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A RUNGE-KUTTA FEHLBERG METHOD WITH PHASE-LAG OF ORDER INFINITY FOR INITIAL-VALUE PROBLEMS WITH OSCILLATING SOLUTION

T. E. SIMOS
Informatics Laboratory, Agricultural University of Athens
Iera Odos 75, Athens 118 55, Greece

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Abstract—A Runge-Kutta method is developed for the numerical solution of initial-value problems with oscillating solution. Based on the Runge-Kutta Fehlberg 2(3) method, a Runge-Kutta method with phase-lag of order infinity is developed. Based on these methods we produce a new embedded Runge-Kutta Fehlberg 2(3) method with phase-lag of order infinity. This method is called as Runge-Kutta Fehlberg Phase Fitted method (RKFPF). The numerical results indicate that this new method is much more efficient, compared with other well-known Runge-Kutta methods, for the numerical solution of differential equations with oscillating solution, using variable step size.

1. INTRODUCTION

We consider the numerical solution of systems of ODEs of the form

$$y' = f(x, y) \tag{1.1}$$

with oscillating solution.

We note that, over the last few years, Runge-Kutta methods [1–4] have been developed for the numerical solution of (1.1), characterized by *the phase-lag property* introduced by Brusa and Nigro [5]. To these methods should be also added *multistep with minimal phase-lag methods,* dealing with the numerical integration of the initial value problem:

$$y'' = f(x, y). \tag{1.2}$$

We are referred to the works [5–8], as well as to the works [9–16]. All these papers deal with methods of various orders and recently works have been published [12,14,16] dealing with the treatment of the same problem but of orders up to *infinity.*

In the present paper, a Runge-Kutta Fehlberg 2(3) method with phase-lag of order infinity is developed. So, a new embedded Runge-Kutta Fehlberg method 2(3) is obtained. It must be noted that for the new method, we must know the frequency of a problem or an approximation of this.

2. THE NEW METHODS

2.1. Formulation of the Method

Conventional Runge-Kutta methods use small step size for the integration of equations describing free oscillations, in order to obtain accurate approximations along the intervals used. In our case, the proposed method is suitable for a long interval integration step, not only when the oscillating equation is subject to *free oscillations of high frequency,* but also in the case of *forced*

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oscillations of low frequency. In general, these methods are much more accurate, compared with other classical ones, used in problems with oscillating solution.

To develop the new method we use the test equation,

$$\frac{dy}{dx} = i\,v\,y, \qquad v \text{ real}. \tag{2.1}$$

Due to the reasons described in [1], we shall confine our considerations to homogeneous phase-lag, and based on its definition given by van der Houwen and Sommeijer [1], we shall use a test equation with an exact solution of the form e^{ivx}. However, as it is shown by the numerical results of Section 6, inhomogeneous problems can be successfully dealt with by increasing the order of homogeneous phase-lag. By comparing the exact and the numerical solution for this equation, and by requiring that these solutions are in phase with maximal order in the step-size h, we derive the so-called *dispersion relation*.

For first-order equations we write the m-stage explicit Runge-Kutta method in the matrix form:

$$\begin{array}{c|ccccc}
0 & 0 & & & & \\
a_1 & b_{10} & & & & \\
a_2 & b_{20} & b_{21} & & & \\
\vdots & \vdots & \vdots & \ddots & & \\
a_m & b_{m0} & b_{m1} & \dots & \dots & b_{mm-1} \\
\hline
 & c_0 & c_1 & \dots & c_{m-2} & \\
 & cc_0 & cc_1 & \dots & \dots & cc_{m-1}
\end{array} \quad . \tag{2.2}$$

Application of (2.2) to (2.1) yields the numerical solution:

$$y = a_*^n\,y \quad \text{and} \quad a_* = A_m(H^2) + i\,H\,B_m(H^2), \quad H = v\,h, \tag{2.3}$$

where A_m and B_m are polynomials in H^2, completely defined by the Runge-Kutta parameters a_i, b_{ij}, and c_i, $i = 1,\dots,3$, $j = 1,\dots,i-1$. The amplification factor is $a_* = a_*(H)$, and y_n denotes the approximation to $y(x_n)$.

A comparison of (2.3) with the solution of (2.1), i.e., $y(x_n) = y_0 \exp(inH)$, leads to the definition of the *dispersion or phase error or phase-lag, and amplification error* [1].

$$t(H) = H - \arg[a_*(H)], \qquad a(H) = 1 - |a_*(H)|. \tag{2.4}$$

If $t(H) = O(H^{r+1})$ and $a(H) = O(H^{s+1})$, then the method is said to be *phase-lag order r and dissipative order s*.

From (2.4), it follows that

$$a(H) = 1 - \sqrt{[{A_m}^2(H^2) + H^2\,{B_m}^2(H^2)]} \tag{2.5}$$

and, hence,

$$\tan(H) - H\left[\frac{B_m(H^2)}{A_m(H^2)}\right] = cH^{r+1}. \tag{2.6}$$

It is obvious that to have phase-lag of order infinity it must be:

$$\frac{\tan(H)}{H} = \left[\frac{B_m(H^2)}{A_m(H^2)}\right]. \tag{2.7}$$

DEFINITION 1. *The interval in which $|a_*| < 1$ is called the interval of imaginary stability.*

3. DERIVATION OF THE APPROXIMATING SCHEMES

Noting that in Fehlberg's method [17], the assumptions for the coefficients of (2.2) are $b_{30} = c_0$, $b_{31} = c_1$, $b_{32} = c_2$, $a_3 = 1$, we derive the coefficients a_i and b_{ij} as it can be seen in the Appendix.

If we apply the method (2.2) with coefficients, a_i, b_{ij}, and c_i, $i = 0, 1, 2$ and $j = 0, 1$, given in the Appendix to the test equation (2.1), we have (2.3) with $m = 2$, where

$$A_2(H^2) = 1 - \frac{H^2}{2} \quad \text{and} \quad B_2(H^2) = 1 - \frac{[a_2\,(3a_1 - 4)]\,H^2}{12\,(a_1 - 2a_2)}. \tag{3.1}$$

From (2.7), we have that to have phase-lag of order infinity it must be:

$$a_1 = -\frac{8a_2\,[H\,(H^2+3)\cos(H) - 3\sin(H)]}{3\,\{H[a_2\,H^2 - 2\,(H^2+2)]\cos(H) + 4\sin(H)\}}. \tag{3.2}$$

From (2.5), it follows that:

$$a(H) = \frac{3a_1\,(2a_2 - 1) - 2a_2}{12\,(a_1 - 2a_2)}\,H_4 + O(H^6). \tag{3.3}$$

Also, from Definition 1, it follows that the *interval of imaginary stability is*

$$(0, \infty) - \{0.514480549119086\}.$$

So, we have the next theorem.

THEOREM 1. *The method (2.2), with the coefficients given in the Appendix, is second order algebraic, has phase-lag of order infinity for the a_1 given in (3.2), is dissipative order four, and has an interval of imaginary stability equal to* $(0, \infty) - \{0.514480549119086\}$.

If we apply the method (2.2), with the coefficients a_i, b_{ij}, and cc_i, $i = 0, 1, 2, 3$ and $j = 0, 1, \ldots, i-1$, given in the Appendix, to the test equation (2.1), we have (2.3) with $m = 3$ where

$$A_3(H^2) = 1 - \frac{H^2}{2} + \frac{a_2\,(3a_2 - 2)\,(3a_1 - 4)\,H^4}{72\,(a_1 - 2a_2)\,(a_2 - 1)} \quad \text{and} \quad B_3(H^2) = 1 - \frac{H^2}{6}. \tag{3.4}$$

From (2.7), we have that to have phase-lag of order infinity it must be:

$$a_1 = -\frac{4a_2\,\{H\,[3a_2\,W_1(H) - 4W_2(H)]\,\cos(H) - 36\,(a_2 - 1)\,\sin(H)\}}{3\,\{H\,[3{a_2}^2\,H^4 - 2a_2\,W_3(H) + 4W_4(H)]\,\cos(H) + 24\,(a_2 - 1)\,\sin(H)\}},$$

where (3.5)

$$W_1(H) = H^4 + 4H^2 + 12, \qquad W_3(H) = 3H^4 + 4H^2 + 12,$$
$$W_2(H) = H^4 + 3H^2 + 9, \qquad W_4(H) = H^4 + 2H^2 + 6.$$

From (2.5), it follows that:

$$a(H) = \frac{3a_1\,(3{a_2}^2 - 3a_2 + 1) - 2a_2(3a_2 - 1)}{36\,(a_1 - 2a_2)\,(a_2 - 1)}\,H^4 + O(H^6). \tag{3.6}$$

Also from Definition 1, it follows that the *interval of imaginary stability is*

$$(0, \infty) - \{1.26962171393343\}.$$

So, we have the next theorem.

THEOREM 2. *The method (2.2) with the coefficients given in the Appendix is third order algebraic, has phase-lag of order infinity for the a_1 given in (3.6), is dissipative order four and has an interval of imaginary stability equal to* $(0, \infty) - \{1.26962171393343\}$.

4. NUMERICAL ILLUSTRATIONS—DISCUSSION

There are a lot of problems in applied sciences and engineering which are expressed by differential equations with oscillating solutions.

The method described in this paper is used for the solution of the problem (1.1), for problems of higher order which can be analyzed on a set of first order equations, and for inhomogeneous problems.

The new method can be used for cases where we want to use a variable step procedure. So, we consider the second scheme *with order three* to estimate the *local truncation error* of the scheme *with order two.* The calculations in the n^{th} step are following a *truncation error estimate, TEC,* given by TEC $= |{y_n}^{3rd} - {y_n}^{2nd}|$ where 3rd or 2nd is the order of the scheme, respectively, and our variable step procedure is the following

$$
\begin{aligned}
&\text{If TEC} \le \text{TOL}, && \text{then } h_n = 2h_{n-1},\\
&\text{If TOL} < \text{TEC} \le 10\ \text{TOL}, && \text{then } h_n = h_n,\\
&\text{If TEC} > 10\ \text{TOL}, && \text{then } h_n = \frac{1}{2}h_{n-1}.
\end{aligned}
$$

4.1. A Model Problem

Consider the equation

$$\mathbf{y}' = \begin{bmatrix} 0 & \varphi \\ -\varphi & 0 \end{bmatrix} \mathbf{y}, \tag{4.1}$$

with initial condition $\mathbf{y}(0) = (1,\ 0)^{\mathsf{T}}$. The exact solution is given by

$$\mathbf{y} = \begin{bmatrix} \cos(\varphi x) \\ -\sin(\varphi x) \end{bmatrix}. \tag{4.2}$$

In Table 1, we present the maximum absolute errors in $\mathbf{y}(1)$, in units of 10^{-6}, for the integration interval $[0, x_{\text{end}}]$, where $x_{\text{end}} = 100$, for $\varphi = 5$ and for various TOL's and stepsizes h, using our new modified Fehlberg method 2(3), the Fehlberg method 2(3), and the Fehlberg 4(5).

Table 1. Maximum absolute error in the range $[0, 100]$ for Problem (4.1) in units of 10^{-6}. h_0 = initial stepsize = 0.001.

TOL	Maximum stepsize Runge-Kutta method			Maximum absolute error Runge-Kutta method		
	New	Fehlberg 2(3)	Fehlberg 4(5)	New	Fehlberg 2(3)	Fehlberg 4(5)
10^{-2}	0.512	0.128	0.256	90651	100215	83123
10^{-3}	0.512	0.128	0.256	7624	18123	6921
10^{-4}	0.256	0.064	0.128	3352	10105	3153
10^{-5}	0.128	0.032	0.064	153	1120	148
10^{-6}	0.128	0.032	0.064	6	124	6

4.2. Inhomogeneous Equation

Consider the equation

$$y'' = -\varphi^2\, y + (\varphi^2 - 1)\, \sin(t), \qquad t \ge 0 \tag{4.3}$$

with exact solution,

$$y(x) = \cos(\varphi x) + \sin(\varphi x) + \sin(x), \qquad \varphi \gg 1. \tag{4.4}$$

The exact solution of this problem consists of a rapidly and a slowly oscillating function; the slowly varying function is due to the inhomogeneous term. The high-order phase-lag takes care of the rapidly oscillating function, and the algebraic order takes care of the slowly varying component. In Table 2, we present the values of the maximum absolute errors in y, in 10^{-4}, for various TOL's and stepsizes, using the same methods as in Problem (4.1). We call x_{end} the end point of integration.

Table 2. Maximum absolute error in the range $[0, 100]$ for Problem (4.3) in units of 10^{-6}. h_0 = initial stepsize = 0.001.

TOL	Maximum stepsize Runge-Kutta method			Maximum absolute error Runge-Kutta method		
	New	Fehlberg 2(3)	Fehlberg 4(5)	New	Fehlberg 2(3)	Fehlberg 4(5)
10^{-2}	0.512	0.128	0.256	120324	130123	100815
10^{-3}	0.256	0.064	0.128	6421	12218	5913
10^{-4}	0.256	0.064	0.128	2231	8423	1921
10^{-5}	0.128	0.032	0.064	208	1015	190
10^{-6}	0.128	0.032	0.064	20	108	18

4.3. A Nonlinear Problem

We consider the non-linear problem:

$$y'' + 100y = \sin y, \qquad y(0) = 0, \quad y'(0) = 1. \tag{4.5}$$

We solved Problem (4.5) using the same methods as in Problems (4.1) and (4.3). In Table 3, we present the values of the maximum absolute errors in $y(4\pi) = -0.059137849898$, in 10^{-4}, for various TOL's and stepsizes, using the same methods as in Problems (4.1) and (4.3).

Table 3. Maximum absolute error in the range $[0, 4\pi]$ for Problem (4.5) in units of 10^{-4}. h_0 = initial stepsize = 0.001.

TOL	Maximum stepsize Runge-Kutta method			Maximum absolute error Runge-Kutta method		
	New	Fehlberg 2(3)	Fehlberg 4(5)	New	Fehlberg 2(3)	Fehlberg 4(5)
10^{-2}	2.048	0.512	1.024	180324	220123	175437
10^{-3}	2.048	0.256	0.512	70567	100125	65789
10^{-4}	1.024	0.128	0.512	15459	45237	10567
10^{-5}	1.024	0.128	0.256	1356	5678	1134
10^{-6}	0.512	0.128	0.256	231	1167	209
10^{-7}	0.512	0.128	0.256	3	94	3
10^{-8}	0.512	0.128	0.256	0	156	0

4.4. Hyperbolic Equation

$$\begin{aligned} \frac{\partial u}{\partial t} = -\frac{\partial u}{\partial x} \qquad & 0 \le x \le 1, \quad t \ge 0, \\ u(t,x) = 0, \quad & u(0,x) = \sin(\pi^2 x^2). \end{aligned} \tag{4.6}$$

Discretization of $\frac{\partial}{\partial x}$ by symmetric differences at internal grid points, and one-sided differences at the boundary point $x = 1$ yields the system

$$\frac{d\mathbf{y}}{dt} = \frac{1}{2}\Delta x \begin{bmatrix} 0 & -1 & & & & \\ 1 & 0 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 0 & -1 \\ & & & -1 & 4 & -3 \end{bmatrix} \mathbf{y}. \tag{4.7}$$

In order to test the capability of the various methods mentioned above and to stay in phase with the exact solution, we have concentrated on approximating the zeros of the solution y. By choosing $\Delta x = 1/50$, we found that the 20^{th} component of the exact solution vector $\mathbf{y}$ reaches its 500^{th} zero at the point

$$S_{500} = 33.509996948\ldots. \tag{4.8}$$

Its numerical approximation, s_{500}, was obtained by integrating with fixed step size and by applying a cubic spline interpolation based on 10 neighboring step points $t_n = t_0 + n\,h$, where h is the step size in the experiment under consideration. The accuracy of this approximation, relative to the distribution of the successive zeros on the t-axis, was measured by the value:

$$\text{RELERR} = \left| \frac{S_{500} - s_{500}}{S_{501} - S_{500}} \right|, \tag{4.9}$$

where S_{501} denotes the 501^{st} zero of the solution $y^{(20)}$.

In Table 4, we present the RELERR values by the same methods of the previous examples. In this example, we use fixed step size. The step sizes shown in the second column were chosen such that all methods require the *same number of right-hand side evaluations.* Because all the methods are embedded, for this example, we use the high order method of the embedded scheme (for example, we use the Fehlberg third order formula for the scheme Fehlberg 2(3)).

Table 4. Relative errors for Problem (4.6).

h	Runge-Kutta method		
	New	Fehlberg 2(3)	Fehlberg 4(5)
1/90	5.06×10^{-4}	5.0×10^{-1}	
1/60			4.7×10^{-2}
1/180	6.1×10^{-6}	4.1×10^{-2}	
1/120			8.5×10^{-4}
1/360	5.3×10^{-8}	3.5×10^{-3}	
1/240			9.3×10^{-6}

5. CONCLUSION

A modified embedded Runge-Kutta Fehlberg 2(3) method is developed in this paper. These methods have algebraic order two and three, respectively, but phase-lag of order infinity, in contrast with classical Runge-Kutta Fehlberg 2(3) methods, which have phase-lag order two and four, respectively. From the numerical results presented in Section 4, it can be seen that the new embedded method is much more efficient than the classical Fehlberg 2(3) method *and* the classical Fehlberg 4(5) method.

All computations were carried out on a PC 386, with a 387 numeric coprocessor in double precision arithmetic with 14 digits accuracy.

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APPENDIX
THE COEFFICIENTS OF THE EMBEDDED METHOD

i		a_i
1		$S_3(H)$ for second order method and $S_5(H)$ for third order method
2		Arbitrary constant
3		1
$i=1$	$j=0$	a_1
$i=2$	$j=0$	$-\frac{a_2}{2a_1}(a_2-2a_1)$
	$j=1$	$\frac{a_2^2}{2a_1}$
$i=3$	$j=0$	c_0
	$j=1$	c_1
	$j=2$	c_2

The coefficients c_i and cc_i are given below.

i	c_i
0	$\frac{S_1}{6a_1 a_2 S_2}$
1	$\frac{2-3a_2}{3a_1 S_2}$
2	$\frac{3a_1-4}{6a_1 S_2}$
i	cc_i
0	$\frac{3a_2-1}{6a_2}$
1	0
2	$\frac{1}{6a_2(1-a_2)}$
3	$\frac{2-3a_2}{6(1-a_2)}$

Here,

$$S_1 = 3a_1^2(2a_2-1) - 4a_1(3a_2^2-1) + 2a_2(3a_2-2),$$

$$S_2 = a_1 - 2a_2,$$

$$S_3(H) = -\frac{8a_2[H(H^2+3)\cos(H) - 3\sin(H)]}{3\{H[a_2 H^2 - 2(H^2+2)]\cos(H) + 4\sin(H)\}},$$

$$S_5(H) = -\frac{S_6(H)}{S_7(H)}, \text{ where}$$

$$S_6(H) = 4a_2\{H[3a_2(H^4+4H^2+12) - 4(H^4+3H^2+9)]\cos(H) - 36(a_2-1)\sin(H)\}$$

$$S_7(H) = 3\{H[3a_2^2 H^4 - 2a_2(3H^4+4H^2+12) + 4(H^4+2H^2+6)]\cos(H) + 24(a_2-1)\sin(H)\}.$$