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REPEATED BLOCKS IN INDECOMPOSABLE TWOFOLD TRIPLE SYSTEMS

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1. Introduction

A Steiner triple system (STS) (a twofold triple system (TTS)) is a pair (V, B) where V is a v-set, and B is a collection of 3-subsets of V called *blocks* or *triples* such that every 2-subset of V is contained in exactly one [exactly two] blocks. The number v is called the *order* of the STS or TTS. It is well-known that an STS of order v exists if and only if $v \equiv 1$ or 3 (mod 6), and a TTS of order v exists if and only if $v \equiv 0$ or 1 (mod 3).

A TTS(v) (V, B) whose block-set B can be partitioned into two subsets B_1, B_2 such that (V, B_1), (V, B_2) are both STS(v)'s, is called *decomposable*; otherwise, it is *indecomposable*. An indecomposable TTS(v) is known to exist if and only if $v \equiv 0$ or 1 (mod 3) and $v \neq 3, 7$ [2].

If a TTS contains two blocks $b_1 = \{x, y, z\}$, $b_2 = \{x, y, z\}$ that are identical as subsets of V then $\{x, y, z\}$ is said to be a *repeated block*; otherwise, a block is called *nonrepeated*. In a recent paper [5], the author and D. Hoffman gave a complete answer to the following question:

> Given $v \equiv 0$ or 1 (mod 3) and nonnegative integer k, does there exist a TTS(v) with exactly k repeated triples?

One may ask the same question with the added condition that the TTS be indecomposable. Since trivially any TTS(v) with $v \equiv 0$ or 4 (mod 6) is indecomposable, the answer to the expanded question for these orders is the same as that for the original question. It is the purpose of this paper to settle completely this expanded question by providing an answer for the cases when $v \equiv 1$ or 3 (mod 6).

2. Preliminaries, recursive construction, and statement of the main theorem

Following [5], let $R(v) = \{k : \exists \text{TTS}(v)$ with exactly k repeated blocks$\}$. It has

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been shown ([5]; cf. also [3]) that for $v > 12$,

$$R(v) = \begin{cases} \{0, 1, \ldots, b_v - 6, b_v - 4, b_v\}, & \text{if } v \equiv 1 \text{ or } 3 \pmod 6, \\ \{0, 1, \ldots, s_v - 2, s_v\}, & \text{if } v \equiv 0 \text{ or } 4 \pmod{12}, \\ \{0, 1, \ldots, s_v - 2, s_v - 1\}, & \text{if } v \equiv 6 \text{ or } 10 \pmod{12}, \end{cases}$$

where $b_v = \frac{1}{6}v(v-1)$, $s_v = \frac{1}{6}v(v-4)$.

Denote now, by analogy, $R_I(v) = \{k : \exists$ *indecomposable* $\mathrm{TTS}(v)$ with exactly k repeated blocks$\}$. We have trivially $R_I(v) = R(v)$ for $v \equiv 0, 4 \pmod 6$, so from now on we concentrate on the case $v \equiv 1$ or $3 \pmod 6$. For small values of v, we have

$$\begin{aligned} &R(3) = \{1\}, && R_I(3) = \emptyset, \\ &R(7) = \{0, 1, 3, 7\}, && R_I(7) = \emptyset, \\ &R(9) = \{0, 1, 2, 3, 4, 6, 12\}, && R_I(9) = \{0, 1, 2, 4\} \quad \text{(cf., e.g. [4]).} \end{aligned}$$

But it is easy to see that we have $R_I(v) \subsetneq R(v)$ for all $v \equiv 1$ or $3 \pmod 6$: if a $\mathrm{TTS}(v)$ has b_v repeated blocks, it is necessarily decomposable, i.e., $b_v \in R(v)$ but $b_v \notin R_I(v)$.

Let us call a collection T of triples *doubly balanced for pairs* if any pair $\{x, y\}$ is contained in 0 or 2 triples of T. Such a collection is *parity-regular* if elements occurring in at least one triple of T all occur in an even number of triples of T, or all occur in an odd number of triples of T, and is *separable* if it can be partitioned into two subsets T_1, T_2 such that a pair is contained in a triple of T_1 if and only if it is contained in a triple of T_2 (i.e., if it can be partitioned into disjoint mutually balanced partial triple systems, cf. [3]).

Clearly, in an indecomposable $\mathrm{TTS}(v)$ the nonrepeated blocks must form a non-separable, parity-regular collection doubly balanced for pairs.

Denote $L(v) = \{0, 1, \ldots, b_v - 8, b_v - 6\}$.

Lemma 2.1. $R_I(v) \subseteq L(v)$ *if* $v \equiv 1$ *or* $3 \pmod 6$.

Proof. Clearly it suffices to consider $v \geq 13$. As in this case $R(v) = \{0, 1, \ldots, b_v - 6, b_v - 4, b_v\}$, the statement of the lemma is equivalent to saying that neither of $b_v - 7$, $b_v - 4$ and b_v belongs to $R_I(v)$. This has already been shown for b_v. Assume now $b_v - 4 \in R_I(v)$. This means that there exists an indecomposable $\mathrm{TTS}(v)$ with exactly $b_v - 4$ repeated blocks and exactly 8 nonrepeated blocks, and the nonrepeated blocks form a collection doubly balanced for pairs that is nonseparable and parity-regular. It is an easy exercise to establish that there are, up to isomorphism, exactly three collections of 8 blocks doubly balanced for pairs. Of these, one is separable, and the other two are not parity-regular. Thus $b_v - 4 \notin R_I(v)$. Assuming $b_v - 7 \in R_I(v)$, we get that there exists an indecomposable $\mathrm{TTS}(v)$ with exactly 14 nonrepeated blocks which form a collection doubly balanced for pairs. It is tedious but straightforward to verify that each such collection is either separable or not parity-regular. □

In what follows, we determine $R_I(v)$ for $v \equiv 1$ or 3 (mod 6), $v \geq 13$. In particular, we will prove the following

Main Theorem. *Let $v \equiv 1$ or 3 (mod 6), $v \geq 15$. Then there exists an indecomposable* TTS(v) *having exactly k repeated blocks if and only if $k \in L(v)$.*

We also determine $R_I(13)$ where one exceptional value occurs. The proof of the Main Theorem is given in Section 4.

If x is an element and P is a set of pairs, denote

$$x^*P = \{\{x, a, b\} : \{a, b\} \in P\}.$$

Lemma 2.2. *Let Q be a 2-factorization of $2K_{v+1}$, and let q be the total number of 2-cycles of Q. Then $t + q \in R_I(2v + 1)$ for all $t \in R_I(v)$.*

Proof. Let $W = V \cup X$ where $V = \{a_i : i = 1, \ldots, v\}, |X| = v + 1$, $V \cap X = \emptyset$. Let (V, B) be an indecomposable TTS(v) with t repeated triples. If $Q = \{Q_1, \ldots, Q_v\}$ is a 2-factorization of $2K_{v+1}$ on X, and $C = \bigcup_{i=1}^{v} a_i * Q_i$, then $(W, B \cup C)$ is an indecomposable TTS($2v + 1$) with $t + q$ repeated triples. □

Corollary 2.3. *Let $v \equiv 1$ or 3 (mod 6). If $t \in R_I(v)$, then $t + \frac{1}{2}(s(v + 1)) \in R_I(2v + 1)$ for every $s \in \{0, 1, \ldots, v - 2, v\}$.*

Proof. In Lemma 2.2, take $Q_i = F_i \cup F_{i\alpha}$, where $F = \{F_1, \ldots, F_v\}$ is any 1-factorization of K_{v+1} on X, and α is any permutation of $\{1, 2, \ldots, v\}$ fixing exactly s letters. □

Corollary 2.4. *For $v \geq 13$, $v \equiv 1$ or 3 (mod 6), $R_I(v) = L(v)$ implies $R_I(2v + 1) = L(2v + 1)$.*

Before proceeding further, we need to describe an auxiliary construction (cf. Construction B in [3]).

Construction B*. Let $W = V \cup X \cup Z$, where $V = \{a_i : i = 1, \ldots, v\}$, $X = \{b_i : i = 1, \ldots, v\}$, $Z = \{\infty_i : i = 1, \ldots, 7\}$, $|V \cup X \cup Z| = 2v + 7$. Let (V, B) be a TTS(v), and let (Z, C) be a TTS(7). Let $m = \frac{1}{2}(v - 1)$, and let $P = \{(p_r, q_r) : q_r - p_r = r, r = 1, 2, \ldots, m\}$ be an $(m, 1)$-Langford sequence (perfect or hooked) [7]. Let $Y = X \backslash S$, where

$$S = \{b_i : i = p_r \text{ or } q_r, r = 4, 5, \ldots, m; (p_r, q_r) \in P\}.$$

We have $|Y|=7$, so let $Y=\{b_{j_i}: i=1,2,\ldots,7\}$. Let further

$$E=\{\{\infty_i, a_t, b_{j_i+t-1}\}: i=1,2,\ldots,7, t=1,2,\ldots,v\},$$
$$F=\{\{a_t, b_{p_r+t-1}, b_{q_r+t-1}\}: t=1,2,\ldots,v, r=4,5,\ldots,m; (p_r,q_r)\in P\},$$
$$G=\{\{b_i, b_{i+1}, b_{i+3}\}: i=1,2,\ldots,v\},$$

with subscripts reduced modulo v to the range $\{1,2,\ldots,v\}$ whenever necessary. If α is a permutation of V, let E_α, F_α be the sets of triples obtained from E, F, respectively, by replacing every a_t, $t=1,\ldots,v$, with $a_t\alpha$. Further, let

$$G_\delta=\begin{cases} G, & \text{if } \delta=1,\\ \{\{b_i, b_{i+2}, b_{i+3}\}: i=1,\ldots,v\}, & \text{if } \delta=0.\end{cases}$$

Let $D=B\cup C\cup E\cup E_\alpha\cup F\cup F_\alpha\cup G\cup G_\delta$. Then (W,D) is a TTS$(2v+7)$.

Lemma 2.5. *Let $v\geqslant 9$, $v\equiv 1$ or 3 (mod 6). If $k\in R_I(v)$, then $k+\frac{1}{2}(s(v+7)+\delta v+\gamma\in R_I(2v+7)$ for every $s\in\{0,1,\ldots,v-2,v\}$, $\delta\in\{0,1\}$, $\gamma\in\{0,1,3,7\}$.*

Proof. In Construction B* above, take (V,B) indecomposable with k repeated triples, and take α that fixes exactly s elements of V. There are exactly 7 triples of E and exactly $\frac{1}{2}(v-7)$ triples of F containing a fixed element a_t, so

$$|(E\cup F)\cap(E_\alpha\cup F_\alpha)|=\tfrac{1}{2}s(v+7).$$

Further, $|G\cap G_\delta|=\delta v$. Finally, take C to contain γ repeated blocks. □

Corollary 2.6. *For $v\geqslant 15$, $v\equiv 1$ or 3 (mod 6), $R_I(v)=L(v)$ implies $R_I(2v+7)=L(2v+7)$.*

2. Starting the induction

In this section, we determine the sets $R_I(v)$ for several small values of v. We start with a trivial

Lemma 3.1. *If $n\equiv 1$ (mod 2) there exists a decomposition of $2K_n$ into $n-1$ Hamiltonian circuits.*

Corollary 3.2. $k\in R(u)$, $u\equiv 0$ *or* 4 (mod 6)$\rightarrow k\in R_I(2u+1)$.

Lemma 3.3. $\{1,2,3,4,5,7,8,10,12\}\subset R_I(13)$.

Proof. Let $W=V\cup X$, $V=\{a_1,\ldots,a_6\}$, $X=\{1,2,\ldots,7\}$. Let (V,B) be a TTS(6), and let $f_1,f_2,\ldots,f_{17}$ be the 2-factors of $2K_7$ on X given in Table 1. For $t=1,2,3,4,5,7,8,10,12$, let $F^{(t)}=\{F_1^{(t)},F_2^{(t)},F_3^{(t)},F_4^{(t)},F_5^{(t)},F_6^{(t)}\}$ be the 2-

Table 1

2-factors

f_1	(123)(4567)	f_{10}	(157)(26)(34)
f_2	(123)(45)(67)	f_{11}	(1527)(346)
f_3	(123)(47)(56)	f_{12}	(1427536)
f_4	(14625)(37)	f_{13}	(1426357)
f_5	(146)(2537)	f_{14}	(1624357)
f_6	(146)(25)(37)	f_{15}	(1634257)
f_7	(146)(27)(35)	f_{16}	(16257)(34)
f_8	(157)(2436)	f_{17}	(24357)(16)
f_9	(157)(24)(36)		

factorization of $2K_7$ on X given in Table 2. Let $C^{(t)} = \bigcup_{i=1}^{6} a_i * F_i^{(t)}$. Then $(W, B \cup C^{(t)})$ is a TTS(13) with exactly t repeated blocks; clearly, it is indecomposable. □

Lemma 3.4. $\{6, 9, 11, 13, 14, 15, 16, 18\} \subset R_I(13)$.

Proof. Let $W = U \cup Y$, $U = \{b_1, b_2, b_3, b_4\}$, $Y = \{1, 2, \ldots, 9\}$. Let (U, D) be a TTS(4), and let $g_1, g_2, \ldots, g_{21}$ be the 21 2-factors of $2K_9$ on Y given in Table 3. Let E be the following set of triples: $E = \{159, 168, 249, 267, 348, 357\}$. For $t = 0, 3, 5, 7, 8, 9, 10, 12$, let $G^{(t)} = \{G_1^{(t)}, G_2^{(t)}, G_3^{(t)}, G_4^{(t)}\}$ be the set of four 2-factors of $2K_9$ on Y given in Table 4; each $G^{(t)} \cup 2E$ decomposes $2K_9$ on Y. Let $C^{(t)} = \bigcup_{i=1}^{4} b_i * G_i^{(t)}$. Then $(W, D \cup C^{(t)} \cup 2E)$ is an indecomposable TTS(13) with exactly $t + 6$ repeated blocks. □

Lemma 3.5. $20 \in R_I(13)$.

Proof. Elements: $V = \{1, 2, \ldots, 13\}$.

Table 2

2-factorizations

$F^{(t)}$	$F_1^{(t)}$	$F_2^{(t)}$	$F_3^{(t)}$	$F_4^{(t)}$	$F_5^{(t)}$	$F_6^{(t)}$
$F^{(1)}$	f_1	f_1	f_4	f_{11}	f_{12}	f_{14}
$F^{(2)}$	f_1	f_1	f_4	f_{11}	f_{13}	f_{17}
$F^{(3)}$	f_1	f_1	f_4	f_7	f_8	f_{15}
$F^{(4)}$	f_2	f_3	f_5	f_5	f_8	f_8
$F^{(5)}$	f_2	f_3	f_4	f_{11}	f_{12}	f_{14}
$F^{(7)}$	f_2	f_3	f_4	f_7	f_8	f_{15}
$F^{(8)}$	f_2	f_3	f_6	f_7	f_8	f_8
$F^{(10)}$	f_2	f_3	f_4	f_7	f_9	f_{16}
$F^{(12)}$	f_2	f_3	f_6	f_7	f_9	f_{10}

Table 3

2-factors

g_1	(123)(456)(789)	g_{12}	(123654)(789)
g_2	(147)(258)(369)	g_{13}	(12563)(47)(89)
g_3	(123)(45)(69)(78)	g_{14}	(1285647)(39)
g_4	(123)(47)(56)(89)	g_{15}	(1328564)(79)
g_5	(123)(47)(58)(69)	g_{16}	(1364)(789)(25)
g_6	(147)(25)(36)(89)	g_{17}	(23658)(14)(79)
g_7	(123658974)	g_{18}	(258)(17)(39)(46)
g_8	(125478963)	g_{19}	(258)(14)(36)(79)
g_9	(1258974)(36)	g_{20}	(456)(17)(28)(39)
g_{10}	(132587964)	g_{21}	(789)(14)(25)(36)
g_{11}	(1325874)(69)		

Repeated blocks:

$$B_R = \{1\,2\,9,\ 1\,10\,13,\ 1\,11\,12,\ 2\,10\,12,\ 2\,11\,13,\ 3\,6\,9,\ 3\,7\,10,\ 3\,12\,13,\ 4\,6\,13,\ 4\,7\,11,\ 4\,8\,12,\ 4\,9\,10,\ 5\,6\,12,\ 5\,7\,13,\ 5\,8\,10,\ 5\,9\,11,\ 6\,10\,11,\ 7\,9\,12,\ 8\,9\,13\}.$$

Nonrepeated blocks:

$$B_N = \{1\,3\,4,\ 1\,3\,5,\ 1\,4\,5,\ 1\,6\,7,\ 1\,6\,8,\ 1\,7\,8,\ 2\,3\,4,\ 2\,3\,5,\ 2\,4\,5,\ 2\,6\,7,\ 2\,6\,8,\ 2\,7\,8\}. \quad \square$$

Lemma 3.6. $17 \notin R_I(13)$.

Proof. There exists, up to isomorphism, a unique collection D of 18 blocks doubly balanced for pairs which is parity-regular but not separable:

$$12A,\ 12B,\ 13A,\ 13B,\ 23A,\ 23B,\ 45A,\ 45C,\ 46A,\ 46C,\ 56A,\ 56C,\ 78B,\ 78C,\ 79B,\ 79C,\ 89B,\ 89C.$$

Suppose there exists a TTS(13) (V, B) with $V = \{1, 2, \ldots, 9, A, B, C, X\}$, and $B = D \cup 2E$, i.e., X is the only additional element not occurring in triples of D,

Table 4

$G^{(t)}$	$G_1^{(t)}$	$G_2^{(t)}$	$G_3^{(t)}$	$G_4^{(t)}$
$G^{(0)}$	g_1	g_1	g_2	g_2
$G^{(3)}$	g_7	g_8	g_{10}	g_{20}
$G^{(5)}$	g_1	g_9	g_{11}	g_{20}
$G^{(7)}$	g_5	g_{12}	g_{16}	g_{20}
$G^{(8)}$	g_3	g_6	g_{14}	g_{15}
$G^{(9)}$	g_1	g_5	g_{20}	g_{21}
$G^{(10)}$	g_3	g_{13}	g_{17}	g_{18}
$G^{(12)}$	g_3	g_4	g_{18}	g_{19}

and all blocks of B other than those of D are repeated. Consider the pairs, say, $1C$ and $2C$. The triples of E containing these pairs must be $1CX$ and $2CX$, respectively, which is a contradiction. □

Lemma 3.7. $R_I(13) = L(13)\setminus\{17\}$.

Proof. We have $0 \in R_I(13)$ by Corollary 3.2. The rest follows from Lemmas 3.3–3.6. □

Lemma 3.8. $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 15\} \subset R_I(15)$.

Proof. Let $V = X \cup Z_{11}$, $X = \{x_1, x_2, x_3, x_4\}$. Let $h_1, \ldots, h_{28}$ be the 2-factors of $2K_{11}$ on Z_{11} given in Table 5. Note that h_1 has a total of four 2-cycles, h_2 and h_3 have three 2-cycles, h_4, h_5, h_6, h_7 have two 2-cycles, $h_8, \ldots, h_{15}$ have one 2-cycle each, while $h_{16}, \ldots, h_{28}$ have no 2-cycles. For $t = 0, 1, 2, 3, 4, 5, 6, 7, 8, 15$, let $F^{(t)} = \{F_1^{(t)}, F_2^{(t)}, F_3^{(t)}, F_4^{(t)}\}$ be the following sets of four 2-factors:

$$F^{(0)} = \{h_{27}, h_{27}, h_{28}, h_{28}\}, \qquad F^{(1)} = \{h_8, h_{20}, h_{25}, h_{26}\},$$
$$F^{(2)} = \{h_{12}, h_{13}, h_{24}, h_{26}\}, \qquad F^{(3)} = \{h_4, h_{11}, h_{16}, h_{18}\},$$
$$F^{(4)} = \{h_6, h_7, h_{19}, h_{19}\}, \qquad F^{(5)} = \{h_1, h_{10}, h_{18}, h_{23}\},$$
$$F^{(6)} = \{h_2, h_3, h_{21}, h_{22}\}, \qquad F^{(7)} = \{h_1, h_5, h_9, h_{17}\},$$
$$F^{(8)} = \{h_2, h_3, h_{14}, h_{15}\}.$$

Further, let $C = \{\{i, i+1, i+6\}, \{i, i+3, i+7\} : i \in Z_{11}\}$, $D_1 = \{\{i, i+2, i+6\} : i \in Z_{11}\}$, $D_2 = \{\{i, i+4, i+6\} : i \in Z_{11}\}$, $D = D_1 \cup D_2$, $D' = \{\{i, i+1, i+3\}, \{i, i+2, i+3\} : i \in Z_{11}\}$, and let (X, E) be a TTS(4). Put $B = C \cup Y \cup \bigcup_{i=1}^{4} x_i * F_i^{(t)}$, where

$$Y = \begin{cases} C, & \text{if } t = 1, 3, 5, 7, \\ D, & \text{if } t = 2, 4, 6, \\ D' & \text{if } t = 0, \\ 2D_1, & \text{if } t = 15. \end{cases}$$

Table 5

2-factors on Z_{11}

h_1	$(01T)(24)(35)(68)(79)$	h_{11}	$(08T241369)(57)$	h_{21}	$(01965874327T)$
h_2	$(03678)(12)(45)(9T)$	h_{12}	$(01234589T)(67)$	h_{22}	$(01985674327T)$
h_3	$(03698)(14)(25)(7T)$	h_{13}	$(01234789T)(56)$	h_{23}	$(0258T134769)$
h_4	$(01T)(2435)(68)(79)$	h_{14}	$(01967432T)(58)$	h_{24}	$(03691452T78)$
h_5	$(456)(789T)(02)(13)$	h_{15}	$(01987432T)(56)$	h_{25}	$(09753124 68T)$
h_6	$(0123458)(67)(9T)$	h_{16}	$(02358T19)(467)$	h_{26}	$\{\{i, i+3\} : i \in Z_{11}\}$
h_7	$(0123458)(69)(7T)$	h_{17}	$(03258T19)(467)$	h_{27}	$\{\{i, i+4\} : i \in Z_{11}\}$
h_8	$(023456789)(1T)$	h_{18}	$(0213)(456)(789T)$	h_{28}	$\{\{i, i+5\} : i \in Z_{11}\}$
h_9	$(08T214369)(57)$	h_{19}	$(03652T)(14789)$		
h_{10}	$(08T236419)(57)$	h_{20}	$(013579T8642)$		

It is then straightforward to verify that (V, B) is an indecomposable TTS(15) with exactly t repeated blocks. □

Lemma 3.9. $\{20, 21, 23, 27\} \subset R_I(15)$.

Proof. Let $V = X \cup Z_8$, $X = \{a_1, a_2, \ldots, a_7\}$. Let $F = \{F_1, F_2, F_3, F_4, F_5\}$ be a maximal set of five 1-factors of K_8 on Z_8 whose leave (=complement) is a 2-factor Q consisting of one triangle and one pentagon; e.g., $Q = (123)(45678)$; such a set F is well known to exist (cf., e.g. [6]). For $i = 1, 2, 3, 4, 5$, let Q_i be the 2-factor on Z_8 obtained by "doubling" each edge of F_i. Let $(X, C^{(t)})$ be a TTS(7) with t repeated triples, $t \in R(7) = \{0, 1, 3, 7\}$. Put $B = \bigcup_{i=1}^{5} a_i * Q_i \cup a_6 * Q \cup a_7 * Q$. Then $(V, B \cup C^{(t)})$ is a TTS(15) with exactly $t + 20$ repeated triples, which is clearly indecomposable. □

Lemma 3.10. $\{9, 10, \ldots, 29\} \setminus \{15, 23\} \subset R_I(15)$.

Proof. Let again $V = X \cup Z_8$, $X = \{a_1, \ldots, a_7\}$ but this time let $F = \{F_1, F_2, F_3, F_4\}$ be a maximal set of four 1-factors of K_8 on Z_8 whose leave is the cubic graph K in Fig. 1. Such a maximal set is easily seen to exist (cf. also [6]).

Let Q_i be the 2-factor on Z_8 obtained by doubling each edge of F_i, $i = 1, 2, 3, 4$. Then $B = \bigcup_{i=1}^{4} a_i * Q_i$ is a set containing 16 repeated triples. Let $2K$ be the graph obtained from K by doubling each of its edges. Consider the following 2-factors of $2K$:

$$
\begin{array}{lll}
k_1 = (126783)(45) & k_4 = (138754)(26) & k_7 = (12387654) \\
k_2 = (126754)(38) & k_5 = (123)(567)(48) & k_8 = (13265784) \\
k_3 = (132654)(78) & k_6 = (12384)(567) &
\end{array}
$$

Then for $s = 1, 2, 3$, $L^{(s)} = \{L_1^{(s)}, L_2^{(s)}, L_3^{(s)}\}$, where $L^{(1)} = \{k_1, k_6, k_8\}$, $L^{(2)} = \{k_4, k_5, k_7\}$, $L^{(3)} = \{k_2, k_3, k_5\}$, is a decomposition of $2K$ into 2-factors. Put $D^{(s)} = \bigcup_{i=5}^{7} a_i * L_i^{(s)}$, and let $(X, C^{(t)})$ be a TTS(7) with t repeated triples. Then $(V, B \cup C^{(t)} \cup D^{(s)})$ is an indecomposable TTS(15) with $16 + t + s$ repeated triples. Since $t \in \{0, 1, 3, 7\}$ and $s \in \{1, 2, 3\}$, this gives $\{17, 18, 19, 20, 21, 22, 24, 25, 26\} \subset R_I(15)$.

Let now $Q_1' = Q_2' = F_3 \cup F_4$. If B' is obtained from B by replacing Q_1, Q_2 with Q_1', Q_2', respectively, then $(V, B' \cup C^{(t)} \cup D^{(s)})$ is an indecomposable TTS(15)

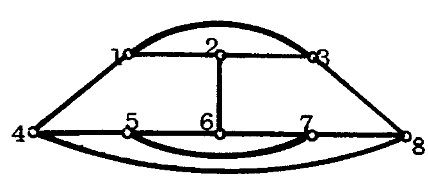

Fig. 1. Graph K.

with $8+t+s$ repeated triples. This implies $\{9, 10, 11, 12, 13, 14, 16, 17, 18\} \subset R_I(15)$ which completes the proof of the lemma. $\square$

Lemma 3.11. $29 \in R_I(15)$.

Proof. Elements: $V = \{1, 2, \ldots, 15\}$.
Nonrepeated blocks:

$$B_N = \{134, 135, 145, 167, 168, 178, 234, 235, 245, 267, 268, 278\}.$$

Repeated blocks:

$$B_R = \{1\,2\,9,\ 1\,10\,13,\ 1\,11\,14,\ 1\,12\,15,\ 2\,10\,15,\ 2\,11\,12,\ 2\,13\,14,\ 3\,6\,10,\ 3\,7\,11,\ 3\,8\,12,\ 3\,9\,13,\ 3\,14\,15,\ 4\,6\,13,\ 4\,7\,14,\ 4\,8\,15,\ 4\,9\,11,\ 4\,10\,12,\ 5\,6\,11,\ 5\,7\,15,\ 5\,8\,13,\ 5\,9\,12,\ 5\,10\,14,\ 6\,9\,15,\ 6\,12\,14,\ 7\,9\,10,\ 7\,12\,13,\ 8\,9\,14,\ 8\,10\,11,\ 11\,13\,15\}.\ \square$$

Lemma 3.12. $R_I(15) = L(15)$.

Proof. Lemmas 3.8–3.11. $\square$

Lemma 3.13. $5s + t + 2 \in R_I(19)$ *for* $s \in \{0, 1, 2, 3, 4, 6\}$, $t \in R(9)$.

Proof. Let $V = X \cup Z_{10}$, $X = \{x_i : i = 1, 2, \ldots, 9\}$. Let $F = \{F_1, F_2, F_3, F_4, F_5, F_6\}$ be a set of six 1-factors of K_{10} on Z_{10} such that its leave is the graph C in Fig. 2. (It is easily seen that such a set exists.) From the set F, form a set Q of six 2-factors $Q_1, \ldots, Q_6$ on Z_{10} by taking unions $F_i \cup F_j$ of two 1-factors in such a way that

(i) Q is a decomposition of $2\bar{C}$ (the complement of $2C$) into 2-factors, and
(ii) exactly s of the Q_i's consist of double edges only (i.e., are obtained by taking the union $F_i \cup F_i$ for some i).

Clearly, such a set Q exists if and only if $s \in \{0, 1, 2, 3, 4, 6\}$. Let further $Q_7 = (12987)(456)(03)$, $Q_8 = (01764392)(58)$, $Q_9 = (012)(3456789)$. It is easy to see that $\{Q_7, Q_8, Q_9\}$ is a decomposition of $2C$ into 2-factors. Let now $B = \bigcup_{i=1}^{9} x_i * Q_i$, and let $(X, D^{(t)})$ be a TTS(9) with t repeated triples. Then $(V, B \cup D^{(t)})$ is an indecomposable TTS(19) with $5s + t + 2$ repeated triples. $\square$

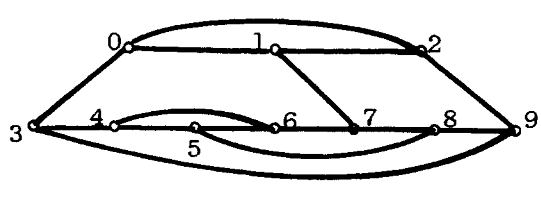

Fig. 2. Graph C.

Lemma 3.14. *$t+q+30 \in R_I(19)$ for $t \in R(9)$, $q \in \{6, 9\}$.*

Proof. Let again $V = X \cup Z_{10}$, $X = \{x_i : i = 1, \ldots, 9\}$. Let $F = \{F_1, \ldots, F_6\}$ be a set of six 1-factors of K_{10} on Z_{10} whose leave is the graph D in Fig. 3. It is again easily seen that such a set F exists (consider, e.g. the (unique) cyclic 1-factorization of K_{10}, cf. [1]). For $i = 1, 2, \ldots, 6$, let Q_i be the 2-factor obtained from F_i by doubling each of its edges. Consider the following 2-factors of the graph $2D$: $g_1 = (05)(12)(34)(67)(89)$, $g_2 = (3467)(05)(12)(89)$, $g_3 = (019)(456)(23)(78)$, $g_4 = (019)(456)(28)(37)$, $g_5 = (019)(34567)(28)$. If we take for the set $\{Q_7, Q_8, Q_9\}$ the set $\{g_1, g_3, g_4\}$ or $\{g_2, g_3, g_5\}$, this set of 2-factors will contain 9 or 6 double edges, respectively. If $(X, D^{(t)})$ is a TTS(9) with t repeated triples and $B = \bigcup_{i=1}^{9} x_i * Q_i$ then $(V, B \cup D^{(t)})$ is an indecomposable TTS(19) with $30+q+t$ repeated triples, where $q = 6$ or 9. □

Lemma 3.15. $R_I(19) = L(19)$.

Proof. Corollary 2.3 implies $k \in R_I(19)$ for all $k \in L(19)$ except for $k = 3$, 8, 13, 18, 23, 28, 33, 38, 40, 41, 42, 43, 44, 48, 51. Taking in Lemma 3.13 consecutively $s = 0, 1, 2, 3, 4, 6$ and $t = 1$ gives $3, 8, 13, 18, 23, 33 \in R_I(19)$; taking $s = 4$, $t = 6$ gives $28 \in R_I(19)$, and taking $s = 6$, $t = 12$ gives $44 \in R_I(19)$. Taking in Lemma 3.14 $q = 6$, $t = 6$, 12 gives $38, 48 \in R_I(19)$, and taking $q = 9$, $t = 1, 2, 3, 4, 6, 12$ gives $40, 41, 42, 43, 45, 51 \in R_I(19)$. □

Lemma 3.16. $T = \{8, 18, 28, 38, 48, 50, 58, 59, 60, 62\} \subset R_I(21)$.

Proof. Let $V = X \cup Z_{12}$, $X = \{a_1, \ldots, a_9\}$. Let G be the graph with $V(G) = Z_{12}$, $E(G) = \{\{x, y\} : |x-y| \in \{1, 2, 3, 5, 6\}\}$, and let $F = \{F_1, \ldots, F_9\}$ be any 1-factorization of G. Put $Q_i = F_i \cup F_{i\alpha}$, where α is any permutation of $\{1, 2, \ldots, 9\}$ fixing exactly s letters. Let $C = \{\{i, i+4, i+8\} : i \in Z_{12}\}$, and let (X, D) be an indecomposable TTS(9) with t repeated blocks ($t \in \{0, 1, 2, 4\}$). Then $(V, 2C \cup D \cup \bigcup_{i=1}^{9} a_i * Q_i)$ is an indecomposable TTS(21) with $6s+t+4$ repeated triples ($s \in \{0, 1, \ldots, 7, 9\}$). This implies $k \in R_I(21)$ for all $k \in T$. □

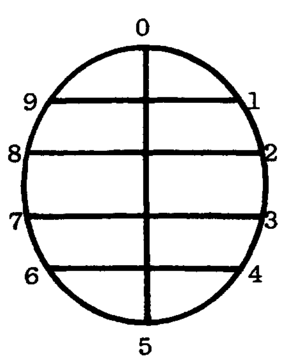

Fig. 3. Graph D.

Lemma 3.17. $U = \{51, 52, 53, 54, 56\} \subset R_I(21)$.

Proof. Let $V = Z_7 \times Z_3$, and consider the direct product P of an STS(7) on Z_7 with an STS(3) on Z_3. If $\{x, y, z\}$, $\{x, u, v\}$ are two triples of the STS(7), replace the sub-STS(9) of P on $\{(x, i), (y, i), (z, i) : i \in Z_3\}$ with an indecomposable TTS(9) having 4 repeated blocks, and replace the sub-STS(9) of P on $\{(x, i), (u, i), (v, i) : i \in Z_3\}$ with a TTS(9) having t repeated blocks ($t \in \{1, 2, 3, 4, 6, 12\}$) in such a way that $\{(x, 0), (x, 1), (x, 2)\}$ is a repeated block; "double" all other blocks of P. The result is an indecomposable TTS(21) with $t + 50$ repeated blocks which implies $k \in R_I(21)$ for all $k \in U$. □

Lemma 3.18. $\{57, 64\} \in R_I(21)$.

Proof. Let $V = X \cup Z_{14}$, $X = \{a_1, \ldots, a_7\}$. Let $C = \{\{i, i+3, i+8\} : i \in Z_{14}\}$. Let G be the graph with $V(G) = Z_{14}$, $E(G) = \{\{x, y\} : |x - y| = 4 \text{ or } 7\}$, and let $F = \{F_1, F_2, F_3\}$ be a decomposition of G into three 1-factors (which exists by [8]). Double each edge of F_i to obtain a 2-factor Q_i, $i = 1, 2, 3$. Let further $Q_4 = (0\,1\,13)(678)(23)(45)(9\,10)(11\,12)$, $Q_5 = (0\,1\,13)(678)(24)(35)(9\,11)(10\,12)$, $Q_6 = (02)(13)(46)(57)(89)(10\,11)(12\,13)$, $Q_7 = (12)(34)(56)(79)(8\,10)(11\,13)(0\,12)$. and let (X, D) be a TTS(7) with t repeated blocks. Then it is straightforward to verify that $(V, 2C \cup D \cup \bigcup_{i=1}^{7} a_i * Q_i)$ is a TTS(21) with $t + 57$ repeated blocks. Taking $t = 0$ and 7, respectively, gives $57, 64 \in R_I(21)$. □

Lemma 3.19. $61 \in R_I(21)$.

Proof. Elements: $V = \{A, B, C, 1, 2, \ldots, 9, \bar{1}, \bar{2}, \ldots, \bar{9}\}$.

Repeated blocks:

$$B_R = \{ABC, A1\bar{1}, A2\bar{2}, A3\bar{3}, B4\bar{4}, B5\bar{5}, B6\bar{6}, C7\bar{7}, C8\bar{8}, C9\bar{9}, 14\bar{7}, 1\bar{4}7, \bar{1}47, 25\bar{8}, 2\bar{5}8, \bar{2}58, 36\bar{9}, 3\bar{6}9, \bar{3}69, 159, 267, 348, 168, 249, 357\} \cup B'_R,$$

where B'_R consists of 36 repeated blocks given in Table 6 (if the entry in row

Table 6

	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$
$\bar{1}$		9	8	C	6	5	B	3	2
$\bar{2}$	9		7	6	C	4	3	B	1
$\bar{3}$	8	7		5	4	C	2	1	B
$\bar{4}$	C	6	5		3	2	A	9	8
$\bar{5}$	6	C	4	3		1	9	A	7
$\bar{6}$	5	4	C	2	1		8	7	A
$\bar{7}$	B	3	2	A	9	8		6	5
$\bar{8}$	3	B	1	9	A	7	6		4
$\bar{9}$	2	1	B	8	7	A	5	4	

labelled $\bar{i}$ and column labelled $\bar{j}$ is x, form triple $\bar{i}\bar{j}x$).

Nonrepeated blocks:

$$B_N = \{A45, A46, A56, A78, A79, A89, B12, B13, B23, B78, B79, B89, C12, C13, C23, C45, C46, C56\}. \quad \square$$

Lemma 3.20. $55 \in R_I(21)$.

Proof. Elements: $V = \{a, b, c, x, y, z\} \cup \{1, 2, \ldots, 15\}$.
Repeated blocks:

$$\begin{aligned} B_R = \{&a\,b\,x,\ a\,c\,y,\ b\,c\,z,\ a\,z\,11,\ b\,y\,6,\ c\,x\,5,\ a\,12\,13,\ a\,14\,15,\ b\,7\,8,\\ &b\,9\,10,\ c\,1\,2,\ c\,3\,4,\ x\,1\,10,\ x\,2\,3,\ x\,4\,12,\ x\,6\,7,\ x\,8\,13,\ x\,9\,14,\\ &x\,11\,15,\ y\,1\,5,\ y\,2\,13,\ y\,3\,14,\ y\,4\,7,\ y\,8\,9,\ y\,10\,15,\ y\,11\,12,\\ &z\,1\,15,\ z\,2\,8,\ z\,3\,9,\ z\,4\,5,\ z\,6\,10,\ z\,7\,12,\ z\,13\,14,\ x\,y\,z\} \cup (T \setminus T_1),\end{aligned}$$

where T is the set of blocks of any transversal design TD(3, 5) whose groups are $G_1 = \{1, 2, 3, 4, 5\}$, $G_2 = \{6, 7, 8, 9, 10\}$, $G_3 = \{11, 12, 13, 14, 15\}$ and such that $T_1 = \{1\,10\,15,\ 2\,8\,13,\ 3\,9\,14,\ 4\,7\,12\} \subset T$.

Nonrepeated blocks:

$$\begin{aligned} B_R = \{&\{a, i, i+1\}, \{b, i, i+1\} : i = 1, 2, 3, 4, 5; 5+1 = 1\}\\ \cup\ \{&\{a, i, i+1\}, \{c, i, i+1\} : i = 6, 7, 8, 9, 10; 10+1 = 6\}\\ \cup\ \{&\{b, i, i+1\}, \{c, i, i+1\} : i = 11, 12, 13, 14, 15; 15+1 = 11\}. \quad \square\end{aligned}$$

Lemma 3.21. $R_I(21) = L(21)$.

Proof. Consider the following 2-factorizations of $2K_{11}$ on $Z_{10} \cup \{\infty\}$:

$$\begin{aligned} Q^{(0)} &= \{(\infty\, 0192837465)\ (\mathrm{mod}\ 10)\},\\ Q^{(1)} &= \{(01)(\infty\, 38)(264759)\ (\mathrm{mod}\ 10)\},\\ Q^{(2)} &= \{(01)(25)(\infty\, 648397)\ (\mathrm{mod}\ 10)\},\\ Q^{(3)} &= \{(01)(35)(69)(\infty\, 4827)\ (\mathrm{mod}\ 10)\},\\ Q^{(4)} &= \{(01)(35)(48)(69)(\infty\, 27)\ (\mathrm{mod}\ 10)\}.\end{aligned}$$

Clearly, each $Q^{(i)}$ has $q = 10i$ 2-cycles. Thus, applying Lemma 2.2 gives $10i + t \in R_I(21)$ for $i \in \{0, 1, 2, 3, 4\}$, $t \in R_I(10)$. Lemmas 3.16–3.20 yield $k \in R_I(21)$ for all remaining $k \in L(21)$. $\square$

Lemma 3.22. $8s + r + t + 56 \in R_I(25)$ *for* $s \in \{0, 1, 2\}$, $r \in \{0, 1, 2, 3, 4, 5, 7, 8\}$, $t \in R(9) = \{0, 1, 2, 3, 4, 6, 12\}$.

Proof. Let $V = (Z_8 \times \{1, 2\} \cup X,\ X = \{a_1, \ldots, a_9\}$. Let C be the following set of triples on $Z_8 \times \{1, 2\}$:

$$C = \{\{(x, 1), (x+3, 1), (x+4, 2)\}, \{(x, 1), (x, 2), (x+3, 2)\}\ x \in Z_8\}.$$

For $r = 0, 1, 2, 3, 4, 5, 7, 8$, let $Q^{(r)} = \{Q_1^{(r)}, Q_2^{(r)}, \ldots, Q_9^{(r)}\}$ be the 2-factorization of the graph, that is obtained by removing from the complete multigraph $2K_{16}$ on $Z_8 \times \{1, 2\}$ all edges occurring in the triples of $2C$, given by:

$$Q_1^{(r)} = \{\{(x, i), (x+4, i)\} : i \in \{1, 2\}, x \in Z_8\},$$
$$Q_2^{(r)} = \{\{(x, 1), (x+2, 2)\} : x \in Z_8\},$$
$$Q_3^{(r)} = \{\{(x, 1), (x+5, 2)\} : x \in Z_8\},$$
$$Q_4^{(r)} = \{\{(x, 1), (x+6, 2)\} : x \in Z_8\},$$
$$Q_5^{(r)} = \{\{(x, 1), (x+7, 2) : x \in Z_8\},$$

for all $r = 0, 1, 2, 3, 4, 5, 7, 8$ (all edges here are "repeated").

As for 2-factors $Q_j^{(r)}$, $j = 6, 7, 8, 9$, these are given by $Q_j^{(r)} = Q_{j1}^{(r)} \cup Q_{j2}^{(r)}$, where $Q_{j1}^{(r)}$ are from Table 7(b), and $Q_{j2}^{(r)}$ are from Table 8 (any of the possibilities P_0, P_1, P_2 may be taken; note that P_s yields a total of $8s$ 2-cycles).

Let (X, D) be any TTS(9) with t repeated triples. Then $(V, 2C \cup D \cup \bigcup_{i=1}^{9} a_i * Q_i)$ is an indecomposable TTS(25) with exactly $8s + r + t + 56$ repeated triples. □

Lemma 3.23. $8s + t + 16 \in R_I(25)$ *for* $s \in \{0, 1, \ldots, 7, 9\}$, $t \in R_I(9) = \{0, 1, 2, 4\}$.

Table 7

(a) List of 2-factors on $Z_8 \times \{1\}$ (second coordinate 1 omitted)

$q_1 = (123)(45786)$,
$q_2 = (17653428)$,
$q_3 = (17532468)$,
$q_4 = (12867543)$,
$q_5 = (1768)(2354)$,
$q_6 = (123)(456)(78)$,
$q_7 = (128643)(57)$,
$q_8 = (235764)(18)$,
$q_9 = (234)(567)(18)$,
$q_{10} = (243568)(17)$,
$q_{11} = (1243)(57)(68)$,
$q_{12} = (2354)(18)(67)$,
$q_{13} = (2468)(17)(35)$,
$q_{14} = (3465)(17)(28)$,
$q_{15} = (17)(28)(34)(56)$,
$q_{16} = (18)(24)(35)(67)$,

(b) 2-factors $Q_{j1}^{(r)}$

r	$Q_{61}^{(r)}$	$Q_{71}^{(r)}$	$Q_{81}^{(r)}$	$Q_{91}^{(r)}$
0	q_1	q_1	q_2	q_2
1	q_2	q_3	q_4	q_6
2	q_2	q_5	q_6	q_7
3	q_4	q_6	q_8	q_{10}
4	q_4	q_6	q_9	q_{13}
5	q_6	q_7	q_{10}	q_{12}
7	q_6	q_{11}	q_{12}	q_{14}
8	q_1	q_1	q_{15}	q_{16}

Table 8

(a) List of 1-factors on $Z_8 \times \{2\}$

$f_1 = \{\{(0,2),(1,2)\},\{(2,2),(3,2)\},\{(4,2),(5,2)\},\{(6,2),(7,2)\}\}$,
$f_2 = \{\{(1,2),(2,2)\},\{(3,2),(4,2)\},\{(5,2),(6,2)\},\{(7,2),(0,2)\}\}$,
$f_3 = \{\{(0,2),(2,2)\},\{(1,2),(3,2)\},\{(4,2),(6,2)\},\{(5,2),(7,2)\}\}$,
$f_4 = \{\{(2,2),(4,2)\},\{(3,2),(5,2)\},\{(6,2),(7,2)\},\{(7,2),(1,2)\}\}$,

(b) 2-factors $Q_{j2}^{(r)}$

$P_0 = \{f_1 \cup f_2, f_1 \cup f_2, f_3 \cup f_4, f_3 \cup f_4\}$,
$P_1 = \{2f_1, 2f_2, f_3 \cup f_4, f_3 \cup f_4\}$,
$P_2 = \{2f_1, 2f_2, 2f_3, 2f_4\}$.

Proof. Let $V = Z_{16} \cup X$, $X = \{a_1, \ldots, a_9\}$. Let $C = \{\{i, i+1, i+3\} : i \in Z_{16}\}$, and let $F = \{F_1, \ldots, F_9\}$ be a 1-factorization of the graph G obtained by removing all edges of C from the complete graph K_{16} on Z_{16}. Let α be a permutation of $\{1, 2, \ldots, 9\}$ fixing exactly s letters. Put $Q_i = F_i \cup F_{i\alpha}$. Let further (X, D) be an indecomposable TTS(9) with t repeated triples. Then $(V, 2C \cup D \cup \bigcup_{i=1}^{9} a_i * Q_i)$ is an indecomposable TTS(25) with $8s + t + 16$ repeated triples. □

Lemma 3.24. $94 \in R_I(25)$.

Proof. Elements: $V = \{A, B, X, Y, Z, a, b, c, d, e, f, 1, 2, \ldots, 7, \bar{1}, \bar{2}, \ldots, \bar{7}\}$.

Repeated blocks:

$$\begin{aligned}B_R = \{&X14, X25, X\overline{12}, X3\bar{3}, X\overline{45}, X\overline{67}, Y\overline{17}, Y\overline{23}, Y4\bar{4}, Y5\bar{5}, Y6\bar{6}, Z1\bar{1},\\ &Z2\bar{2}, Z\overline{34}, Z\overline{56}, Z7\bar{7}, XY7, XZ6, YZ3, Xad, Xbe, Xcf, Yae, Ybf,\\ &Ycd, Zaf, Zbd, Zce, XAB, A26, B1\bar{7}, a15, b1\bar{2}, c13, d16, e1\bar{6},\\ &f1\bar{5}, 12\bar{4}, A37, B36, a2\bar{1}, b2\bar{3}, c24, d27, e2\bar{7}, f2\bar{6}, 23\bar{5}, A5\bar{4},\\ &B47, a3\bar{2}, b3\bar{4}, c57, d35, e3\bar{1}, f3\bar{7}, 34\bar{6}, A\overline{15}, B\overline{14}, a4\bar{3}, b5\bar{6},\\ &c6\bar{7}, d4\bar{5}, e4\bar{2}, f4\bar{1}, 45\bar{7}, A\overline{26}, B\overline{25}, a6\bar{5}, b46, c\overline{16}, d\overline{13}, e5\bar{3},\\ &f5\bar{2}, 56\bar{1}, A\overline{37}, B\overline{36}, a7\bar{6}, b7\bar{1}, c\overline{24}, d\overline{27}, e6\bar{4}, f6\bar{3}, 67\bar{2}, A1Y,\\ &B2Y, a4\bar{7}, b\overline{57}, c\overline{35}, d\overline{46}, e7\bar{5}, f7\bar{4}, 71\bar{3}, A4Z, B5Z\}.\end{aligned}$$

Nonrepeated blocks:

$$B_N = \{Aab, Aac, Abc, Ade, Adf, Aef, Bab, Bac, Bbc, Bde, Bdf, Bef\}.$$

□

Lemma 3.25. $R_I(25) = L(25)$.

Proof. Consider the following 2-factorizations of $2K_{13}$ on $Z_{12} \cup \{\infty\}$:

$$Q^{(0)} = \{(\infty\,0\,1\,11\,2\,10\,3\,9\,4\,8\,5\,7\,6)\ (\text{mod}\ 12)\},$$
$$Q^{(1)} = \{(0\,1)(\infty\,4\,9\,3\,10\,2\,11\,7\,5\,8\,6)\ (\text{mod}\ 12)\},$$
$$Q^{(2)} = \{(0\,11)(6\,8)(\infty\,1\,5\,2\,10\,3\,9\,4\,7)\ (\text{mod}\ 12)\},$$
$$Q^{(3)} = \{(2\,10)(3\,4)(5\,8)(\infty\,9\,7\,0\,6\,1\,11)\ (\text{mod}\ 12)\},$$
$$Q^{(4)} = \{(1\,10)(2\,9)(3\,5)(4\,8)(\infty\,7\,6\,0\,11)\ (\text{mod}\ 12)\}.$$

Here each $Q^{(i)}$ has $q = 12i$ 2-cycles. Applying Lemma 2.2 gives $12i + t \in R_I(25)$ for $i \in \{0, 1, 2, 3, 4\}$, $t \in R_I(12) = \{0, 1, \ldots, 13, 16\}$. In particular, it gives $\{0, 1, \ldots, 61\} \in R_I(25)$. Lemma 3.22 gives $k \in R_I(25)$ for all $k \in \{56, 57, \ldots, 92\}$ except for $k = 90$. Applying Lemma 3.23 with $s = 9$, $t = 2$ gives $90 \in R_I(25)$. Finally, Lemma 3.24 gives $94 \in R_I(25)$. □

Lemma 3.26. $108 \in R_I(27)$.

Proof. Elements: $V = \{A, B, C, X, Y, Z, a, b, c, d, e, f, g, h, j, x, y, z, 1, 2, \ldots, 9\}$.

Repeated blocks:

$$\begin{aligned} B_R = \{&aX1, cX2, bX3, gX4, jX5, hX6, dX7, fX8, eX9, AXx, BXy, CXz,\\ &aY2, cY3, bY1, gY5, jY6, hY4, dY8, fY9, eY7, AYy, BYz, CYx,\\ &aZ3, cZ1, bZ2, gZ6, jZ4, hZ5, dZ9, fZ7, eZ8, AZz, BZx, CZy,\\ &ax4, cx5, bx6, gx7, jx8, hx9, dx1, fx2, ex3, A14, B17, C47,\\ &ay5, cy6, by4, gy8, jy9, hy7, dy2, fy3, ey1, A25, B28, C58,\\ &az6, cz4, bz5, gz9, jz7, hz8, dz3, fz1, ez2, A36, B39, C69,\\ &a78, c89, b79, g12, j23, h13, d45, f56, e46, Aa9, Bd6, Cg3,\\ &adg, aej, afh, beh, bfg, bdj, cfj, cdh, ceg, Ab8, Be5, Ch2,\\ &ABC, XYZ, xyz, 159, 267, 348, 168, 249, 357, Ac7, Bf4, Cj1\}.\end{aligned}$$

Nonrepeated blocks:

$$\begin{aligned} B_N = \{&Ade, Adf, Aef, Agh, Agj, Ahj, Bab, Bac, Bbc, Bgh, Bgj, Bhj,\\ &Cab, Cac, Cbc, Cde, Cdf, Cef\}.\end{aligned}$$ □

Lemma 3.27. $R_i(27) = L(27)$.

Proof. Applying Lemma 2.2 to $R_I(13)$ gives $k \in R_I(27)$ for all $k \in L(27)$ except for $k = 108$ in which case Lemma 3.26 applies. □

Lemma 3.28. $176 \in R_I(33)$.

Proof. Let $V = Z_{18} \cup X$, $X = \{a_1, \ldots, a_{15}\}$. Let (X, D) be an indecomposable TTS(15) with 26 repeated blocks (which exists by Lemma 3.10). Let $C = \{\{i, i+6, i+12\} : i \in Z_{18}\}$, and let $F = \{F_1, \ldots, F_{15}\}$ be a 1-factorization of $2K_{18}$ on Z_{18} from which the edges of $2C$ have been removed. Let Q_i be the 2-factor obtained by doubling F_i, $i = 1, \ldots, 15$. Then $(V, 2C \cup D \cup \bigcup_{i=1}^{15} a_i * Q_i)$ is an indecomposable TTS(33) with $6 + 26 + 135 = 167$ repeated blocks. □

Lemma 3.29. $R_I(33) = L(33)$.

Proof. Lemma 2.5 applied to $R_I(13)$ gives $k \in R_I(33)$ for all $k \in L(33)$ except for $k = 167$ in which case Lemma 3.28 applies. □

4. Proof of the Main Theorem

Let $v \geq 15$, $v \equiv 1$ or $3 \pmod 6$. The necessity was proved in Lemma 2.1. As for sufficiency, for $v = 15, 19, 21, 25, 27, 33$ it was shown in Section 3. Also applying Lemma 2.2 with $v = 15$ gives $R_I(31) = L(31)$, so we may assume that $v \geq 37$. Assume that for all $u \leq v$ $(u \geq 15)$, $R_I(u) = L(u)$. If $v \equiv 3$ or $7 \pmod{12}$ then $u = \frac{1}{2}(v-1) \equiv 1$ or $3 \pmod 6$, $u \geq 15$, and so $R_I(u) = L(u)$. By Lemma 2.2, we get $R_I(v) = L(v)$. If $v \equiv 1$ or $9 \pmod{12}$, then $u = \frac{1}{2}(v-7) \equiv 1$ or $3 \pmod 6$, $u \geq 15$. Therefore $R_I(u) = L(u)$, and by Lemma 2.5, we get $R_I(v) = L(v)$ as well. This completes the proof. □

References

[1] A. Hartman and A. Rosa, Cyclic one-factorization of the complete graph, Europ. J. Combin. 6 (1985) 45–48.

[2] E.S. Kramer, Indecomposable triple systems, Discrete Math. 9 (1974) 173–180.

[3] C.C. Lindner and A. Rosa, Steiner triple systems having a prescribed number of triples in common, Canad. J. Math. 27 (1975) 1166–1175; Corrigendum: Canad. J. Math. 30 (1978) 896.

[4] R. Mathon and A. Rosa, A census of Mendelsohn triple systems of order 9, Ars Combin. 4 (1977) 309–315.

[5] A. Rosa and D. Hoffman, The number of repeated blocks in twofold triple systems, J. Combin. Theory Ser. A 41 (1986) 61–88.

[6] A. Rosa and W.D. Wallis, Premature sets of 1-factors, or How not to schedule round robin tournaments, Discrete Appl. Math. 4 (1982) 291–297.

[7] J.E. Simpson, Langford sequences: perfect and hooked, Discrete Math. 44 (1983) 97–104.

[8] G. Stern and H. Lenz, Steiner triple systems with given subspaces; another proof of the Doyen–Wilson theorem, Boll. Un. Mat. Ital. A (5) 17 (1980) 109–114.