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Best quadrature formula on Sobolev class with Chebyshev weight ☆

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Abstract

Using best interpolation function based on a given function information, we present a best quadrature rule of function on Sobolev class $KW^r[-1,1]$ with Chebyshev weight. The given function information means that the values of a function $f \in KW^r[-1,1]$ and its derivatives up to $r-1$ order at a set of nodes $\mathbf{x}$ are given. Error bounds are obtained, and the method is illustrated by some examples.

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1. Introduction

Consider the integral

$$I_\omega(f) = \int_{-1}^{1} \omega(t) f(t)\,\mathrm{d}t.$$

To approximate $I_\omega(f)$, Gauss quadrature formula is often preferred. Let $\xi_1, \ldots, \xi_n$ be zeros of the nth orthogonal polynomial with respect to the weight function $\omega(t)$ (Gaussian nodes for brevity), then there exist weights $\lambda_1, \ldots, \lambda_n$ such that the numerical quadrature of the type

$$\int_{-1}^{1} \omega(t) f(t)\,\mathrm{d}t = \sum_{i=1}^{n} \lambda_i f(\xi_i) + R_n(f)$$

is exact for $f \in P_{2n-1}$ (P_N is the set of polynomials of degree at most N), i.e., the remainder term $R_n(f) = 0$ for all $f \in P_{2n-1}$.

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Furthermore, besides the values of function *f*, its derivative values are known. The quadrature formula of the following form

$$\int_{-1}^{1} \omega(t) f(t)\,\mathrm{d}t \cong \sum_{i=1}^{n} \sum_{j=0}^{2s} \lambda_{ij} f^{(j)}(\xi_i) \tag{1.1}$$

is more suitable, which is called Gauss–Turán quadrature formula and is exact for all $f \in P_{2(s+1)n-1}$. Especially when $s = 0$, the corresponding Gauss–Turán quadrature formula is the classical Gauss quadrature formula.

It was more than 100 years after Gauss published his famous method of approximate integration that there appeared the idea of numerical quadrature rules involving multiple nodes. Gauss–Turán formulas, or quadrature formulas with the highest degree of algebraic precision with multiple nodes, have extensively been studied in the last decades from both an algebraic and numerical point of view. Numerically stable methods for constructing nodes ξ_i and coefficients λ_{ij} can be found in [7,12]. For more details on Gauss–Turán quadratures and corresponding orthogonal polynomials, see the book [3]. Some interesting results concerning L^1 and L^∞-error bounds of formula (1.1) with generalized Chebyshev weight functions, and for analytic integrands, have been given recently in [5,6,8].

Let $\omega(t) = (1 - t^2)^{-1/2}$, then $\xi_1, \ldots, \xi_n$ are the zeros of the nth-degree Chebyshev polynomial of the first kind $T_n(t)$, i.e., $\xi_i = \cos((2i-1)\pi/2n)$, $i = 1, \ldots, n$. The explicit formula for λ_{ij} attracts many scholars to investigate. For related work, see [1,4,11,22] and references cited therein. Moreover, there is no quadrature using a linear combination of values of *f* and its derivatives such that (1.1) holds for all polynomials of degree $2(s+1)n$.

Now, suppose we only know the values of f and its derivatives at a set of nodes $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, not all values are needed. These available values are usually obtained from scientific and engineering computing terminal and are thus expensively priced, therefore, any disuse of them is not economic or judicious. That is to say, we may not have freedom to choose the quadrature nodes and it is difficult to obtain the values of f at any other nodes except $\mathbf{x}$. Gauss–Turán quadrature formula seems to be not suitable for such problems, since the nodes $x_1, \ldots, x_n$ do not always coincide with Gaussian nodes $\xi_1, \ldots, \xi_n$. In this paper, we obtain a method to solve such problems.

In order to state our results, here and henceforth we assume that $\mathbf{x}$ be a fixed set of nodes in $[-1, 1]$:

$$-1 = x_0 \leqslant x_1 < x_2 < \cdots < x_n \leqslant x_{n+1} = 1.$$

For any natural number r and any positive number K, denote by $KW^r[-1, 1]$ the Sobolev class consisting of every function whose $(r-1)$st derivative $f^{(r-1)}$ is absolutely continuous on the interval $[-1, 1]$ and its rth derivative $f^{(r)}$ satisfies

$$|f^{(r)}(t)| \leqslant K \quad \text{a.e. } t \in [-1, 1].$$

For any $f \in KW^r[-1, 1]$, the following $r \times n$ matrix

$$\mathscr{H}^r_{\mathbf{x}}(f) := (f^{(j)}(x_i))_{r\times n}, \quad i = 1, 2, \ldots, n, \quad j = 0, 1, \ldots, r-1$$

defines a Lagrange–Hermite (Hermite for brevity) information operator $\mathscr{H}^r_{\mathbf{x}}: KW^r[-1, 1] \to \mathbb{R}^{r\times n}$ at $\mathbf{x}$ and $\mathscr{H}^1_{\mathbf{x}}$ is just the Lagrange information operator. If there is a single node, i.e., $\mathbf{x} = (\eta)$, we shall write $\mathscr{H}^r_\eta$ instead of $\mathscr{H}^r_{(\eta)}$.

Suppose that the given Hermite information $\mathscr{H}_{\mathbf{x}}(f)$ of a function $f \in KW^r[-1, 1]$ at $\mathbf{x}$ is obtained expensively. It is also difficult to obtain the information of *f* at any other nodes. In view of this, the classical best quadrature formula in the sense of Sard [10] and of Nikolskii [9] seems to be not very suitable. To solve such problem, Wang and Mi [14] first proposed a definition for the best quadrature based on the given information. Some relations for these three quadrature formulae have been addressed in [2].

Any quadrature formula Q for approximating the integration $I_\omega(\overline{f})$, $\overline{f} \in KW^r[-1, 1]$, can be viewed as a functional (not necessarily linear) acting on $\mathscr{H}_{\mathbf{x}}(KW^r[-1, 1])$. Its error bound is defined by

$$E(\mathscr{H}^r_{\mathbf{x}}(f); Q) := \max_{\substack{\bar{f}\in KW^r[a,b] \\ \mathscr{H}^r_{\mathbf{x}}(\bar{f}) = \mathscr{H}^r_{\mathbf{x}}(f)}} |I_\omega(\bar{f}) - Q(\mathscr{H}^r_{\mathbf{x}}(f))|.$$

Definition 1.1. Of all quadrature formulae based on the information $\mathscr{H}_{\mathbf{x}}^r(f) \in \mathscr{H}_{\mathbf{x}}^r(KW^r[-1,1])$, the one $Q^*(\mathscr{H}_{\mathbf{x}}^r(f))$ which minimizes the error bound

$$E(\mathscr{H}_{\mathbf{x}}^r(f); Q^*) = \min_Q E(\mathscr{H}_{\mathbf{x}}^r(f); Q) =: R(\mathscr{H}_{\mathbf{x}}^r(f))$$

is said to be the best quadrature formula (if it exists) based on the given information $\mathscr{H}_{\mathbf{x}}^r(f)$ for the integration $I_\omega(f)$. Correspondingly, the error bound $R(\mathscr{H}_{\mathbf{x}}^r(f))$ is called the radius of the Hermite information $\mathscr{H}_{\mathbf{x}}^r(f)$ for the integration $I_\omega(f)$.

Wang and Mi [14] recently proposed a best quadrature formula based on the given Hermite information $\mathscr{H}_{\mathbf{x}}^2(f)$ on $KW^2[a,b]$. In [16], Wang and Yang extend the approaches of [14] to rth Sobolev class $KW^r[a,b]$ based on the information $\mathscr{H}_{\mathbf{x}}^r(f)$. We followed the thoughts of [14] and [16,18–20] to present a best quadrature formula with a piecewise weight function on $KW^r[a,b]$, see [21]. For related materials, see also [13]. Best quadrature formulas given by [13,14,16,21] are shown to be realized by a nonlinear functional for $r \geqslant 2$, while in the best quadrature formula of Sard and Nikolskii or Gauss rule is linear.

In this paper, we develop a best quadrature formula based on the given information $\mathscr{H}_{\mathbf{x}}^r(f)$ with the Chebyshev weight functions of the first kind and the second kind on Sobolev class $KW^r[-1,1]$. Section 2 contains some auxiliary lemmas. We give the method to obtain the best quadrature formula and the error bound in Section 3. In Section 4, some numerical experiments which compared our method with Gauss–Turán quadrature are presented.

2. Auxiliary lemmas

In this paper, we suppose $e = \pm 1$, $W^r := 1W^r[0,1]$.

First, it can be straightforwardly shown [16] that the following two lemmas are concerned with the perfect spline interpolation problem and the extremal properties of the perfect spline interpolation by an elementary analysis and zero counting argument.

Lemma 2.1. *For any $\eta \in [0,1]$, suppose that $\mathscr{H}_\eta^r(\varphi) \in \mathscr{H}_\eta^r(W^r)$ is given. Setting*

$$P_{e,r}(t) := P_{e,r}(t;\eta,\varphi) = \sum_{j=0}^{r-1} \frac{\varphi^{(j)}(\eta)}{j!}(t-\eta)^j + \frac{e}{r!}|t-\eta|^r, \tag{2.1}$$

we have

$$\mathscr{H}_\eta^r(P_{e,r}) = \mathscr{H}_\eta^r(\varphi), \quad i.e., \; P_{e,r}^{(j)}(\eta) = \varphi^{(j)}(\eta), \quad j = 0,1,\ldots,r-1,$$

and

$$P_{-,r}(t) \leqslant \varphi(t) \leqslant P_{+,r}(t), \quad t \in [0,1]. \tag{2.2}$$

Lemma 2.2. *Suppose that the information $\mathscr{H}_{(0,1)}^r(\varphi) \in \mathscr{H}_{(0,1)}^r(W^r)$ is given. Let the perfect spline of degree r defined by*

$$S_{e,r}(t;\varphi) = \sum_{j=0}^{r-1} \frac{\varphi^{(j)}(1)}{j!}(t-1)^j + \frac{e}{r!}\left\{(1-t)^r - 2\sum_{i=1}^{r}(-1)^i (t-\xi_{e,i})_-^r\right\} \tag{2.3}$$

satisfy

$$0 \leqslant \xi_{e,1} \leqslant \xi_{e,2} \leqslant \cdots \leqslant \xi_{e,r} \leqslant 1,$$

$$S_{e,r}^{(j)}(0) = \varphi^{(j)}(0), \quad j = 0,1,\ldots,r-1.$$

Then we have

$$\sum_{i=1}^{r}(-1)^{r-i}\xi_{e,i}^{j}=p_{e,j},\quad j=1,\ldots,r, \tag{2.4}$$

and

$$S_{-,r}(t)\leqslant\varphi(t)\leqslant S_{+,r}(t),\quad t\in[0,1], \tag{2.5}$$

where

$$p_{e,j}=\tfrac{1}{2}(1-e(-1)^{r}j!\varphi^{(r-j)}[0,1^{j}]). \tag{2.6}$$

Here, as usual, we have put

$$u_{-}^{r}=\begin{cases}0 & \text{if } u\geqslant 0,\\ u^{r} & \text{if } u<0,\end{cases}$$

and $\varphi[0,1^j]$ denotes the divided differences of the function φ at the points 0, 1, where 1 is repeated j times. $\varphi^{(r-j)}$ is the $(r-j)$th derivative of φ and $\varphi[0,1^j]$ is equivalent to

$$\varphi[0,1^{j}]=(-1)^{j}\left(\varphi(0)-\sum_{l=0}^{j-1}(-1)^{l}\frac{\varphi^{(l)}(1)}{l!}\right).$$

Therefore, we can construct the best interpolation formula from the perfect spline, which are defined in the form of (2.1) and (2.3). However, this needs to solve the nonlinear equations (2.4) (if $r\geqslant 2$). For $r>4$, it involves solving two algebraic equations, one of which is of degree at least three, therefore, explicit solutions are not available for $r>4$. For details, see [15–17,19]. Here, we list the solutions of (2.4) when $r\leqslant 4$ in the following three lemmas.

Lemma 2.3. *Let the Hermite information $\mathscr{H}^2_{(0,1)}(\varphi)\in\mathscr{H}^2_{(0,1)}(W^2)$, and $p_{e,j}$ be defined by* (2.6). *Then we have*

$$\xi_{e,1}=\frac{1}{2}\left(\frac{p_{e,2}}{p_{e,1}}-p_{e,1}\right),\quad \xi_{e,2}=\frac{1}{2}\left(\frac{p_{e,2}}{p_{e,1}}+p_{e,1}\right). \tag{2.7}$$

Lemma 2.4. *Let Hermite information $\mathscr{H}^3_{(0,1)}(\varphi)\in\mathscr{H}^3_{(0,1)}(W^3)$ and $p_{e,j}$ be defined by* (2.6). *Then we have*

$$\xi_{e,1}=\tfrac{1}{2}(p_{e,1}+a_{e,3}-\sqrt{(p_{e,1}-a_{e,3})^{2}+2(p_{e,2}-p_{e,1}^{2})}),$$

$$\xi_{e,2}=a_{e,3},$$

$$\xi_{e,3}=\tfrac{1}{2}(p_{e,1}+a_{e,3}+\sqrt{(p_{e,1}-a_{e,3})^{2}+2(p_{e,2}-p_{e,1}^{2})}),$$

where

$$a_{e,3}:=\frac{p_{e,1}^{3}-3p_{e,1}p_{e,2}+2p_{e,3}}{3(p_{e,2}-p_{e,1}^{2})}. \tag{2.8}$$

Lemma 2.5. *Let Hermite information $\mathscr{H}^4_{(0,1)}(\varphi) \in \mathscr{H}^4_{(0,1)}(W^4)$ and $p_{e,j}$ be defined by* (2.6). *Then we have*

$$\xi_{e,1} = \tfrac{1}{2}(a_{e,4} - \sqrt{a_{e,4}^2 - 4\overline{a}_{e,4}}),$$
$$\xi_{e,2} = \tfrac{1}{2}(p_{e,1} + a_{e,4} - \sqrt{(p_{e,1} - a_{e,4})^2 - 2(p_{e,1}^2 - p_{e,2} + 2\overline{a}_{e,4})}),$$
$$\xi_{e,3} = \tfrac{1}{2}(a_{e,4} + \sqrt{a_{e,4}^2 - 4\overline{a}_{e,4}}),$$
$$\xi_{e,4} = \tfrac{1}{2}(p_{e,1} + a_{e,4} + \sqrt{(p_{e,1} - a_{e,4})^2 - 2(p_{e,1}^2 - p_{e,2} + 2\overline{a}_{e,4})}),$$

where

$$a_{e,4} := \frac{1}{2(p_{e,1}^4 + 3p_{e,2}^2 - 4p_{e,1}p_{e,3})}(-p_{e,1}^5 + 2p_{e,1}^3 p_{e,2} + 4p_{e,1}^2 p_{e,3} + 4p_{e,2}p_{e,3} - 3p_{e,1}(p_{e,2}^2 + 2p_{e,4})),$$

$$\overline{a}_{e,4} := \frac{1}{12(p_{e,1}^4 + 3p_{e,2}^2 - 4p_{e,1}p_{e,3})}(p_{e,1}^6 - 3p_{e,1}^4 p_{e,2} + 9p_{e,2}^3 - 8p_{e,1}^3 p_{e,3}$$
$$- 24p_{e,1}p_{e,2}p_{e,3} + 16p_{e,3}^2 - 18p_{e,2}p_{e,4} + 9p_{e,1}^2(p_{e,2}^2 + 2p_{e,4})).$$

Remark 2.1. If $p_{e,1} = \frac{1}{2}(1 - e\varphi'[0,1]) = 0$ in Lemma 2.3, then

$$\int_0^1 (e - \varphi''(t))\,\mathrm{d}t = e - \varphi'[0,1] = 0,$$

implying $\varphi''(t) = e$ a.e. since $e - \varphi''(t)$ keeps constant sign a.e. over [0, 1], thus leading to a degenerate case, and in this case, $\varphi(t)$ is indeed a polynomial of degree two. For the case of statement, we tacitly assume without loss of generality that $\varphi'[0,1] \neq e$ so that the solution in (2.7) is unambiguous. Similarly, in (2.8) $p_{e,2} - p_{e,1}^2 = 0$ is equivalent to $\xi_{e,1} = \xi_{e,2} = \xi_{e,3} = p_{e,1}$ and it also leads to a degenerate case. Therefore, the assumption $p_{e,2} - p_{e,1}^2 \neq 0$ is reasonable, and the same applies to $p_{e,1}^4 + 3p_{e,2}^2 - 4p_{e,1}p_{e,3} \neq 0$.

3. Quadrature rule

In this section, we deal with the best quadrature formula following from the best interpolation formula. From Lemmas 2.1 and 2.2, we easily have the following theorem (see [16,18,20]). For $r = 2$, it can be found in [14].

Theorem 3.1. *For the given Hermite information $\mathscr{H}^r_{\mathbf{x}}(f) \in \mathscr{H}_{\mathbf{x}}(KW^r[-1,1])$, there exists $\psi_{e,r} \in KW^r[-1,1]$ such that*

$$\mathscr{H}^r_{\mathbf{x}}(\psi_{-,r}) = \mathscr{H}^r_{\mathbf{x}}(\psi_{+,r}) = \mathscr{H}^r_{\mathbf{x}}(f)$$

and

$$\psi_{-,r}(t) \leqslant f(t) \leqslant \psi_{+,r}(t), \quad \forall t \in [-1,1]. \tag{3.1}$$

Here

$$\psi_{e,r}(t)=\begin{cases}\displaystyle\sum_{j=0}^{r-1}\frac{f^{(j)}(x_1)}{j!}(t-x_1)^j+\frac{Ke}{r!}(x_1-t)^r, & -1\leqslant t\leqslant x_1,\\ \displaystyle\sum_{j=0}^{r-1}\frac{f^{(j)}(x_{i+1})}{j!}(t-x_{i+1})^j+\frac{Ke}{r!} & \\ \quad\displaystyle\left((x_{i+1}-t)^r-2\sum_{v=1}^{r}(-1)^v(t-x_i-\Delta x_i\xi_{e,i,v})_-^r\right), & x_i\leqslant t\leqslant x_{i+1},\\ & i=1,2,\dots,n-1,\\ \displaystyle\sum_{j=0}^{r-1}\frac{f^{(j)}(x_n)}{j!}(t-x_n)^j+\frac{Ke}{r!}(t-x_n)^r, & x_n\leqslant t\leqslant 1,\end{cases}$$

and $\xi_{e,i,v}$ *satisfies*

$$\sum_{v=1}^{r}(-1)^{r-v}\xi_{e,i,v}^{j}=p_{e,i,j},\quad j=1,2,\dots,r, \tag{3.2}$$

where $\Delta x_i=x_{i+1}-x_i$, *and*

$$p_{e,i,j}=\frac{1}{2}\left(1-e(-1)^r j!\frac{f^{(r-j)}[x_i,x_{i+1}^j]}{K}\right).$$

Proof. First, we divide the interval $[-1,1]$ into $n+1$ subintervals $[x_i,x_{i+1}]$ $(i=0,1,\dots,n)$. Then we transform each subinterval into [0, 1] and use the perfect spline by Lemmas 2.1 and 2.2.

Setting

$$\varphi_i(t)=\frac{1}{K\,\Delta x_i^r}f(x_i+\Delta x_i t),\quad i=0,1,\dots,n, \tag{3.3}$$

we have $\varphi_i\in W^r$ and $\mathscr{H}^r_{(x_i,x_{i+1})}(f)=\mathscr{H}^r_{(0,1)}(\varphi_i)$. To construct the expression of $\psi_{e,r}$ in the subintervals $[x_0,x_1]$ and $[x_n,x_{n+1}]$, we replace η by 1 and 0 in Lemma 2.1, respectively.

Furthermore, for any $1\leqslant i\leqslant n-1$, let $\xi_{e,i,1},\ \xi_{e,i,2},\dots,\xi_{e,i,r}$ satisfy (2.4) with respect to φ_i and $p_{e,i,j}$, i.e., they satisfy

$$\sum_{v=1}^{r}(-1)^{(r-v)}\xi_{e,i,v}^{j}=p_{e,i,j}=\tfrac{1}{2}(1-e(-1)^r j!\varphi_i^{(r-j)}[0,1^j]),$$
$$i=1,2,\dots,n-1,\quad j=1,2,\dots,r. \tag{3.4}$$

The following two equalities are easy to check:

$$\varphi_i^{(r-j)}[0,1^j]=(-1)^j\left(\varphi_i^{(r-j)}(0)-\sum_{l=0}^{j-1}(-1)^l\frac{\varphi_i^{(r-j+l)}(1)}{l!}\right),$$

$$f^{(r-j)}[x_i,x_{i+1}^j]=(-\Delta x_i)^{-j}\left(f^{(r-j)}(x_i)-\sum_{l=0}^{j-1}(-\Delta x_i)^l\frac{f^{(r-j+l)}(x_{i+1})}{l!}\right).$$

These together with (3.3) and (3.4) successively give (3.2). The expression of $\psi_{e,r}$ in the subintervals $[x_i,x_{i+1}]$ $(i=1,2,\dots,n-1)$ follows from Lemmas 2.2 and (3.3) directly. The proof is completed. □

Actually, Theorem 3.1 provides the best interpolation formula at a point $t \in [-1, 1]$ based on the given information $\mathscr{H}^r_{\mathbf{x}}(f)$, and it is unique. Indeed, we have (see [15,16,18,20]):

Theorem 3.2. *If $\mathscr{H}^r_{\mathbf{x}}(f) \in \mathscr{H}^r_{\mathbf{x}}(KW^r[-1, 1])$, then*

$$i_t^*(\mathscr{H}^r_{\mathbf{x}}(f)) := \frac{\psi_{+,r}(t) + \psi_{-,r}(t)}{2}$$

is the unique best interpolation formula which makes the error bound

$$e(\mathscr{H}^r_{\mathbf{x}}(f); i_t) := \max_{\substack{\bar{f} \in KW^r[-1,1] \\ \mathscr{H}^r_{\mathbf{x}}(\bar{f}) = \mathscr{H}^r_{\mathbf{x}}(f)}} |\bar{f}(t) - i_t(\mathscr{H}^r_{\mathbf{x}}(f))|,$$

attains its minimum

$$e(\mathscr{H}^r_{\mathbf{x}}(f); i_t^*) = \frac{\psi_{+,r}(t) - \psi_{-,r}(t)}{2} = \min_{i_t} e(\mathscr{H}^r_{\mathbf{x}}(f); i_t) := r_t(\mathscr{H}^r_{\mathbf{x}}(f)).$$

The error bound $r_t(\mathscr{H}^r_{\mathbf{x}}(f))$ is called the radius of information $\mathscr{H}^r_{\mathbf{x}}(f)$ for interpolation at t.

From (3.1) and Theorem 3.2, we obtain the following two theorems.

Theorem 3.3. *Let $\omega(t) = (1 - t^2)^{-1/2}$, then the best quadrature formula based on the given Hermite information $\mathscr{H}^r_{\mathbf{x}}(f) \in \mathscr{H}^r_{\mathbf{x}}(KW^r[-1, 1])$ for $I_\omega(f)$ is*

$$\begin{aligned} Q^*_\omega(\mathscr{H}^r_{\mathbf{x}}(f)) = \sum_{j=0}^{r-1} \frac{1}{j!} &\left(\sum_{i=0}^{n-1} f^{(j)}(x_{i+1}) T_j(x_{i+1}, x_i, x_{i+1}) + f^{(j)}(x_n) T_j(x_n, x_n, 1) \right) \\ &- \frac{K}{r!} \sum_{i=1}^{n-1} \sum_{v=1}^{r} (-1)^v (T_r(x_i + \Delta x_i \xi_{+,i,v}, x_i, x_i + \Delta x_i \xi_{+,i,v}) \\ &- T_r(x_i + \Delta x_i \xi_{-,i,v}, x_i, x_i + \Delta x_i \xi_{-,i,v})), \end{aligned} \tag{3.5}$$

and its radius of the Hermite information is

$$\begin{aligned} R_\omega(\mathscr{H}^r_{\mathbf{x}}(f)) = \frac{K}{r!} &\left((-1)^r \sum_{i=0}^{n-1} T_r(x_{i+1}, x_i, x_{i+1}) + T_r(x_n, x_n, 1) \right) \\ &- \frac{K}{r!} \sum_{i=1}^{n-1} \sum_{v=1}^{r} (-1)^v (T_r(x_i + \Delta x_i \xi_{+,i,v}, x_i, x_i + \Delta x_i \xi_{+,i,v}) \\ &+ T_r(x_i + \Delta x_i \xi_{-,i,v}, x_i, x_i + \Delta x_i \xi_{-,i,v})), \end{aligned} \tag{3.6}$$

where

$$T_j(\alpha, a, b) = \int_a^b (1 - t^2)^{-1/2} (t - \alpha)^j \, dt,$$

and $\xi_{e,i,v}$, $i = 1, \ldots, n - 1$, $v = 1, \ldots, r$, satisfy (3.2).

Proof. From (3.1) and for any $t \in [-1, 1]$, we have

$$(1 - t^2)^{-1/2} \psi_{-,r}(t) \leqslant (1 - t^2)^{-1/2} f(t) \leqslant (1 - t^2)^{-1/2} \psi_{+,r}(t), \tag{3.7}$$

since $\omega(t)=(1-t^2)^{-1/2}$ is a positive weight function on $(-1,1)$. The following best quadrature formula based on the given information $\mathscr{H}_{\mathbf{x}}^r(f)$ for the integration $I_\omega(f)$ follows from (3.7) and Theorem 3.2:

$$Q_\omega^*(\mathscr{H}_{\mathbf{x}}^r(f))=I_\omega(i_t^*)=I_\omega\left(\frac{\psi_{+,r}+\psi_{-,r}}{2}\right), \tag{3.8}$$

and its radius is

$$R_\omega(\mathscr{H}_{\mathbf{x}}^r(f))=I_\omega(r_t)=I_\omega\left(\frac{\psi_{+,r}-\psi_{-,r}}{2}\right). \tag{3.9}$$

Combining (3.8) and $\psi_{e,r}$ gives

$$\begin{aligned} Q_\omega^*(\mathscr{H}_{\mathbf{x}}^r(f)) = {} & \sum_{j=0}^{r-1}\left(\sum_{i=0}^{n-1}\frac{f^{(j)}(x_{i+1})}{j!}\int_{x_i}^{x_{i+1}}(1-t^2)^{-1/2}(t-x_{i+1})^j\,\mathrm{d}t\right.\\ & \left.+\frac{f^{(j)}(x_n)}{j!}\int_{x_n}^{1}(1-t^2)^{-1/2}(t-x_n)^j\,\mathrm{d}t\right)-\frac{K}{r!}\sum_{i=1}^{n-1}\sum_{v=1}^{r}(-1)^v\\ & \times\left(\int_{x_i}^{x_i+\Delta x_i\xi_{+,i,v}}(1-t^2)^{-1/2}(t-x_i-\Delta x_i\xi_{+,i,v})^r\,\mathrm{d}t\right.\\ & \left.-\int_{x_i}^{x_i+\Delta x_i\xi_{-,i,v}}(1-t^2)^{-1/2}(t-x_i-\Delta x_i\xi_{-,i,v})^r\,\mathrm{d}t\right). \end{aligned}$$

Let

$$\int_a^b(1-t^2)^{-1/2}(t-\alpha)^j\,\mathrm{d}t:=T_j(\alpha,a,b), \tag{3.10}$$

we can immediately derive the conclusion as desired in (3.5). Applying (3.9), together with (3.10) and $\psi_{e,r}$ in Theorem 3.1, we prove the remainder of the theorem. □

The value of $T_j(\alpha,a,b)$ can be calculated by a recurrence formula. Using the following integral relation

$$\int(1-t^2)^{-1/2}(t-\alpha)^j\,\mathrm{d}t=-\alpha\int(1-t^2)^{-1/2}(t-\alpha)^{j-1}\,\mathrm{d}t-\int(t-\alpha)^{j-1}\,\mathrm{d}\sqrt{1-t^2},$$

and integrating by parts for the last one on its right-hand side, yields the recurrence relations

$$T_0(\alpha,a,b)=\arcsin b-\arcsin a,$$

$$T_1(\alpha,a,b)=\sqrt{1-a^2}-\sqrt{1-b^2}-\alpha T_0(\alpha,a,b),$$

$$\begin{aligned} T_j(\alpha,a,b)= {} & \frac{1}{j}\left(\sqrt{1-a^2}(a-\alpha)^{j-1}-\sqrt{1-b^2}(b-\alpha)^{j-1}\right)\\ & -\frac{2j-1}{j}\alpha T_{j-1}(\alpha,a,b)+\frac{j-1}{j}(1-\alpha^2)T_{j-2}(\alpha,a,b),\quad j\geqslant 2. \end{aligned}$$

Theorem 3.4. *Let* $\omega(t)=\sqrt{1-t^2}$, *then the best quadrature formula based on the given Hermite information* $\mathscr{H}_{\mathbf{x}}^r(f) \in \mathscr{H}_{\mathbf{x}}^r(KW^r[-1,1])$ *for* $I_\omega(f)$ *is*

$$Q_\omega^*(\mathscr{H}_{\mathbf{x}}^r(f)) = \sum_{j=0}^{r-1} \frac{1}{j!}\left(\sum_{i=0}^{n-1} f^{(j)}(x_{i+1})S_j(x_{i+1}, x_i, x_{i+1}) + f^{(j)}(x_n)S_j(x_n, x_n, 1)\right)$$
$$- \frac{K}{r!}\sum_{i=1}^{n-1}\sum_{v=1}^{r}(-1)^v(S_r(x_i + \Delta x_i\xi_{+,i,v}, x_i, x_i + \Delta x_i\xi_{+,i,v})$$
$$- S_r(x_i + \Delta x_i\xi_{-,i,v}, x_i, x_i + \Delta x_i\xi_{-,i,v})),$$

and its radius of the Hermite information is

$$R_\omega(\mathscr{H}_{\mathbf{x}}^r(f)) = \frac{K}{r!}\left((-1)^r\sum_{i=0}^{n-1} S_r(x_{i+1}, x_i, x_{i+1}) + S_r(x_n, x_n, 1)\right)$$
$$- \frac{K}{r!}\sum_{i=1}^{n-1}\sum_{v=1}^{r}(-1)^v(S_r(x_i + \Delta x_i\xi_{+,i,v}, x_i, x_i + \Delta x_i\xi_{+,i,v})$$
$$+ S_r(x_i + \Delta x_i\xi_{-,i,v}, x_i, x_i + \Delta x_i\xi_{-,i,v})),$$

where

$$S_j(\alpha, a, b) = \int_a^b \sqrt{1-t^2}(t-\alpha)^j \, \mathrm{d}t,$$

and $\xi_{e,i,v}, i = 1, \ldots, n-1, v = 1, \ldots, r$, *satisfy* (3.2).

Proof. The proof is similar to Theorem 3.3. We omit the trivial details. □

For $S_j(\alpha, a, b)$, we give the recurrence relations

$$S_0(\alpha, a, b) = \tfrac{1}{2}(b\sqrt{1-b^2} - a\sqrt{1-a^2} + \arcsin b - \arcsin a),$$

$$S_1(\alpha, a, b) = \tfrac{1}{3}((1-a^2)^{3/2} - (1-b^2)^{3/2}) - \alpha S_0(\alpha, a, b),$$

$$S_j(\alpha, a, b) = \frac{1}{j+2}((1-a^2)^{3/2}(a-\alpha)^{j-1} - (1-b^2)^{3/2}(b-\alpha)^{j-1})$$
$$- \frac{2j+1}{j+2}\alpha S_{j-1}(\alpha, a, b) + \frac{j-1}{j+2}(1-\alpha^2)S_{j-2}(\alpha, a, b), \quad j \geqslant 2.$$

4. Numerical results

In this section, we compare the best quadrature proposed in Section 3 with the Gauss–Tur*á*n quadrature formula using stochastic experiments. Suppose $\omega(t) = (1-t^2)^{-1/2}$ is the Chebyshev weight function of the first kind and the function class is on $KW^4[-1,1]$. The corresponding Gauss–Tur*á*n quadrature formula ($s=1$) is in the following form:

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = \frac{\pi}{n}\left(\sum_{i=1}^{n} f(\xi_i) + \frac{1}{4^n} f'[\xi_1^2, \ldots, \xi_n^2]\right),$$

Table 1
Stochastic experiment results for the best quadrature and Gauss–Turán quadrature

k	N	n	Error$^{\mathrm{GT}}$	Error*	$R_\omega(\mathscr{H}^4_{\mathbf{x}}(f))$
1	12	5	2.31×10^{-6}	2.48×10^{-6}	1.87×10^{-5}
2	18	7	3.80×10^{-7}	2.29×10^{-7}	5.23×10^{-6}
3	22	8	2.30×10^{-7}	1.46×10^{-7}	1.66×10^{-6}
4	25	10	5.80×10^{-8}	7.68×10^{-8}	1.30×10^{-6}
5	30	12	2.36×10^{-8}	3.89×10^{-8}	5.23×10^{-7}
6	38	16	3.41×10^{-9}	3.58×10^{-9}	2.27×10^{-7}
7	45	19	2.82×10^{-9}	1.85×10^{-9}	8.63×10^{-8}
8	50	20	3.61×10^{-10}	8.06×10^{-10}	7.16×10^{-8}

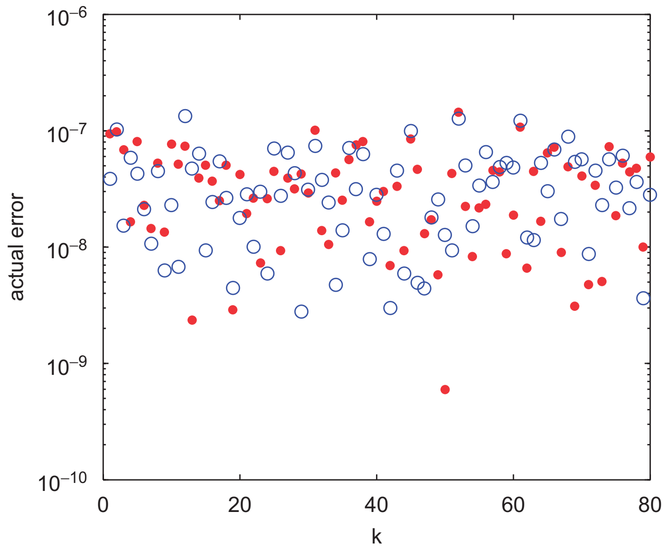

Fig. 1. The case $N=30$ and $n=12$, Error$^{\mathrm{GT}}$ (circles) and Error* (dots) from 80 stochastic experiments.

which is equivalent to

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t=\frac{\pi}{n}\sum_{i=1}^{n}\left(f(\xi_i)-\frac{1-\xi_i^2}{2n^2}\sum_{\substack{k=1\\k\neq i}}^{n}\frac{1}{\xi_i-\xi_k}f'(\xi_i)+\frac{1-\xi_i^2}{4n^2}f''(\xi_i)\right), \tag{4.1}$$

or [4]

$$\int_{-1}^{1}\frac{f(t)}{\sqrt{1-t^2}}\,\mathrm{d}t=\frac{\pi}{n}\sum_{i=1}^{n}\left(f(\xi_i)-\frac{\xi_i}{4n^2}f'(\xi_i)+\frac{1-\xi_i^2}{4n^2}f''(\xi_i)\right).$$

Function f_k (k is the serial number) used in stochastic experiments is randomly chosen from the function class $1W^4[-1,1]$ $(K=1)$ and given in the following:

$$f_k(t)=\sum_{i=1}^{4}A_it^{i-1}+\frac{1}{24}A_5(t_1-t)_-^4+\sum_{i=2}^{N}\frac{A_{i+4}-A_{i+3}}{24}(t_i-t)_-^4,$$

where A_i $(i=1,2,\ldots,N+4)$ are independent random numbers uniformly distributed on $[-1,1]$ and $t_1\leqslant t_2\leqslant\cdots\leqslant t_N$ are reset of random numbers uniformly distributed on $[-1,1]$. The number of quadrature points n is chosen from $N/3$ to $N/2$.

Table 2
Stochastic experiment results for the best quadrature and Gauss–Turán quadrature

k	N	n	Error^{GT}	Error^{*}	$R_\omega(\mathscr{H}^4_{\mathbf{x}}(f))$
1	12	5	8.76×10^{-3}	1.77×10^{-4}	1.41×10^{-3}
2	18	7	7.15×10^{-4}	3.60×10^{-5}	2.53×10^{-4}
3	22	8	4.82×10^{-4}	3.59×10^{-5}	2.10×10^{-4}
4	25	10	1.19×10^{-4}	1.01×10^{-5}	6.66×10^{-5}
5	30	12	6.72×10^{-5}	7.54×10^{-6}	4.08×10^{-5}
6	38	16	1.60×10^{-5}	6.97×10^{-7}	2.65×10^{-6}
7	45	19	8.06×10^{-6}	1.87×10^{-7}	2.74×10^{-6}
8	50	20	1.89×10^{-6}	9.80×10^{-8}	9.47×10^{-7}

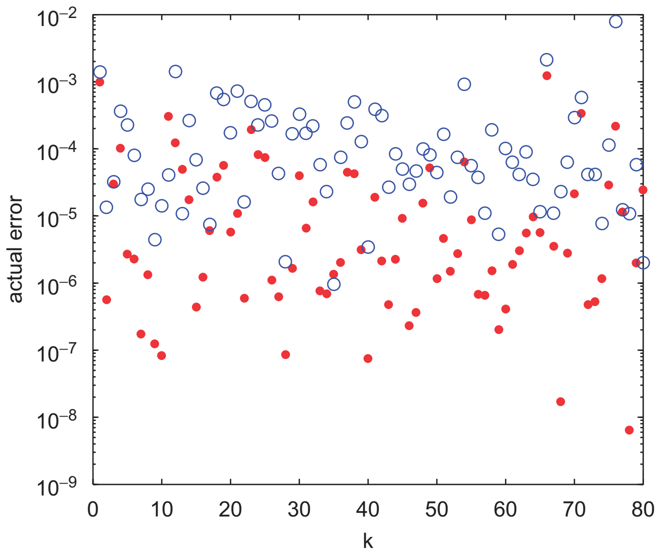

Fig. 2. The case $N = 30$ and $n = 12$, Error^{GT} (circles) and Error^{*} (dots) from 80 stochastic experiments.

As mentioned in Section 1, the given information $\mathscr{H}^4_{\mathbf{x}}(f)$ may be obtained from scientific and engineering computing practices and are thus expensively priced. In that case, it is difficult to obtain the information of f at any other nodes except $\mathbf{x}$. Especially when the quadrature nodes $x_1, \ldots, x_n$ do not coincide with Gaussian nodes $\xi_1, \ldots, \xi_n$, (4.1) is not applicable directly. Therefore, the nodes $x_1, \ldots, x_n$ in our numerical experiments are chosen in the following two ways, which are discussed in Examples 4.1 and 4.2. Note that in the tables, Error^{GT} and Error^{*} denote the actual error (|Approximation value − Exact value|) of the Gauss–Turán quadrature and the best quadrature, respectively. $R_\omega(\mathscr{H}^4_{\mathbf{x}}(f))$ is the theoretic error or the worst error of the best quadrature based on the given information $\mathscr{H}^4_{\mathbf{x}}(f)$.

Example 4.1. In this example, we suppose that $x_1, x_2, \ldots, x_n$ are chosen as the reset of $\xi_1, \xi_2, \ldots, \xi_n$, i.e., $x_i = \cos(\pi - ((2i-1)/2n)\pi)$, $i = 1, 2, \ldots, n$. In this way, the approximation value of the Gauss–Turán quadrature formula is obtained from (4.1) directly, since the information of f at the set of Gaussian nodes is given. We list some of the results in Table 1. Fig. 1 shows the values of Error^{GT} and Error^{*} from 80 stochastic experiments in the case of $N = 30$ and $n = 12$.

Example 4.2. In this example, we suppose that the nodes $x_1, x_2, \ldots, x_n$ of the quadrature formula are chosen as the reset of random numbers uniformly distributed on $[-1, 1]$. On the assumption that we only know the information of f at $x_1, x_2, \ldots, x_n$, whereas using the Gauss–Turán quadrature we need its information at Gaussian nodes $\xi_i = \cos((2i -$

$1)\pi/2n)$, $i = 1, \ldots, n$. We may choose in several ways to obtain its information. Here, we use piecewise cubic spline interpolation which interpolates the function f at the points $x_1, x_2, \ldots, x_n$. We give some results in Table 2 and Fig. 2 shows the values of Error^{GT} and Error^* from 80 stochastic experiments in the case of $N = 30$ and $n = 12$.

Our stochastic experiments show that for a single function f the best quadrature is always the same as the Gauss–Turán quadrature, when we choose the nodes $x_1, x_2, \ldots, x_n$ as the Gaussian nodes, see Table 1 and Fig. 1. However, when choosing the nodes $x_1, x_2, \ldots, x_n$ of the quadrature formula as the random numbers uniformly distributed on $[-1, 1]$, the best quadrature always performs better than the Gauss–Turán quadrature, see Table 2 and Fig. 2.

All computations have been performed using a 16-digit arithmetic.

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