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ORIGINAL ARTICLE

Delayed system control in presence of actuator saturation

A. Mahjoub [a,*], F. Giri [b], N. Derbel [a]

[a] CEM Laboratory, National School of Engineers of Sfax, 3038 Sfax, Tunisia
[b] GREYC Lab, UMR CNRS, University of Caen, Caen 14000, France

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Abstract The paper is introducing a new design method for systems' controllers with input delay and actuator saturations and focuses on how to force the system output to track a reference input not necessarily saturation-compatible. We propose a new norm based on the way we quantify tracking performance as a function of saturation errors found using the same norm. The newly defined norm is related to signal average power making possible to account for most common reference signals e.g. step, periodic. It is formally shown that, whatever the reference shape and amplitude, the achievable tracking quality is determined by a well defined reference tracking mismatch error. This latter depends on the reference rate and its compatibility with the actuator saturation constraint. In fact, asymptotic output-reference tracking is achieved in the presence of constraint-compatible step-like references.

1. Introduction

Controlling linear systems with input saturation has been much studied especially over the last two decades, see e.g. [1] and references list therein. The solutions proposed so far have been developed following two main paths, called, respectively, anti-windup compensator (AWC) synthesis and direct control design (DCD). The first approach consists of designing a controller that ensures satisfactory control performances in the absence of actuator saturation case. Then, a static or dynamic compensator is designed to minimize the effect of actuator saturation on the closed-loop performances. In the DCD method, the input constraint is taken into account at the controller design phase. In addition to actuator saturation, physical systems are also subject to (less or more significant) dead-times [2].

The conjunction of these two ubiquitous factors, if it is not appropriately accounted for in the control design stage, may cause drastic deterioration of control performances. The point is that relatively few works have dealt with the problem of controlling delayed systems with saturating actuators. In this respect, a number of global bounded stabilization [3–7] and asymptotic local stabilization results [8–10] have been reached using direct control designs. Stabilization results have also been achieved using anti-windup based designs [11,12]. For instance, in [12] the AWC is designed to ensure L_2-stability

* Corresponding author. Tel.: +216 22190185.
E-mail address: mahjoub_adel@yahoo.fr (A. Mahjoub).
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of the operator relating the control saturation error $\tilde{u} = u - sat(u)$ (difference between the control input signal generated with and without input saturation) to $\tilde{z} = z - \bar{z}$ (error between the control system performance outputs with and without input saturation).

This L_2-stability result is certainly interesting due to its generality. But, the class of admissible inputs (references, disturbances) for which the condition $\tilde{u} \in L_2$ holds is not explicitly defined. Furthermore, it is not clear how the constrained closed-loop system behaves when the inputs are not constraint-compatible so that $\tilde{u} \notin L_2$.

In this note, the focus is made on global asymptotic tracking of arbitrary-shape reference signals for systems with constrained input with delay. We will show that asymptotic reference tracking is achievable for constraint-compatible step-like reference signals. To this end, a saturating controller is developed within the ring of pseudo-polynomial using the (finite-spectrum) pole-placement technique [13,14].

The point is that, in practical situations, reference compatibility may be difficult to check or even lost due to model uncertainties. Then, it is of practical interest to analyze the tracking capability of the proposed controller facing constraint-incompatible reference signals of arbitrary shape. This issue has never been investigated in the context of input-constrained dead-time systems. It is presently dealt with, for the considered class of saturated controllers, making use of available input-output L_2-stability tools [15]. The novelty is that the obtained tracking performance is assessed using a new, more suitable, norm representing signal average power (rather than energy). The new norm induces a normed space, denoted L_{2a}, that contains all bounded signals (while the energy-related L_2-space does not). Then, L_{2a} turns out to be a quite suitable framework to address the tracking issue in the presence of arbitrary-shape (possibly not constraint-compatible) references. Making use of the L_{2a}-norm, it is formally shown that the proposed saturated controller features quite interesting output-reference matching properties. Accordingly, the tracking accuracy is related to a reference tracking mismatch error, depending on reference rate and its constraint-compatibility error. The smaller the tracking mismatch error, the better the average tracking quality. This holds independently of the input shape and amplitude.

The present paper is an improved and more complete version of the conference paper [17]. It is organized as follows: Section 2 is devoted to formulating the control problem; the controller is designed in Section 3 and analyzed in Section 4; the corresponding tracking performances are illustrated by simulation in Section 5.

2. Control problem statement

We are interested in controlling input-delayed linear systems of the form:

$$A(s)\hat{y}(s) = B(s)e^{-s\tau}\hat{u}(s) \tag{1}$$

with:

$$\begin{aligned} A(s) &= s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \\ B(s) &= b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 \end{aligned} \tag{2}$$

in the presence of the input constraint:

$$|u(t)| \leqslant u_M \tag{3}$$

where $\hat{u}(s)$ and $\hat{y}(s)$ are the Laplace transforms of $u(t)$ and $y(t)$, the system input and output (respectively); u_M denotes the maximal allowed amplitude of the control signal; the integer n and the real numbers (a_i, b_i) are the system order and parameters, respectively. It is supposed that $A(s)$ is Hurwitz polynomial and $(sA(s), B(s))$ are coprime. These assumptions guarantee system controllability even in the presence of the input limitation (3). The only assumption on $B(s)$ is that $b_0 \neq 0$ so that the system static gain is nonzero. That is, $B(s)$ may be Hurwitz or not, allowing the system (1) to be nonminimum phase [16]. The aim of the study was to develop a controller that makes the tracking error,

$$e_y = y - y^* \tag{4}$$

as small as possible, whatever the initial conditions, where y^* denotes an arbitrary-shape bounded reference signal. The point is that the system nonminimum phase nature makes perfect tracking (e.g. $e_y \in L_2$) unachievable (even in the unconstrained case) in the presence of arbitrary-shape reference signals. In fact, perfect tracking is only achievable (in the unconstrained case) for reference signals generated by a model of the form $D(s)\hat{y}^*(s) = 0$ with $D(s)$ is any polynomial with simple zeros on the imaginary axis. Then, it is well known that perfect tracking can be achieved by incorporating $D(s)$ in the control law, in accordance with the internal model principle [16]. Presently, we make the common choice $D(s) = s$ which leads to control laws with integral action, allowing for perfect tracking of constant references (as $D(s)\hat{y}^*(s) = 0$ is then equivalent to $\dot{y}^*(t) = 0$).

In turn, the input constraint (3) introduces a structural limitation of the class of references that can be perfectly matched. Specifically, perfect matching cannot be achieved if the reference y^* is not constraint-compatible. In the case of constant references, constraint-compatibility is simply characterized by the condition $|u^*(t)| \leqslant u_M$ with:

$$u^*(t) = \frac{A(0)}{B(0)}y^*(t+\tau) = \frac{a_0}{b_0}y^*(t+\tau)$$

where $A(0)/B(0)$ is nothing other than the inverse of the system static gain. An equivalent formulation of reference constraint-compatibility is that $u^* - sat(u^*) = 0$ where $sat(\cdot)$ denotes the function of saturation defined by:

$$sat(z) = \min(u_M, |z|)\ sgn(z), \quad z \in \boldsymbol{R} \tag{5}$$

Now, the controller we seek must be able to guarantee perfect asymptotic tracking in the presence of constant constraint-compatible references. Otherwise, the tracking quality must depend on how much the reference is deviating from the ideal shape defined by the equations $\dot{y}^*(t) = 0$ and $u^* - sat(u^*) = 0$. The instantaneous deviation is conveniently represented by the output-reference mismatch error $|\dot{y}^*| + |u^* - sat(u^*)|$. The smaller this error is the better must be the tracking quality. This objective is presently formalized requiring that the performance operator,

$$|\dot{y}^*| + |u^* - sat(u^*)| \rightarrow |e_y| \tag{6a}$$

is L_2 stable. Accordingly [e.g. 15], there exists a pair of positive real constants (α, β) such that one has, for all bounded input y^* and any real $T > 0$:

$$\left(\int_0^T |e_y(t)|^2 dt\right)^{1/2} \leqslant \alpha\left(\int_0^T (|\dot{y}^*(t)| + |(u^* - sat(u^*)|)^2 dt\right)^{1/2} + \beta$$
$$\leqslant \alpha\left(\int_0^T |\dot{y}^*(t)|^2 dt\right)^{1/2} + \alpha\left(\int_0^T |(u^* - sat(u^*)|^2 dt\right)^{1/2} + \beta \tag{6b}$$

where the second inequality is obtained using the Schwartz inequality[1] and x_T designates the truncation of a signal x at T.[2] Each of the two statements (6a) and (6b) represents the L_2-tracking performance. In the case of ideal shape references, one has $|\dot{y}^*| + |u^* - sat(u^*)| \in L_2$ and the L_2-performance (6b) yields (by letting $T \to \infty$) $e_y \in L_2$. In the case of arbitrary-shape and/or not constraint-compatible references, $|\dot{y}^*| + |u^* - sat(u^*)| \notin L_2$ and so (6b) cannot be expressed using the L_2-norm. In this situation one gets, dividing both sides of (6b) by $T^{1/2}$ and letting $T \to \infty$:

$$\limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T |e_y(t)|^2 dt\right)^{1/2} \leqslant \alpha \limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T |\dot{y}^*(t)|^2 dt\right)^{1/2} + \alpha \limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T |(u^* - sat(u^*)|^2 dt\right)^{1/2} \tag{6c}$$

Note that the quantity $\limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T |x(t)|^2 dt\right)^{1/2}$ is nothing other than the average power of the signal x: $\boldsymbol{R}^+ \to \boldsymbol{R}$ and this power is finite if the signal is bounded. The above observations motivate the introduction of a new and larger normed space, denoted L_{2a}, described in Definition 1 and Lemma 1.

Definition 1

(1) L_{2a} is the set of all bounded signals x: $R^+ \to R$.
(2) $\|\cdot\|_{a2}$ denotes the mapping: $L_{2a} \to R$, $x \to \|x\|_{2a}$ with $\|x\|_{2a} \overset{def}{=} \limsup_{T\to\infty}\left(\frac{1}{T}\int_0^T |x(t)|^2 dt\right)^{1/2}$.

Lemma 1. *L_{2a} is a space vector over the real space R and the mapping $\|.\|_{a2}$ is a norm in L_{2a}. Accordingly, any element of*

$O_{2a} \overset{def}{=} \{x \in L_{2a} : \|x\|_{2a} = 0\}$ *is a null vector in* L_{2a}

Remark 1

(1) The proof of Lemma 1 is straightforward; it is omitted mainly for space limitation.
(2) In summary, the normed space L_{2a} is the set of all bounded signals with finite average power, in which no distinction is made between two signals x_1 and x_2 such that $x_1 - x_2$ is of null average power.
(3) The space L_{2a} is much broader than the energy-related L_2 space. For instance, the former contains step-like and periodic signals while the latter does not. On the other hand, it is well known that any finite energy signal is necessarily zero average power. It turns out that $L_2 \subset O_{2a}$. The inverse is not true.For instance, the signal $x(t) = 1/\sqrt{t+1}$ ($t \in \boldsymbol{R}^+$) belongs to O_{2a} but not to L_2. More generally, an asymptotically vanishing signal (whatever its convergence rate) is necessarily zero average power but not finite energy.
(4) Compared to the normed space L_∞, the space L_{2a} contains the same elements (i.e. all bounded signals). However, the two spaces are deeply different by the corresponding norms. Indeed, the null element in L_∞ reduces to the sole zero signal (i.e. $x(t) = 0$, $\forall t$) whereas the null element in L_{2a} refers to any signal $x \in O_{2a}$ (i.e. with null mean power). For instance, the signals $x(t) = 1/\sqrt{t+1}$ and $x(t) = \sin(t)/t$ ($t \in R^+$) are elements of O_{2a} but are not zeros in L_∞.
(5) Using the L_{2a} norm, the performance, (6c) writes simply: $\|e_y\|_{2a} \leqslant \alpha(\|u^* - sat(u^*)\|_{2a} + \|\dot{y}^*\|_{2a})$.

The control problem at hand consists in designing, for the system (1) operating under the constraint (3), a controller that guarantees the L_2-stability of the performance operator (6a) so that the L_2 tracking performance (6b) is ensured, with some real α. This performance entails the two following L_2/L_{2a} features:

$$\|e_y\|_2 \leqslant \alpha(\|u^* - sat(u^*)\|_2 + \|\dot{y}^*\|_2) \text{ when } |\dot{y}^*| + |u^* - sat(u^*)| \in L_2 \tag{7a}$$

$$\|e_y\|_{2a} \leqslant \alpha\left(\|u^* - sat(u^*)\|_{2a} + \|\dot{y}^*\|_{2a}\right) \text{ when } |\dot{y}^*| + |u^* - sat(u^*)| \in L_{2a} \tag{7b}$$

3. Controller design

The control design represents an extension to the input-constrained case of the finite-spectrum assignment (FSA) method [13,14]. The FSA approach is itself an extension to time-delay systems case of the standard pole placement design technique [16]. Presently, this design technique is used because it enjoys at least three appealing features. First, it applies to nonminimum phase systems class while other methods (e.g. model reference) do not. Second, despite the system dead-time, the FSA design leads to a closed-loop system with a finite number of poles whose positions are arbitrarily chosen by the designer; this is not necessarily the case with standard methods (Remark 2, Part 3). Finally, the FSA design will prove (see analysis of Section 4) to be useful in perfectly facing the input limitation caused by actuator saturation. The FSA design approach relies on specific notions from the algebra of pseudo-polynomials. A summary of these notions is given in Appendix A, but the reader might consult [13,14] for an exhaustive presentation. Just as in the standard pole assignment method, the first point is an arbitrary choice, by the designer, of a pair of Hurwitz polynomials of the form:

$$C(s) = s^n + c_{n-1}s^{n-1} + \cdots + c_1 s + c_0$$
$$A(s) = s^n + \lambda_{n-1}s^{n-1} + \cdots + \lambda_1 s + \lambda_0 \tag{8}$$

[1] Tus, $\int_0^T (|\dot{y}^*(t)||(u^* - sat(u^*))|)^2 dt \leqslant \left(\int_0^T |\dot{y}^*(t)|^2 dt\right)^{1/2} \times \left(\int_0^T (u^* - sat(u^*))^2 dt\right)^{1/2}$.

[2] The truncation of x at T, designated by x_T, is defined as follows: $x_T(t) = x(t)$ for $0 \leqslant t \leqslant T$ and $x_T(t) = 0$ for $t > T$.

As $sA(s)$ and $B(s)$ are coprime on $\mathbf{R}[s]$ (ring of polynomials with real coefficients), they are also coprime on $\mathbf{E}$, the ring of pseudo-polynomials (Appendix A, Definition A3). Then, there exists a pair of pseudo-polynomials $R'(s)$ and $P'(s)$ satisfying the Bezout equation (Appendix A, Remark A1):

$$R'(s)sA(s) + P'(s)B(s)e^{-s\tau} = C(s)\Lambda(s) \tag{9}$$

Here, one might notice two points: (i) the operators $R'(s)$ and $P'(s)$ are pseudo-polynomials (not standard real polynomials) with unknown degrees and are not unique; (ii) considering in (9) the product $sA(s)$ (rather than just$A(s)$) entails an integral action in the final regulator. Following [13,14], one divides $P'(s)$ by the polynomial $sA(s)$, which is monic, and gets $P'(s) = Q(s)sA(s) + P(s)$ with deg $P(s) \leqslant n$. Then, letting $R(s) = R'(s) + Q(s)B(s)e^{-s\tau}$, (9) rewrites:

$$R(s)sA(s) + P(s)B(s)e^{-s\tau} = C(s)\Lambda(s) \tag{10}$$

As $\deg(P(s)B(s)e^{-s\tau}) \leqslant 2n-1$, it follows that deg $(R(s)sA(s)) = \deg(C(s)\Lambda(s)) = 2n$ which implies that deg $(R(s)) = n-1$, because $\deg(sA(s)) = n+1$. Furthermore, as $sA(s)$ and $C(s)\Lambda(s)$ are monic, $R(s)$ must in turn be monic. In light of the above observations, it is readily seen that $R(s)$ and $P(s)$ are uniquely expressed as follows:

$$\begin{aligned} R(s) &= s^{n-1} + \sum_{i=0}^{n-2} R_i(e^{-s\tau})s^i + R_{-1}(s) \\ P(s) &= \sum_{i=0}^{n} P_i(e^{-s\tau})s^i + P_{-1}(s) \end{aligned} \tag{11}$$

where $R_{-1}(s)$ and $P_{-1}(s)$ belong to $\mathbf{G}$, the set of transfer functions of distributed and punctual delay operators (Appendix A). For $i \geqslant 0$, $R_i(e^{-s\tau})$ and $P_i(e^{-s\tau})$ belong to $\mathbf{R}[e^{-s\tau}]$, the set of polynomials in $e^{-s\tau}$. Unlike the case of non-delayed systems, the (finite-degree) operators $R(s)$ and $P(s)$ are presently pseudo-polynomials and, consequently, are analytical functions of s (Appendix A, Remark A1). Now, let us temporarily suppose that the system (1) and (2) is not subject to the constraint (3). Then, the FSA control method suggests the control law $\hat{u}(s) = -\frac{P(s)}{sR(s)}\hat{e}_y(s)$. Clearly, this corresponds to a regulator featuring an unitary-feedback and integral action. With the above notations, the saturated pole-placement regulator is given the alternative expression:

$\hat{u}(s) = \frac{\Lambda(s)-sR(s)}{\Lambda(s)}\hat{u}(s) - \frac{P(s)}{\Lambda(s)}\hat{e}_y(s)$; where the involved transfer functions are asymptotically stable (because $\Lambda(s)$ is Hurwitz) and causal (Remark 2, Part 1). As the system input is subject to the constraint (3), the above regulator is modified so that it generates a control action not exceeding the constraint limits. Specifically, the following saturated controller is considered and illustrated by the block diagram below (see Fig. 1):

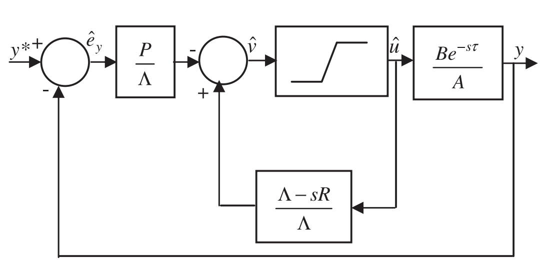

Figure 1 Block diagram of the closed-loop system.

$$\hat{v}(s) = \frac{\Lambda(s) - sR(s)}{\Lambda(s)}\hat{u}(s) - \frac{P(s)}{\Lambda(s)}\hat{e}_y(s) \tag{12a}$$

$$u(t) = \mathrm{sat}(v(t)) = sgn(v(t))\min(|v(t)|, u_M) \tag{12b}$$

Remark 2

(1) Note that the controller (12a) and (12b) is causal (i.e. $v(t)$ and $u(t)$ depend only on measurements available at time t) because $(\Lambda(s) - sR(s))/\Lambda(s)$ is a strictly proper transfer function and also $P(s)/\Lambda(s)P(s)/\Lambda(s)$ is proper. Indeed, it is readily seen from (11) and (8) that:

$$\frac{\Lambda(s) - sR(s)}{\Lambda(s)} = \frac{\lambda_{n-1}s^{n-1} + \cdots + \lambda_1 s + \lambda_0}{\Lambda(s)} - \sum_{i=0}^{n-2} R_i(e^{-s\tau})\frac{s^{i+1}}{\Lambda(s)} - R_{-1}(s)\frac{s}{\Lambda(s)}$$

The first fraction on the right side of this equality is clearly strictly proper (because the degree of $\Lambda(s)$ equals n). Furthermore, all fractions $s^{i+1}/\Lambda(s)$ are also strictly proper because $i+1 < n$ for $i = 0 \ldots n-2$). This implies that the transfer functions $R_i(e^{-s\tau})s^{i+1}/\Lambda(s)$ are causal because $R_i(e^{-s\tau})$ are polynomials in $e^{-s\tau}$. The last transfer function $R_{-1}(s)s/\Lambda(s)$ is also causal because $s/\Lambda(s)$ is strictly proper and $R_{-1}(s)$ belongs to $\mathbf{G}$ i.e. $R_{-1}(s)$is the transfer function of a distributed delay operator (Appendix A, property A1). A similar argument can be reproduced to prove that $P(s)/\Lambda(s)$ is in turn causal.

(2) If the control signal stops saturating for a long time then the controller (12a) and (12a) reduces to the standard linear pole placement regulator $\hat{u}(s) = -(P(s)/sR(s))\hat{e}_y(s)$. Then, it is easily checked that the tracking error undergoes the equation:

$$\hat{e}_y(s) = -(R(s)A(s)/\Lambda(s)C(s))s\hat{y}^*(s)$$

and so vanishes exponentially fast, in the case of constant references, because $C(s)\Lambda(s)$ is Hurwitz.

(3) The above remark also shows that the closed-loop system turns out to be linear with finite-spectrum in the absence of saturation (i.e. its transfer function has a finite number of poles coinciding with the zeros of the polynomials $\Lambda(s)C(s)$). This is an important characteristic of the used control design method. This characteristic is not necessarily ensured with more standard design techniques.To illustrate this, consider a simple proportional regulator $u(t) = -Ke_y(t)$ with K represents a real constant. Putting this regulator in closed-loop with the system (1) leads to closed-loop system:

$$\hat{e}_y(s) = -(A(s)/(A(s) + KB(s)e^{-\tau s}))\hat{y}^*(s)$$

Clearly, the closed-loop transfer function is infinite spectrum as its denominator $A(s) + KB(s)e^{-\tau s}$ has an infinite number of zeros. This is a well known fact in delayed systems theory, e.g. [22,23] and references therein.

(4) The controller (12a) and (12b) contains a distributed delay that can only approximately be implemented using digital means. This issue has been investigated in [19–21].

4. Controller tracking capability analysis

The closed-loop control system composed of the constrained system (1)–(4) and the saturated regulator (12a) and (12b) will now be analyzed. The aim was to show how the design parameters should be selected in order to achieve the control objective of Section 2. The analysis is progressively conducted in three major steps. First, it is shown in Section 4.1 that all signal rates (i.e. $\dot{u}, \dot{v}, \dot{y}$) are related to the reference rate $\dot{y}^*$ through L_2 stable operators. Then, it is established in Section 4.2 that the input saturation error $v - u$ is related to the tracking mismatch error $|\dot{y}^*| + |u^* - sat(u^*)|$ through a L_2 stable operator. This will prove to be a crucial ingredient to prove, in Section 4.3, the achievement of L_2/L_{2a} tracking performance (7a) and (7b). These results are reached by making use of absolute stability tools [15] which are particularly useful when facing static nonlinearities like the saturation function. The present approach, which is not very usual in constrained dead-time systems [7–12], finds some roots in recent works on non-delayed constrained systems e.g. [17].

4.1. Signal rate analysis

In this subsection, the focus is made on the first time-derivatives of all signals. It is shown that signal derivatives are related to the reference signal rate through L_2 stable operators. Combining (1) and (12a) so that e_y is eliminated, one gets the following relation between signals rate:

$$s\hat{v}(s) = -\frac{C(s) - A(s)}{A(s)}(s\hat{u}(s)) + \frac{P(s)}{A(s)}(s\hat{y}^*(s)) \tag{13}$$

This equation fits the feedback representation of Fig. 2 where:

$$\hat{\delta}_1(s) = \frac{P(s)}{A(s)}(s\hat{y}^*(s)) \tag{14}$$

This feedback schema allows application of absolute stability theorems [15]. The aim was to establish a sufficient condition (on the polynomial $C(s)$) that ensures the L_2-stability of the feedback in order to get bounding properties on the derivatives of u and v. To this end, we begin firstly by showing that the nonlinear operator Ψ (which represents the mapping: $\dot{v} \to \dot{u}$) lies in some conic sector[3] ([0 1]). It is subject of proposition 1 whose proof is placed in Appendix B.

Proposition 1. *Let the polynomial $C(s)$ in (10) be chosen such that:*

$$\inf_{0 \leqslant \omega < +\infty} Re\left(\frac{C(j\omega)}{A(j\omega)}\right) > 0 \tag{15}$$

Then, the feedback of Fig. 2 is L_2-stable. Consequently, the following mappings: $\dot{y}^* \to \dot{u}^*$, $\dot{y}^* \to \dot{v}$, $\dot{y}^* \to \dot{u}$, $\dot{y}^* \to \dot{y}$, $\dot{y}^* \to \dot{e}_y$ are all L_2-stable.

4.2. Control saturation error analysis

In this subsection, a key result concerning the control saturation error $v - u$ will be established. It consists in showing that the mapping $|\dot{y}^*| + |u^* - sat(u^*)| \to |v - u|$ is L_2-stable.

Proposition 2. *Consider the control system composed of the system (1) and (2), subject to the saturation constraint (3), in closed-loop with the controller (12a) and (12b) where the polynomial $C(s)$ in (10) satisfies (15). Then, the mapping $|\dot{y}^*| + |u^* - sat(u^*)| \to |v - u|$ is L_2-stable.*

The proof of this proposition is placed in Appendix C.

4.3. L_2 tracking performance achievement

Theorem 1. *Consider the closed-loop control system of Proposition 2. Then, the performance operator (6a), $|\dot{y}^*| + |u^* - sat(u^*)| \to |e_y|$, is actually L_2- stable. Consequently, the controller (12a) and (12b) features the L_2/L_{2a} tracking performance described by (7a) and (7b)*

Proof. Using (1a) and operating sRA on both sides of (4) give:

$$\begin{aligned} RAs\hat{e}_y(s) &= RAs\hat{y}(s) - RAs\hat{y}^*(s) \\ &= RBe^{-s\tau}s\hat{u}(s) - RAs\hat{y}^*(s) \end{aligned} \tag{16}$$

Similarly, operating $\Lambda Be^{-s\tau}$ on both sides of (12a) yields:

$$\Lambda Be^{-s\tau}\hat{v}(s) = \Lambda Be^{-s\tau}\hat{u}(s) - RBe^{-s\tau}s\hat{u}(s) - BPe^{-s\tau}\hat{e}_y(s) \tag{17}$$

Adding (16) and (17) yields, using (10):

$$\Lambda C\hat{e}_y(s) = \Lambda Be^{-s\tau}(\hat{u}(s) - \hat{v}(s)) - RA(s\hat{y}^*(s)) \tag{18}$$

Recall that the mapping $|\dot{y}^*| + |u^* - sat(u^*)| \to u - v$ is L_2-stable (by Proposition 2). Also, both $Be^{-s\tau}/C$ and $RA/\Lambda C$ are L_2-stable. Then, one immediately gets from (18) that the mapping $|\dot{y}^*| + |u^* - sat(u^*)| \to |e_y|$ is L_2 stable. □

Remark 3. Interestingly, the result of Theorem 1 holds whatever the reference shape and even if this reference is not constraint-compatible. This is an original result, compared to existing literature, especially that on direct control design for delayed input-constrained systems [8–10] which generally dealt with stabilization problems (in the presence of null reference). The result of Theorem 1 is also in progress with respect to the conference paper [18] as, there, the reference signal was supposed to be constraint-compatible (in which case $u^* - sat(u^*) = 0$). The notions of tracking mismatch error (i.e. the quantity $|\dot{y}^*| + |u^* - sat(u^*)|$), the L_{2a}-norm and corresponding normed space are also new features with respect to [18].

5. Simulation

The system (1) and (2) is simulated in Matlab/Simulink using the following numerical values:

$$\begin{aligned} &A(s) = s^2 + 1.25s + 0.25, \quad B(s) = s + 0.7, \quad \tau = 2s \\ &u_M = 1 \end{aligned} \tag{19}$$

First, let us make some comparison with the simulation example in [11] (although the system considered there is a two-input). Clearly, our example is more interesting as it is a second order and the delay is important (nearly 45% of the system equivalent time constant). In [11], the system is a first order and the delay is merely 10% of the system time-constant.

Applying the control design of Section 3 to the example (19), a controller like (12a) and (12b) is obtained by solving the Bezout Eq. (10)using the following Hurwitz polynomials:

[3] Recall that ψ in Fig. 2 belongs to a sector $[a, b]$ if: $a(\dot{v})^2 \leqslant \dot{u}\dot{v} \leqslant b(\dot{v})^2$.

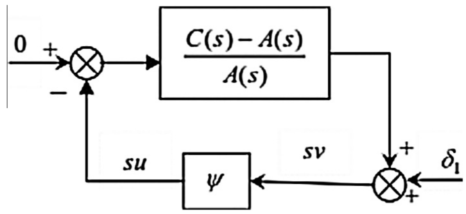

Figure 2 Nonlinear feedback relating sv and su.

$$C(s) = s^2 + 1.85s + 0.71 \quad \text{and} \quad A(s) = s^2 + 2.25s + 1.125$$

It is readily checked (e.g. plotting the Nyquist plot of $C(s)/A(s)$) that condition (15) is satisfied. No condition is imposed on $A(s)$. Solving Eq. (10), one gets $P(s) = 3.13s^2 + 4.26s + 1.12$ and

$$R(s) = s + 2.85 - 3.16e^{-s\tau} + 0.233\left(\frac{1 - e^{-(s+1)\tau}}{s+1}\right) - 1.03\left(\frac{1 - e^{-(s+0.25)\tau}}{s+0.25}\right) + 3.20\left(\frac{1 - e^{-s\tau}}{s}\right)$$

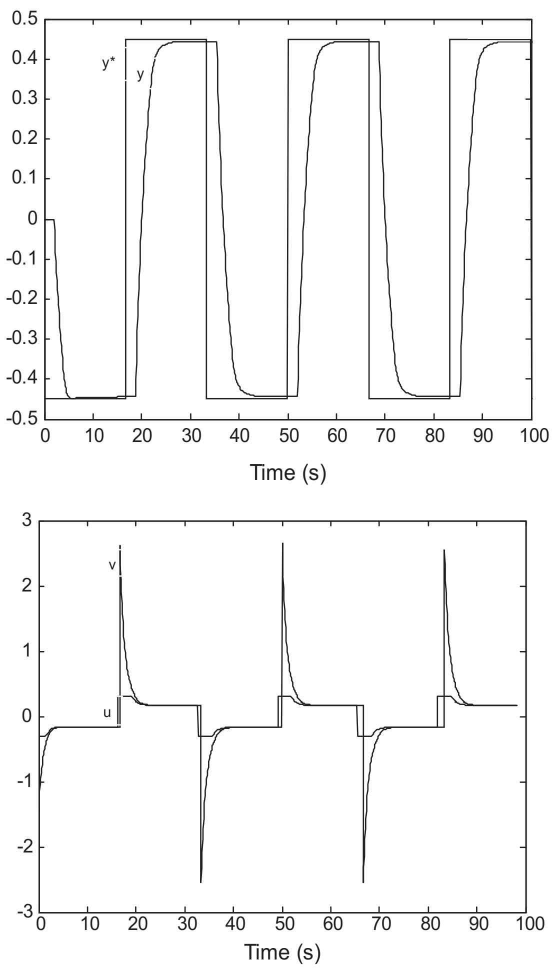

Figure 3 Controller tracking performance in presence of a constraint-compatible reference when condition (15) is satisfied. Top: the system output (solid), reference signal (dashed). Bottom: the computed control (dashed) and applied control (solid).

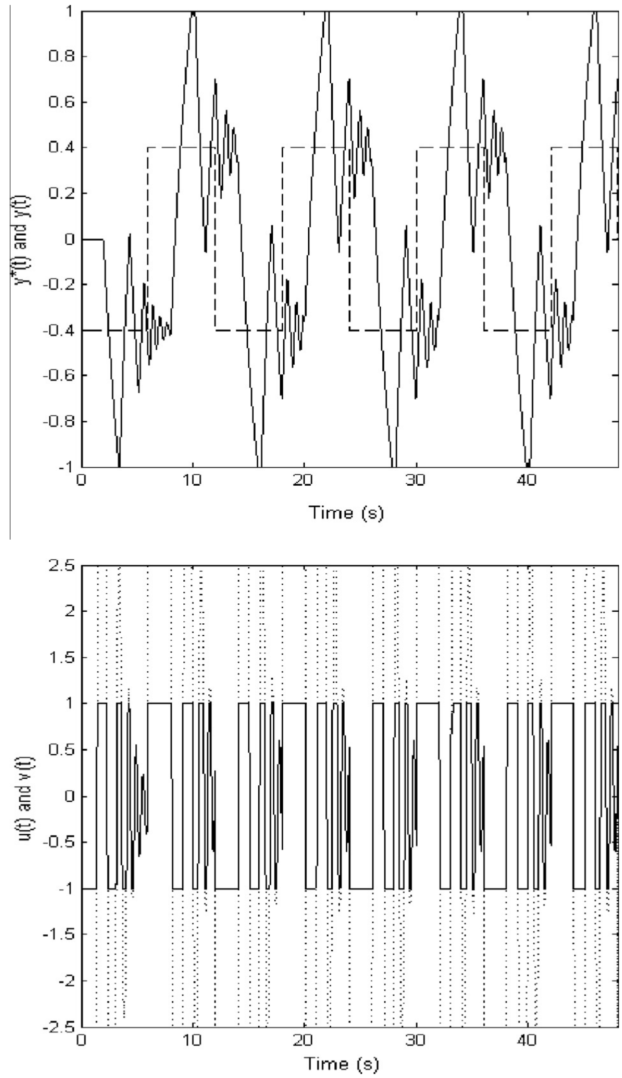

Figure 4 Controller tracking performance in presence of a constraint-compatible reference when condition (15) is not satisfied. Top: the system output (solid) and reference signal (dashed). Bottom: the computed control (dashed) and applied control (solid).

As pointed out in the general case (Remark 2, Part 1), the controller transfer functions $(A(s) - sR(s))/A(s)$ and $P(s)/A(s)$ are strictly proper. The resulting tracking quality is illustrated by Fig. 3 considering a constraint-compatible square reference. It is seen that the tracking performance is quite satisfactory where the system output tracks well its reference trajectory. Furthermore, note that the output-reference tracking rapidity performance is structurally limited by the presence of the input dead-time. This observed dead-time in the output-reference matching is structural as no linear state or output feedback can compensate a system dead-time. To better appreciate the tracking performance of (12a) and (12b), and to check the importance of condition (15), let us now take $C(s) = s^2 + 2s + 100$ and keep all remaining controller parameters unchanged. It is easily checked that condition (15) is no longer satisfied. The new closed-loop system responses are plotted in Fig. 4 which shows a clear deterioration of output-reference tracking performance.

6. Conclusion

Control problem of input-delayed nonminimum-phase linear systems (1) and (2) is considered in the presence of actuator saturation (3). The control design is performed within the ring of pseudo-polynomials, using the (finite-spectrum) pole-placement design technique. Using input-output stability tools, it is formally shown that the controller (12a) and (12b) enjoys the appealing L_2/L_{2a} tracking features described by (7a) and (7b). Accordingly, the tracking quality depends on the L_2/L_{2a} norm of the tracking mismatch error $|\dot{y}^*| + |u^* - sat(u^*)|$. The smaller this error, the better the output-reference tracking quality.

This result is quite powerful as it holds, whatever the bounded reference signal.. To the author's knowledge, no equivalent result is currently available in the context of actuator saturation control of input-delayed systems. The present work can be extended in many directions including generalization to multivariable systems, distributed delay systems, and uncertain delay.

Appendix A. Pseudo-polynomials algebra

This appendix gives a brief presentation of relevant concepts and results of pseudo-polynomial algebra. For a more exhaustive presentation, the author might consult the references [13,14].

Definition A1. 2D-polynomials are bi-variable polynomials of the form $f(s,x) = \sum_{i=0}^{n} f_i(x)s^i$ where $f_i \in R[x]$, the ring of real polynomials in x. The set of 2D-polynomials with variables s and x is denoted $R[s, x]$, the degree of a 2D-polynomial $f(s, x)$ in the variable s is the largest integer i such that $f_i \neq 0$ and is denoted $\deg_s(f)$.

The 2D-polynomials considered in this paper involve the Laplace variable s and the delay operator $e^{-\tau s}$ as a second variable. Though the two variables s and $e^{-\tau s}$ are analytically related, they are algebraically independent i.e. $f(s, e^{-\tau s}) = 0$ (for all $s \in \boldsymbol{C}$) $\Rightarrow$ all coefficients of $f(s, e^{-\tau s})$ are null.

Definition A2. The set G is the ring of all rational complex functions of the form $g(s) = n(s, e^{-\tau s})/d(s)$, with $n \in R[s, e^{-\tau s}]$, $d(s) \in R[s]$ and $\deg_s(n) < \deg(d)$, such that $g(s)$ is analytical on C.

Property A1. *The elements of G are transfer functions of distributed delay operators G: $x \rightarrow G[x]$ with $G[x](t) = \int_{p\tau}^{q\tau} g(\theta) x(t-\theta)d\theta$, where g is a real function and p, q are positive integers.*

Property A2. *Let $F = R[e^{-\tau s}] + G$. Then, F is the ring of all rational complex functions of the form $g(s) = \frac{n(s,e^{-\tau s})}{d(s)}$, with $n \in R[s, e^{-\tau s}]$, $d \in R[s]$, $\deg_s(n) \leqslant \deg_s(d)$ and such that g(s) is analytical on C. Consequently, the elements of F are transfer functions of operators H: $x \rightarrow H[x]$ such that: $H[x](t) = \sum_{i=0}^{m} h_i x(t - i\tau) + \int_{p\tau}^{q\tau} h(\theta)x(t-\theta)d\theta$, for some real numbers $h_0, \ldots, h_m$, integers m, p, q and real function h.*

That is, $\boldsymbol{F}$ is the set of transfer functions of all punctual and distributed delay operators.

Definition A3. Let E be the ring of polynomials in s with coefficients in F i.e. $E = F[s]$. Then, E is the ring of all analytical functions of the form $g(s) = n(s, e^{-\tau s})/d(s)$, with $n \in R[s, e^{-\tau s}]$, $d(s) \in R[s]$ (without condition on the relative degrees of n and d).

The elements of $\boldsymbol{E}$ are called pseudo-polynomials.

Remark A1. In the above definitions, $g(s) = \frac{n(s,e^{-s\tau})}{d(s)}$ is said to be analytical means that this function is well defined on the whole complex plane. It turns out that, all zeros of $d(s)$ are also zeros of $n(s, e^{-\tau s})$ with at least the same multiplicity.

Definition A4. The degree in E is defined by: $\deg\left(\frac{n(s,e^{-\tau s})}{d(s)}\right) = \deg_s(n(s, e^{-\tau s})) - \deg_s(d(s))$.

Note that for $g(s) \in \boldsymbol{E}$, if $\deg(g) \leqslant 0$ then, $g(s) \in \boldsymbol{F}$ and if $\deg(g) < 0$ then $g(s)$ also belongs to $\mathbf{G}$.

Property A3. *Any $f \in E$ can be uniquely decomposed as $f(s) = \sum_{i=0}^{\deg(f)} f_i(e^{-\tau s})s^i + g(s)$ with $g(s) \in G$ and $f_i(e^{-\tau s}) \in R[e^{-\tau s}]$.*

Definition A5. Let $f \in \mathbf{E}$ be of degree n. Then f is monic if $f_n(e^{-\tau s})$ is constant equal to 1, where the notations are as in Property A3.

Property A4. *Let f, $g \in E$ be any pair of pseudo-polynomials*

1. *The pair f, g has a unique greater common divisor (GCD) (up to a multiplicative real factor). The two elements are coprime if their GCD is invertible in E i.e. the GCD is a real number.*
2. *E is a Bezout ring. That is, any ideal generated by a finite set of elements of E is principal i.e. it is generated by the GCD's of these elements.*
3. *For any f, $g \in E$, there exist a pair of elements, x (s) and y (s), of E such that:*

$$x(s)f(s) + y(s)g(s) = GCD(f(s), g(s)) \tag{A1}$$

4. *For any f, $g \in E$, with g(s) monic, there exists a unique pair of elements q, $r \in E$ such that*

$$f(s) = q(s)g(s) + r(s), \quad \text{with } \deg_s(r(s)) < \deg_s(g(s)).$$

Property A5. *A fraction N (s)/D(s) of elements of E is proper if D (s) is monic and $\deg_s(D) > \deg_s(N)$. A proper fraction is realizable by a set of delay differential equations (with punctual and distributed delay operators).*

Appendix B. Proof of proposition 1

The first step is to show that the nonlinear mapping ψ, in the feedback of Fig. 2, belongs to the sector [0 1] in the sense that: $0 \leqslant \dot{v} \times \dot{u} \leqslant (\dot{v})^2$. By definition, one has $\text{sat}(x) = sgn(x)\min(|x|, u_M)$, for all $x \in (-\infty, +\infty)$. This implies that:

$$\frac{d\text{sat}}{dx}(x) = 1 \;\text{ for } |x| \leqslant 1 \text{ and } \frac{d\text{sat}}{dx}(x) = 0, \;\text{ for } |x| > 1 \tag{B1}$$

On the other hand, one gets from (12b):

$$\dot{u}(t) = \frac{d\text{sat}}{dv}(v(t)) \times \dot{v}(t) \tag{B2}$$

Multiplying both sides of (B2) by $\dot{v}(t)$ gives $\dot{v} \times \dot{u} = \frac{dsat}{dv}(v)\dot{v}^2$. This, together with (B1), implies that $0 \leqslant \dot{v} \times \dot{u} \leqslant (\dot{v})^2$ which proves that ψ actually belongs to the sector [0 1]. Then, it follows applying the circle criterion (see e.g. [15]) that the feedback of Fig. 2 is L_2-stable provided that:

$\inf_{0 \leqslant \omega < +\infty} \mathrm{Re}((C(j\omega) - \Lambda(j\omega))/\Lambda(j\omega)) > -1$. But this condition is nothing other than (15). Then, it follows from Fig. 2 that the two mappings $\delta_1 \to \dot{v}$ and $\delta_1 \to \dot{u}$ are L_2-stable. As the transfer function $S(s)/\Lambda(s)$ is asymptotically stable, one has from (14) that, the mapping $\dot{y}^* \to \delta_1$ is also L_2-stable. Combining the above results, one gets that $\dot{y}^* \to \dot{v}$ and $\dot{y}^* \to \dot{u}$ are both L_2-stable.

On the other hand, it readily follows from (1) that:

$$s\hat{y}(s) = \frac{B(s)e^{-\tau s}}{A(s)} s\hat{u}(s) \tag{B3}$$

As $A(s)$ is Hurwitz and $B(s)e^{-\tau s}/A(s)$ is proper, it follows that the mapping $\dot{u} \to \dot{y}$ is L_2-stable. We have already shown that $\dot{y}^* \to \dot{u}$ is L_2-stable. Then, one gets that $\dot{y}^* \to \dot{y}$ is L_2-stable.

Then, Eq. (4) implies that the mapping $\dot{y}^* \to \dot{e}_y = \dot{y} - \dot{y}^*$ is in turn L_2-stable. Finally, by definition, one has $u^*(t) = (a_0/b_0)y^*(t+\tau)$ which clearly shows that the mapping $\dot{y}^* \to \dot{u}^*$ is L_2-stable.

Appendix C. Proof of proposition 2

From (12a) one gets, using (4) and Definition 1 (Part 1):

$$\begin{aligned}\hat{v}(s) - \hat{u}(s) &= -\frac{R}{\Lambda} s\hat{u}(s) - \frac{P}{\Lambda}\hat{e}_y(s)\\ &= -\frac{R}{\Lambda} s\hat{u}(s) - \frac{P}{\Lambda}\left(\frac{Be^{-s\tau}}{A}\hat{u}(s) - \frac{b_0}{a_0}e^{\tau s}\hat{u}^*(s)\right)\end{aligned}$$

This implies successively that:

$$\begin{aligned}\hat{v}(s) - \hat{u}(s) = &-\frac{R}{\Lambda} s\hat{u}(s) - \frac{PBe^{-s\tau}}{\Lambda A}(\hat{u}(s) - \hat{u}^*(s))\\ &+ \left(\frac{b_0}{a_0} - \frac{B}{A}\right)\frac{Pe^{-s\tau}}{\Lambda}\hat{u}^*(s)\end{aligned}$$

$$\hat{v}(s) - \hat{u}(s) = -\frac{R}{\Lambda} s\hat{u}(s) - \frac{PBe^{-s\tau}}{\Lambda A}(\hat{u}(s) - \hat{u}^*(s)) + \hat{\delta}_2(s)$$

with $\hat{\delta}_2(s) = \left(\frac{b_0}{a_0} - \frac{B}{A}\right)\frac{Pe^{-s\tau}}{\Lambda}\hat{u}^*(s)$

$$\hat{v}(s) - \hat{u}(s) = -\frac{p_0 Be^{-s\tau}}{\Lambda A}(\hat{u}(s) - \hat{u}^*(s)) + \hat{\delta}_2(s) + \hat{\delta}_3(s),$$

with $\hat{\delta}_3(s) = -\frac{R}{\Lambda} s\hat{u}(s) - \frac{(P-p_0)Be^{-\tau s}}{sA\Lambda}(s\hat{u}(s) - s\hat{u}^*(s))$

$$\hat{v}(s) - \hat{u}(s) = \frac{p_0 b_0}{\lambda_0 a_0}(\hat{u}^*(s) - \hat{u}(s)) + \hat{\delta}_2(s) + \hat{\delta}_3(s) + \hat{\delta}_4(s)$$

with $\hat{\delta}_4 = \left(\frac{s_0 Be^{-s\tau}}{\Lambda A} - \frac{s_0 b_0}{\lambda_0 a_0}\right)(\hat{u}^*(s) - \hat{u}(s))$

Going back to time context, one gets:

$$\begin{aligned}v(t) - u(t) = &\frac{p_0 b_0}{\lambda_0 a_0}(u^*(t) - sat(u^*(t))) + \frac{p_0 b_0}{\lambda_0 a_0}(sat(u^*(t)) - u)\\ &+ \delta_2(t) + \delta_3(t) + \delta_4(t)\end{aligned}$$

$$v(t) - u(t) = \frac{c_0}{a_0}(sat(u^*(t)) - u(t)) + \delta_2(t) + \delta_3(t) + \delta_4(t) + \delta_5(t)$$

with $\delta_5 = \frac{c_0}{a_0}(sat(u^*) - u^*)$

$$v(t) - u(t) = \frac{c_0}{a_0}(sat(u^*(t)) - u(t)) + \delta_6(t) \tag{C1}$$

with $\delta_6 = \delta_2 + \delta_3 + \delta_4 + \delta_5$

where $p_0 = P(0)$ and this result is obtained using the fact that $p_0 b_0 = \lambda_0 c_0$ by letting $s = 0$ in (10). Let us demonstrate that all mappings $\mu \to \delta_i$ $(i = 2 \ldots 5)$ are L_2-stable where:

$$\mu \stackrel{def}{=} |\dot{y}^*| + |u^* - sat(u^*)| \tag{C2}$$

To this end, we will make intensive use of Proposition 1.

Mapping $\mu \to \delta_2$. By definition one has:

$$\begin{aligned}\hat{\delta}_2(s) &= \left(\frac{b_0}{a_0} - \frac{B}{A}\right)\frac{Pe^{-s\tau}}{\Lambda}\hat{u}^*(s)\\ &= \left(\frac{b_0 A(s) - a_0 B(s)}{a_0 A(s)}\right)\frac{Pe^{-s\tau}}{\Lambda}\hat{u}^*(s)\end{aligned}$$

$$\hat{\delta}_2(s) = \frac{D(s)}{a_0 A(s)}\frac{Pe^{-s\tau}}{\Lambda}s\hat{u}^*(s) \tag{C3}$$

where

$$D(s) = b_0(s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1) - a_0(b_{n-1}s^{n-2} + \cdots + b_1)$$

The passage from the second to the third equality in (C3) is performed using the fact that:

$$\begin{aligned}&b_0(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0) - a_0(b_{n-1}s^{n-1} + \cdots + b_1 s\\ &\quad + b_0) = D(s) \times s\end{aligned}$$

In view of (2), it is checked that the transfer function $D(s)/a_0 A(s)$ is strictly proper and asymptotically stable (because $A(s)$ is Hurwitz). Owing to $Pe^{-s\tau}/\Lambda$, recall that (see Section 3) $P(s)$ is a pseudo-polynomial and so it is an analytical function i.e. it has no poles on the complex plane. Therefore, the only poles of $Pe^{-s\tau}/\Lambda$ are the zeros of $\Lambda(s)$ which is a Hurwitz polynomial in s. Hence, the linear transfer function $Pe^{-s\tau}/\Lambda$ is asymptotically stable. Using these observations, it follows from (C2) that the mapping $\dot{u}^* \to \delta_2$ is L_2-stable. We know by Proposition 1 that the mapping $\dot{y}^* \to \dot{u}^*$ is L_2-stable. Then, one immediately gets that $\mu \to \delta_2$ is L_2-stable.

Mapping $\mu \to \delta_3$. R/Λ is L_2-stable because is Λ Hurwitz and the mapping $\dot{y}^* \to \dot{u}^*$ is L_2-stable by Proposition 1. Then, one gets that the mapping:

$$\mu \to \frac{R}{\Lambda}\dot{u} \text{ is } L_2\text{-stable} \tag{C4}$$

Similarly, as $(P - p_0)/s$ is a pseudo-polynomial with degree $n - 1 (= \deg(P) - 1)$, the transfer function $(P - p_0)/(s\Lambda)$ turns out to be L_2-stable because Λ is Hurwitz. By Proposition 1, the mappings $\dot{y}^* \to \dot{u}$ and $\dot{y}^* \to \dot{u}^*$ are L_2-stable. Then one gets, letting $\hat{\chi}(s) \stackrel{def}{=} \frac{(P-p_0)Be^{-\tau s}}{sA\Lambda}(s\hat{u}(s) - s\hat{u}^*(s))$, that:

$$\mu \to \chi \text{ is } L_2\text{-stable} \tag{C5}$$

From (C4) and (C5) one gets: $\mu \to \delta_3$ is L_2-stable.

Mapping $\mu \to \delta_4$. First, let us perform the following decomposition:

$$\frac{1}{s}\left(\frac{p_0Be^{-s\tau}}{\Lambda A}-\frac{p_0b_0}{\lambda_0a_0}\right)=\frac{1}{s}\left(\frac{p_0B\lambda_0a_0e^{-s\tau}-p_0b_0\Lambda A}{\lambda_0a_0\Lambda A}\right)$$
$$=T_1(s)+T_2(s) \tag{C6}$$

where $T_1(s)=\frac{\lambda_0a_0p_0e^{-s\tau}\sum_{i=1}^{n-1}b_is^{i-1}-p_0b_0\sum_{i=1}^{2n}\alpha_is^{i-1}}{\lambda_0a_0\Lambda A}$,

$$T_2(s)=\frac{p_0b_0}{\Lambda A}\left(\frac{e^{-s\tau}-1}{s}\right)\quad \text{with } \alpha_i=\sum_{j=\max(0,i-n)}^{\min(i,n)}\lambda_ja_{i-j} \tag{C7}$$

In view of (C6), $\delta_4$is rewritten as follows:

$$\hat{\delta}_4(s)=(T_1(s)+T_2(s))(s\hat{u}-s\hat{u}^*) \tag{C8}$$

Let us demonstrate that both transfer functions T_1 and T_2 are L_2-stable. This is clearly the case for T_1 because it is strictly proper and its denominator ΛA is Hurwitz. Owing to T_2, notice that the pseudo-polynomial $(e^{-s\tau}-1)/s$ is the transfer function of the linear operator H: $x\to Hx$ with:

$$(Hx)(t)=\int_0^\tau x(t-\theta)\mathrm{d}\theta=\int_0^t h(\theta)x(t-\theta)\mathrm{d}\theta$$

with $h(\theta)\overset{def}{=}\begin{cases}1 & \text{if } 0\leqslant\theta\leqslant\tau\\ 0 & \text{otherwise}\end{cases}$.

Clearly the impulse response h belongs toL_1. Then, the linear operator H is asymptotically and L_2-stable [15]. Furthermore, $p_0b_0/\Lambda A$ is clearly L_2-stable (because ΛA is Hurwitz). Then, it follows from (C7) that T_2 is L_2-stable. Again, by Proposition 1 the mappings $\dot{y}^*\to\dot{u}$ and $\dot{y}^*\to\dot{u}^*$ are L_2-stable. Then one gets from (C8) that $\mu\to\ \delta_4$ is L_2-stable.

Mapping $\mu\to\delta_5$. By definition $\delta_5=\frac{c_0}{a_0}(sat(u^*)-u^*)$. Then, one gets from (C2) that $\mu\to\delta_5$ is L_2-stable.

Mapping $\mu\to\delta_6$. As $\delta_6=\delta_2+\delta_3+\delta_4+\delta_5$, it follows from the previous steps that the mapping $\mu\to\delta_6$ is in turn L_2-stable.

The rest of the proof consists in showing that:

$$|v(t)-u(t)|\leqslant|\delta_6(t)|,\ \text{for all } t \tag{C9}$$

It is clear that (C9) holds when $|v(t)|\leqslant u_M$ because one has $v(t)-u(t)=0$, due to (12b). So let us establish (C9) in the case where $|v(t)|>u_M$. As A and C are Hurwitz, their coefficients a_0 and c_0 are positive (by e.g. Routh criterion). Furthermore, as $|v(t)|>u_M$, one has from (12b) that $|u(t)|=u_Msgn(v(t))$ which implies that $|u(t)|>|sat(u^*(t))|$. Then, multiplying both sides of (C1) by $sgn(v(t))$ and using the fact that $sgn(v(t))=sgn(u(t))$, one gets for all t:

$$|v(t)-u(t)|=\frac{c_0}{a_0}(sat(u^*(t))sgn(v(t))-|u(t)|)$$
$$+\delta_6(t)sgn(v(t))\leqslant\delta_6(t)sgn(v(t)) \tag{C10}$$

This implies, on the one hand, that $\delta_6(t)sgn(v(t))$ is nonnegative; and, on the other hand, that $|v(t)-u(t)|\leqslant|\delta_6(t)|$. Hence, inequality (C10) does hold which proves Proposition 2.

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