$. Suppose that $\lambda_{0} \in D$ and that $C$ is a non-void open and closed subset of $K\left(\lambda_{0}\right)$. For every disjoint open sets $U, V$ such that $C \subset U$ and $K\left(\lambda_{0}\right) \backslash C \subset V$, there exists $r>0$ such that $\left|\lambda-\lambda_{0}\right|<r$ implies $\lambda \in D$ with $K(\lambda) \subset U \cup V$ and $K(\lambda) \cap U \neq \varnothing$. Then $\lambda \rightarrow K(\lambda) \cap U$ is analytic multivalued on $B\left(\lambda_{0}, r\right)$.

Proof. Doing a translation we can suppose that $0 \notin V$. By upper semicontinuity we have $K(\lambda) \subset U \cup V$ for $\left|\lambda-\lambda_{0}\right|<r$ if $r$ is small enough. We apply Theorem 3.9 with $u(\lambda, z)=z$ on $B\left(\lambda_{0}, r\right) \times U$ and $u(\lambda, z)=0$ on $B\left(\lambda_{0}, r\right) \times V$. We get an analytic multivalued function $\lambda \rightarrow L(\lambda)$, defined on $B\left(\lambda_{0}, r\right)$, such that $C \subset L\left(\lambda_{0}\right) \subset C \cup\{0\}$ and $L(\lambda) \subset U \cup\{0\}$. Let $E=$ $\left\{\lambda\left|\left|\lambda-\lambda_{0}\right|<r\right.\right.$ and $\left.K(\lambda) \subset V\right\}$; by upper semi-continuity this set is open. If this set is empty the first part of the theorem is proved. So we suppose $E$ non-void. In this case $L(\lambda)=\{0\}$ on $E$ and in particular on each component $E_{1}$ of $E$. It is impossible that $E_{1}=B\left(\lambda_{0}, r\right)$ because $\lambda_{0} \notin E$; so $E_{1}$ has a boundary point $\alpha$ in $B\left(\lambda_{0}, r\right)$. By Theorem 3.13, there exists a sequence $\left(\lambda_{n}\right)$ converging to $\alpha$ with $\lambda_{n} \neq \alpha, \lambda_{n}$ in $E_{1}$ and $\lim _{n \rightarrow \infty} \Delta\left(L(\alpha), L\left(\lambda_{n}\right)\right)=0$. So $L(\alpha)=\{0\}$. But $\alpha$ is not in $E$, so $L(\alpha)$ has a point in $U$ which is not zero, so we have a contradiction. Hence the first part of the theorem is true. By upper semi-continuity of $\lambda \rightarrow K(\lambda)$ we have upper semi-continuity of $\lambda \rightarrow K(\lambda) \cap U$ on $B\left(\lambda_{0}, r\right)$. We have only to prove that $\Omega^{\prime}=\left\{(\lambda, z)| | \lambda_{0}-\lambda_{0} \mid<r\right.$, $z \notin K(\lambda) \cap U\}$ is pseudoconvex. If $K\left(\lambda_{0}\right) \backslash C$ is empty it is obvious. If not we have $L\left(\lambda_{0}\right)=C \cup\{0\}$ and $L(\lambda)=(K(\lambda) \cap U) \cup\{0\}$ for $\left|\lambda-\lambda_{0}\right|<r$. By Theorem 3.9, $\Omega^{\prime \prime}=\left\{(\lambda, z)| | \lambda \lambda_{0} \mid<r, z \notin L(\lambda)\right\}$ is pseudoconvex. Of course $\Omega^{\prime}=\Omega^{\prime \prime} \cup\left\{(\lambda, z)| | \lambda-\lambda_{0} \mid<r, z=0\right\}$. A small technical argument
shows that $\Omega^{\prime}$ is locally pseudoconvex at each of its boundary points. So it is pseudoconvex.

Corollary 3.15. With the hypotheses of Theorem 3.14, if $K\left(\lambda_{0}\right)$ is totally disconnected then $\lambda \rightarrow K(\lambda)$ is continuous at $\lambda_{0}$.

We come now to the more important results of this section. They had been conjectured, without connection to spectral theory and fibers theory, by K. Oka [32] in order to generalize Hartogs' theorem. But they have been proved for the first time by T. Nishino [30]. Their proofs, mainly the proof of Lemma 3.16, are too complicated to be given. Consequently we refer the reader to $[30,42]$.

We shall say that $\alpha_{0}$ in $K\left(\lambda_{0}\right)$ is a first kind isolated point if for $s>0$ given such that $K\left(\lambda_{0}\right) \cap B\left(\lambda_{0}, s\right)=\left\{a_{0}\right\}$ there exists $r>0$ such that $\left|\lambda-\lambda_{0}\right|<r$ implies $\#\left(K(\lambda) \cap B\left(\lambda_{0}, s\right)\right)<\infty$. By Theorems 3.7 and 3.8 this means that the graph of $\lambda \rightarrow K(\lambda)$, restricted to a small neighbourhood of ( $\lambda_{0}, \alpha_{0}$ ), is an analytic variety. We then define $D K(\lambda)$ by $K(\lambda)$ minus the set of first kind isolated points of $K(\lambda)$. Obviously $D K(\lambda)$ is closed and $K(\lambda)^{\prime} \subset D K(\lambda)$.

Lemma 3.16 (Oka-Nishino). Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$. Then:
$1^{\circ}$ either $K(\lambda)$ is finite for every $\lambda$ in $D$ and then $D K(\lambda)$ is always void,
$2^{\circ}$ or $D K(\lambda)$ is non-void for every $\lambda$ in $D$ and $\lambda \rightarrow D K(\lambda)$ is analytic multivalued on $D$.

For every transfinite number $\alpha$ it is then possible to define $D^{\alpha} K(\lambda)$ by
(a) $D^{\alpha} K(\lambda)=D\left(D^{\alpha-1} K(\lambda)\right)$, if $\alpha$ is not a limit ordinal,
(b) $D^{\alpha} K(\lambda)=\bigcap_{B<\alpha} D^{\beta} K(\lambda)$, if $\alpha$ is a limit ordinal.

If $D^{\alpha} K(\lambda)$ is non-void, then by the previous lemma $\lambda \rightarrow D^{\alpha} K(\lambda)$ is an analytic multivalued function.

Theorem 3.17 (Oka-Nishino). Let $\lambda \rightarrow K(\lambda)$ be an analytic multivalued function defined on a domain $D$ of $\mathbb{C}$. We suppose that $K(\lambda)$ is countable for every $\lambda$ in a set $E$ of positive outer capacity. Then $K(\lambda)$ is countable for every $\lambda$ in $D$. More precisely there exists $\alpha \in \mathscr{F}$ such that the $\alpha$-topological derivative of $K(\lambda)$ is void for every $\lambda$ in $D$.

Corollary 3.18. Conjecture 3 is true.
Proof. By Theorem 3.2, $\lambda \rightarrow \operatorname{Sp} f(\lambda)$ is analytic multivalued on $D$, so we apply the previous theorem.

As mentioned in Section 2 we can globalize Theore 2.8, when $A$ is separable, in the form of Theorem 2.9.

Corollary 3.19. Theorem 2.9 is true.
Proof. With the hypotheses of Theorem 2.9 and using Theorem 3.17, we obtain that $\lambda \rightarrow K_{g}(\lambda)$ is countable for every $\lambda$ in $W$. If $A$ is separable, $A_{\lambda}=A /(f-\lambda) A$ is also separable and commutative. If $f \in A$ then $\operatorname{Sp} \bar{g}^{\lambda}=K_{g}(\lambda)$ is countable. But if for every element $x$ of a commutative separable Banach algebra $B$ we have $\operatorname{Sp} x$ countable, then the set of characters $\mathscr{M}(B)$ is countable. Consequently $\bar{f}^{1}(\lambda)$, which can be identified to $\mathscr{M}\left(A_{\lambda}\right)$, is countable, for every $\lambda$ in $D$. The proof is now finished as in [7] or [8].

If $A$ is not separable it may happen that $K_{g}(\lambda)$ is countable for every $f$ in $A$, with $\bar{f}^{1}(\lambda)$ uncountable, but very "thin." For example, let us take $X$ a onepoint compactification of a discrete uncountable set, $Y=X \times \bar{\Delta}$, with $\Delta=\{\lambda| | \lambda \mid<1\}$ and $A$ the algebra of continuous functions $f$ on $Y$ such that $\lambda \rightarrow f(x, \lambda)$ is holomorphic on $\Delta$ for every $x \in X$. Of course $\mathscr{A}(A)=Y$. If we take $f:(x, \lambda) \rightarrow \lambda$, for a fixed $\lambda_{0}$ in $\Delta$ we have $\bar{f}^{1}\left(\lambda_{0}\right)=X \times\left\{\lambda_{0}\right\}$ uncountable and $K_{g}(\lambda)$ countable, because it is compact with only one limit point.

To conclude this paper I would like to say that the method of multivalued analytic functions seems powerful because it reduces some spectral and fiber problems to the study of the geometry of domains of holomorphy, for which we have a lot of classical results. Perhaps it may also apply to other fields, for instance, cluster sets theory, differential equations, etc.

For more information see $[13,53,58,59,61]$.

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