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Normal families and shared values of meromorphic functions ☆

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ABSTRACT

Let k ($\geqslant 2$) be a positive integer, let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros have multiplicity at least $k+1$, and let $a(z)$ ($\not\equiv 0$), $h(z)$ ($\not\equiv 0$) be two holomorphic functions on D. If, for each $f \in \mathcal{F}$, $f = a(z) \Leftrightarrow f^{(k)} = h(z)$, then $\mathcal{F}$ is normal in D.

1. Introduction

Let D be a domain in the whole complex plane C and $\mathcal{F}$ a family of meromorphic functions defined in D. $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_j}\}$ which converges spherically locally uniformly in D, to a meromorphic function or ∞ (see Hayman [7], Schiff [8], Yang [9]).

Let f and g be meromorphic functions on a domain D, and let a and b be two complex numbers. If $g(z) = b$ whenever $f(z) = a$, we write

$$f(z) = a \quad \Rightarrow \quad g(z) = b.$$

If $f(z) = a \Rightarrow g(z) = b$ and $g(z) = b \Rightarrow f(z) = a$, we write

$$f(z) = a \quad \Leftrightarrow \quad g(z) = b.$$

If $f(z) = a \Leftrightarrow g(z) = a$, we say that f and g share a on D.

Schwick [1] was the first to draw a connection between values shared by functions in $\mathcal{F}$ and the normality of the family $\mathcal{F}$. Specifically, he proved the following theorem.

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, and let a_1, a_2, a_3 be three distinct complex numbers. If, for each $f \in \mathcal{F}$, f and f' share a_1, a_2, a_3, then $\mathcal{F}$ is normal in D.*

Fang and Zalcman [2] proved the following theorem.

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, let k be a positive integer, and let a, b be two nonzero complex numbers. If, for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $k+1$, and $f = a \Leftrightarrow f^{(k)} = b$, then $\mathcal{F}$ is normal in D.*

In this paper, we extend Theorem B as follows.

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Theorem 1. *Let k $(\geqslant 2)$ be a positive integer, let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros have multiplicity at least $k+1$, and let $a(z)$ $(\neq 0)$, $h(z)$ $(\not\equiv 0)$ be two holomorphic functions on D. If, for each $f \in \mathcal{F}$, $f = a(z) \Leftrightarrow f^{(k)} = h(z)$, then $\mathcal{F}$ is normal in D.*

In [2], an example was given to shows that the condition in Theorem 1 that $h(z) \not\equiv 0$ is necessary.

Example 1. Let m, k be positive integers; let $D = \{z\colon |z| < 1\}$; and let $\mathcal{F} = \{f_n\}$, where $f_n(z) = nz^{m+k}$, $a(z) = z^{m+k}$, $h(z) = z^m$. Clearly, $\mathcal{F}$ fails to be normal at the origin. However, all the zeros of f_n have multiplicity $k+m$, and $f_n = a(z) \Leftrightarrow f_n^{(k)} = h(z)$ on D. This shows that the condition in Theorem 1 that $a(z) \neq 0$ is necessary.

Remark. The proof of this result follows the general lines of the proof of the main result in [4], with important elaborations based on the argument in the recent paper [10].

We write $\Delta = \{z\colon |z| < 1\}$, $\Delta_r = \{z\colon |z| < r\}$ and $\Delta_r' = \{z\colon 0 < |z| < r\}$.

2. Some lemmas

In order to prove our theorems, we require the following results.

Lemma 1. *(See [3].) Let k be a positive integer, let $\mathcal{F}$ be a family of functions meromorphic on the unit disc Δ, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geqslant 1$ such that $|f^{(k)}(z)| \leqslant A$ whenever $f(z) = 0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0 \leqslant \alpha \leqslant k$,*

(a) *points $z_n \in \Delta$, $z_n \to z_0$;*
(b) *functions $f_n \in \mathcal{F}$; and*
(c) *positive numbers $\rho_n \to 0$*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on C, all of whose zeros have multiplicity at least k, such that $g^{\#}(\zeta) \leqslant g^{\#}(0) = kA + 1$. In particular, g has order at most 2.

Lemma 2. *(See [4].) Let $g(z)$ be a transcendental meromorphic function of finite order on C, and let $P(z)$ be a polynomial, $P(z) \not\equiv 0$. Suppose that all zeros of $g(z)$ have multiplicity at least $k+1$. Then $g^{(k)}(z) - P(z)$ has infinitely many zeros.*

Lemma 3. *(See [5].) Let m, k be two positive integers, and let $Q(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0 + \frac{q(z)}{p(z)}$, where $a_m, a_{m-1}, \ldots, a_0$ are constants with $a_m \neq 0$, and $q(z)$ $(\not\equiv 0)$, $p(z)$ are coprime polynomials with $\deg q(z) < \deg p(z)$. If $Q^{(k)}(z) \neq 1$ for $z \in C$, then*

$$Q(z) = \frac{z^k}{k!} + \cdots + a_0 + \frac{1}{(az+b)^n},$$

where $a \neq 0$, and n is a positive integer. Additionally, if all zeros of $Q(z)$ have multiplicity at least $k+1$, then $Q(z) = \frac{(cz+d)^{k+1}}{az+b}$, where c, d are constants with $c \neq 0$.

Lemma 4. *(See [6].) Let m, k be two positive integers with $m \geqslant 2$, $k \geqslant 2$, and let $Q(z)$ be a rational function, all of whose zeros have multiplicity at least $k+1$, and all of whose poles are multiple with the possible exception of $z = 0$. Then $Q^{(k)}(z) = z^m$ has a solution in C.*

Lemma 5. *(See [10].) Let $Q(z)$ be a rational function, all of whose poles are multiple and whose zeros all have multiplicity at least $k+1$. If $Q^{(k)}(z) \neq z^m$, $z \in C$ for some integer $m \geqslant 1$, then either*

(i) *$k = 1$ or*
(ii) *$m = 1$ and $Q(z) = \frac{(z+c)^{k+1}}{(k+1)!}$*

for some nonzero constant c.

Lemma 6. *Let k be a positive integer, let $a_n(z)(\neq 0)$ be holomorphic functions with $\{|a_n(z)|\}$ being locally uniformly bounded away from 0, and let $\{f_n\}$ be a sequence of meromorphic functions in a domain D, all of whose zeros of f_n have multiplicity at least $k+1$. Let $\{h_n(z)\}$ be a sequence of functions holomorphic on D such that $h_n \to h$ locally uniformly on D, where $h(z) \neq 0$ and $\not\equiv \infty$ for $z \in D$. Suppose that for each n, $f_n = a_n(z) \Leftrightarrow f_n^{(k)} = h_n(z)$, then $\{f_n\}$ is normal on D.*

Proof. Suppose that $\{f_n\}$ is not normal at z_0. We may assume that $D=\Delta$ and $h(z_0)=1$. By Lemma 1, after choosing appropriate subsequences we may assume that there exist $z_n \to z_0$, and $\rho_n \to 0^+$ such that

$$\rho_n^{-k} f_n(z_n+\rho_n\zeta)=g_n(\zeta)\to g(\zeta),$$

spherically uniformly on compact subsets of C, where $g(\zeta)$ is a nonconstant meromorphic function on C, all of whose zeros have multiplicity at least $k+1$ and g has order at most 2.

We claim that

(a) $g^{(k)} \neq 1$; and
(b) no poles of g are simple.

Suppose now that $g^{(k)}(\zeta_0)=1$. We claim that $g^{(k)} \not\equiv 1$. Otherwise, g must be a polynomial of exact degree k, which contradicts the fact that each zero of g has multiplicity at least $k+1$. Since $g^{(k)}(\zeta_0)=1=h(z_0)$ but $g^{(k)}\not\equiv 1$, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large)

$$f_n^{(k)}(z_n+\rho_n\zeta_n)=g_n^{(k)}(\zeta_n)=h_n(z_n+\rho_n\zeta_n).$$

It follows that $f_n(z_n+\rho_n\zeta_n)=a_n(z_n+\rho_n\zeta_n)$, so that

$$g_n(\zeta_n)=\frac{f_n(z_n+\rho_n\zeta_n)}{\rho_n^k}=\frac{a_n(z_n+\rho_n\zeta_n)}{\rho_n^k}.$$

Thus $g(\zeta_0)=\lim_{n\to\infty} g_n(\zeta_n)=\infty$, which contradicts $g^{(k)}(\zeta_0)=1$. This proves (a).

Next we prove (b). Suppose $g(\zeta_0)=\infty$. There exists a closed disc $K=\{\zeta\colon |\zeta-\zeta_0|\leqslant\delta\}$ on which $1/g$ and $1/g_n$ are holomorphic (for n sufficiently large) and $1/g_n \to 1/g$ uniformly. Hence, $\frac{1}{g_n(\zeta)}-\frac{\rho_n^k}{a_n(z_n+\rho_n\zeta)}\to\frac{1}{g(\zeta)}$ uniformly on K; and since $1/g$ is nonconstant, there exist ζ_n, $\zeta_n\to\zeta_0$, such that (for n large enough)

$$\frac{1}{g_n(\zeta_n)}-\frac{\rho_n^k}{a_n(z_n+\rho_n\zeta_n)}=0.$$

Hence $f_n(z_n+\rho_n\zeta_n)=a_n(z_n+\rho_n\zeta_n)$. Thus we have

$$g_n^{(k)}(\zeta_n)=f_n^{(k)}(z_n+\rho_n\zeta_n)=h_n(z_n+\rho_n\zeta_n). \tag{2.1}$$

If $k=1$, then we have by (2.1)

$$\left(\frac{1}{g(\zeta)}\right)'\bigg|_{\zeta=\zeta_0}=-\frac{g'(\zeta_0)}{g^2(\zeta_0)}=\lim_{n\to\infty}\left[-\frac{g_n'(\zeta_n)}{g_n^2(\zeta_n)}\right]=0,$$

so that ζ_0 is a multiple pole of $g(\zeta)$. Thus no poles of g are simple.

Similarly, if $k=2$, then we have by (2.1)

$$\begin{aligned}\left(\frac{1}{g(\zeta)}\right)''\bigg|_{\zeta=\zeta_0}&=-\frac{g''(\zeta_0)}{g^2(\zeta_0)}+2\frac{[g'(\zeta_0)]^2}{g^3(\zeta_0)}\\&=\lim_{n\to\infty}\left[-\frac{g_n''(\zeta_n)}{g_n^2(\zeta_n)}+2\frac{[g_n'(\zeta_n)]^2}{g_n^3(\zeta_n)}\right]\\&=-\lim_{n\to\infty}\frac{g_n''(\zeta_n)}{g_n^2(\zeta_n)}+2\lim_{n\to\infty}\frac{[g_n'(\zeta_n)]^2}{g_n^3(\zeta_n)}\\&=2\lim_{n\to\infty}\left\{\left[-\frac{g_n'(\zeta_n)}{g_n^2(\zeta_n)}\right]^2 g_n(\zeta_n)\right\}.\end{aligned} \tag{2.2}$$

Since $\lim_{n\to\infty} g_n(\zeta_n)=\infty$, by (2.2) we have

$$\lim_{n\to\infty}\left[-\frac{g_n'(\zeta_n)}{g_n^2(\zeta_n)}\right]^2=0.$$

Thus $(1/g(\zeta))'|_{\zeta=\zeta_0}=0$, so that ζ_0 is a multiple pole of $g(\zeta)$. Hence no poles of g are simple.

If $k\geqslant 3$, mathematical induction shows that

$$\left(\frac{1}{u}\right)^{(k)}=-\frac{u^{(k)}}{u^2}+k!\frac{(u')^k}{u^{k+1}}+\sum_{0\leqslant i\leqslant k-2}A_i[u]u^i, \tag{2.3}$$

where $A_i[u]$ is a polynomial of $(1/u)', (1/u)'', \ldots, (1/u)^{(k-1)}$ for each u meromorphic in D.

Thus by (2.1) and (2.3),

$$\begin{aligned}\left(\frac{1}{g(\zeta)}\right)^{(k)}\bigg|_{\zeta=\zeta_0} &= \lim_{n\to\infty}\left(\frac{1}{g_n(\zeta)}\right)^{(k)}\bigg|_{\zeta=\zeta_n}\\ &= \lim_{n\to\infty}\left[-\frac{g_n^{(k)}(\zeta_n)}{g_n^2(\zeta_n)} + k!\frac{(g_n'(\zeta_n))^k}{g_n^{k+1}(\zeta_n)} + \sum_{0\leqslant i\leqslant k-2} A_i[g_n]g_n^i(\zeta_n)\right]\\ &= \lim_{n\to\infty}\left[k!\frac{(g_n'(\zeta_n))^k}{g_n^{k+1}(\zeta_n)} + \sum_{0\leqslant i\leqslant k-2} A_i[g_n]g_n^i(\zeta_n)\right]\\ &= \lim_{n\to\infty}\left[k!\frac{(g_n'(\zeta_n))^k}{g_n^{k+1}(\zeta_n)} + \sum_{1\leqslant i\leqslant k-2} A_i[g_n]g_n^i(\zeta_n)\right] + A_0[g](\zeta_0)\\ &= \lim_{n\to\infty}\left[k!\left(-\frac{(g_n'(\zeta_n))}{g_n^2(\zeta_n)}\right)^k(-1)^k g_n^{k-2}(\zeta_n) + \sum_{1\leqslant i\leqslant k-2} A_i[g_n]g_n^{i-1}(\zeta_n)\right]g_n(\zeta_n)\\ &\quad + A_0[g](\zeta_0). \end{aligned} \tag{2.4}$$

Since $\lim_{n\to\infty} g_n(\zeta_n) = \infty$, by (2.4) we get

$$\lim_{n\to\infty}\left[k!\left(-\frac{(g_n'(\zeta_n))}{g_n^2(\zeta_n)}\right)^k(-1)^k g_n^{k-2}(\zeta_n) + \sum_{1\leqslant i\leqslant k-2} A_i[g_n]g_n^{i-1}(\zeta_n)\right] = 0.$$

Similarly, we have

$$\lim_{n\to\infty}\left[k!\left(-\frac{(g_n'(\zeta_n))}{g_n^2(\zeta_n)}\right)^k(-1)^k g_n^{k-3}(\zeta_n) + \sum_{2\leqslant i\leqslant k-2} A_i[g_n]g_n^{i-2}(\zeta_n)\right] = 0.$$

Proceeding inductively, we obtain at last

$$\lim_{n\to\infty}\left[-\frac{g_n'(\zeta_n)}{g_n^2(\zeta_n)}\right]^k = 0.$$

It follows that $(1/g(\zeta))'\big|_{\zeta=\zeta_0} = 0$, so that ζ_0 is a multiple pole of $g(\zeta)$. Hence no poles of g are simple. This proves (b). □

By Lemma 2, g is a rational function. By (a), (b) and Lemma 3, g is a constant, a contradiction. Thus $\{f_n\}$ is normal on D.

3. Proof of Theorem 1

We may assume that $D = \Delta$. We only need to show that $\mathcal{F}$ is normal at a point z_0, for each $z_0 \in \Delta$. Suppose that $h(z_0) \neq 0$. Then by Lemma 6, we get that $\mathcal{F}$ is normal at z_0.

We now prove that $\mathcal{F}$ is normal at a point z_0 with $h(z_0) = 0$. Without loss of generality, we may assume that $z_0 = 0$. Making standard normalization, we may assume that

$$h(z) = z^m + a_{m+1}z^{m+1} + \cdots = z^m b(z), \quad z \in \Delta,$$

$m \geqslant 1$, $b(0) = 1$, and $h(z) \neq 0$ for $0 < |z| < 1$.

We argue by contradiction. Choosing a sequence $\{f_n\}$ of $\mathcal{F}$ and renumbering, we may assume that no subsequence of $\{f_n\}$ is normal at 0.

Let $\mathcal{H} = \{F_n\colon\ F_n(z) = \frac{f_n(z)}{z^m}\}$. We claim that $f_n(0) \neq 0$. Otherwise, we assume that $f_n(0) = 0$. Then, since all zeros of f_n have multiplicity at least $k+1$, also $f_n^{(k)}(0) = 0 = h(0)$. By the value sharing assumption of the theorem this would imply $f_n(0) = a(0) \neq 0$, a contradiction. Hence $f_n(0) \neq 0$. Thus, $F_n(0) = \infty$. In fact, each F_n has a pole of order m at 0.

Suppose that we have shown that $\mathcal{H}$ is normal at 0. Next, we prove that $\mathcal{F}$ is normal at 0. Since $\mathcal{H}$ is normal at $z = 0$, there exist $\Delta_\delta = \{z\colon\ |z| < \delta\}$ and a subsequence of $\{F_n(z)\}$ such that $\{F_n(z)\}$ converges uniformly to a meromorphic function $F(z)$ or ∞ on Δ_δ. Noting that $F(0) = \infty$, we can find a $\varepsilon \in [0; \delta]$ and a positive constant M such that $|F(z)| \geqslant M$ for all $z \in \Delta_\varepsilon$. Therefore, for sufficiently large n, we obtain that $|F_n(z)| \geqslant \frac{M}{2}$. Thus $f_n(z) \neq 0$ for sufficiently large n and all $z \in \Delta_\varepsilon$. Therefore $\frac{1}{f_n}$ is analytic in Δ_ε. Thus, for sufficiently large n, we have

$$\left|\frac{1}{f_n(z)}\right| = \left|\frac{1}{F_n(z)}\frac{1}{|z|^m}\right| \leqslant \frac{2^m}{\varepsilon^m}\frac{2}{M}, \quad |z| = \frac{\varepsilon}{2}.$$

By the Maximum Principle and Montel's theorem, $\mathcal{F}$ is normal at $z = 0$.

We now turn to prove $\mathcal{H}$ is normal at 0. Suppose not. By Lemma 1, after choosing appropriate subsequences we may assume that there exist $z_n \to 0$, and $\rho_n \to 0^+$ such that

$$\rho_n^{-k} F_n(z_n + \rho_n \zeta) = g_n(\zeta) \to g(\zeta),$$

spherically uniformly on compact subsets of C, where $g(\zeta)$ is nonconstant meromorphic function on C, all of whose zeros have multiplicity at least $k+1$.

We consider two cases.

Case 1. We may suppose that $\frac{z_n}{\rho_n} \to \infty$. We have

$$\begin{aligned} f_n^{(k)}(z) &= z^m F_n^{(k)}(z) + \sum_{l=1}^{k} \binom{k}{l} (z^m)^{(l)} F_n^{(k-l)}(z) \\ &= z^m F_n^{(k)}(z) + \sum_{l=1}^{k} c_l z^{m-l} F_n^{(k-l)}(z), \end{aligned} \tag{3.1}$$

where

$$c_l = \begin{cases} \binom{k}{l} m(m-1)\cdots(m-l+1), & l \leqslant m, \\ 0, & l > m. \end{cases}$$

Since $\rho_n^l g_n^{(k-l)}(\zeta) = F_n^{(k-l)}(z_n + \rho_n \zeta)$, $l = 0, 1, \ldots, k$, we obtain

$$\frac{f_n^{(k)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} = \left[g_n^{(k)}(\zeta) + \sum_{l=1}^{k} c_l \frac{g_n^{(k-l)}(\zeta)}{(\frac{z_n}{\rho_n} + \zeta)^l} \right] \frac{1}{b(z_n + \rho_n \zeta)}. \tag{3.2}$$

Now

$$\lim_{n \to \infty} \frac{c_l}{(\frac{z_n}{\rho_n} + \zeta)^l} = 0, \quad l = 1, 2, \ldots, k, \tag{3.3}$$

and

$$\lim_{n \to \infty} \frac{1}{b(z_n + \rho_n \zeta)} = 1. \tag{3.4}$$

By (3.2), (3.3) and (3.4), we have

$$\frac{f_n^{(k)}(z_n + \rho_n \zeta)}{h(z_n + \rho_n \zeta)} \to g^{(k)}(\zeta),$$

uniformly on compact subsets of C disjoint from the poles of g.

We claim that

(i) $g^{(k)} \neq 1$; and
(ii) no poles of g are simple.

Suppose now that $g^{(k)}(\zeta_0) = 1$. We claim that $g^{(k)} \not\equiv 1$. Otherwise, g must be a polynomial of exact degree k, which contradicts the fact that each zero of g has multiplicity at least $k+1$. Since $g^{(k)}(\zeta_0) = 1$ but $g^{(k)} \not\equiv 1$, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n sufficiently large) $f_n^{(k)}(z_n + \rho_n \zeta_n) = h(z_n + \rho_n \zeta_n)$. It follows that $f_n(z_n + \rho_n \zeta_n) = a(z_n + \rho_n \zeta_n)$, so that

$$g_n(\zeta_n) = \frac{f_n(z_n + \rho_n \zeta_n)}{\rho_n^k (z_n + \rho_n \zeta_n)^m} = \frac{a(z_n + \rho_n \zeta_n)}{\rho_n^k (z_n + \rho_n \zeta_n)^m}.$$

Thus $g(\zeta_0) = \lim_{n \to \infty} g_n(\zeta_n) = \infty$, which contradicts $g^{(k)}(\zeta_0) = 1$. This proves (i).

Next we prove (ii). Suppose $g(\zeta_0) = \infty$. There exists a closed disc $K = \{\zeta\colon |\zeta - \zeta_0| \leqslant \delta\}$ on which $1/g$ and $1/g_n$ are holomorphic (for n sufficiently large) and $1/g_n \to 1/g$ uniformly. Hence, $\frac{1}{g_n(\zeta)} - \frac{\rho_n^k (z_n + \rho_n \zeta)^m}{a(z_n + \rho_n \zeta)} \to \frac{1}{g(\zeta)}$ uniformly on K; and since $1/g$ is nonconstant, there exist ζ_n, $\zeta_n \to \zeta_0$, such that (for n large enough)

$$\frac{1}{g_n(\zeta_n)} - \frac{\rho_n^k (z_n + \rho_n \zeta_n)^m}{a(z_n + \rho_n \zeta_n)} = 0.$$

Hence $f_n(z_n + \rho_n \zeta_n) = a(z_n + \rho_n \zeta_n)$. Thus we have

$$f_n^{(k)}(z_n + \rho_n \zeta_n) = h(z_n + \rho_n \zeta_n). \tag{3.5}$$

By (3.2) and (3.5) we can obtain

$$g_n^{(k)}(\zeta_n)=\left[\frac{f_n^{(k)}(z_n+\rho_n\zeta_n)}{h(z_n+\rho_n\zeta_n)}b(z_n+\rho_n\zeta_n)-\sum_{l=1}^{k}c_l\frac{g_n^{(k-l)}(\zeta_n)}{(\frac{z_n}{\rho_n}+\zeta_n)^l}\right]\to 1. \tag{3.6}$$

Using a similar fashion as Lemma 6, by (2.2), (2.3), (2.4) and (3.6), we can prove (ii).

By Lemma 2, g is a rational function. By (i), (ii) and Lemma 3, g is a constant, a contradiction. Thus $\{f_n\}$ is normal on D.

Case 2. So we may assume that $\frac{z_n}{\rho_n}\to\alpha$, a finite complex number. We have

$$\frac{F_n(\rho_n\zeta)}{\rho_n^k}=\frac{F_n(z_n+\rho_n(\zeta-\frac{z_n}{\rho_n}))}{\rho_n^k}\to g(\zeta-\alpha),$$

the convergence being spherically uniform on compact sets of C. Clearly, all zeros of $g(\zeta-\alpha)$ have multiplicity at least $k+1$, and the pole of $g(\zeta-\alpha)$ at $\zeta=0$ has multiplicity at least m. Now

$$G_n(\zeta)=\frac{f_n(\rho_n\zeta)}{\rho_n^{m+k}}=\frac{F_n(\rho_n\zeta)}{\rho_n^k}\frac{(\rho_n\zeta)^m}{\rho_n^m}\to\zeta^m g(\zeta-\alpha)=G(\zeta), \tag{3.7}$$

uniformly on compact subsets of C. Since $g(\zeta-\alpha)$ has a pole of multiplicity at least m at $\zeta=0$, $G(0)\neq 0$ and all zeros of $G(\zeta)$ have multiplicity at least $k+1$.

We claim that

(iii) $G^{(k)}(\zeta)\neq\zeta^m$, $\zeta\in C$;
(iv) no poles of g are simple.

Indeed, suppose that $G^{(k)}(\zeta_0)=\zeta_0^m$. Then $G(\zeta)$ is holomorphic at ζ_0, and

$$\frac{f_n^{(k)}(\rho_n\zeta)-h(\rho_n\zeta)}{\rho_n^m}=G_n^{(k)}(\zeta)-\frac{h(\rho_n\zeta)}{\rho_n^m}\to G^{(k)}(\zeta)-\zeta^m.$$

First we assume that $G^{(k)}(\zeta)\equiv\zeta^m$. Then G is a nonconstant polynomial. Therefore G has a zero ς_0. Since all zeros of G have multiplicity at least $k+1$, we deduce $\varsigma_0^m=G^{(k)}(\varsigma_0)=0$, hence $\varsigma_0=0$. This contradicts $G(0)\neq 0$. Thus $G^{(k)}(\zeta)\not\equiv\zeta^m$. Suppose that $G^{(k)}(\zeta_0)=\zeta_0^m$. By Hurwitz theorem, there exist ζ_n, $\zeta_n\to\zeta_0$, such that (for n sufficiently large) $f_n^{(k)}(\rho_n\zeta_n)-h(\rho_n\zeta_n)=0$. It follows that $f_n(\rho_n\zeta_n)=a(\rho_n\zeta_n)$. Thus $G(\zeta_0)=\lim_{n\to\infty}G_n(\zeta_n)=\infty$, which contradicts $G^{(k)}(\zeta_0)=\zeta_0^m$. This proves (iii).

Next we prove (iv). Suppose $G(\zeta_0)=\infty$. There exists a closed disc $K=\{\zeta\colon |\zeta-\zeta_0|\leqslant\delta\}$ on which $1/G$ and $1/G_n$ are holomorphic (for n sufficiently large) and $1/G_n\to 1/G$ uniformly. Hence, $\frac{1}{G_n(\zeta)}-\frac{\rho_n^{k+m}}{a(\rho_n\zeta)}\to\frac{1}{G(\zeta)}$ uniformly on K; and since $1/G$ is nonconstant, there exist ζ_n, $\zeta_n\to\zeta_0$, such that (for n large enough)

$$\frac{1}{G_n(\zeta_n)}-\frac{\rho_n^{k+m}}{a(\rho_n\zeta_n)}=0.$$

Hence $f_n(\rho_n\zeta_n)=a(\rho_n\zeta_n)$. Thus we have

$$f_n^{(k)}(\rho_n\zeta_n)=h(\rho_n\zeta_n).$$

By (3.7) we can obtain

$$G_n^{(k)}(\zeta_n)=\frac{f_n^{(k)}(\rho_n\zeta_n)}{\rho_n^m}=\frac{h(\rho_n\zeta_n)}{\rho_n^m}=b(\rho_n\zeta_n)\zeta_n^m\to\zeta_0^m. \tag{3.8}$$

Using a similar fashion as Lemma 6, by (2.2), (2.3), (2.4) and (3.8), we can prove (iv).

Firstly, Lemma 2 implies that $G(\zeta)$ is rational.

Suppose that $m\geqslant 2$. It follows from Lemma 4 and (iv) that $G^{(k)}(\zeta)=\zeta^m$ has a solution in C. This contradicts with (iii). Thus by Lemma 5, we have $m=1$ and

$$G(\zeta)=\frac{(\zeta+c)^{k+1}}{(k+1)!},\quad c\neq 0. \tag{3.9}$$

It then follows from (3.7) and (3.9) that there exist points $\zeta_n\to -c$ such that $f_n(\rho_n\zeta_n)=0$. In fact, $\rho_n\zeta_n$ are zeros of f_n of exact multiplicity $k+1$.

We suppose that the functions f_n are all holomorphic in some fixed disc Δ_ρ. Recall that the sequence $\{f_n\}$ is not normal at 0; on the other hand, by Lemma 6, it is normal on Δ'_ρ, since $h(z)\neq 0$ there.

We claim that the sequence $\{f_n\}$ tends to ∞ locally uniformly on Δ'_ρ. In fact, since $\{f_n\}$ is normal on Δ'_ρ, $\{f_n\}$ is normal in $C_{\rho/2} = \{z\colon\ |z| = \rho/2\}$. Thus there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges uniformly to a holomorphic function $f(z)$ or ∞ on $C_{\rho/2}$.

If $f_{n_k}(z) \to f(z)$, then there exist an integer N and a positive number M such that

$$\left|f_{n_k}(z)\right| \leqslant M$$

for all $k \geqslant N$, $z \in C_{\rho/2}$. By the maximum modulus theorem, we have

$$\left|f_{n_k}(z)\right| \leqslant M$$

for all $k \geqslant N$, $|z| \leqslant \rho/2$. Hence $\{f_{n_k}\}$ is normal in $\{z\colon\ |z| \leqslant \rho/2\}$ by Montel's normality criterion (see [7]). This contradicts with our assumption. Hence $\{f_n\}$ tends to ∞ locally uniformly on Δ'_ρ.

Suppose first that there exists $0 < \delta < \rho$ such that each f_n has only the single zero $\xi_n = \rho_n \zeta_n$ in Δ_δ. Put

$$H_n(z) = \frac{f_n(z)}{(z-\xi_n)^{k+1}}. \tag{3.10}$$

Then $\{H_n\}$ is a sequence of nonvanishing holomorphic functions on Δ_δ and tending to ∞ locally uniformly on Δ'_δ. It follows that the sequence $\{1/H_n\}$ of holomorphic functions tends to 0 locally uniformly on Δ'_δ and hence, by the maximum principle, on Δ_δ. In particular, $H_n(2\rho_n\zeta_n) \to \infty$. But by (3.7), (3.9) and (3.10),

$$H_n(2\rho_n\zeta_n) = \frac{f_n(2\rho_n\zeta_n)}{(\rho_n\zeta_n)^{k+1}} = \frac{G_n(2\zeta_n)}{\zeta_n^{k+1}} \to \frac{G(-2c)}{(-c)^{k+1}} = \frac{1}{(k+1)!},$$

a contradiction. Thus, we may assume that for any $\delta > 0$, f_n has at least two distinct zeros in Δ_δ for n sufficiently large. Choose η_n such that $f_n(\eta_n) = 0$ and f_n has no zeros on $\{z\colon\ 0 < |z - \xi_n| < |\eta_n - \xi_n|\}$, then $\eta_n \to 0$. We claim that $\eta_n/\rho_n \to \infty$. Otherwise, taking a subsequence if necessary, from (3.7) and (3.9), we could deduce $\eta_n/\rho_n \to -c$. So G_n would have zeros of multiplicity at least $k+1$ in ζ_n and η_n/ρ_n, and both sequences $\{\zeta_n\}$ and $\{\eta_n/\rho_n\}$ converge to $-c$ which implies that G has a zero of multiplicity at least $2k+2$ in $-c$, a contradiction. Since $\eta_n/\rho_n \to \infty$, $\xi_n/\eta_n = \rho_n\zeta_n/\eta_n \to 0$. Put

$$K_n(z) = \frac{f_n((\eta_n - \xi_n)z)}{(\eta_n - \xi_n)^{k+1}}, \qquad \widetilde{h}_n(z) = \frac{h_n((\eta_n - \xi_n)z)}{\eta_n - \xi_n}.$$

Then $\{K_n\}$ is a sequence of functions holomorphic on each bounded set of C for large enough n, all of whose zeros have multiplicity at least $k+1$. Similarly, the sequence of holomorphic functions $\{\widetilde{h}_n\}$ is defined for each $z \in C$ for n sufficiently large, and $\widetilde{h}_n(z) \to z$ locally uniformly on C. Clearly,

$$K_n(z) = \frac{a((\eta_n - \xi_n)z)}{(\eta_n - \xi_n)^{k+1}} \quad \Leftrightarrow \quad K_n^{(k)}(z) = \widetilde{h}_n(z).$$

Hence, by Lemma 6, $\{K_n\}$ is normal on $C - \{0\}$. We claim that $\{K_n\}$ is also normal at 0. Indeed, otherwise $K_n \to \infty$ locally uniformly on $C - \{0\}$. But this is impossible, as $K_n(\eta_n/(\eta_n - \xi_n)) = 0$ and $\eta_n/(\eta_n - \xi_n) \to 1$. Thus $\{K_n\}$ is normal on C. Taking a subsequence and renumbering, we have $K_n \to K$ locally uniformly on C, for an entire function K, all of whose zeros have multiplicity at least $k+1$. Suppose that $K^{(k)}(z) \equiv z$. Thus $K(z) = z^{k+1}/(k+1)!$. But $K_n(\eta_n/(\eta_n - \xi_n)) = 0$ and $\eta_n/(\eta_n - \xi_n) \to 1$, so that $K(1) = 0$, a contradiction. We claim that $K^{(k)} \neq z$. Otherwise, we may suppose that $K^{(k)}(z_0) = z_0$. By Hurwitz theorem, there exist z_n, $z_n \to z_0$, such that (for n sufficiently large) $K_n^{(k)}(z_n) - \widetilde{h}_n(z_n) = 0$. It follows that $f_n((\eta_n - \xi_n)z_n) = a((\eta_n - \xi_n)z_n)$. Thus $K(z_0) = \lim_{n\to\infty} K_n(z_n) = \infty$, which contradicts $K^{(k)}(z_0) = z_0$. This proves $K^{(k)} \neq z$. But $K_n(\xi_n/(\eta_n - \xi_n)) = 0$ and $\xi_n/(\eta_n - \xi_n) \to 0$, so that $K(0) = 0$ and hence $K^{(k)}(0) = 0$, a contradiction. The contradiction shows that $\mathcal{H}$ is normal at 0.

It remains to prove Theorem 1 in the general case, in which the functions f_n need not be holomorphic in any fixed disc about the origin. Thus, taking a subsequence if necessary, we may assume that for any $\delta > 0$, f_n has both a zero and a pole in Δ_δ for n sufficiently large. Choose ω_n such that $f_n(\omega_n) = \infty$ and f_n has no poles on $\{z\colon\ 0 < |z - \xi_n| < |\omega_n - \xi_n|\}$, then $\omega_n \to 0$. By (3.7) and (3.9), $\omega_n/\rho_n \to \infty$, so that $\xi_n/\omega_n = \rho_n\zeta_n/\omega_n \to 0$. Put

$$L_n(z) = \frac{f_n((\omega_n - \xi_n)z)}{(\omega_n - \xi_n)^{k+1}}, \qquad \widehat{h}_n(z) = \frac{h_n((\omega_n - \xi_n)z)}{\omega_n - \xi_n}.$$

Then $\{L_n\}$ is a sequence of meromorphic functions for large enough n, all of whose zeros have multiplicity at least $k+1$. Similarly, the sequence of holomorphic functions $\{\widehat{h}_n\}$ is defined for each $z \in C$ for n sufficiently large, and $\widehat{h}_n(z) \to z$ locally uniformly on C. Clearly,

$$L_n(z) = \frac{a((\omega_n - \xi_n)z)}{(\omega_n - \xi_n)^{k+1}} \quad \Leftrightarrow \quad L_n^{(k)}(z) = \widehat{h}_n(z).$$

Hence, by Lemma 6, $\{L_n\}$ is normal on $C - \{0\}$. Since $\xi_n/\omega_n \to \infty$, the functions L_n are holomorphic on $\Delta_{1/2}$ for large n. Thus we may apply the fact (already proved) that Theorem 1 holds for functions holomorphic in a neighborhood of 0 to conclude that $\{L_n\}$ is normal on $\Delta_{1/2}$. Thus $\{L_n\}$ is normal on C. Taking a subsequence if necessary and renumbering, we have $L_n \to L$ locally uniformly on C, for a meromorphic function L, all of whose zeros have multiplicity at least $k+1$. Suppose that $L^{(k)}(z) \equiv z$. Thus $L(z) = z^{k+1}/(k+1)!$. But $K_n(\omega_n/(\omega_n - \xi_n)) = \infty$ and $\omega_n/(\omega_n - \xi_n) \to 1$, so that $K(1) = \infty$, a contradiction. We claim that $L^{(k)} \neq z$. Otherwise, we may suppose that $L^{(k)}(z_0) = z_0$. By Hurwitz theorem, there exist z_n, $z_n \to z_0$, such that (for n sufficiently large) $L_n^{(k)}(z_n) - \widehat{h}_n(z_n) = 0$. It follows that $f_n((\omega_n - \xi_n)z_n) = a((\omega_n - \xi_n)z_n)$. Thus $L(z_0) = \lim_{n\to\infty} L_n(z_n) = \infty$, which contradicts $L^{(k)}(z_0) = z_0$. This proves $L^{(k)} \neq z$. But $L_n(\xi_n/(\omega_n - \xi_n)) = 0$ and $\xi_n/(\omega_n - \xi_n) \to 0$, so that $L(0) = 0$ and hence $L^{(k)}(0) = 0$, a contradiction. The contradiction shows that $\mathcal{H}$ is normal at 0. It then follows, exactly as before, that $\mathcal{F}$ is normal at 0. This completes the proof of Theorem 1.

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