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Plancherel Formula for Berezin Deformation of L^2 on Riemannian Symmetric Space

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$\mathrm{U}(p,q)/(\mathrm{U}(p)\times\mathrm{U}(q))$. Berezin representations of $\mathrm{O}(p,q)$ are the restrictions of such representations to the subgroup $\mathrm{O}(p,q)$. We obtain the explicit Plancherel formula for the Berezin representations. The support of the Plancherel measure is a union of many series of representations. The density of the Plancherel measure on each piece of the support is an explicit product of Γ-functions. We also show that the Berezin representations give an interpolation between L^2 on noncompact symmetric space $\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q)$ and L^2 on compact symmetric space $\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)$.

0. INTRODUCTION

0.1. *Kernel Representations*

Let G be a *classical* real Lie group and let K be its maximal compact subgroup. Consider the Riemannian noncompact symmetric space G/K. There exists a hermitian symmetric space

$$\tilde{G}/\tilde{K} \supset G/K$$

such that

$$\dim_{\mathbb{R}} G/K = \tfrac{1}{2}\dim_{\mathbb{R}} \tilde{G}/\tilde{K}$$

and G/K is a totally real submanifold in $\tilde{G}/\tilde{K}$ (the list of embeddings $G/K \to \tilde{G}/\tilde{K}$ is given in Section 6). We say that the symmetric space $\tilde{G}/\tilde{K} \supset G/K$ is the *hermitization* of the symmetric space G/K.

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We define a *kernel representation* ρ of the group G as the restriction of a unitary highest weight representation $\tilde{\rho}$ of $\widetilde{G}$ to the subgroup G. By a well-known Harish–Chandra construction, the highest weight representations of $\widetilde{G}$ are natural representations in spaces of (scalar-valued or vector-valued) holomorphic functions on $\widetilde{G}/\widetilde{K}$. We say that ρ is a *scalar valued kernel representation* if $\tilde{\rho}$ is realized in scalar-valued holomorphic functions on $\widetilde{G}/\widetilde{K}$.

The kernel representations are deformations of $L^2(G/K)$ in some precise sense explained in Subsection 1.13.

The purpose of this paper[2] is to obtain the Plancherel formula for scalar-valued kernel representations (see formulas (2.6)–(2.15)).

There were several reasons for the interest that was attracted by kernel representations in the past 5 years (see [2, 7, 8, 33–36, 38, 39, 40, 52, 56]), and we will formulate reasons that are the most close to the author. First, there are many explicit analytical formulas related to the kernel representations (I hope that this paper also confirms this statement; see also [57, 58]). Second, spectra of the kernel representations are very rich.[3] Third, the kernel representations have some interaction with function theory;[4] see [34, 36, 38]. Fourth, the kernel representations also are closely related to Olshanskii constructions of representations of the finite dimensional groups $\mathrm{U}(p, \infty)$, $\mathrm{O}(p, \infty)$, $\mathrm{Sp}(p, \infty)$ (see [38]).

0.2. *Bibliographical Comments*

Let G/K itself be an hermitian symmetric space (i.e., $G = \mathrm{U}(p, q)$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}^*(2n)$, $\mathrm{SO}(n, 2)$). Then its hermitization $\widetilde{G}/\widetilde{K}$ is $G/K \times G/K$. A kernel representation of G in this case is a tensor product of a highest weight representation ρ_α of G and a lowest weight representation ρ^*_β of G. In a short paper [4] published in 1978 Berezin announced a nice Plancherel formula for the *sufficiently large parameter* $\alpha = \beta$ *of highest*

[2] This work is a continuation of works [35, 36] but logically it is independent of these papers. Our main result was announced in [36].

[3] The most interesting spectral problems of noncommutative harmonic analysis that were intensively investigated in last 20 years are

— L^2 on pseudoriemannian symmetric spaces

— Howe dual pairs (and the problem of decomposition of L^2 on Stiefel manifolds, which are in some sense equivalent to Howe dual pairs)

Each representation that occurs in the spectrum of a Howe dual pair occurs in the spectrum of some kernel representation. The converse statement is false. The *a priori* explanation of this phenomenon is contained in [33]. I think that spectra of the kernel representations and spectra of L^2 on pseudoriemannian symmetric spaces essentially differ. *A priori* embedding of spectra of $L^2(\mathrm{U}(p, q, \mathbb{K})/\mathrm{U}(r, \mathbb{K}) \times \mathrm{U}(p-r, q, \mathbb{K}))$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ into spectra of the kernel representations is discussed in [33, 38].

[4] For instance, in [38] we use functional-theoretic arguments for construction of singular unitary representations of groups $\mathrm{U}(p, q, \mathbb{K})$.

weight (see below Sections 1.10–1.11). In this case[5] the kernel representation is equivalent to the representation of G in $L^2(G/K)$. Berezin died soon after this and the proof never was published.[6] Berezin's work did not attract serious interest at this time (see only papers [15, 46] on related subjects).

Second, the kernel representations appeared in G. I. Olshanskii's and my work, which was partially announced in [30, 43] and partially published in [38]. This work concerned vector valued kernel representations of the groups $G = \mathrm{O}(p, q)$, $\mathrm{U}(p, q)$, $\mathrm{Sp}(p, q)$ for *small values* of the highest weight. The main topic of our work was investigation of the discrete part of spectra of the kernel representations and a construction of "exotic" unitary representations of G with simple functional theoretical tools.

In the middle 1990s interest in the kernel representations increased (we list some publications: [2, 7, 8, 33, 39, 40, 52, 56]). In 1994 Upmeier and Unterberger [52] published a proof of the Berezin formula (see also [2]).[7] Van Dijk and Hille [7] obtained the complete Plancherel formula for rank 1 groups. Olafsson and Ørsted [39] proved that for a large highest weight a scalar valued kernel representation of G is equivalent to the representation of G in $L^2(G/K)$.

In paper [35] there was defined the B-function for an arbitrary classical noncompact Riemannian symmetric space. For the symmetric cones $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$, $\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$, and $\mathrm{GL}(n, \mathbb{H})/\mathrm{Sp}(n)$ these B-integrals coincide with the Gindikin B-function constructed in [11] (1964) (see also the exposition in [9]). The construction of [35] for special cases of parameters gives some integrals of Siegel [48], Hua Loo Keng [21], Unterberger and Upmeier [52], and Arazy and Zhang [2]. The Plancherel formula for scalar-valued kernel representations of all classical groups for large values of parameters is easily reduced to these B-integrals. In this case a kernel representation is equivalent to the representation of G in $L^2(G/K)$ and the spectrum of the kernel representation is supported by the principal nondegenerate unitary series.

The case of small values of a highest weight was discussed in paper [36]. In this case the spectrum of a kernel representation is quite intricate and work [36] contains the natural decomposition of a kernel representation on subrepresentations having relatively simple spectra.

The purpose of the present paper is to obtain the complete Plancherel formula for the kernel representations in the scalar-valued case.

[5] Tensor products for $\mathrm{SL}_2(\mathbb{R}) = \mathrm{U}(1, 1)$ were earlier investigated by Pukanszky [45] and by Vershik, Gelfand, and Graev [53]; see also [27].

[6] Twenty years later I heard some reminiscences about this proof but I cannot reconstruct the proof itself. It seems that it essentially differs from the Unterberger–Upmeier [52] proof and my proof [35].

[7] Their result also covers groups E_6, E_7.

0.3. *Contents*

Main part (Sections 1–5) of the paper deals with the series $G = \mathrm{O}(p, q)$.[8]

Section 1 of the paper contains preliminaries. We discuss the definition of the kernel representations and simple a priori properties of the Plancherel formula. We also formulate some necessary properties of spherical functions.

Basic results are formulated in Section 2. For large values of α (where α is the parameter of a highest weight) the Plancherel measure ν_α has the form

$$E(\alpha) \prod_{k=1}^{p} |\Gamma(\tfrac{1}{2}(\alpha + (p+q)/2 + s_k))|^2 \, \mathfrak{R}(s) \, ds,$$

where $\mathfrak{R}(s)$ is the Gindikin–Karpelevich density (see (1.42)–(1.43)),

$$s_1, ..., s_p \in i\mathbb{R} \tag{0.1}$$

(this notation means that $\operatorname{Re} s_j = 0$), and $E(\alpha)$ is a meromorphic factor.

Assume $q - p$ is sufficiently large. Let us move the parameter α from $+\infty$ to 0. After passing across the point $\alpha = \frac{1}{2}(p+q) - 1$ an additional piece of support of the Plancherel measure appears. This piece is defined by the conditions

$$s_1 = \alpha - \tfrac{1}{2}(p+q) + 1; \qquad s_2, ..., s_p \in i\mathbb{R}. \tag{0.2}$$

After passing across the point $\alpha = \frac{1}{2}(p+q) - 2$ the third piece of support of the Plancherel measure appears:

$$s_1 = \alpha - \tfrac{1}{2}(p+q) + 1, \qquad s_2 = \alpha - \tfrac{1}{2}(p+q) + 2; \qquad s_3, ..., s_p \in i\mathbb{R}. \tag{0.3}$$

After passing across the point $\alpha = \frac{1}{2}(p+q) - 3$ we obtain two additional components of the support,

$$\begin{aligned} &s_1 = \alpha - \tfrac{1}{2}(p+q) + 1, \qquad s_2 = \alpha - \tfrac{1}{2}(p+q) + 2, \\ &s_3 = \alpha - \tfrac{1}{2}(p+q) + 3, \qquad s_4, ..., s_p \in i\mathbb{R}, \end{aligned}$$

and

$$s_1 = \alpha - \tfrac{1}{2}(p+q) + 3, \qquad s_2, ..., s_p \in i\mathbb{R}, \tag{0.4}$$

[8] This case is the most complicated and all difficulties existing for other series exist also for $\mathrm{O}(p, q)$. For all other series our proof is more simple.

etc. After passing across the point $\alpha=(q-p)/2$ we obtain the first one-point piece,

$$s_1=\alpha-\tfrac{1}{2}(p+q)+1, \qquad s_2=\alpha-\tfrac{1}{2}(p+q)+2, ..., s_p=\alpha-\tfrac{1}{2}(p+q)+p.$$

This means that our representation has a subrepresentation entering discretely.

At the point $\alpha=p-1$ the component (0.1) of the support disappears. At the point $\alpha=p-2$ the components (0.2), (0.4) also disappear, etc.

Theorems 2.2–2.4 contain the complete description of this process and give the Plancherel density on each component of the support. Interpretation of these pieces is given in [36]; in the present paper this is not discussed.

A nature of spectra of the scalar-valued kernel representations is explained in [36].[9]

For an integer negative α our construction gives the Plancherel formula for some finite dimensional representation of $\mathrm{O}(p,q)$ (see Section 2.6 of the paper).

Section 3 is based on [35] and contains an evaluation of the B-integral (see formulas (3.2)–(3.4)). For instance, in the case $p=q$ our B-integral is given by

$$\int_{R+R^t>0}\prod_{j=1}^{p}\frac{\det[(R+R^t)/2]_j^{\lambda_j-\lambda_{j+1}}}{\det[1+R]_j^{\sigma_j-\sigma_{j+1}}}\cdot\det(R+R^t)^{-p}\,dR$$
$$=\mathrm{const}\cdot\frac{\Gamma(\lambda_k-(p+k)/2+1)\,\Gamma(\sigma_k-\lambda_k-(p-k)/2)}{\Gamma(\sigma_k-p+k)}, \qquad (0.5)$$

where the integration is given over the space of dissipative $p\times p$ real matrices R and the symbol $[A]_j$ denotes the left upper $j\times j$ block of a matrix A.

The evaluation of B-integral gives a possibility to obtain the Plancherel formula for $\alpha>\frac{1}{2}(p+q)-1$. In Section 4 we construct the analytic continuation of the Plancherel formula to arbitrary $\alpha\in\mathbb{C}$.

In Section 5 we prove positive definiteness of spherical functions that appear in the right side of the Plancherel formula.

Section 6 contains a discussion of other series of classical groups. The B-integrals for other series of real classical groups are evaluated in [35]

[9] The discrete part of spectra of kernel representations consists of singular unitary representations having quite interesting properties. For instance, these infinite dimensional (non-highest-weight) representations have Gelfand–Tsetlin bases; see [43, 28]. The problems of decomposition of restrictions and tensor products for these representations also seem rich; see [30, 38]). In [43, 38] it was shown that these representations admit inductive limits as $q\to\infty$. Certain representations of this type appear in spectra discussed in [50, 25].

and a generalization of the consideration of Sections 1, 4, 5 to other series is quite trivial. Hence we give only short remarks and also give the Plancherel formula in form which slightly differs from Theorem 2.2. The author intentionally considers the series $\mathrm{O}(p, q)$ (and not the so-called "general case") to keep the exposition more or less self-enclosed. I try to avoid formal logical dependence on recent papers and also minimize using the machinery of representation theory of semisimple groups as far as it is possible;[10] I also try to avoid notation demanding long explanations.

1. PRELIMINARIES

A. *Positive Definite Kernels*

The subject of the paper is analysis in a family of Hilbert spaces defined by positive definite kernels. The notion of a positive definite kernel and the associated machinery are quite old (see [6, 26, 51]) but not widely known. In this section we briefly discuss some elementary properties of the positive definite kernels and associated Hilbert spaces.

1.1. *Positive Definite Kernels*

Let H be a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$; let X be a subset in H. Consider the function $L(x, y)$ on $X \times X$ defined by

$$L(x, y) = \langle x, y \rangle.$$

Obviously, for all $x_1, ..., x_n \in X$ we have

$$\det \begin{pmatrix} L(x_1, x_1) & \cdots & L(x_1, x_n) \\ \cdots & \cdots & \cdots \\ L(x_n, x_1) & \cdots & L(x_n, x_n) \end{pmatrix} \geqslant 0. \tag{1.1}$$

Let X be an abstract set. A function $L(x, y)$ on $X \times X$ is called a *positive definite kernel* if it satisfies the conditions

1. $L(x, y) = \overline{L(y, x)}$.
2. For any $x_1, x_2, ..., x_n \in X$ inequality (1.1) holds.

[10] We need some basic properties of spherical functions; all necessary information is contained in Helgason [17, Chap. 4].

Let $L(x, y)$ be a positive definite kernel on X. Then there exist a Hilbert space $H = H[L]$ and a system of vectors $v_x \in H$ enumerated by points $x \in X$ such that

1. $\langle v_x, v_y \rangle_H = L(x, y)$.
2. The linear span of the vectors v_x is dense in H.

The family v_x is called a *supercomplete basis.*[11]

This construction is natural in the following sense. Let H' be another Hilbert space and let $v'_x \in H'$ be another system of vectors satisfying the same conditions. Then there exists the unique unitary operator $U\colon H \to H'$ such that $Uv_x = v'_x$ for all $x \in X$.

If X is a separable metric space and the kernel $L(x, y)$ is continuous, then the Hilbert space $H[L]$ is separable and the map $x \mapsto v_x$ is continuous.

In Sections 1.2–1.3 and 1.4 we discuss two ways of "materialization" of the space $H[L]$.

1.2. *Scalar Product in the Space of Complex-Valued Measures*

Assume X is a separable complete metric space. Let μ be a complex-valued measure (charge) on X with a compact support. Consider a vector

$$v(\mu) = \int_X v_x \, d\mu(x) \in H[L].$$

Thus, we obtain a way to represent elements of $H[L]$. Obviously,

$$\langle v(\mu), v(\nu) \rangle_{H[L]} = \int_{X \times X} L(x, y) \, d\mu(x) \, \overline{d\nu(y)}.$$

Let us state the same construction more formally. Consider the linear space $\mathcal{M}(X)$ of all compactly supported complex-valued measures on X. Consider the scalar product in $\mathcal{M}(X)$ defined by the formula

$$\langle \mu, \nu \rangle = \int_{X \times X} L(x, y) \, d\mu(x) \, \overline{d\nu(y)}.$$

We obtain a structure of a pre-Hilbert space in $\mathcal{M}(X)$. The space $H[L]$ is the Hilbert space associated with the pre-Hilbert space $\mathcal{M}(X)$ (elements of the supercomplete basis correspond to measures supported at points).

[11] Other terms for v_x are *overfilled basis* or *system of coherent states.*

1.3. *Scalar Products in Spaces of Distributions*

Assume X is a smooth manifold and the kernel $L(x, y)$ is smooth. Denote by $\mathscr{D}$ the space of compactly supported distributions on X. Consider the scalar product in $\mathscr{D}$ given by the formula

$$\langle \chi, \varphi \rangle = \{L(x, y), \chi(x) \otimes \overline{\varphi(y)}\}, \tag{1.2}$$

where the brackets $\{\cdot, \cdot\}$ denote the pairing of smooth functions and distributions. Consider the Hilbert space H associated with the pre-Hilbert space $\mathscr{D}$. Denote the δ-distribution supported at a point x by δ_x. Obviously,

$$\langle \delta_x, \delta_y \rangle = L(x, y).$$

Hence, we can identify H with $H[L]$ and the vectors $\delta_x \in H$ with elements of the supercomplete basis $v_x \in H[L]$.

Remark 1. The space of distributions equipped with the scalar product (1.2) is not complete. This means that some vectors of $H[L]$ cannot be represented by distributions.

Remark 2. The scalar product (1.2) in $\mathscr{D}$ can be degenerate. This means that a vector $h \in H$ can be represented by a distribution in various ways.

1.4. *The Embedding of $H[L]$ to the Space of Functions on X*

For any $h \in H[L]$ we consider the function

$$f_h(x) := \langle h, v_x \rangle_{H[L]}$$

on the space X. Obviously, the map $h \mapsto f_h$ is an embedding of $H[L]$ to the space of all functions on X. We denote the image of the embedding by $H^\circ[L]$. By construction, the space $H^\circ[L]$ has the structure of a Hilbert space:

$$\langle f_h, f_{h'} \rangle_{H^\circ[L]} := \langle h, h' \rangle_{H[L]}.$$

LEMMA 1.1. *Assume X is a separable metric space and let the kernel $L(x, y)$ be continuous. Let a sequence $h_j \in H[L]$ converge to h. Then f_{h_j} converges to f_h uniformly on compacts.*

Proof. Let $Y \subset X$ be a compact set. Let $x \in Y$. Then

$$\begin{aligned} |f_{h_j}(x) - f_h(x)| &= |\langle (h_j - h), v_x \rangle_{H[L]}| \\ &\leqslant \|h_j - h\| \cdot \|v_x\| = \|h_j - h\| \cdot \sqrt{L(x, x)} \\ &\leqslant \|h_j - h\| \cdot \sqrt{\max_{x \in Y} L(x, x)}. \quad \blacksquare \end{aligned}$$

Obviously, the function $\varphi_a(x) \in H^\circ[L]$ associated with the vector $v_a \in H[L]$ is given by the formula

$$\varphi_a(x) = L(x, a). \tag{1.3}$$

LEMMA 1.2. *Let X be a locally compact metric space and let the kernel $L(x, y)$ be continuous. Then functions $f_h \in H^\circ[L]$ are continuous.*

Proof. The linear span of the functions φ_a is dense in $H^\circ[L]$. Then we apply Lemma 1.1. ∎

Remark. Finite linear combinations of the elements of the supercomplete basis v_x are dense in $H[L]$. Hence finite linear combinations $\sum_j^N c_j \varphi_{h_j}(x)$ are dense in $H^\circ[L]$. Hence any element of $H^\circ[L]$ can be approximated uniformly on compacts by functions having the form $\sum_j^N c_j \varphi_{h_j}(x)$.

LEMMA 1.3 (Reproducing Property). *For any $f \in H^\circ[L]$, $x \in X$, the following identity holds*:

$$f(x) = \langle f, \varphi_x \rangle_{H^\circ[L]}. \tag{1.4}$$

Proof. Let $f = f_h$. Then

$$\langle f_h, \varphi_x \rangle_{H^\circ[L]} = \langle h, v_x \rangle_{H[L]} = f_h(x). \quad ∎$$

Remark. Equation (1.4) gives a nonexplicit description of the scalar product in $H^\circ[L]$ and this description is sufficient for many purposes. Another description is the following identity. Let $e_n(x) \in H^\circ[L]$ be an orthonormal basis. Then

$$L(x, y) = \sum \overline{e_n(x)}\, e_n(y).$$

(*Proof.* consider $\langle \varphi_x, e_n \rangle$.)

LEMMA 1.4. *Let Ω be an open domain in $\mathbb{C}^n$ and let $L(z, u)$ be a positive definite kernel on Ω. Let $L(z, u)$ be holomorphic in the variable u and anti-holomorphic in the variable z. Then all elements of the space $H^\circ[L]$ are holomorphic functions on Ω.*

Proof. This is a corollary of Lemma 1.1. ∎

1.5. *Operations with Positive Definite Kernels*

LEMMA 1.5. (a) *Let $L_1(x, y)$, $L_2(x, y)$ be positive definite kernels on X. Then $L_1(x, y) + L_2(x, y)$ is a positive definite kernel*

(b) *Let $L_1(x, y)$, $L_2(x, y)$ be positive definite kernels on X. Then the kernel $L_1(x, y)\, L_2(x, y)$ is positive definite*

(c) *Let $L_j(x, y)$ be positive definite kernels and suppose the sequence $L_j(x, y)$ converges to $L(x, y)$ point-wise. Then $L(x, y)$ is a positive definite kernel.*

(d) *Let $L_m(x, y)$ be a family of positive definite kernels enumerated by points of some measure space M with positive measure μ. Assume the integral*

$$L^*(x, y) = \int_M L_m(x, y)\, d\mu(m)$$

be absolutely convergent for all $x, y \in X$. Then $L^*(x, y)$ is a positive definite kernel.

(e) *Let $L(x, y)$ be a positive definite kernel and let $\lambda(x)$ be a function on X. Then the kernel $M(x, y) = \lambda(x)\, \overline{\lambda(y)}\, L(x, y)$ is positive definite.*

Proof. (a) Let v_x (resp. w_x) be the supercomplete basis in $H[L_1]$ (respectively $H[L_2]$). We consider the system of vectors $v_x \oplus w_x \in H[L_1] \oplus H[L_2]$. Then

$$L_1(x, y) + L_2(x, y) = \langle v_x \oplus w_x, v_y \oplus w_y \rangle.$$

(b) Proof is similar, $H[L_1 L_2] \subset H[L_1] \otimes H[L_2]$

(d) This is a consequence of (a) and (c).

(e) Indeed, $H[M] = H[L]$ and the supercomplete basis in $H[M]$ consists of vectors $\gamma(x)\, v_x$, where v_x is the supercomplete basis in $H[L]$. ∎

1.6. *Positive Definite Kernels on Homogeneous Spaces*

Let Γ be a group acting on X and let a positive definite kernel $L(x, y)$ be Γ-invariant,

$$L(gx, gy) = L(x, y) \qquad \text{for all} \quad g \in \Gamma,\ x, y \in X.$$

Obviously, for each $g \in \Gamma$ there exists the unique unitary operator $U(g)$: $H[L] \to H[L]$ such that

$$U(g)\, v_x = v_{gx} \qquad \text{for all} \quad x \in X.$$

Then

$$U(g_1 g_2) = U(g_1)\, U(g_2).$$

Hence, $U(g)$ is a unitary representation of Γ.

Let X be a Γ-homogeneous space, $X=\Gamma/K$; let x_0 be an K-fixed point.

Let $L(x, y)$ be a Γ-invariant function. We claim that L is uniquely determined by the function

$$l(y) := L(x_0, y).$$

Indeed, let $u, z \in X$. Then $u = gx_0$ for some element g in Γ and

$$L(u, z) = L(gx_0, z) = L(x_0, g^{-1}z) = l(g^{-1}z).$$

Moreover, for any $\gamma \in K$ we have

$$l(y) = L(x_0, y) = L(\gamma x_0, \gamma y) = L(x_0, \gamma y) = l(\gamma y)$$

We see that the function l is a K-invariant function on $X = \Gamma/K$.

We also can consider a K-invariant function $l(y)$ as a function on double cosets $K\backslash\Gamma/K$.

We see that *there is canonical correspondence between the three following sets*:

— *Γ-invariant functions on $\Gamma/K \times \Gamma/K$*

— *K-invariant functions on Γ/K*

— *functions on $K\backslash\Gamma/K$.*

We say that a K-invariant function on Γ/K or a function on $K\backslash\Gamma/K$ is *positive definite* if the associated kernel on $\Gamma/K \times \Gamma/K$ is positive definite.

1.7. *On K-Invariant Vectors in Representations of Γ*

Let Γ, K, x_0 be the same as above. Let ρ be a unitary representation of Γ in a Hilbert space H. Assume that there exists a K-invariant vector $v \in H$ and assume v to be a *cyclic vector*.[12]

Consider the map $\Gamma/K \to H$ given by the formula

$$gx_0 \mapsto \rho(g)\, v$$

(the image of the map is the Γ-orbit of the vector v). Then the function

$$L(g_1 x_0, g_2 x_0) := \langle \rho(g_1)\, v, \rho(g_2)\, v \rangle_H$$

is a Γ-invariant positive definite kernel on Γ/K.

Hence, we can identify the Hilbert space H with the space $H[L]$; the supercomplete basis in H consists of vectors $\rho(g)\, v$.

[12] This means that the linear span of vectors $\rho(g)\, v$ is dense in H.

The function $l(y)$ in our case is the matrix element $\langle \rho(g)\, v, v\rangle$ and our construction (Segal–Gelfand–Naimark construction) reconstructs the representation ρ by its matrix element.

B. *Kernel Representations*

Assume $p \leqslant q$.

1.8. *Pseudoorthogonal Group* O(p, q)

Consider the linear space $\mathbb{C}^p \oplus \mathbb{C}^q$ equipped with the indefinite hermitian form

$$J((x, y), (u, v)) = \sum_{j=1}^{p} x_j \bar{u}_j - \sum_{k=1}^{q} y_k \bar{v}_k; \qquad (x, y), (u, v) \in \mathbb{C}^p \oplus \mathbb{C}^q. \tag{1.5}$$

The *pseudounitary group* $\mathrm{U}(p, q)$ is the group of all linear operators $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ in $\mathbb{C}^p \oplus \mathbb{C}^q$ preserving the form $J(\cdot, \cdot)$. In other words, a matrix $g \in \mathrm{U}(p, q)$ satisfies the condition

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.6}$$

The *pseudoorthogonal group* $\mathrm{O}(p, q)$ is the subgroup of $\mathrm{U}(p, q)$ consisting of real matrices. Below in Sections 1–5 by the symbol G we denote the group

$$G = \mathrm{O}(p, q).$$

By K we denote the subgroup $\mathrm{O}(p) \times \mathrm{O}(q) \subset G$ consisting of matrices having the form $\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}$. It is a maximal compact subgroup in G.

1.9. *Matrix Balls*

By $\mathrm{B}_{p,q}(\mathbb{C})$ we denote the space of all complex $p \times q$ matrices z having norm < 1 (where *a norm* is the norm of the operator $v \mapsto vz$ from the euclidean space $\mathbb{C}^p$ to the euclidean space $\mathbb{C}^q$; recall that $\|z\|^2$ is the maximal eigenvalue of $z^* z$).

By $\mathrm{B}_{p,q}(\mathbb{R})$ we denote the space of all real $p \times q$ matrices with norm < 1.

The group $\mathrm{U}(p, q)$ acts on the matrix ball $\mathrm{B}_{p,q}(\mathbb{C})$ by *fractional linear* transformations

$$z \mapsto z^{[g]} := (\alpha + z\gamma)^{-1} (\beta + z\delta). \tag{1.7}$$

This action is transitive and the stabilizer of the point $z=0$ is the subgroup $\mathrm{U}(p)\times\mathrm{U}(q)$. Hence, $\mathrm{B}_{p,q}(\mathbb{C})$ is the symmetric space

$$\mathrm{B}_{p,q}(\mathbb{C})=\mathrm{U}(p,q)/\mathrm{U}(p)\times\mathrm{U}(q).$$

In the same way, $\mathrm{B}_{p,q}(\mathbb{R})$ is the symmetric space

$$\mathrm{B}_{p,q}(\mathbb{R})=G/K=\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q).$$

Arbitrary symmetric space admits the unique up to a factor invariant measure. For the space $\mathrm{B}_{p,q}(\mathbb{R})$ the $\mathrm{O}(p,q)$-invariant measure is given by the formula

$$d\lambda(z)=\det(1-z^*z)^{-(p+q)/2}\,d\mu(z), \tag{1.8}$$

where $d\mu(z)$ is the Lebesgue measure on $\mathrm{B}_{p,q}(\mathbb{R})$.

1.10. *Berezin Kernels*

THEOREM 1.6.[13] *The kernel*

$$L_\alpha(z,u)=\det(1-z^*u)^{-\alpha}$$

on the matrix ball $\mathrm{B}_{p,q}(\mathbb{C})$ *is positive definite if and only if*

$$\alpha=0,1,2,...,p-1 \qquad \text{or} \qquad \alpha>p-1. \tag{1.9}$$

Thus, for α satisfying the Berezin condition (1.9), we obtain the Hilbert spaces $H_\alpha:=H[L_\alpha]$ and $H_\alpha^\circ:=H^\circ[L_\alpha]$. The function $L_\alpha(z,u)$ is anti-holomorphic in z and hence by Lemma 1.4 the space H_α° consists of holomorphic functions on the matrix ball $\mathrm{B}_{p,q}(\mathbb{C})$.

Remark. For $\alpha>p+q-1$ the scalar product in H_α° can be represented in the form

$$\langle f,g\rangle_\alpha=C(\alpha)\int_{\mathrm{B}_{p,q}} f(z)\,\overline{g(z)}\,\det(1-z^*z)^{\alpha-p-q}\,d\mu(z),$$

where $d\mu(z)$ is the Lebesgue measure on $\mathrm{B}_{p,q}(\mathbb{C})$ and $C(\alpha)$ is the meromorphic factor determined by the condition $\langle 1,1\rangle_\alpha=1$ (it is evaluated in [21]). In particular, H_{p+q}° is the Bergman space (i.e., the intersection of $L^2(B_{p,q})$ with the space of all holomorphic functions). For

[13] See Berezin [3] (1975); see also Gindikin [11], Rossi and Vergne [47], and Wallach [55]. See also a recent exposition in [9].

$\alpha = q$ we obtain the Hardy space H^2. The scalar product in this case is given by the formula

$$\langle f(z), g(z) \rangle_q = \int_{z^* z = 1} f(z)\, \overline{g(z)}\, dv(z),$$

where $dv(z)$ is the unique $\mathrm{U}(p) \times \mathrm{U}(q)$-invariant measure on the set[14] $zz^* = 1$. For other values of the parameter α there exist integral formulas including partial derivatives but they are not simple (see [1]).

Remark. For $\alpha > p-1$ the space H_α contains all polynomials on $\mathrm{B}_{p,q}(\mathbb{C})$. For $\alpha = 0, 1, ..., p-1$ this is false. Consider the matrix

$$\mathscr{A} = \begin{pmatrix} \frac{\partial}{\partial z_{11}} & \cdots & \frac{\partial}{\partial z_{1q}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{p1}} & \cdots & \frac{\partial}{\partial z_{pq}} \end{pmatrix}.$$

Let $k = 0, 1, 2, ..., p-1$. We claim that each function $f \in H_k^\circ$ satisfies the system of partial differential equations

$$\{ D_{j_1, ..., j_k}^{i_1, ..., i_k} f(z) = 0 \qquad \text{for all} \quad (k+1) \times (k+1) \text{ minors } D_{j_1, ..., j_k}^{i_1, ..., i_k} \quad \text{of } \mathscr{A}$$

It is sufficient to verify these conditions for the functions $f(z) = \det(1 - za)^{-k}$. In particular, the space H_0° contains only constants.

Proposition 1.7 [3, 47]. (a) *For any* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{U}(p, q)$ *the operator*

$$\widetilde{T}_\alpha(g)(z)\, f(z) = f((a + zc)^{-1}\, (b + zd)) \det(a + zc)^{-\alpha} \tag{1.10}$$

is unitary in H_α°.

Remark. If α is not integer, then

$$\begin{aligned} \det(a + zc)^{-\alpha} &= \det a^{-\alpha} \det(1 + zca^{-1})^{-\alpha} \\ &= |\det a|^{-\alpha} \cdot e^{-\alpha(i \arg \det a + 2\pi k i)} \det(1 + zca^{-1})^{-\alpha} \end{aligned} \tag{1.11}$$

is a multi-valued function. It is readily seen that $\|ca^{-1}\| < 1$. Hence $(1 + zca^{-1})^{-\alpha}$ is a well-defined single-valued function on the matrix ball

[14] Stiefel manifold.

$B_{p,q}(\mathbb{C})$. Hence the expression (1.11) has a countable family of holomorphic branches on $B_{p,q}(\mathbb{C})$ and formula (1.10) defines a countable family of well-defined operators, which differs by constant factors $e^{2\pi k\alpha i}$.

Proof. Consider the supercomplete basis $\varphi_x(z) = \det(1 - x^* z)^{-\alpha}$ in H_α°. A calculation show that

$$\tilde{T}_\alpha(g^{-1})\, \varphi_x(z) = \det(a + zc)^\alpha\, \varphi_{x^{[g]}}(z).$$

The simple identity

$$\det(1 - x^{[g]}(y^{[g]})^*) = \det(1 - xy^*) \det(a + xc)^{-1}\, \overline{\det(a + yc)}^{\,-1}$$

implies

$$\langle \varphi_x, \varphi_y \rangle_{H^\circ_\alpha} = \langle \tilde{T}_\alpha(g^{-1})\, \varphi_x, \tilde{T}_\alpha(g^{-1})\, \varphi_y \rangle_{H^\circ_\alpha}. \quad \blacksquare$$

Obviously,

$$\tilde{T}_\alpha(g_1)\, \tilde{T}_\alpha(g_2) = e^{2\pi m\alpha i} \tilde{T}_\alpha(g_1 g_2), \qquad \text{where} \qquad m \in \mathbb{Z}$$

If α is integer, then $\tilde{T}_\alpha$ is a linear representation of $U(p, q)$. If α is not integer, then $\tilde{T}_\alpha$ is a projective representation of $U(p, q)$ or a linear representation of the universal covering group $U(p, q)^\sim$ of the group $U(p, q)$.

1.11. *Kernel Representations of* $O(p, q)$

The kernel representation T_α of the group $G = O(p, q)$ is the restriction of the representation $\tilde{T}_\alpha$ to the subgroup $O(p, q)$. We also say that the function $f(z) = 1$ is *the marked vector* in H_α°. We denote this vector by Ξ.

Remark. A kernel representation is a linear representation. Indeed, for real matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we can write $|\det a|^{-\alpha} (1 + zca^{-1})^{-\alpha}$ instead of (1.11).

LEMMA 1.8. *The vector* Ξ *is* $O(p, q)$*-cyclic.*

Proof. Let $Q \subset H_\alpha$ be a subspace containing the G-orbit of Ξ. This orbit consists of the functions (1.11) and hence the functions $\det(1 + zca^{-1})^{-\alpha}$ are contained in Q. But the point ca^{-1} is the image of 0 under the fractional linear transformation (1.7). Since the action of $O(p, q)$ is transitive on $B_{p,q}(\mathbb{R})$, the subspace Q contains all functions $\varphi_u(z) = \det(1 - zu^*)^{-\alpha}$, where $u \in B_{p,q}(\mathbb{R})$. Furthermore, since the family φ_u depends on u antiholomorphically, $\varphi_u \in Q$ for all $u \in B_{p,q}(\mathbb{C})$. But φ_u is the supercomplete basis in H_α°. Hence, $Q = H_\alpha^\circ$. $\blacksquare$

LEMMA 1.9. *Any* $\mathrm{O}(p,q)$*-invariant subspace in* H_α° *contains an* $\mathrm{O}(p,q)$*-invariant vector.*

Proof. Assume $H_\alpha^\circ = R \oplus Q$, where R, Q are invariant subspaces. Assume that R does not contain an $\mathrm{O}(p) \times \mathrm{O}(q)$-invariant vector. Then the projection of Ξ to R is zero, and hence $\Xi \in Q$. But Ξ is cyclic. Thus, $Q = H_\alpha^\circ$. ∎

1.12. *Another Description of the Kernel-Representations*

Let α satisfy the Berezin conditions (1.9). By Lemma 1.5(e), the kernel

$$M_\alpha(z,u) = \frac{\det(1-zz^*)^{\alpha/2}\det(1-uu^*)^{\alpha/2}}{\det(1-zu^*)^\alpha} \tag{1.12}$$

on $\mathrm{B}_{p,q}(\mathbb{R})$ is positive definite. A simple calculation shows that the kernel M_α is $\mathrm{O}(p,q)$-invariant. Hence (see Section 1.6) we obtain a unitary representation of the group $\mathrm{O}(p,q)$ in the Hilbert space $H[M_\alpha] \simeq H^\circ[M_\alpha]$ (see Section 1.6). The group $\mathrm{O}(p,q)$ acts in $H^\circ[M_\alpha]$ by the substitutions

$$f(z) \mapsto f((a+zc)^{-1}(b+zd)). \tag{1.13}$$

The *marked vector* Ξ in this model is the element of the supercomplete basis corresponding to the point $0 \in \mathrm{B}_{p,q}(\mathbb{R})$; i.e., $\Xi(z) = \det(1-zz^*)^{\alpha/2}$.

We claim that two constructions of the kernel representations are equivalent. The canonical unitary $\mathrm{O}(p,q)$-intertwining operator

$$A\colon H^\circ[L_\alpha] \to H^\circ[M_\alpha]$$

is defined by the formula

$$Af(z) = f(z)\det(1-zz^*)^{\alpha/2}, \qquad \text{where} \qquad f \in H^\circ[L_\alpha], \qquad z \in B_{p,q}(\mathbb{R}).$$

This map takes elements of the supercomplete basis in $H^\circ[L_\alpha]$ to elements of the supercomplete basis in $H^\circ[M_\alpha]$. It also takes the marked vector $\Xi \in H^\circ[L_\alpha]$ to the marked vector $\Xi \in H^\circ[M_\alpha]$.

1.13. *Limit as* $\alpha \to \infty$

Let λ be the $\mathrm{O}(p,q)$-invariant measure on $\mathrm{B}_{p,q}(\mathbb{R})$ (see (1.8)). Denote by C_0 the space of compactly supported continuous functions on $\mathrm{B}_{p,q}(\mathbb{R})$. If $f \in C_0$, then $f(z)\,\lambda(z)$ is a complex valued measure on $\mathrm{B}_{p,q}$. Hence (see Section 1.2) we obtain the scalar product in the space C_0 given by

$$\langle f_1, f_2\rangle = A_\alpha \int_{\mathrm{B}_{p,q}(\mathbb{R}) \times \mathrm{B}_{p,q}(\mathbb{R})} M_\alpha(z,u)\, f_1(z)\,\overline{f_2(u)}\, d\lambda(z)\, d\lambda(u). \tag{1.14}$$

Let us define the normalization constant A_α by the condition

$$A_\alpha = \left(\int_{\mathrm{B}_{p,q}(\mathbb{R})} (1 - zz^*)^\alpha \, d\lambda(z) \right)^{-1}$$

(it is a Hua Loo Keng integral; see (3.5)). Obviously, $M_\alpha(z, z) = 1$ and $M_\alpha(z, u) < 1$ if $z \neq u$. It is readily seen that the sequence $A_\alpha M_\alpha(z, u)$ approximates the δ-distribution $\delta(z - u)$ on $\mathrm{B}_{p,q}(\mathbb{R}) \times \mathrm{B}_{p,q}(\mathbb{R})$. Thus, the limit of the scalar products (1.14) as $\alpha \to \infty$ is

$$\langle f_1, f_2 \rangle = \int_{\mathrm{B}_{p,q}(\mathbb{R})} f_1(z) \overline{f_2(z)} \, d\lambda(z).$$

In this sense *the limit of kernel representations as* $\alpha \to \infty$ *is the space* $L^2(G/K)$.

Remark. We emphasis that the action of $\mathrm{O}(p, q)$ in $L^2(\mathrm{O}(p, q)/\mathrm{O}(p) \times \mathrm{O}(q))$ and in all the spaces $H^\circ[M_\alpha]$ is given by the same formula (1.13) and only scalar product in the space of functions varies. We will see that the spectrum of a kernel representation T_α and the structure of the Plancherel formula essentially depend on α.

1.14. *Preliminary Remarks on the Plancherel Formula*

Our purpose is to obtain a decomposition of the kernel representation T_α on irreducible representations.

An irreducible representation of $G = \mathrm{O}(p, q)$ is called *spherical* if it contains a K-fixed vector. This vector is called *spherical vector*. Recall that the space of K-fixed vectors for G has dimension 0 or 1 (Gelfand theorem; see for instance [17, Theorem 4.3.1 and Lemma 4.36]). Denote the set of all unitary spherical representations of $\mathrm{O}(p, q)$ by $\hat{G}_{sph}$.

Remark. An explicit description of the set $\hat{G}_{sph}$ is not known. The parametrization of all (generally speaking nonunitary) spherical representations of $\mathrm{O}(p, q)$ is simple and it is given below in Section 1.17).

By H_ρ we denote the space of a spherical representation ρ; by $\xi(\rho)$ we denote the spherical vector in H_ρ whose length is 1.

LEMMA 1.10. *The decomposition of the kernel-representation* T_α *on irreducible representations has the form*

$$T_\alpha(g) = \int_{\rho \in \hat{G}_{sph}} \rho(g) \, d\nu_\alpha(\rho), \tag{1.15}$$

where ν_α *is a Borel measure on* $\hat{G}_{sph}$.

Remark. For a definition of direct integrals of representations and the abstract Plancherel formula see, for instance, [23, 8.4].

Proof. By Lemma 1.9, the decomposition contains only spherical representations. Hence, by the abstract Plancherel theorem the representation $T_\alpha(g)$ has the form

$$T_\alpha(g) = \bigoplus_{j=1}^{\kappa} R_j,$$

where

$$R_j = \int_{\rho \in \hat{G}_{sph}} \rho(g)\, d\nu_\alpha^j(\rho)$$

and the measure ν_α^{j+1} is absolutely continuous with respect to ν_α^j for all j. The number κ can be 1, 2, ..., ∞. We must prove that $\kappa = 1$.

All K-fixed vectors in R_j are functions having the form $\varphi_j(\rho)\, \xi(\rho)$, where $\varphi_j(\rho)$ is a ν_α^1-measurable function on $\hat{G}_{sph}$.

Consider the projection $\Xi^{(1,2)}$ of the marked vector Ξ to $R_1 \oplus R_2$. Since the vector Ξ is cyclic in whole space, its projection must be cyclic in $R_1 \oplus R_2$. The vector $\Xi^{(1,2)}$ has the form

$$(\varphi_1(\rho)\, \xi(\rho), \varphi_2(\rho)\, \xi(\rho)) \in R_1 \oplus R_2.$$

Obviously, the cyclic span of $\Xi^{(1,2)}$ in $R_1 \oplus R_2$ contains only vectors

$$(q_1(\rho)\, \xi(\rho), q_2(\rho)\, \xi(\rho))$$

satisfying the condition

$$\varphi_2(\rho)\, q_1(\rho) = \varphi_1(\rho)\, q_2(\rho).$$

If $\nu_\alpha^2 \neq 0$ we obtain a contradiction, since the cyclic span of $\Xi^{(1,2)}$ is a proper subspace in $R_1 \oplus R_2$. ∎

1.15. *Normalization of the Plancherel Measure*

The measure ν_α in (1.15) is defined up to equivalence of measures.[15]

[15] Measures μ, ν are equivalent if there exists a function χ such that $\chi \neq 0$ almost everywhere (in sense of ν) and $\mu = \chi\nu$.

The image of the marked vector Ξ in the direct integral (1.15) is some function $\varphi(\rho)\,\xi(\rho)$, where $\xi(\rho)$ is a unit K-fixed vector in H_ρ. It is convenient to assume

$$\varphi(\rho)=1. \tag{1.16}$$

This assumption uniquely determines the measure ν_α.

Remark. Assumption (1.16) is not restrictive. Indeed, let us assume that the image of Ξ in (1.15) is a function $\gamma(\rho)\,\xi(\rho)$. Then the Plancherel measure is completely defined by this assumption and it equals to $(1/\sqrt{|\gamma|})\,\nu_\alpha$, where ν_α is our normalized measure.

After the normalization (1.16) we obtain the equality of matrix elements

$$\langle T_\alpha(g)\,\Xi,\Xi\rangle_{H_\alpha}=\left\langle\left[\int_{\rho\in\hat{G}_{sph}}\rho(g)\,d\nu_\alpha(\rho)\right]\cdot 1,1\right\rangle \tag{1.17}$$

or

$$\langle T_\alpha(g)\,\Xi,\Xi\rangle_{H_\alpha}=\int_{\rho\in\hat{G}_{sph}}\langle\rho(g)\,\xi(\rho),\xi(\rho)\rangle_{H_\rho}\,d\nu_\alpha(\rho) \tag{1.18}$$

(recall that H_α is the space of the kernel representation and H_ρ are the spaces of spherical representations).

Conversely, assume that we know a measure ν_α on $\hat{G}_{sph}$ satisfying condition (1.18). Then it satisfies condition (1.17). Hence the representations T_α and $\int_{\hat{G}_{sph}}\rho(g)\,d\nu_\alpha$ have the same matrix elements, and therefore they are canonically equivalent (see Section 1.7).

The marked vector Ξ is K-invariant; therefore (see Section 1.6) we can consider the matrix element

$$\mathscr{B}_\alpha(g):=\langle T_\alpha(g)\,\Xi,\Xi\rangle,\qquad g\in G$$

as a function on the symmetric space $B_{p,q}=G/K$ or a function on double cosets $K\backslash G/K$. The vectors Ξ and $T_\alpha(g)\,\Xi$ are elements of the supercomplete basis in $H[M_\alpha]$; therefore the function $\mathscr{B}_\alpha$ can be easily evaluated.

In the matrix ball model of G/K the function $\mathscr{B}_\alpha$ is given by the formula

$$\mathscr{B}_\alpha(z)=\det(1-zz^*)^{\alpha/2};\qquad z\in \mathrm{B}_{p,q}. \tag{1.19}$$

Let us obtain the formula for $\mathscr{B}_\alpha$ as a function on $K\backslash G/K$.

First let us give a natural parametrization of the double cosets $K\backslash G/K$. Denote by a_t the element of $\mathrm{O}(p, q)$ given by the matrix

$$a_t = \begin{pmatrix} \cosh t_1 & & & \sinh t_1 & & & 0 & \cdots & 0 \\ & \ddots & & & \ddots & & & \ddots & \\ & & \cosh t_p & & & \sinh t_p & 0 & \cdots & 0 \\ \sinh t_1 & & & \cosh t_1 & & & 0 & \cdots & 0 \\ & \ddots & & & \ddots & & & \ddots & \\ & & \sinh t_p & & & \cosh t_p & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & \\ & \ddots & & & \ddots & & & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & \cdots & 1 \end{pmatrix}. \tag{1.20}$$

Obviously,

$$a_{t+s} = a_t a_s, \qquad t, s \in \mathbb{R}^p.$$

We denote by $\mathscr{A}$ the subgroup in $\mathrm{O}(p, q)$ consisting of all elements a_t.

It is readily seen that an arbitrary element g of $G = \mathrm{O}(p, q)$ can be represented in the form

$$g = k_1 a_t k_2, \qquad \text{where} \qquad k_1, k_2 \in K, \qquad a_t \in \mathscr{A}.$$

The collection of the parameters $t = (t_1, ..., t_p)$ is uniquely determined up to permutations of t_j and reflections

$$(t_1, ..., t_p) \mapsto (\sigma_1 t_1, ..., \sigma_p t_p), \tag{1.21}$$

where $\sigma_j = \pm 1$.

We denote by D_p the hyperoctahedral group,[16] i.e. the group of transformations of $\mathbb{R}^p$ generated by permutations of coordinates and the reflections (1.21). *We identify the set $K\backslash G/K$ with the set of D_p-orbits on $\mathscr{A}$.*[17] In the coordinates $(t_1, ..., t_p)$ the matrix element $\mathscr{B}_\alpha$ is given by the formula

$$\mathscr{B}_\alpha(t_1, ..., t_p) = \prod_{k=1}^{p} \cosh^{-\alpha} t_k. \tag{1.22}$$

[16] It is a Weyl group. The series $\mathrm{O}(p, p)$ is not an exception since we consider $\mathrm{O}(p, p)$ and not $\mathrm{SO}(p, p)$.

[17] We also can identify the set $K\backslash G/K$ with the subset in $\mathbb{R}^p$ defined by the inequalities $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_p \geqslant 0$.

Thus, we must obtain the expansion (1.18) of the function $\mathscr{B}_\alpha$ given by formula (1.19) or (1.22) in positive definite spherical functions.

Our purpose in Section C is to give an expression for the spherical functions of the group $O(p, q)$.

C. *Spherical Representations and Spherical Transform*

In this section we give a description of spherical representations of $O(p, q)$, this description is explicit for representations of general position and semi-explicit for some exceptional values of parameters. We also present Harish–Chandra integral formula for spherical functions and preliminaries on spherical transform.

We also describe another model of the symmetric space $O(p, q)/O(p) \times O(q)$, since it is necessary for our calculations.

1.16. *Parabolic Subgroup and Flag Manifold*

Consider the space $\mathbb{R}^p \oplus \mathbb{R}^q$ equipped with the indefinite symmetric form J defined by formula (1.5). Denote by $e_1, ..., e_{p+q}$ the standard basis in $\mathbb{R}^{p+q}$.

A subspace $V \subset \mathbb{R}^p \oplus \mathbb{R}^q$ is called *isotropic* if the form J is zero on V. An *isotropic flag* $\mathscr{V}$ in $\mathbb{R}^p \oplus \mathbb{R}^q$ is a family of isotropic subspaces

$$\mathscr{V}\colon V_1 \subset V_2 \subset \cdots \subset V_p, \qquad \text{where} \qquad \dim V_j = j.$$

The *flag manifold* $\mathscr{F}$ is the space of all isotropic flags in $\mathbb{R}^p \oplus \mathbb{R}^q$.

The space $\mathscr{F}$ is an $O(p, q)$-homogeneous space. A *minimal parabolic subgroup* in $O(p, q)$ is the stabilizer of a point $\mathscr{V} \in \mathscr{F}$. Let us give more explicit description of the minimal parabolic subgroup.

For this let us consider the basis $v_1, ..., v_p, w_1, ..., w_{q-p}, v'_1, ..., v'_p$ in $\mathbb{R}^p \oplus \mathbb{R}^q$ defined by

$$v_j = \frac{1}{\sqrt{2}}(e_j + e_{q+j}); \qquad v'_j = \frac{1}{\sqrt{2}}(e_j - e_{q+j}), \qquad w_k = e_{p+k}. \quad (1.23)$$

Then

$$J(v_k, v'_k) = 1, \qquad J(w_k, w_k) = 1,$$

and the scalar products of all other pairs of the basic vectors are zero. In the new basis, matrices $g \in O(p, q)$ satisfy the condition

$$g \mathscr{J} g^t = \mathscr{J},$$

where $\mathcal{J}$ is the $(p+(q-p)+p)\times(p+(q-p)+p)$ matrix given by

$$\mathcal{J}=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

The Lie algebra $\mathfrak{g}$ of the group $G=\mathrm{O}(p, q)$ consists of matrices S satisfying the condition

$$S\mathcal{J}+\mathcal{J}S^t=0.$$

The explicit form of this condition is

$$S=\begin{pmatrix}\alpha&\beta&\gamma\\\varepsilon&\delta&-\beta^t\\\mu&-\varepsilon^t&-\alpha^t\end{pmatrix},\qquad \gamma^t=-\gamma,\quad \delta^t=-\delta,\quad \mu^t=-\mu.$$

Denote by L_k the subspace in $\mathbb{R}^p\oplus\mathbb{R}^q$ generated by the last k basic vectors $v'_{p-k+1}, ..., v'_p$. We denote by $P\subset\mathrm{O}(p, q)$ the stabilizer in $\mathrm{O}(p, q)$ of the isotropic flag

$$\mathscr{L}\colon L_1\subset\cdots\subset L_p \tag{1.24}$$

(we consider the right action of $\mathrm{O}(p, q)$ on vector rows). The subgroup P is a minimal parabolic subgroup in $\mathrm{O}(p, q)$ and

$$\mathscr{F}\simeq\mathrm{O}(p, q)/P.$$

Elements of the parabolic subgroup P in the basis (1.23) have the form

$$\begin{pmatrix}A^{t-1}&*&*\\0&C&*\\0&0&A\end{pmatrix}, \tag{1.25}$$

where A is an upper triangular matrix and $C\in\mathrm{O}(q-p)$.

Elements of the subgroup $\mathscr{A}$ (see Section 1.15) in the basis (1.23) are diagonal matrices with eigenvalues

$$e^{-t_1}, ..., e^{-t_p}, 1, ..., 1, e^{t_1}, ..., e^{t_p}.$$

Remark. Let us change the order of the basis elements (1.23) to $v_p, \ldots, v_1, w_1, \ldots, w_{q-p}, v'_1, \ldots, v'_p$. The Lie algebra $\mathfrak{g}$ in this basis consists of matrices

$$S = \begin{pmatrix} \alpha & \beta & \gamma \\ \varepsilon & \delta & -\beta^t\theta \\ \mu & -\theta\varepsilon^t & -\theta\alpha^t\theta \end{pmatrix}, \qquad \gamma^t = -\theta\gamma\theta, \quad \delta^t = -\delta, \quad \mu^t = -\theta\mu\theta, \tag{1.25a}$$

where θ is the $p \times p$ matrix having units on the second diagonal and zeroes outside the second diagonal.

Elements of the parabolic subgroup P in this basis are upper triangular matrices

$$\begin{pmatrix} \sigma_1 e^{-t_p} & * & \cdots & * & * & * & \cdots & * & * \\ 0 & \sigma_2 e^{-t_{p-1}} & \cdots & * & * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_p e^{-t_1} & * & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & C & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & 0 & \sigma_p e^{t_1} & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \sigma_2 e^{t_{p-1}} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \sigma_1 e^{t_p} \end{pmatrix}, \tag{1.25b}$$

where $\sigma_j = \pm 1$ and $C \in \mathrm{O}(q-p)$.

1.17. *Spherical Representations*

Denote by Gr_k the space of all k-dimensional isotropic subspaces in $\mathbb{R}^p \oplus \mathbb{R}^q$. Consider the tautological embedding of the flag space $\mathscr{F}$ into the product of the Grassmannians $\times_{k=1}^p \mathrm{Gr}_k$ (to each point $\mathscr{V}$: $V_1 \subset \cdots \subset V_p$ we assign the point $(V_1, \ldots, V_p) \in \times_{k=1}^p \mathrm{Gr}_k$).

Consider the natural action of $\mathrm{O}(p, q)$ on Gr_k. For $g \in G$ we denote by $j_k(g, V)$ the Jacobian of the transformation g at the point $V \in \mathrm{Gr}_k$. By $J(g, \mathscr{V})$ we denote the Jacobian of the transformation g on the flag space $\mathscr{F}$.

Remark. Let us evaluate $J(g, \mathscr{V})$ and $j_k(g, V)$ for $\mathscr{V} = \mathscr{L}$ and for $g \in P$ given by formula (1.25b). In the first case we must evaluate the determinant of the transformation g on the tangent space at the point $\mathscr{L}$. For the flag manifold $\mathscr{F} = G/P$, the tangent space at $\mathscr{L}$ is the quotient of the associated

Lie algebras $\mathfrak{g}/\mathfrak{p}$. The Lie algebra $\mathfrak{p}$ of P is the algebra of upper triangular block matrices having the block structure (1.25.b). The space $\mathfrak{g}/\mathfrak{p}$ can be identified with the space of lower triangular matrices (or more formally with the space of matrices (1.25.a) defined up to an addition of elements of $\mathfrak{p}$; the group of matrices (1.24.b) acts on this quotient space by conjugations). The eigenvalues of the transformation (1.25.b) on $\mathfrak{g}/\mathfrak{p}$ are

— $e^{t_l - t_k}$, where $k < l$ (the block α)

— e^{t_k} with multiplicities $(q-p)$ (the block ε)

— $e^{t_k + t_l}$, where $k > l$ (the block μ).

This gives the expression

$$J(g, \mathscr{L}) = \exp\left\{\sum t_j(q-p+2j-2)\right\}$$
$$= \exp\left\{(q+p-2)(t_1 + \cdots + t_p) - 2\sum_{k=1}^{p-1} (t_1 + \cdots + t_k)\right\}.$$

In the same way, points of the tangent space to the Grassmannian Gr_k at the point L_k can be identified with left lower $(k \times (q+p-k)$ blocks of matrices (1.25.a). This implies the following formula for Jacobians:

$$j_k(g, \mathscr{L}) = \exp\{(q+p-k-1)(t_{p-k+1} + \cdots + t_p)\}, \qquad \text{where} \qquad g \in P.$$

Fix $s_1, ..., s_p \in \mathbb{C}$. Assume $s_0 = 0$. We define the representation $\tilde{\pi}_s = \tilde{\pi}_{s_1, ..., s_p}$ of the group $\mathrm{O}(p, q)$ in the space of functions on $\mathscr{F}$ by the formula

$$\tilde{\pi}_s(g)\, f(V_1, ..., V_p) = f(gV_1, ..., gV_p)\, J(g, \mathscr{V})^{1/2} \prod_{k=1}^{p} j_k(g, V_k)^{(s_j - s_{j-1})/(q+j)}.$$

Remark. The representation $\tilde{\pi}_s$ is a Harish–Chandra module[18] and hence a topology in the space of functions on $\mathscr{F}$ is not essential. For instance, we can consider the space $L^2(\mathscr{F})$, the space of smooth functions $C^\infty(\mathscr{F})$, the space of distributions $\mathscr{D}(\mathscr{F})$, the space of hyperfunctions, etc. For us it is more convenient to consider the space of distributions.

Remark. Consider the δ-function $\delta_{\mathscr{L}}$ supported at the point $\mathscr{L} \in \mathscr{F}$ (see (1.24)). Obviously, the delta-function $\delta_{\mathscr{L}}$ is an eigenfunction with respect

[18] This means (for instance, see [24]) that the spectrum of the maximal compact subgroup K in the space of functions on $\mathscr{F}$ has finite multiplicities. The finiteness of multiplicities is a corollary of $\mathrm{O}(p) \times \mathrm{O}(q)$-homogeneity of $\mathscr{F}$.

to the parabolic subgroup P. Our calculations of Jacobians show that for a matrix $g \in P$ given by (1.25.b) we have

$$\tilde{\pi}_s(g)\,\delta_{\mathscr{L}}(\mathscr{V}) = \exp\left\{-\sum_{j=1}^{p} t_j(s_j + (q-p)/2 + j - 1)\right\}\delta_{\mathscr{L}}(\mathscr{V}). \tag{1.26}$$

We want to define a *canonical irreducible subquotient* π_s in $\tilde{\pi}_s$.

Remark. For generic $s \in \mathbb{C}^n$ the representation $\tilde{\pi}_s$ is irreducible and hence $\pi_s \simeq \tilde{\pi}_s$.

Consider the function $f_0(\mathscr{V}) = 1$ on the space $\mathscr{F}$, it is the unique K-invariant function on $\mathscr{F}$ (since $\mathscr{F}$ is K-homogeneous). Denote by S the G-cyclic span of f_0. Denote by R the sum of all proper $\mathrm{O}(p, q)$-submodules in S.

Lemma 1.11. $R \neq S$.

Proof. Indeed, there is the unique K-fixed vector in S and this vector is cyclic. Hence it cannot be an element of a proper submodule. Hence a proper submodule in S doesn't contain a K-fixed vector. Thus, R also has no K-fixed vectors and hence $f_0 \notin R$. ∎

We define the $\mathrm{O}(p, q)$-module π_s by

$$\pi_s = S/R.$$

Theorem 1.12.[19] *The representations π_s are precisely all spherical representations of* $\mathrm{O}(p, q)$. *Moreover,*

$$\pi_s \simeq \pi_{s'} \qquad \textit{iff there exists} \qquad \gamma \in D_p \qquad \textit{such that} \qquad \gamma s = s'.$$

Hence we can consider our Plancherel measure ν_α as a measure on $\mathbb{C}^p/D_p$. It will be more convenient for us to consider the Plancherel measure as a D_p-invariant measure on $\mathbb{C}^p$ or any measure on $\mathbb{C}^p$ whose D_p-average is ν_α.

1.18. *Unitary Spherical Representations*

Lemma 1.13. *Assume a representation π_s be unitary. Then for any j*

$$\operatorname{Re} s_j = 0 \qquad \textit{or} \qquad \operatorname{Im} s_j = 0. \tag{1.27}$$

Proof. The representation dual to π_s is π_{-s}. The complex conjugate representation to π_s is $\pi_{\bar{s}}$. If π_s is unitary, then the dual representation is equivalent to the complex conjugate representation. Hence $-s = \gamma\bar{s}$ for some $\gamma \in D_p$. ∎

[19] For instance, see [17, Theorem 4.4.3].

It is readily seen that for pure imaginary $s_1, ..., s_p$ the representation π_s is unitary in $L^2(\mathcal{F})$. These representations are called *representations of the principal nondegenerate series.*

For some other values of s representations π_s also are unitary, but scalar products in these cases are more complicated.

THEOREM 1.14 (See [17, 4.8.1]). *Denote by ρ the vector*

$$((q-p)/2, (q-p)/2+1, ..., (q+p)/2-1)) \in \mathbb{R}^p.$$

Denote by Q the convex polyhedron in $\mathbb{R}^p$ with vertices $\gamma\rho$, where $\gamma \in D_p$. Then for any unitary representation π_s of $\mathrm{O}(p, q)$

$$(\mathrm{Re}\, s_1, ..., \mathrm{Re}\, s_p) \in Q. \tag{1.28}$$

Moreover, the spherical function of a spherical representation π_s is bounded if and only if condition (1.28) *holds.*

Our next purpose is to give an integral formula for the spherical functions in an explicit form. For this we must present another realization of the symmetric space G/K.

1.19. *Matrix Wedges*

First consider the case $p=q$. Consider the matrix ball $\mathrm{B}_{q,q}(\mathbb{R})$. Consider the *Cayley transform*

$$\mathrm{Cay}\colon z \mapsto \frac{1-z}{1+z}. \tag{1.29}$$

The image of the matrix ball $\mathrm{B}_{q,q}(\mathbb{R})$ under the Cayley transform Cay is the wedge W_q consisting of matrices R satisfying the condition[20]

$$R + R^t > 0$$

(where the notation $Q > 0$ means that a matrix Q is positive definite). It is convenient to write R in the form

$$R = T + S, \qquad \text{where} \qquad T = T^t > 0; \qquad S = -S^t.$$

The group $\mathrm{O}(q, q)$ acts on $\mathrm{B}_{q,q}(R)$ and hence it acts on W_q. For a description of the latter action we consider the basis (1.23) in $\mathbb{R}^q \oplus \mathbb{R}^q$. In our case $p=q$, hence the basis elements w_j are missing. Hence $\mathrm{O}(q, q)$ in

[20] A matrix satisfying the condition $R + R^t > 0$ is called *dissipative*.

this basis is the group of all real $(q+q)\times(q+q)$-matrices $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The group $\mathrm{O}(q,q)$ acts on W_q by fractional linear transformations

$$R\mapsto R^{[g]}:=(a+Rc)^{-1}\,(b+Rd).$$

In this model, the parabolic subgroup $P\subset \mathrm{O}(q,q)$ becomes the group of real matrices having the form

$$\begin{pmatrix} a & b \\ 0 & a^{t-1} \end{pmatrix}; \qquad \begin{array}{c}\text{where } a \text{ is upper triangular}\\ \text{and } a^{t-1}b=b^ta^{-1}.\end{array} \tag{1.30}$$

Hence, the parabolic subgroup acts on W_q by affine transformations

$$R\mapsto a^{t-1}Ra^{-1}+a^{t-1}b.$$

We emphasis that $a^{-1}b$ is a skew-symmetric matrix.

Consider the action of the group $\mathrm{O}(q,q)$ in the space of functions on W_q by substitutions

$$U(g)\,f(R)=f(R^{[q]}).$$

We want to obtain eigenfunctions of the parabolic subgroup P on the wedge W_q. For arbitrary collection $s_1,\dots,s_p\in\mathbb{C}$, $s_{p+1}=0$ we define the function

$$\Psi_{s_1,\dots,s_q}(R)=\prod_{j=1}^{q}\det[T]_j^{(\theta_j+s_j-s_{j+1})/2}$$
$$\text{where}\quad \theta_1=\cdots=\theta_{p-1}=-1,\qquad \theta_p=p-1. \tag{1.31}$$

Here the symbol $[T]_j$ denotes the left upper $j\times j$ block of the matrix $T=\frac{1}{2}(R+R^t)$.

Consider $g\in P$ given by formula (1.30). Let $e^{t_1},\dots,e^{t_p}$ be the absolute values of the diagonal elements of the block a. Then

$$\Psi_{s_1,\dots,s_q}(R^{[g]})=\exp\left\{-\sum_{j=1}^{q}(j-1+s_j)\,t_j\right\}\Psi_{s_1,\dots,s_q}(R). \tag{1.32}$$

Remark. Compare (1.32) and (1.26).

1.20. *Sections of Wedges*

Consider arbitrary group $O(p, q)$. Let us represent a point $z \in B_{p,q}(\mathbb{R})$ as a block $p \times (p+(q-p))$-matrix $z = (z_1\ z_2)$. Consider the block $(p+(q-p)) \times (p+(q-p))$ matrix

$$\tilde{z} = \begin{pmatrix} 0 & 0 \\ z_1 & z_2 \end{pmatrix} \in B_{q,q}(\mathbb{R}).$$

Thus, we realized the matrix ball $B_{p,q}(\mathbb{R})$ as a submanifold of $B_{q,q}(\mathbb{R})$. The image of $B_{q,q}$ under the Cayley transform (1.29) is the matrix wedge W_q. The image $SW_{p,q}$ of $B_{p,q}(\mathbb{R})$ under the Cayley transform is the set $SW_{p,q}$ of all $(p+(q-p)) \times (p+(q-p))$-matrices $R \in W_q$ having block structure

$$R = \begin{pmatrix} 1 & 0 \\ Q & H \end{pmatrix}. \tag{1.33}$$

The condition $R + R^t > 0$ for a matrix (1.33) is equivalent to the condition

$$\tfrac{1}{2}(H + H^t) - QQ^t > 0 \tag{1.34}$$

(the spaces $SW_{p,q}$ are real sections of so-called Siegel domains of the second type; see [44]).

We will write matrices $R \in SW_{p,q}$ in the form

$$R = \begin{pmatrix} 1 & 0 \\ 2L & M+N \end{pmatrix}, \qquad M = M^t,\ N = -N^t.$$

In these notations condition (1.34) has the form

$$M - LL^t > 0 \qquad \text{or} \qquad \begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix} > 0. \tag{1.35}$$

Clearly, general fractional linear transformations of the wedge W_q do not preserve the section $SW_{p,q}$. It can be easily checked that the group of fractional linear transformations of W_q preserving $SW_{p,q}$ is $O(q-p) \times O(p, q)$ and the subgroup $O(q-p) \subset O(q-p) \times O(p, q)$ acts on $SW_{p,q}$ trivially.[21]

[21] *Proof.* The subgroup in $O(q, q)$ preserving the section $B_{p,q}$ of $B_{q,q}$ is $O(q-p) \times O(p, q)$. On the other hand, $O(q-p) \times O(p, q)$ is a maximal subgroup in $O(q, q)$.

Consider the natural action of the group $O(p, q)$ in the space of functions on $SW_{p,q}$. The eigenfunctions of the parabolic subgroup $P \subset O(p, q)$ in this model are given by the formula

$$\Psi_{s_1, \ldots, s_p}(R) = \prod_{j=1}^{p} \det \left[\begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix} \right]_{q-p+j}^{-(\theta_j + s_j - s_{j+1})/2}$$

$$= \prod_{j=1}^{p} \det[M - LL^t]_j^{(\theta_j + s_j - s_{j+1})/2}, \tag{1.36}$$

where

$$\theta_1 = \cdots = \theta_{p-1} = -1, \qquad \theta_p = \tfrac{1}{2}(q+p) - 1$$

$$s_1, \ldots, s_p \in \mathbb{C}, \qquad s_{p+1} = 0.$$

Consider an element g of the parabolic subgroup P given by (1.25). Let e^{t_j} be absolute values of the eigenvalues of the matrix a. The transformation g takes $\Psi_{s_1, \ldots, s_p}$ to $\lambda \Psi_{s_1, \ldots, s_p}$ where

$$\lambda = \exp\left\{ -\sum ((q-p)/2 + j - 1 + s_j)\, t_j \right\}$$

(see the simple calculations in [35]).

The *Berezin kernel* L_α (see Section 1.10) in the models W_q, $SW_{p,q}$ is given by the formula

$$L_\alpha(R_1, R_2) = \frac{\det(R_1 + R_1^t)^{\alpha/2} \det(R_2 + R_2^t)^{\alpha/2}}{\det(R_1 + R_2^t)^\alpha}.$$

This gives the following expression for the function $\mathscr{B}_\alpha(R)$ in our coordinates:

$$\mathscr{B}_\alpha(R) = L_\alpha(R, 1) = \frac{2^{p\alpha} \det \begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix}^{\alpha/2}}{\det(1 + M + N)^\alpha}. \tag{1.37}$$

The $O(p, q)$-invariant measure on $SW_{p,q}$ is

$$\det \begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix}^{-(p+q)/2} dL\, dM\, dN,$$

where dL, dM, dN are Lebesgue measures on the spaces of matrices.

1.21. *Canonical Embedding of the Spherical G-Module π_s into the Space $C^\infty(G/K)$*

Here we preserve the notations of Section 1.17.

First we define the canonical intertwining operator J_s from $\tilde{\pi}_s$ to $C^\infty(G/K)$. This operator is uniquely defined by the property

$$J_s\colon \delta_{\mathscr{L}} \mapsto \Psi_s,$$

where the P-eigenfunctions $\delta_{\mathscr{L}}$, Ψ_s were defined in Sections 1.17, 1.19 and 1.20. By the intertwining property, we obtain

$$J_s \delta_{g\mathscr{L}}(R) = \Psi_s(R^{[g]})$$

and this defines the operator J_s on all δ-functions. Then we extend J_s by linearity and continuity to the operator from the space of distributions on $\mathscr{F}$ to the space of smooth functions on G/K.

LEMMA 1.15. (a) *The operator J_s induces an embedding of the canonical subquotient π_s to $C^\infty(G/K)$.*

Let us denote by $d\mathbf{k}$ the Haar measure on $K = \mathrm{O}(p) \times \mathrm{O}(q)$. We assume that the measure of the whole group is 1.

Proof. Let R, S be the same as in Section 1.17. Let $Q \subset C^\infty(G/K)$ be a G-invariant closed subspace. Then for any function $f \in R$, its *average*

$$f^K(R) := \int_{\mathbf{k} \in K} f([R]^{\mathbf{k}})\, d\mathbf{k}$$

is contained in Q. Hence, Q contains a K-invariant function.

For this reason, J_s takes the submodule R to 0 (since R has no K-invariant vector). Suppose that J_s is zero on S. Then J_s is an operator from $\tilde{\pi}_s/S$ to $C^\infty(G/K)$. But the module $\tilde{\pi}_s/S$ has no K-invariant vectors. Hence J_s is identical zero and this contradicts its definition. ▮

Let us denote by

$$\Phi_s = \Phi_{s_1, \ldots, s_p}$$

the image of the spherical vector v of the representation π_s under the map J_s.

LEMMA 1.16. *The function Φ_s on G/K coincides with the spherical function of the representation π_s.*

Proof. Denote the spherical function of π_s by Φ'_s. By uniqueness of a spherical vector,

$$(\pi_s(g)\, v)^K := \int_K \pi_s(\mathbf{k}g)\, v\, d\mathbf{k} = \Phi'_s(g)\, v.$$

The K-average of the function $\Phi_s(R^{[\mathbf{k}g]})$ also is proportional to $\Phi_s(R)$. The point $1 \in \mathrm{SW}_{p,q} = G/K$ is K-fixed and hence the coefficient of proportionality is $\Phi_s(1^{[g^{-1}]})$. But J_s takes $(\pi_s(g)\,v)^K$ to $\Phi_s(R^{[g]})^K$ and hence the coefficients of proportionality coincide; i.e., $\Phi'_s(g) = \Phi_s(g^{-1})$. It remains to note that g and g^{-1} are elements of the same double K-coset. ▮

Obviously, the K-fixed function $f_0 = 1$ on $\mathscr{F}$ can be represented in the form

$$f_0(\mathscr{V}) = \int_{\mathbf{k} \in K} \delta_{\mathscr{L}}(\mathbf{k}\mathscr{V})\, d\mathbf{k}.$$

Hence, its image under J_s is the K-average of Ψ_s. This gives the integral formula for spherical functions given in the next subsection.

1.22. *Harish–Chandra Integral Formula for Spherical Functions*

Spherical functions are K-averages of P-eigenfunctions on G/K,

$$\Phi_{s_1, \ldots, s_p}(R) = \int_{\mathbf{k} \in \mathrm{O}(p) \times \mathrm{O}(q)} \Psi_{s_1, \ldots, s_p}(R^{[\mathbf{k}]})\, d\mathbf{k}. \tag{1.38}$$

LEMMA 1.17.

$$|\Phi_{s_1, \ldots, s_p}(t)| \leqslant \Phi_{\mathrm{Re}\, s_1, \ldots, \mathrm{Re}\, s_p}(t). \tag{1.39}$$

Proof. is obvious.

1.23. *Spherical Transform*

Let $f(z)$ be a K-invariant function on G/K. Then the *spherical transform* of f is defined by the formula

$$\hat{f}(s) = \int_{G/K} \Phi_s(z)\, f(z)\, d\lambda(z), \tag{1.40}$$

where λ is the G-invariant measure on G/K.

If $f \in L^2 \cap L^1(G/K)$, then the *Gindikin–Karpelevich inversion formula* (see [9, 13, 14, 17]) is valid,

$$f(z) = C \cdot \int_{i\mathbb{R}^p} \hat{f}(s)\, \Phi_s(z)\, \mathfrak{R}(s)\, ds, \tag{1.41}$$

where C is a known constant (see [17, formula (4.6.40)]) and $\mathfrak{R}(s)$ is the Gindikin–Karpelevich density. For $G = \mathrm{O}(p, q)$ the density is

$$\mathfrak{R}(s) = \prod_{k=1}^{p} \frac{\Gamma((q-p)/2+s_k)\,\Gamma((q-p)/2-s_k)}{\Gamma(s_k)\,\Gamma(-s_k)} \tag{1.42}$$

$$\times \prod_{1 \leqslant k < l \leqslant p} \frac{\Gamma\left(\frac{1}{2}(1+s_l+s_k)\right)\Gamma\left(\frac{1}{2}(1+s_l-s_k)\right)}{\Gamma\left(\frac{1}{2}(s_l+s_k)\right)\Gamma\left(\frac{1}{2}(s_l-s_k)\right)} \tag{1.43a}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(1-s_l+s_k)\right)\Gamma\left(\frac{1}{2}(1-s_l-s_k)\right)}{\Gamma\left(\frac{1}{2}(-s_l+s_k)\right)\Gamma\left(\frac{1}{2}(-s_l-s_k)\right)} \tag{1.43b}$$

Remark. This expression is an elementary function. For instance, using the complement formula for Γ-function, we reduce the factor (1.43) to the form

$$\prod_{1 \leqslant k < l \leqslant p} (s_k^2 - s_l^2) \tan \pi(s_k + s_l) \tan \pi(s_k - s_l). \tag{1.44}$$

If $q-p$ is even, then (1.42) equals

$$\prod_{k=1}^{p} \left\{ \prod_{\tau=0}^{(q-p)/2-1} (\tau^2 - s_k^2) \right\}. \tag{1.45}$$

If $(q-p)$ is odd, then (1.42) equals

$$\prod_{k=1}^{p} \left\{ s_k \tan \pi s_k \prod_{\tau=0}^{(q-p-3)/2} ((\tau+1/2)^2 - s_k^2) \right\}. \tag{1.46}$$

For pure imaginary s we can replace (1.42) and (1.43) by

$$\prod_{k=1}^{p} \left| \frac{\Gamma((q-p)/2+s_k)}{\Gamma(s_k)} \right|^2 \prod_{1 \leqslant k < l \leqslant p} \left| \frac{\Gamma\left(\frac{1}{2}(1+s_l+s_k)\right)\Gamma\left(\frac{1}{2}(1+s_l-s_k)\right)}{\Gamma\left(\frac{1}{2}(s_l+s_k)\right)\Gamma\left(\frac{1}{2}(s_l-s_k)\right)} \right|^2.$$

Nevertheless the long expression (1.42) and (1.43) is more convenient for our calculations.

1.24. *Another Formula for the Spherical Transform*

By the integral formula (1.38) for spherical functions, we can write the spherical transform (1.40) of a K-invariant function in the following form:

$$\hat{f}(s) = \int_{G/K} f(z)\, \Psi_s(z)\, d\lambda(z). \tag{1.47}$$

1.25. *Further Structure of the Paper*

We want to obtain an expansion of the function $\mathscr{B}_\alpha(z)$ in spherical functions. If $\mathscr{B}_\alpha \in L^1 \cap L^2(\mathrm{B}_{p,q}(\mathbb{R}))$ (or $\alpha > p+q-1$), then it is sufficient to evaluate the spherical transform of the function $\mathscr{B}_\alpha(z)$, and the Gindikin–Karpelevich inversion formula gives the required expansion.

In Section 3 we evaluate the spherical transform of $\mathscr{B}_\alpha(z)$ using formula (1.47). The final result is given in Theorem 2.1. Then in Section 4 we construct the analytic continuation of our formula to arbitrary α. As result, we obtain an expansion of $\mathscr{B}_\alpha$ in spherical functions. In Section 5 we prove the positive definiteness of these spherical functions.

D. *Deformation of* L^2 *on Riemannian Compact Symmetric Spaces* $\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)$ *and Kernel Representations of* $\mathrm{O}(p+q)$

This subject is a supplement to the main topic of the paper.

1.26. *The Symmetric Spaces* $\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$ *and* $\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)$

Consider the group $\mathrm{U}(p+q)$ consisting of all complex block $(p+q)\times(p+q)$-matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ satisfying the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the subgroup $\mathrm{O}(p+q) \subset \mathrm{U}(p+q)$ consisting of real matrices.

Consider the Grassmannians $\mathrm{Gr}_{p,q}(\mathbb{C})$ and $\mathrm{Gr}_{p,q}(\mathbb{R})$ consisting of p-dimensional subspaces in $\mathbb{C}^{p+q}$ and $\mathbb{R}^{p+q}$, respectively. Obviously, these Grassmannians are the symmetric spaces

$$\mathrm{Gr}_{p,q}(\mathbb{C}) = \mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$$

$$\mathrm{Gr}_{p,q}(\mathbb{R}) = \mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q).$$

Denote by $\mathrm{Mat}_{p,q}(\mathbb{C})$ (resp. $\mathrm{Mat}_{p,q}(\mathbb{R})$) the space of all $p\times q$-matrices over $\mathbb{C}$ (resp. over $\mathbb{R}$). For any $z\in\mathrm{Mat}_{p,q}$ we define its graph $\mathrm{GRAPH}_z\subset \mathrm{Gr}_{p,q}$. Obviously, the map $z\mapsto \mathrm{GRAPH}_z$ is an embedding $\mathrm{Mat}_{p,q}\to\mathrm{Gr}_{p,q}$ and the image of the embedding is dense in $\mathrm{Gr}_{p,q}$.

In the coordinate $z\in\mathrm{Mat}_{p,q}$, the action of the group $\mathrm{U}(p+q)$ on the Grassmannian $\mathrm{Gr}_{p,q}$ is given by the formula

$$z\mapsto z^{[g]}=(a+zc)^{-1}\,(b+zd) \tag{1.48}$$

coinciding with formula (1.7).

1.27. *Representations* $\tilde{T}_{-n}$

Fix $n=0, 1, 2, \ldots$. Denote by $\varphi_u(z)$ the polynomial on $\mathrm{Mat}_{p,q}$ given by the formula

$$\varphi_u(z)=\det(1+u^*z)^n, \qquad \text{where}\quad a\in\mathrm{Mat}_{p,q}.$$

Denote by H_{-n} the linear span of all polynomials $\varphi_u(z)$. Obviously, the space H_{-n} is finite dimensional (since the degree of the polynomial $\varphi_u(z)$ is $\leqslant pn$).

Consider the action of the group $\mathrm{U}(p,q)$ in the space H_{-n} given by

$$\tilde{T}_{-n}\begin{pmatrix} a & b\\ c & d\end{pmatrix} f(z)=f((a+zc)^{-1}\,(b+zd))\det(a+zc)^n \tag{1.49}$$

coinciding with formula (1.10). Then

$$T_{-n}(g)\,\varphi_u(z)=\varphi_{u^{[g-1]}}\det(a+bu^*)^n.$$

Hence the transformations $\tilde{T}_{-n}(g)$ preserve the space H_{-n}.

Consider the scalar product in H_{-n} given by

$$\langle f_1(z), f_2(z)\rangle_{-n}=C_n\int_{\mathrm{Mat}_{p,q}} f_1(z)\overline{f_2(z)}\det(1+z^*z)^{-n-p-q}\,dz,$$

where the normalization constant C_n is defined by the condition $\langle 1, 1\rangle_{-n}=1$ (it is one of the Hua Loo Keng integrals, see [21]).

It is easy to check that the operators $\tilde{T}_{-n}(g)$ are unitary with respect to this scalar product and

$$\langle\varphi_u, \varphi_v\rangle_{-n}=\det(1+u^*v)^n. \tag{1.50}$$

Hence, the (finite dimensional) Hilbert space H_{-n} is the Hilbert space H° defined by the positive definite kernel

$$L_{-n}(u, v) = \det(1 + u^* v)^n; \qquad a, b \in \mathrm{Mat}_{p,q}. \tag{1.51}$$

1.28. *Kernel Representations of* $\mathrm{O}(p+q)$

The *kernel representation* T_{-n} of the group $\mathrm{O}(p+q)$ is the restriction of the representation $\tilde{T}_{-n}$ to the subgroup $\mathrm{O}(p+q)$.

We also define the *marked vector*

$$\Xi\colon f(z) = 1.$$

1.29. *Limit as* $n \to \infty$

Let us consider the kernel

$$M_{-n}(z, u) = \frac{\det(1 + z^t u)^n}{\det(1 + z^t z)^{n/2} \det(1 + u^t u)^{n/2}}$$

on $\mathrm{Mat}_{p,q}(\mathbb{R})$. By Lemma 1.5(e), the kernel M_{-n} is positive definite. A simple calculation shows, that the kernel is $\mathrm{O}(p+q)$-invariant. Consider the Hilbert space $H^\circ[M_{-n}]$. The operator

$$Af(z) = \det(1 + z^* z)^{n/2} f(z)$$

defines the canonical unitary $\mathrm{O}(p+q)$-intertwining operator $H^\circ[L_\alpha] \to H^\circ[M_\alpha]$.

Obviously, $M_{-n}(z, z) = 1$ and $M_{-n}(z, u) < 1$ for $z \neq u$. The arguments given in Section 1.13 show that a natural limit of the spaces H_{-n} as $n \to \infty$ is

$$L^2(\mathrm{O}(p+q)/\mathrm{O}(p) \times \mathrm{O}(q)).$$

1.30. *Preliminary Remarks on the Plancherel Formula*

By Section 1.7, the matrix element

$$\mathscr{B}_{-n}(g) = \langle T_{-n}(g)\, \Xi, \Xi \rangle_{H_{-n}}$$

is a function on $\mathrm{O}(p+q)/\mathrm{O}(p) \times \mathrm{O}(q) \simeq \mathrm{Gr}_{p,q}(\mathbb{R})$. In the coordinate $z \in \mathrm{Mat}_{p,q}(\mathbb{R})$ it is given by

$$\mathscr{B}_{-n}(z) = \det(1 + z z^t)^{-n/2}.$$

Denote by $\widehat{\mathrm{O}(p+q)}_{sph}$ the set of all irreducible representations of $\mathrm{O}(p+q)$ having an $\mathrm{O}(p) \times \mathrm{O}(q)$-invariant vector (*spherical vector*); a description of this (countable) set is given by the Helgason theorem [17, Theorem 5.4.1]. By H_ρ we denote the space of a spherical representation $\rho \in \widehat{\mathrm{O}(p+q)}_{sph}$. Denote by ξ_ρ the spherical vector in H_ρ having unit length.

Arguments given in Sections 1.13 and 1.14 show that the decomposition of T_{-n} in irreducible representations has the form

$$T_{-n}(g) = \bigoplus_{\rho \in \Delta_n} \rho(g),$$

where Δ_n is a finite subset in $\widehat{\mathrm{O}(p+q)}_{sph}$.

The scalar product in $\bigoplus_{\rho \in \Delta_n} H_\rho$ has the form

$$\left\langle \bigoplus_{\rho \in \Delta_n} v_\rho, \bigoplus_{\rho \in \Delta_n} w_\rho \right\rangle = \sum_{\rho \in \Delta_n} \nu_\rho^n \langle v_\rho, w_\rho \rangle_{H_\rho}, \tag{1.52}$$

where ν_ρ^n are positive constants and $v_\rho, w_\rho \in H_\rho$. Formula (1.52) is called the *Plancherel formula*.

We normalize the constants ν_ρ^n by the assumption

$$\text{the image of} \quad \Xi \quad \text{in} \quad \bigoplus_{\rho \in \Delta_n} H_\rho \quad \text{is} \quad \bigoplus_{\rho \in \Delta_n} \xi_\rho.$$

The constants ν_ρ^n are evaluated in Section 2 as a corollary of the Plancherel formula for kernel-representations of $\mathrm{O}(p, q)$.

E. *An Interpolation between $L^2(\mathrm{O}(p,q))/\mathrm{O}(p)\times\mathrm{O}(q))$ and $L^2(\mathrm{O}(p+q))/\mathrm{O}(p)\times\mathrm{O}(q))$?*

The purpose of the section is a formulation of a strange problem.

1.31. *General Representations $\tilde{T}_\alpha$*

Denote by $Hol(\mathrm{B}_{p,q})$ the space of holomorphic functions in $\mathrm{B}_{p,q}(\mathbb{C})$ equipped with the topology of uniform convergence on compacts.

Consider arbitrary $\alpha \in \mathbb{C}$ and consider the action of $\mathrm{U}(p, q)$ in $Hol(\mathrm{B}_{p,q})$ given by the formula

$$\hat{T}_\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = f((a+zc)^{-1}(b+zd)) \det(a+zc)^{-\alpha}. \tag{1.53}$$

Denote by $\mathscr{H}_\alpha^\circ$ the cyclic span of the function $f(z) = 1$. Denote by $\tilde{T}_\alpha$ the restriction of $\hat{T}_\alpha$ to $\mathscr{H}_\alpha^\circ$. It is easy to observe that $\tilde{T}_\alpha$ is an irreducible Harish–Chandra module.

Let us denote by H_α^{fin} the space of polynomials contained in $\mathscr{H}_\alpha^\circ$. Consider the action of the Lie algebra $u(p, q)$ in H_α^{fin}. The space H_α^{fin} is an irreducible $u(p, q)$-module with a highest weight. For $\alpha \in \mathbb{R}$ there exists the unique $u(p,q)$-invariant hermitian form in H_α^{fin} in (it is called the *Shapovalov form*[22]). In general this form is indefinite.

If $\alpha \in \mathbb{R}$ satisfies the Berezin conditions (1.9), then the Shapovalov form is positive definite. In this case the Shapovalov form coincides with the Berezin scalar product, and the representations $\widetilde{T}_\alpha$ coincides with the representations $\widetilde{T}_\alpha$ constructed in Section 1.10. If α is a negative integer, then $\widetilde{T}_\alpha$ is finite dimensional. By the unitary Weyl trick, there is no difference between finite dimensional representations of $\mathrm{U}(p,q)$, holomorphic finite dimensional representations of $\mathrm{GL}(p+q,\mathbb{C})$ and finite dimensional representations of $\mathrm{U}(p+q)$. The representations $\widetilde{T}_\alpha$ for negative integer α differ from the representations $\widetilde{T}_{-n}$ defined in Section 1.27 by a nonessential change of notations.

1.32. *Nonunitary Kernel Representations of* $\mathrm{O}(p,q)$?

Consider the restriction T_α of $\widetilde{T}_\alpha$ to the subgroup $\mathrm{O}(p,q)$. It is a well-defined representation of the group $\mathrm{O}(p,q)$ in the space $\mathscr{H}_\alpha^\circ$.

We have seen that

$$\lim_{\alpha\to+\infty} T_\alpha \simeq L^2(\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q))$$

$$\lim_{n\to-\infty} T_n \simeq L^2(\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)).$$

It seems that the Plancherel formula (2.5)–(2.15) gives the decomposition of the kernel representation T_α for any complex α. Unfortunately it is a result of "mathematical physics level." This is the solution of a problem which has no satisfactory formulation (since the definition of the abstract Plancherel formula doesn't exist for nonunitary representations).

Remark. For the case $p=1$ the space $\mathscr{H}_\alpha^\circ$ equipped with the Shapovalov form is a Pontryagin space[23] and in this case our Plancherel formula is really the Plancherel formula for arbitrary real α.

The questions of this type were discussed for a long time and they arise in deep Molchanov work [27] (1980) containing the Plancherel decomposition of tensor products of unitary representations of $\mathrm{SL}(2,\mathbb{R})$ (see also [8] and references in this paper). It was clear that the Molchanov formula gives a formal interpolation between tensor products of unitary representations of $\mathrm{SL}(2,\mathbb{R})$ and tensor products of finite dimensional representations. Unfortunately before our time a quite satisfactory group-theoretical interpretation of this interpolation did not exist.

[22] Its definition of highest weight modules over various algebras is uniform; see for instance [32].

[23] This means that the negative inertia index of the Shapovalov form is finite.

2. FORMULATION OF RESULTS

2.1. *Large* α

THEOREM 2.1. *Let* $\alpha > (p+q)/2 - 1$. *Then the spectrum of the kernel-representation* T_α *of the group* $\mathrm{O}(p, q)$ *is supported by nondegenerate principal series and the Plancherel decomposition is given by the formula*

$$\prod_{j=1}^{p} \cosh^{-\alpha} t_j = C \cdot 2^{p\alpha} \frac{1}{\prod_{j=1}^{p} \Gamma(\alpha - j + 1)} \tag{2.1}$$

$$\times \int_{i\mathbb{R}^p} \prod_{k=1}^{p} \left\{ \Gamma\left(\frac{1}{2}(\alpha - (p+q)/2 + 1 + s_k)\right)\right. \tag{2.2a}$$

$$\left.\times \Gamma\left(\frac{1}{2}(\alpha - (p+q)/2 + 1 - s_k)\right)\right\} \tag{2.2b}$$

$$\times \prod_{k=1}^{p} \frac{\Gamma((q-p)/2 + s_k)\, \Gamma((q-p)/2 - s_k)}{\Gamma(s_k)\, \Gamma(-s_k)} \tag{2.3}$$

$$\times \prod_{1 \leqslant k < l \leqslant p} \frac{\Gamma\left(\frac{1}{2}(1 + s_l + s_k)\right) \Gamma\left(\frac{1}{2}(1 + s_l - s_k)\right)}{\Gamma\left(\frac{1}{2}(s_l + s_k)\right) \Gamma\left(\frac{1}{2}(s_l - s_k)\right)} \tag{2.4a}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(1 - s_l + s_k)\right) \Gamma\left(\frac{1}{2}(1 - s_l - s_k)\right)}{\Gamma\left(\frac{1}{2}(-s_l + s_k)\right) \Gamma\left(\frac{1}{2}\, 1/2(-s_l - s_k)\right)} \tag{2.4b}$$

$$\times \Phi_{s_1, \ldots, s_p}(t_1, \ldots, t_p)\, ds_1\, ds_2 \cdot ds_p, \tag{2.5}$$

where C *is a constant.*

Remark. The factor (2.3)–(2.4) is the Gindikin–Karpelevich density. It is an elementary function; see (1.44)–(1.46).

2.2. *Analytic Formula for Arbitrary* α

Fix $m = 0, 1, \ldots, p$. Consider nonnegative integers

$$u_1 \leqslant u_2 \leqslant \cdots \leqslant u_m$$

satisfying the condition

$$\alpha+2u_m+m<\tfrac{1}{2}(p+q)$$

(if $m=0$, then a collection $\{u\}$ is empty).

THEOREM 2.2. *Let* $p\neq q$ *and* α *be arbitrary, or* $p=q$ *and* $\alpha\in\mathbb{R}\setminus\{1,2,\dots,p-1\}$. *Then*

$$\prod_{j=1}^{p}\cosh^{-\alpha}t_j$$
$$=C\cdot\sum_{\substack{m;\\ u_1\leqslant\cdots\leqslant u_m<1/4(p+q)-m/2-(1/2)\alpha}}E_m(\alpha,u)$$
$$\times\int_{i\mathbb{R}^{p-m}}Y_m(\alpha,u;s)\,\mathfrak{R}_m(s)\,\Phi_{\alpha-(p+q)/2+1+2u_1,\dots,\alpha-(p+q)/2+m+2u_m,s_{m+1},\dots,s_p}$$
$$\times(t_1,\dots,t_p)\times ds_{m+1}\cdots ds_p, \tag{2.6}$$

where C *is the same as above*;

$$E_m(\alpha,u)=(2\pi)^m\frac{p!}{m!}\,2^{p\alpha}\prod_{j=1}^{p}\frac{1}{\Gamma(\alpha-j+1)} \tag{2.7}$$

$$\times\prod_{\tau=1}^{m}\frac{\left(\begin{array}{c}\left(-\alpha+\frac{1}{2}(p+q)-2u_\tau-\tau\right)\Gamma(\alpha-p+\tau-2u_\tau)\\ \times\Gamma(-\alpha+q-\tau-2u_\tau)\end{array}\right)}{(u_\tau-u_{\tau-1})!\,\Gamma\left(-\alpha+\frac{1}{2}(p+q)-\tau+1-u_\tau-u_{\tau-1}\right)} \tag{2.8}$$

$$\times\prod_{1\leqslant\sigma<\tau\leqslant m}\left\{\left(-\alpha+\frac{1}{2}(p+q)-\frac{1}{2}(\tau+\sigma)-u_\sigma-u_\tau\right)\right.$$

$$\times\left(\frac{1}{2}(\tau-\sigma)+u_\tau-u_\sigma\right) \tag{2.9}$$

$$\times\frac{\Gamma\left(\frac{1}{2}(\tau-\sigma+1)+u_\tau-u_\sigma\right)}{\Gamma\left(\frac{1}{2}(\tau-\sigma)+u_\tau-u_{\sigma-1}\right)} \tag{2.10a}$$

$$\left.\times\frac{\Gamma\left(-\alpha+\frac{1}{2}(p+q)-\frac{1}{2}(\tau+\sigma)-u_\tau-u_\sigma+\frac{1}{2}\right)}{\Gamma\left(-\alpha+\frac{1}{2}(p+q)-\frac{1}{2}(\tau+\sigma)-u_\tau-u_\sigma\right)}\right\} \tag{2.10b}$$

$$Y_m(\alpha, u; s) = \prod_{k=m+1}^{p} \left\{ \Gamma\left(\frac{1}{2}\left(\alpha - \frac{1}{2}(p+q) + m + 1 + s_k\right)\right) \right. \tag{2.11a}$$

$$\left. \times \Gamma\left(\frac{1}{2}\left(\alpha - \frac{1}{2}(p+q) + m + 1 - s_k\right)\right) \right\} \tag{2.11b}$$

$$\times \prod_{\tau \leqslant m;\, k > m} \left\{ \left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - \tau - 2w_\tau + s_k\right)\right) \right. \tag{2.12a}$$

$$\times \left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - \tau - 2w_\tau - s_k\right)\right) \tag{2.12b}$$

$$\times \frac{\Gamma\left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - (\tau - 1) - 2u_\tau + s_k\right)\right)}{\Gamma\left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - \tau + 2 + 2w_{\tau-1} + s_k\right)\right)} \tag{2.13a}$$

$$\left. \times \frac{\Gamma\left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - (\tau + 1) - 2u_\tau - s_k\right)\right)}{\Gamma\left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - \tau + 2 + 2w_{\tau-1} - s_k\right)\right)} \right\} \tag{2.13b}$$

and

$$\mathfrak{R}_m(s) = \prod_{k=m+1}^{p} \frac{\Gamma((q-p)/2 + s_k)\, \Gamma((q-p)/2 - s_k)}{\Gamma(s_k)\, \Gamma(-s_k)} \tag{2.14}$$

$$\times \prod_{m+1 \leqslant k < l \leqslant p} \frac{\Gamma\left(\frac{1}{2}(1 + s_l + s_k)\right) \Gamma\left(\frac{1}{2}(1 + s_l - s_k)\right)}{\Gamma\left(\frac{1}{2}(s_l + s_k)\right) \Gamma\left(\frac{1}{2}(s_l - s_k)\right)} \tag{2.14a}$$

$$\times \frac{\Gamma\left(\frac{1}{2}(1 - s_l + s_k)\right) \Gamma\left(\frac{1}{2}(1 - s_l - s_k)\right)}{\Gamma\left(\frac{1}{2}(-s_l + s_k)\right) \Gamma\left(\frac{1}{2}(-s_l - s_k)\right)} \tag{2.15}$$

Remarks. (a) The factor $\mathfrak{R}_m(s)$ is an elementary function.

(b) More convenient notations are used in Section 4 (see 4.13).

(c) A formula that is not so explicit, but short is given in Section 6.

Remark. The summand corresponding $m=0$ coincides with the integral (2.1)–(2.5). For summands corresponding $m=p$, the integration is given by one point set and hence these summands are spherical functions $\Phi_{\dots}$ with some coefficients.

2.3. *The Case* $\alpha = p-1, p-2, \dots, 1$

In this case some summands disappear.

PROPOSITION 2.3. *Let* $\alpha = p-h$, *where* $h \leqslant p$. *Then the factor* $E_m(\alpha, u)$ *is nonzero if and only if*

$$m \geqslant h; \qquad u_1 = u_2 = \cdots = u_h = 0.$$

Proof. Vanishing of $E_m(\alpha, u)$ is completely defined by the behavior of the factor

$$\frac{\prod_{\tau=1}^{m} \Gamma(\alpha - p + \tau + 2u_\tau)}{\prod_{j=1}^{p} \Gamma(\alpha - j + 1)}. \tag{2.16}$$

The denominator has a pole of order h at $\alpha = p-h$. If the fraction is non-vanishing, then the numerator has a pole of order h at the same point. ∎

2.4. *The Case* $\alpha = -1, -2, -3, \dots$

Assume $E_m(\alpha, u) \neq 0$. The denominator of (2.16) has a pole of order p in α. Hence, the numerator also has a pole of order p. Hence,

$$m = p.$$

This means that all integrals in Plancherel formula (2.6) vanish and we have only a finite sum of spherical functions with some coefficients. The coefficient $E_m(\alpha, u)$ is nonzero iff

$$m + 2u_m \leqslant -\alpha.$$

2.5. *The Plancherel Formula for the Kernel Representations* T_α *of* $\mathrm{O}(p, q)$

THEOREM 2.4. *Let* α *satisfy Berezin conditions* (1.9). *Then*

(a) *If* $E_m(\alpha, u) \neq 0$ (*see Section* 2.3), *then all spherical functions*

$$\Phi_{\alpha-(p+q)/2+1+2u_1, \dots, \alpha-(p+q)/2+m+2u_m, s_{m+1}, \dots, s_p}$$

are positive definite.

(b) *Formula* (2.6)–(2.15) *is really the Plancherel formula.*

2.6. *The Plancherel Formula for Kernel-Representations of* $\mathrm{O}(p+q)$

For a negative integer $\alpha = -n$ (see Section 2.4 above) formula (2.6) gives the expansion of $\det(1-z^*z)^n$ in $\mathrm{O}(p)\times\mathrm{O}(q)$-spherical functions of $\mathrm{O}(p,q)$ and this is equivalent to the Plancherel formula for the kernel representations of $\mathrm{O}(p+q)$.

Remark. This formula is related to Pickrell's expansion [59]. Pickrell's case corresponds to the symmetric spaces $U(2n)/U(n)\times U(n)$. For detail comments, see [58].

2.7. *The Case of Indefinite Shapovalov Form*

For noninteger $\alpha < p-1$ we obtain the problem discussed in Section 1.E.

3. B-FUNCTION OF THE SPACE $\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q)$

In this section we construct a matrix imitation of the B-integral

$$\mathrm{B}(x,y) = \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt$$

for the symmetric spaces $\mathrm{O}(p,q)/\mathrm{O}(p)\times\mathrm{O}(q)$. For symmetric spaces $\mathrm{GL}(n,\mathbb{K})/\mathrm{U}(n,\mathbb{K})$ the B-integrals were defined by Gindikin [11] (see also exposition in [9]); for other symmetric spaces B-integrals were obtained in [35].

3.1. *B-Integral*

Let

$$\lambda_1, \ldots, \lambda_p, \sigma_1, \ldots, \sigma_p \in \mathbb{C}.$$

We also assume

$$\lambda_{p+1} = \sigma_{p+1} = 0.$$

Let $\mathrm{SW}_{p,q}$ be the section of wedge defined in Section 1.20.

THEOREM 3.1. *Let* λ_k, σ_k *satisfy the inequalities*

$$\tfrac{1}{2}(q+k)-1 < \lambda_k < \sigma_k - \tfrac{1}{2}(p-k). \tag{3.1}$$

Then

$$\int_{\mathrm{sw}_{p,q}(\mathbb{R})} \prod_{j=1}^{p} \frac{\det[M-LL^t]_j^{\lambda_j-\lambda_{j+1}}}{\det[1+M+N]_j^{\sigma_j-\sigma_{j+1}}} \cdot \det(M-LL^t)^{-(p+q)/2}\, dM\, dN\, dL$$

$$= \int_{\substack{M-LL^t>0\\ N=-N^t}} \prod_{j=1}^{n} \frac{\det \begin{bmatrix} 1 & L^t \\ L & M \end{bmatrix}_{q-p+j}^{\lambda_j-\lambda_{j+1}}}{\det[1+M+N]_j^{\sigma_j-\sigma_{j+1}}} \tag{3.2}$$

$$\times \det \begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix}^{-(p+q)/2} dL\, dM\, dN \tag{3.3}$$

$$= \pi^{(1/2)\, p(q-1)} \prod_{k=1}^{p} \frac{\Gamma(\lambda_k-(q+k)/2+1)\, \Gamma(\sigma_k-\lambda_k-(p-k)/2)}{\Gamma(\sigma_k-p+k)}. \tag{3.4}$$

The proof of the theorem is given in Sections 3.2–3.6.

Remark. If $p=q$ then the block L is missing and the integral (3.2), (3.3) has more simple form, see (0.5). In this case the calculation given below also is simpler. The main simplification is the expression for the matrix (3.13): the first block row and the first block column are missing. An evaluation of its determinant also is simpler.

Remark. We have $M=M^t>0$, $N=-N^t$. Hence,

$$\det(1+M+N)>0.$$

Indeed, for any $v \in \mathbb{C}^p$ we have $\mathrm{Re}\, v(M+N)\, v^* = vMv^* > 0$. Hence, the eigenvalues λ_j of $M+N$ satisfy the condition $\mathrm{Re}\, \lambda_j > 0$. Hence, the eigenvalues of $1+M+N$ are nonzero.

Remark. Hua Loo Keng in [21] evaluated the integrals[24]

$$\int_{\mathbf{B}_{p,q}(\mathbb{R})} \det(1-zz^*)^\tau\, dz. \tag{3.5}$$

Cayley transform reduces the *Hua integral* to the following special case of our integral

$$\mathrm{const} \cdot \int_{\mathrm{sw}_q(\mathbb{R})} \frac{\det \begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix}^\tau}{\det(1+M+N)^{2\tau}}\, dL\, dM\, dN.$$

Our calculation in this case is not homotopic to Hua calculations.

[24] Hua integrals also can be reduced to Selberg B-integrals by integration over $K \backslash G/K$.

3.2. *Replacement of Notations*

First we call to mind the standard formula (see [10]) for the determinant of a block $(m+n)\times(m+n)$-matrix

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det(D - CA^{-1}B) \tag{3.6}$$

Let us represent M, N as block $((p-1)+1)\times((p-1)+1)$ matrices, and L as a block $((p-1)+1)\times(q-p)$ matrix:

$$M = \begin{pmatrix} P & q^t \\ q & r \end{pmatrix}; \qquad N = \begin{pmatrix} A & -b^t \\ b & 0 \end{pmatrix}; \qquad L = \begin{pmatrix} H \\ l \end{pmatrix}.$$

Then for $j \leqslant p-1$

$$\begin{bmatrix} 1 & L^t \\ L & M \end{bmatrix}_{q-p+j} \quad \text{coincides with} \quad \begin{bmatrix} 1 & H^t \\ H & P \end{bmatrix}_{q-p+j}$$

$$[1+M+N]_j \quad \text{coincides with} \quad [1+P+A]_j$$

and by (3.6)

$$\det\begin{pmatrix} 1 & L^t \\ L & M \end{pmatrix} = \det\begin{pmatrix} 1 & H^t & l^t \\ H & P & q^t \\ l & q & r \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix} \cdot \left[r - (l \quad q) \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \begin{pmatrix} l^t \\ q^t \end{pmatrix} \right]$$

$$\det(1+M+N) = \det\begin{pmatrix} 1+P+A & q^t - b^t \\ q+b & 1+r \end{pmatrix}$$

$$= \det(1+P+A)\cdot(1+r-(q+b)(1+P+A)^{-1}(q^t+b^t)).$$

By the Sylvester criterion, the domain of integration $\left(\begin{smallmatrix} 1 & L^t \\ L & M \end{smallmatrix}\right) > 0$ (see (1.35)) in new notation has the form

$$\begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix} > 0; \qquad r - (l \quad q) \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \begin{pmatrix} l^t \\ q^t \end{pmatrix} > 0. \tag{3.7}$$

By the remark given in Section 3.1,

$$\det(1+P+A)>0$$

3.3. *Substitution*

Let us replace the variable r by

$$u=r-(l \quad q)\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{-1}\begin{pmatrix}l^t\\ q^t\end{pmatrix}$$

(all other variables are the same). By (3.7), we have $u>0$. The Jacobian of the substitution is 1. Thus, our integral is converted to the form

$$\int_{P-HH^t>0} dP\,dA\,dH\left(\Xi(A,P,H)\times\int_{u>0;\, q,\, b\in\mathbb{R}^{p-1};\, l\in\mathbb{R}^{q-p}} u^{\lambda_p-(p+q)/2}\right. \tag{3.8}$$

$$\times\left\{1+u+(l \quad q)\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{-1}\begin{pmatrix}l^t\\ q^t\end{pmatrix}+(q \quad b)\right. \tag{3.9}$$

$$\left.\left.\times\begin{pmatrix}-(1+P+A)^{-1} & -(1+P+A)^{-1}\\ (1+P+A)^{-1} & (1+P+A)^{-1}\end{pmatrix}\begin{pmatrix}q^t\\ b^t\end{pmatrix}\right\}^{-\sigma_p} du\,dl\,dq\,db\right), \tag{3.10}$$

where

$$\Xi(A,P,H)=\prod_{j=1}^{p-2}\frac{\det\begin{bmatrix}1 & H^t\\ H & P\end{bmatrix}_{q-p+j}^{\lambda_j-\lambda_{j+1}}}{\det[1+P+A]_j^{\sigma_j-\sigma_{j+1}}}\cdot\frac{\det\begin{pmatrix}1 & H^t\\ H & P\end{pmatrix}^{\lambda_{p-1}-(p+q)/2}}{\det(1+P+A)^{\sigma_{p-1}}} \tag{3.11}$$

is an expression independent of u, b, l, q.

First we want to evaluate the interior integral (3.9)–(3.10).

3.4. *Transformation of the Integrand*

Denote by S the expression

$$S=1+P+A.$$

Let us represent the expression in the curly brackets in (3.10) in the form

$$\left\{1+u+(l \quad q \quad b)\,X\begin{pmatrix}l^t\\ q^t\\ b^t\end{pmatrix}\right\}, \tag{3.12}$$

where

$$X = \begin{pmatrix} \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 0 \\ 0 & -S^{-1} \end{pmatrix} & \begin{pmatrix} 0 \\ -S^{-1} \end{pmatrix} \\ \begin{pmatrix} 0 & S^{-1} \end{pmatrix} & S^{-1} \end{pmatrix} \tag{3.13}$$

(X is a block matrix whose elements are block matrices itself). The last summand in the curly brackets is a quadratic form in the variables b, q, l. But the matrix X is not symmetric and it is more natural to re-write expression (3.12) in the form

$$\left\{ 1 + u + \begin{pmatrix} l & q & b \end{pmatrix} \frac{1}{2} (X + X^t) \begin{pmatrix} l^t \\ q^t \\ b^t \end{pmatrix} \right\}. \tag{3.14}$$

3.5. *Separation of Variables*

LEMMA 3.2.

$$\det(\tfrac{1}{2}(X + X^t)) = \det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \cdot \det(1 + P + A)^{-2}.$$

Proof.

$\det(\frac{1}{2}(X + X^t))$

$$= \det \begin{pmatrix} \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{2} S^{t-1} - \frac{1}{2} S^{-1} \end{pmatrix} & \begin{pmatrix} 0 \\ -\frac{1}{2} S^{-1} + \frac{1}{2} S^{t-1} \end{pmatrix} \\ \begin{pmatrix} 0 & \frac{1}{2} S^{-1} - \frac{1}{2} S^{t-1} \end{pmatrix} & \frac{1}{2} S^{-1} + \frac{1}{2} S^{t-1} \end{pmatrix}.$$

Adding the third row to the second row and the third column to the second column, we obtain

$$\det \begin{pmatrix} \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} & \begin{pmatrix} 0 \\ (1 + P - A)^{-1} \end{pmatrix} \\ \begin{pmatrix} 0 & (1 + P + A)^{-1} \end{pmatrix} & \frac{1}{2} (1 + P + A)^{-1} + \frac{1}{2} (1 + P - A)^{-1} \end{pmatrix}.$$

Formula (3.6) reduces the determinant to the form

$$\det\begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \cdot \det\Bigg(\frac{1}{2}(1+P+A)^{-1} + \frac{1}{2}(1+P-A)^{-1}$$

$$-(0 \quad (1+P+A)^{-1})\begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}\begin{pmatrix} 0 \\ (1+P-A)^{-1} \end{pmatrix}\Bigg)$$

$$= \det\begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{-1} \det(1+P+A)^{-1} \det(1+P-A)^{-1}$$

$$\times \det\left(\frac{1}{2}(1+P-A) + \frac{1}{2}(1+P+A) - P\right).$$

The last factor is 1. We also observe

$$(1+P+A)^t = 1+P-A$$

and hence their determinants coincide. ∎

LEMMA 3.3. $X + X^t > 0$.

Proof. In the identity

$$\det(1+M+N) = \det(1+P+A) \cdot [1 + r - (q+b)(1+P+A)^{-1}(q^t - b^t)]$$

we have $\det(1+M+N) > 0$, $\det(1+P+A) > 0$. Hence, the factor in the square brackets is positive. Hence, the expression (3.12) is positive for all $u > 0$, and all q, b, l. The quantity (3.12) coincides with the quantity (3.14). Hence, the matrix $X + X^t$ is nonnegative definite. By Lemma 3.2, its determinant is nonzero and we obtain the required statement. ∎

Consider the linear substitution

$$(l \quad q \quad b)\sqrt{\tfrac{1}{2}(X+X^t)} = h \in \mathbb{R}^{q-p} \oplus \mathbb{R}^{p-1} \oplus \mathbb{R}^{p-1}$$

to the interior integral (3.9)–(3.10). Its Jacobian is

$$\det\begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{1/2} \cdot \det(1+P+A)$$

and hence the interior integral converts to the form

$$\det\begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{1/2} \cdot \det(1+P+A) \tag{3.15}$$

$$\times \int_{u>0,\, h \in \mathbb{R}^{q+p-2}} u^{\lambda_p - (p+q)/2} \{1 + u + |h|^2\}^{-\sigma_p}\, du\, dh. \tag{3.16}$$

The first factor (3.15) adds to the product $\Xi(A, P, H)$ (see (3.11)) and we reduce our B-integral (3.3) to the product of the integrals

$$\int_{P-HtH>0,\, A=-A^t} \Xi(A, P, H) \det \begin{pmatrix} 1 & H^t \\ H & P \end{pmatrix}^{1/2} \cdot \det(1+P+A)\, dA\, dP\, dH$$

$$\times \int_{u>0,\, h\in\mathbb{R}^{q+p-2}} u^{\lambda_p-(p+q)/2} \{1+u+|h|^2\}^{-\sigma_p}\, du\, dh.$$

Let us denote the B-integral (3.3) by

$$I_{p,q}(\alpha_1, ..., \alpha_p; \sigma_1, ..., \sigma_p) \tag{3.17}$$

and let us denote the factor (3.16) by $J_{p,q}(\alpha_p; \sigma_p)$. We obtain the recurrence identity

$$I_{p,q}(\alpha_1, ..., \alpha_p; \sigma_1, ..., \sigma_p)$$
$$= I_{p-1,q-1}(\alpha_1-\tfrac{1}{2}, ..., \alpha_{p-1}-\tfrac{1}{2}; \sigma_1-1, ..., \sigma_{p-1}-1)\, J_{p,q}(\alpha_p; \sigma_p).$$

3.6. *Evaluation of* $J_{p,q}(\alpha_p; \sigma_p)$

This problem is trivial. First we consider spherical coordinates in $\mathbb{R}^{p+q-2}$ in the variable h. Then $J_{p,q}(\alpha_p; \sigma_p)$ converts to the form

$$\frac{2\pi^{(p+q)/2-1}}{\Gamma((p+q)/2-1)} \int_{u>0} \int_{r>0} u^{\lambda_p-(p+q)/2} r^{p+q-3} \{1+u+r^2\}^{-\sigma_p}\, dr\, du.$$

The substitution $v=r^2$ reduces our integral to a special case of the Dirichlet B-integral

$$\int_{u>0,\, v>0} \frac{u^{a-1}v^{b-1}}{(1+u+v)^{a+b+c}}\, du\, dv = \frac{\Gamma(a)\,\Gamma(b)\,\Gamma(c)}{\Gamma(a+b+c)}.$$

Finally, we obtain

$$I_{p,q} = \pi^{(p+q)/2-1} \frac{\Gamma(\lambda_p-(p+q)/2+1)\,\Gamma(\sigma_p-\lambda_p)}{\Gamma(\sigma_p)}.$$

This completes the proof of Theorem 3.1.

3.7. *Spherical Transform of* $\mathscr{B}_\alpha$

COROLLARY 3.4. *Let* $\alpha > p+q-1$. *Then the spherical transform of* $\mathscr{B}_\alpha$ *is*

$$\frac{2^{p\alpha}}{\prod_{1\leqslant j\leqslant p} \Gamma(\alpha-j+1)} \prod_{k=1}^{p} \Gamma\left(\frac{1}{2}\left(\alpha-\frac{1}{2}(p+q)+1+s_k\right)\right)$$
$$\times \Gamma\left(\frac{1}{2}\left(\alpha-\frac{1}{2}(p+q)+1-s_k\right)\right). \tag{3.18}$$

Proof. The function $\mathscr{B}_\alpha$ is given by the formula (1.37). By Section 1.24, we must evaluate the integral

$$\int_{G/K} \mathscr{B}_\alpha(z)\, \Psi_s(z)\, d\lambda(z).$$

But the integral is a special case of our B-integral for

$$\lambda_k = \tfrac{1}{2}\alpha + \tfrac{1}{2}s_k + \tfrac{1}{4}(q-p) + \tfrac{1}{2}(k-1)$$
$$\sigma_k = \alpha.$$

3.8. *Proof of Theorem* 2.1

By the Gindikin–Karpelevich inversion formula and Corollary 3.4, we obtain the statement of the theorem for $\alpha > (p+q)-1$.

For $\alpha > (p+q)/2-1$ the statement of the theorem follows from the trivial Lemma 4.1 proved below.

4. FORMAL ANALYTIC CONTINUATION

We proved the Plancherel formula (2.1)–(2.5) for large values of the parameter α. Its left part $\prod \cosh^{-\alpha}(t_j)$ depends analytically on $\alpha \in \mathbb{C}$. The integrand in the right part has singularities on the lines

$$\operatorname{Re} \alpha = \tfrac{1}{2}(p+q) - 1 - 2\kappa, \qquad \text{where} \quad \kappa = 0, 1, 2, \ldots. \tag{4.1}$$

Thus, the right part of formula (2.1)–(2.5) may be nonanalytic for these values of α.

Our next purpose is to construct the analytic continuation of the right part to arbitrary complex α.

4.1. *Analyticity*

Let us denote the right part of the formula (2.1)–(2.5) by

$$\mathfrak{F}(\alpha) := \mathfrak{F}(\alpha; t) = E(\alpha) \int_{i\mathbb{R}^p} Y(\alpha; s)\, \mathfrak{R}(s)\, \Phi_s(t)\, ds, \tag{4.2}$$

where the meromorphic factor $E(\alpha)$ is given by formula (2.1), the factor $Y(\alpha; s)$ is defined by (2.2), and $\mathfrak{R}(s)$ is the Gindikin–Karpelevich density (2.3)–(2.4). *In this section we fix the variable t and we omit the argument t from the notation $\mathfrak{F}(\alpha; t)$.*

Consider domains $\Pi_0, \Pi_1, \ldots$ in $\mathbb{C}$ defined by

$$\Pi_0\colon \quad \operatorname{Re} \alpha > \tfrac{1}{2}(p+q) - 1$$

$$\Pi_k\colon \quad \tfrac{1}{2}(p+q) - 1 - 2k < \operatorname{Re} \alpha < \tfrac{1}{2}(p+q) - 1 - 2(k-1) \qquad \text{where} \quad k > 0.$$

(See Fig. 1.)

Lemma 4.1. *The function $\mathfrak{F}(\alpha)$ is an analytical function on Π_κ for all* $\kappa = 0, 1, 2, \ldots.$

Proof. (a) *Convergence of the integral* (4.2). First, the Gindikin–Karpelevich factor $\mathfrak{R}(s)$ has a polynomial growth in s; see formulas (1.44)–(1.46).

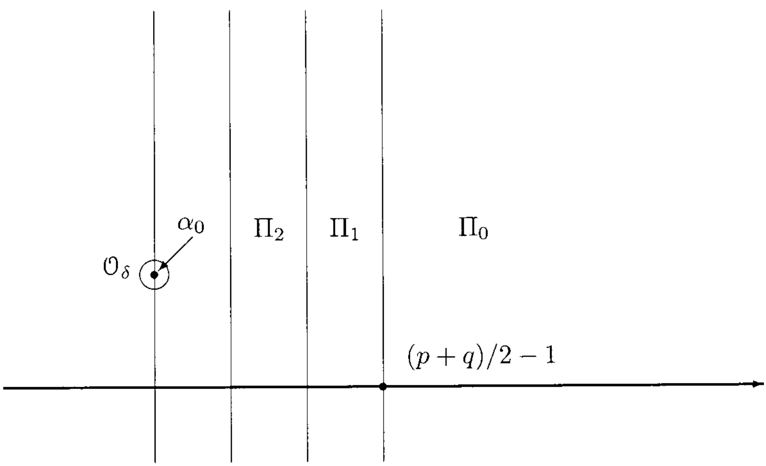

FIG. 1. The complex plane α. The lines $\alpha = (p+q)/2 - 1 - 2\kappa$ and the domains Π_k.

By the formula (see [19, 1.18.6])

$$|\Gamma(a+iy)| = (2\pi)^{1/2}\,|y|^{a-1/2}\exp\{-\tfrac{1}{2}\pi\,|y|\}(1+o(1)), \qquad |y|\to\infty, \tag{4.3}$$

the factor $Y(\alpha; s)$ exponentially decreases for imaginary s.

A spherical function $\Phi_s(t)$ is a spherical function of a unitary representation and hence we have $|\Phi_s(t)| \leqslant 1$.[25]

Hence the integrand exponentially decreases and the integral absolutely converges.

(b) *Existence of* $\frac{\partial}{\partial\alpha}\mathfrak{F}(\alpha)$. It is sufficient to prove uniform convergence of the integral

$$\int_{i\mathbb{R}^p} \frac{\partial}{\partial\alpha}\, Y(\alpha; s)\,\mathfrak{R}(s)\,\Phi_s(t)\,ds \tag{4.4}$$

in a small neighborhood of a fixed point $\tilde{\alpha}$. For this we need uniformity in a of $o(1)$ in (4.3). In fact, the asymptotics is really uniform, but formally we have no possibility to refer to [19]. Formula (4.3) is derived from the Binet formula (see [19, (1.9.4)])

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \int_0^\infty \left[\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right] t^{-1}e^{-tz}\,dt.$$

This formula easily implies a uniform estimate of the form

$$|\Gamma(a+iy)| \leqslant \mathrm{const}\cdot\exp(-(\tfrac{1}{2}\pi-\varepsilon)\,|y|), \qquad |a-\tilde{a}|<\delta.$$

The Cauchy integral for the derivative,

$$2\pi i f'(z) = \int_L \frac{f(z)}{(z-u)^2}\,dz,$$

implies the same estimate for derivative of Γ-function.

We observe that the integrand (4.4) is dominated by some function having the form

$$P(s)\exp\left(-b\sum|s_j|\right)$$

where $P(s)$ is a polynomial and $b>0$.

Thus, the function $\mathfrak{F}(\alpha)$ has a derivative in the complex variable α and this complete the proof. ∎

[25] For the following inductive steps this argument must be replaced by inequality (1.39).

LEMMA 4.2. *Let $p \neq q$. Then the function $\mathfrak{F}(\alpha)$ is continuous on the line $\alpha \in \mathbb{R}$.*

Proof. Let $h = \frac{1}{2}(p+q) - 1 - 2\kappa$ be one of our singular points. Consider the factor

$$\frac{\Gamma\left(\frac{1}{2}\left(\alpha - \frac{1}{2}(p+q) + 1 + s_k\right)\right) \Gamma\left(\frac{1}{2}(q-p) + s_k\right)}{\Gamma(s_k)}$$

on the imaginary axis. This expression exponentially decreases as $is \to \infty$. The singularity at the point $s_k = 0$ has the form

$$C \cdot \frac{s_k}{\alpha - h + s_k}$$

and hence the factor is bounded for imaginary s_k in a neighborhood of the point $s_k = 0$.

By the Lebesgue theorem about dominant convergence, the expression (4.2) is continuous at the point $\alpha = h$. ∎

Remark. The function $\mathfrak{F}(\alpha)$ is continuous at the real points $\alpha = \frac{1}{2}(p+q) - 1 - 2\kappa$ but it is not smooth at these points.

We denote the restriction of the function $\mathfrak{F}(\alpha)$ to the domain Π_κ by

$$\mathfrak{F}_\kappa(\alpha).$$

4.2. *Analytic Continuation of $\mathfrak{F}_\kappa(\alpha)$ through a Point of a Line* $\operatorname{Re}\alpha = \frac{1}{2}(p+q) - 1 - 2\kappa$

The following lemma is the main result in this section. Its proof is given in Sections 4.2–4.5.

LEMMA 4.3. *Let α_0 satisfy the condition*

$$\operatorname{Re}\alpha_0 = \tfrac{1}{2}(p+q) - 1 - 2\kappa; \qquad \operatorname{Im}\alpha_0 \neq 0.$$

Then

(a) *the function $\mathfrak{F}_\kappa(\alpha)$ admits analytic continuation to some small neighborhood*

$$\mathcal{O}_\delta: |\alpha - \alpha_0| < \delta \qquad \text{where} \qquad \delta < \min\{1/1000, |\operatorname{Im}\alpha|/1000\} \tag{4.5}$$

of the point α_0;

(b) *for any*

$$\alpha \in \mathcal{O} \cap \Pi_{\kappa+1}$$

we have

$$\mathfrak{F}_{\kappa+1}(\alpha) - \mathfrak{F}_{\kappa}(\alpha)$$

$$= \frac{\pi p}{\kappa!} \prod_{j=1}^{p} \frac{1}{\Gamma(\alpha - j + 1)} \tag{4.6}$$

$$\times \frac{\left(\left(-\alpha + \frac{1}{2}(p+q) - 2\kappa - 1\right) \Gamma(\alpha - p + 1 + 2\kappa) \times \Gamma(-\alpha + q - 1 - 2\kappa)\right)}{\Gamma(-\alpha + 1/2(p+q) - \kappa)} \tag{4.7}$$

$$\times \int_{i\mathbb{R}^{p-1}} \prod_{2 \leqslant k \leqslant p,\ \pm} \Gamma\left(\frac{1}{2}\left(\alpha - \frac{1}{2}(p+q) + 2 + 2\kappa \pm s_k\right)\right) \tag{4.8}$$

$$\times \prod_{2 \leqslant k \leqslant p,\ \pm} \frac{\left(\left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - 1 \pm s_k\right)\right) \times \Gamma\left(\frac{1}{2}\left(-\alpha + \frac{1}{2}(p+q) - 2\kappa \pm s_k\right)\right)\right)}{\Gamma(1/2(-\alpha + 1/2(p+q) + 1 \pm s_k))} \tag{4.9}$$

$$\times \prod_{2 \leqslant k \leqslant p,\ \pm} \frac{\Gamma\left(\frac{1}{2}(q-p) \pm s_k\right)}{\Gamma(\pm s_k)} \tag{4.10}$$

$$\times \prod_{2 \leqslant k < l \leqslant p,\ \pm} \frac{\Gamma\left(\frac{1}{2}(1 + s_k \pm s_l)\right) \Gamma\left(\frac{1}{2}(1 - s_k \pm s_l)\right)}{\Gamma\left(\frac{1}{2}(s_k \pm s_l)\right) \Gamma\left(\frac{1}{2}(-s_k \pm s_l)\right)} \tag{4.11}$$

$$\times \Phi_{\alpha - (p+q)/2 + 1 + 2\kappa,\, s_2, \ldots, s_p}(t)\, ds_2 \cdots ds_p. \tag{4.12}$$

In the last formula we use the following notation:

$$\prod_{\pm} \Gamma(a \pm s) := \Gamma(a+s)\, \Gamma(a-s). \tag{4.13}$$

4.3. *Existence of Analytic Continuations*

Let us represent the expression (4.2) (or (2.1)–(2.5)) for $\mathfrak{F}_\kappa(\alpha)$ in the form

$$E(\alpha) \int \prod_{k,\, \pm} \Gamma(\tfrac{1}{2}(\alpha - \tfrac{1}{2}(p+q) + 1 \pm s_k)) \cdot \mathfrak{R}(s)\, \Phi_s(t)\, ds.$$

Let $\varepsilon_1 > \varepsilon_2 \cdots > \cdots > \varepsilon_p \geqslant 0$ be very small (for instance $\varepsilon_1 < \delta/10$, where δ was defined in (4.5)). Consider the function

$$\mathfrak{F}_\kappa(\alpha; \varepsilon) = E(\alpha) \int_{i\mathbb{R}^p} \prod_{k,\, \pm} \Gamma(\tfrac{1}{2}(\alpha + \varepsilon_k - \tfrac{1}{2}(p+q) + 1 \pm s_k)) \cdot \mathfrak{R}(s)\, \Phi_s(t)\, ds \tag{4.14}$$

in the domain

$$\Pi_\kappa^\varepsilon\colon\ -2\kappa < \operatorname{Re}(\alpha - \tfrac{1}{2}(p+q) + 1) < -2(\kappa - 1) - \varepsilon_1.$$

LEMMA 4.4. (a) *For any* ε *the function* $\mathfrak{F}_\kappa(\alpha; \varepsilon)$ *admits the holomorphic continuation to the domain* $\mathcal{O}_\delta$ *(see* (4.5)).

(b) *The functions* $|\mathfrak{F}_\kappa(\alpha; \varepsilon)|$ *in* $\mathcal{O}_\delta$ *are bounded by a constant independent on* ε.

Proof. (a) The factor $\mathfrak{R}(s)$ is holomorphic in the domain $|\operatorname{Re}(s_j)| < 1/4$ and its poles are very far from the contour $\mathcal{L}$ that is described below. Consider

$$\alpha \in \mathcal{O}_\delta \cap \Pi_\kappa^\varepsilon. \tag{4.15}$$

The integrand (4.14) has poles on hyperplanes

$$s_k = \pm(\tfrac{1}{2}(p+q) - 1 - \alpha - 2u - \varepsilon_k)), \qquad \text{where} \qquad u = 0, 1, \ldots.$$

If $u = \kappa$, then the poles are lying near points $\pm \operatorname{Im} \alpha_0$. In Fig. 2 the poles are marked as black circles. The arrows show the direction of their motion if $\operatorname{Re} \alpha$ decreases. The white circles show the rough position of the poles when $\alpha \in \mathcal{O}_\delta \cap \Pi_{\kappa+1}^\varepsilon$.

Consider the contour L_k on the complex plane $s_k \in \mathbb{C}$ given by Fig. 2. Let $\mathcal{L} \subset \mathbb{C}^p$ be the product of the contours L_k. Obviously, for $\alpha \in \mathcal{O}_\delta$, we can

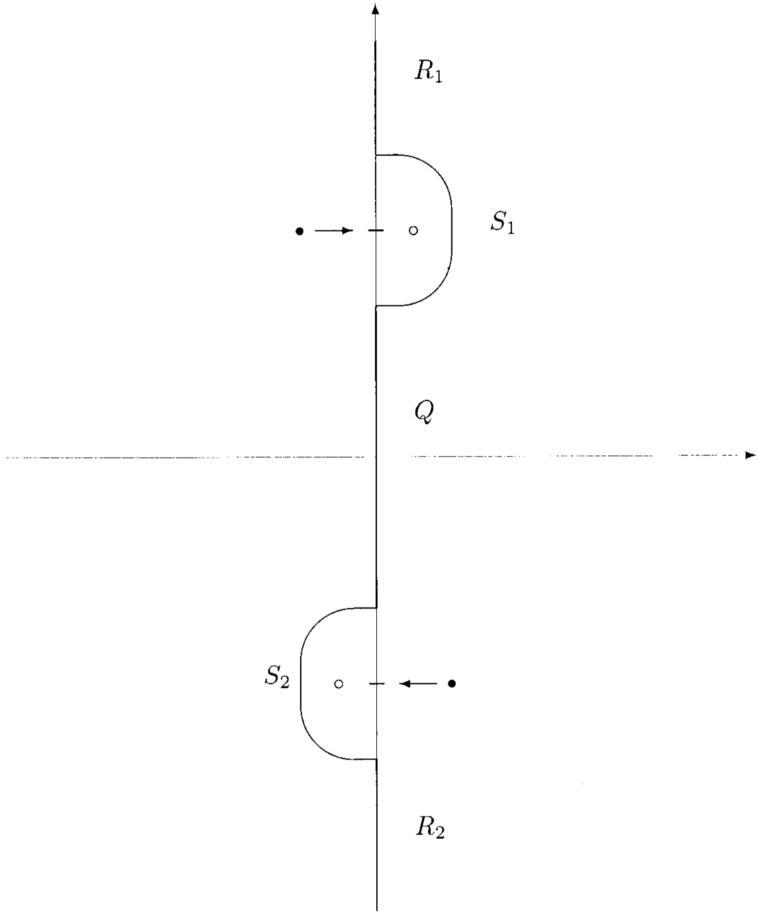

FIG. 2. The contour $L_k = R_1 \cup S_1 \cup Q \cup S_2 \cup R_2$ on the complex plane s_k. The centers of the semicircles S_1, S_2 are $\pm i \operatorname{Im} \alpha_0$; the radius of the semicircles is 10δ.

replace the integration over $i\mathbb{R}^p$ in the formula (4.14) by the integration over $\mathscr{L}$. But the integral

$$\int_{\mathscr{L}} = \int_{\mathscr{L}} \prod_{k,\,\pm} \Gamma\left(\tfrac{1}{2}(\alpha + \varepsilon_k - \tfrac{1}{2}(p+q) + 1 \pm s_k)\right) \cdot \mathfrak{R}(s)\, \Phi_s(t)\, ds$$

is holomorphic with respect to α in the domain $\mathcal{O}_\delta$ (indeed, the surface $\mathscr{L}$ does not intersect the singularities and the integrand exponentially decreases as $|s| \to \infty$).

(b) Consider the parameter $\theta_k := \operatorname{Im} s_k$ on the contour L_k.

LEMMA 4.5. *There exist constants $A = A(t)$, N independent on ε such that*

$$\left|\prod_{k,\,\pm} \Gamma\left(\tfrac{1}{2}(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1\pm s_k)\right)\cdot \mathfrak{R}(s)\, \Phi_s(t)\right|$$

$$\leqslant A \prod_{j=1}^{p} (1+|\theta_j|)^N \exp(-\pi\, |\theta_j|)$$

for all $(s_1, ..., s_p) \in \mathcal{L}$.

Proof of Lemma 4.5. Let us estimate all factors in the left part of the inequality.

(a) *The Gindikin–Karpelevich factor* $\mathfrak{R}(s)$. By formulas (1.45) and (1.46), for any imaginary s_k we have

$$\left|\frac{\Gamma\left(\frac{1}{2}(q-p)+s_k\right)\Gamma\left(\frac{1}{2}(q-p)-s_k\right)}{\Gamma(s_k)\,\Gamma(-s_k)}\right| \leqslant \text{const}\cdot(1+|\theta_k|)^{2(q-p)}.$$

The same expression is bounded on the semicircles S_1, S_2.

We also must estimate the factor (1.44). First

$$\left|\prod_{k=1}^{p} (s_k^2-s_l^2)\right| \leqslant \text{const} \prod_k (1+|\theta_k|)^{2(p-1)}.$$

Second let us estimate the factors

$$\tan(\pi(s_k \pm s_l))$$

of (1.44). If s_k, s_l are imaginary, then $|\tan(\pi(s_k \pm s_l))| < 1$. If $s_k, s_l \in S_1, S_2$, then this expression is bounded (since S_1, S_2 are compact sets). Let s_k be imaginary and $s_l \in S_1, S_2$. Then we obtain a value having the form $|\tan(x+iy)|$, where $x, y \in \mathbb{R}$, $|x| < 10\pi\delta$. Then

$$|\tan(x+iy)| = \left|\frac{\tan x + \tan iy}{1+\tan x \tan iy}\right|$$

$$\leqslant |\tan x + \tan iy| \leqslant |\tan x| + 1 \leqslant \tan(10\pi\delta) + 1.$$

(b) *The Γ-factor* $Y(\alpha; s)$. By formula (4.3), for imaginary s_k we have

$$|\Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1+s_k))\,\Gamma(\tfrac{1}{2}(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1-s_k))| \\ \leqslant \text{const}\cdot(1+|\theta_k|)^{\operatorname{Re}\alpha+\varepsilon_k-1/2(p+q)}\exp(-\pi\,|\theta_k|).$$

For $s_k \in S_1, S_2$ the same expression is bounded (but very large).

(c) *Spherical functions* $\Phi_s(t)$. By estimate (1.39), we have

$$|\Phi_s(t)| \leqslant \Phi_{\operatorname{Re} s}(t) \leqslant \max \Phi_{r_1, \ldots, r_p}(t)$$

where the maximum is given over all real vectors $(r_1, \ldots, r_p)$ satisfying the condition $|r_j| \leqslant 10\delta$. Hence, for a fixed t the spherical function in integrand is dominated by a constant.

This completes the proof of Lemma 4.5. ▌

Now we can complete the proof of Lemma 4.4(b). By Lemma 4.5, we have

$$|\mathfrak{F}_\kappa(\alpha; \varepsilon)| \leqslant A(t)^p \int_{\mathbb{R}^p} \prod_{j=1}^{p} [(1+|\theta_j|)^N \exp(-\pi\,|\theta_j|)] \prod_{j=1}^{p} \chi(\theta_j)\, d\theta_1 \cdots d\theta_p,$$

where the function $\chi(\theta)$ is given by the formula

$$\chi(\theta) := \frac{ds}{d\theta} = \begin{cases} 1 & \text{if } \ |\operatorname{Im}\alpha_0 - |\theta|| \geqslant 10\delta \\ (100\delta^2 - (\operatorname{Im}\alpha_0-\theta)^2)^{-1/2} & \text{if } \ |\operatorname{Im}\alpha_0 - |\theta|| \leqslant 10\delta. \end{cases}$$

Hence

$$|\mathfrak{F}_\kappa(\alpha; \varepsilon)| \leqslant A(t)^p \left(\int_{-\infty}^{\infty} (1+|\theta|)^N \exp(-\pi\,|\theta|)\, \chi(\theta)\, d\theta\right)^p.$$

This complete the proof of uniform boundedness of the functions $\mathfrak{F}_\kappa(\alpha; \varepsilon)$ for a fixed t (Lemma 4.4(b)). ▌

Now we are ready to prove existence of the analytic continuation of the function $\mathfrak{F}_\kappa$.

Proof of Lemma 4.3(b). Let us denote by ε/n the vector $(\varepsilon_1/n, \ldots, \varepsilon_p/n)$. Consider the sequence of functions

$$g_n(\alpha) = \mathfrak{F}_\kappa(\alpha; \varepsilon/n)$$

in the circle $\mathcal{O}_\delta$. Since the functions $g_n(\alpha)$ are uniformly bounded, by the Montel theorem there exists a subsequence g_{n_j} that is uniformly convergent on each smaller circle. Let $g(\alpha)$ be its limit. By Weierstrass theorem, $g(\alpha)$ is holomorphic in $\mathcal{O}_\delta$. It remains to notice that

$$\lim_{n\to\infty} \mathfrak{F}_\kappa(\alpha;\varepsilon/n)=\mathfrak{F}_\kappa(\alpha) \qquad \text{for} \quad \alpha\in\Pi_\kappa$$

for $\alpha\in\Pi_\kappa\cap\mathcal{O}_\delta$. Hence, $g(\alpha)$ is the analytic continuation of $\mathfrak{F}_\kappa(\alpha)$ to the circle $\mathcal{O}_\delta$.

4.4. *Forcing of Poles*

First we want to obtain an explicit formula for the analytic continuation of $\mathfrak{F}_\kappa(\alpha;\varepsilon)$ to the domain $\Pi^\varepsilon_{\kappa+1}$.

Let the contours L_k be the same as above. Let $i\mathbb{R}_k$ be the imaginary axis on the complex plane s_k. Consider the surface

$$\mathcal{L}_k=i\mathbb{R}_1\times\cdots\times i\mathbb{R}_{k-1}\times L_k\times\cdots\times L_p\subset\mathbb{C}^p.$$

We have $\mathcal{L}_1=\mathcal{L}$, $\mathcal{L}_p=i\mathbb{R}^p$.

Consider $\alpha\in\mathcal{O}_\delta\cap\Pi^\varepsilon_{\kappa+1}$. Then

$$\mathfrak{F}_{\kappa+1}(\alpha;\varepsilon)=E(\alpha)\int_{i\mathbb{R}^p}\prod_{k,\,\pm} \times\Gamma\left(\tfrac{1}{2}\left(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1\pm s_k\right)\right)\cdot\mathfrak{R}(s)\,\Phi_s(t)\,ds \tag{4.16}$$

$$\mathfrak{F}_{\kappa}(\alpha;\varepsilon)=E(\alpha)\int_{\mathcal{L}}\prod_{k,\,\pm} \times\Gamma\left(\tfrac{1}{2}\left(\alpha+\varepsilon_k-\tfrac{1}{2}(p+q)+1\pm s_k\right)\right)\cdot\mathfrak{R}(s)\,\Phi_s(t)\,ds. \tag{4.17}$$

Hence,

$$\mathfrak{F}_{\kappa+1}(\alpha;\varepsilon)-\mathfrak{F}_{\kappa}(\alpha;\varepsilon)=\int_{i\mathbb{R}^p}-\int_{\mathcal{L}}=\sum_\sigma\left[\int_{\mathcal{L}_{\sigma+1}}-\int_{\mathcal{L}_\sigma}\right]. \tag{4.18}$$

Looking at Fig. 2 we observe

$$\int_{\mathcal{L}_{\sigma+1}}-\int_{\mathcal{L}_\sigma}=2\pi i\int_{\substack{s_1\in i\mathbb{R}_1,\,\ldots,\,s_{\sigma-1}\in i\mathbb{R}_{\sigma-1}\\ s_{\sigma+1}\in L_{\sigma+1},\,\ldots,\,\sigma_p\in L_p}} \tag{4.19}$$

$$\times\Big[\operatorname*{Res}_{s_\sigma=\alpha+\varepsilon_\sigma-1/2(p+q)+1+2\kappa}-\operatorname*{Res}_{s_\sigma=-\alpha-\varepsilon_\sigma+1/2(p+q)-1-2\kappa}\Big] \tag{4.20}$$

$$\times\,ds_1\cdots ds_{\sigma-1}\,ds_{\sigma+1}\cdots ds_p. \tag{4.21}$$

The integrand in (4.16) and (4.17) is an even function in s_σ and hence two residues in (4.20) and (4.21) differ only by sign. The order of the poles

$$s_\sigma = \pm(\alpha + \varepsilon_\sigma - \tfrac{1}{2}(p+q) + 1 + 2\kappa)$$

of the integrand is 1 and hence the residues can be evaluated by a simple substitution,

$$\begin{aligned}
&\operatorname*{Res}_{s_\sigma = \alpha + \varepsilon_\sigma - 1/2(p+q) + 1 + 2\kappa} \\
&\quad = H_\sigma(\alpha, \varepsilon, s) \\
&\quad := E(\alpha) \left[\frac{\Gamma\left(\frac{1}{2}\left(\alpha + \varepsilon_\sigma - \frac{1}{2}(p+q) + 1 - s_\sigma\right)\right)}{s_\sigma - \alpha - \varepsilon_\sigma + \frac{1}{2}(p+q) - 1 - 2\kappa} \right. \\
&\qquad \times \Gamma\left(\frac{1}{2}\left(\alpha + \sigma_\sigma - \frac{1}{2}(p+q) + 1 + s_\sigma\right)\right) \\
&\qquad \times \prod_{\pm;\, k \neq \sigma} \Gamma\left(\frac{1}{2}\left(\alpha + \varepsilon_k - \frac{1}{2}(p+q) + 1 \pm s_k\right)\right) \\
&\qquad \left. \cdot \Re(s)\, \Phi_s(t) \right] \Bigg|_{s_\sigma = \alpha + \varepsilon_\sigma - 1/2(p+q) + 1 + 2\kappa}
\end{aligned} \tag{4.22}$$

In this way, we reduce the sum (4.18) to

$$2 \sum_{\sigma=1}^{p} \int_{i\mathbb{R}_1 \times \cdots \times i\mathbb{R}_{\sigma-1} \times L_{\sigma+1} \times \cdots \times L_p} H_\sigma(\alpha, \varepsilon, s)\, ds_1 \cdots ds_{\sigma-1}\, ds_{\sigma+1} \cdots ds_p.$$

We obtain an expression for $\mathfrak{F}_{\kappa+1}(\alpha; \varepsilon) - \mathfrak{F}_\kappa(\alpha; \varepsilon)$. Unfortunately, the domains of integration are still complicated. For this reason, we apply the transformation (4.18) to each summand in the last expression. We obtain $p(p-1)/2$ additional summands, which are integrals over $(p-2)$-dimensional surfaces. Each integral can be easily evaluated by residues. After this we apply our arguments again, again, and again.

It is possible to write the final expression (it is slightly long). Fortunately, this is not necessary. The only goal of our interest is

$$\lim_{\varepsilon \to 0} (\mathfrak{F}_{\kappa+1}(\alpha; \varepsilon) - \mathfrak{F}_\kappa(\alpha; \varepsilon)). \tag{4.23}$$

For instance, consider the summand obtained by the substitution

$$\begin{aligned}
s_\sigma &= \alpha + \varepsilon_\sigma - \tfrac{1}{2}(p+q) + 1 + 2\kappa \\
s_\upsilon &= \alpha + \varepsilon_\upsilon - \tfrac{1}{2}(p+q) + 1 + 2\kappa.
\end{aligned}$$

Then the integrand contains the factors

$$\frac{1}{\Gamma\left(\pm\frac{1}{2}(s_\sigma - s_\upsilon)\right)}\Bigg|_{\substack{s_\sigma=\alpha+\varepsilon_\sigma-1/2(p+q)+1+2\kappa\\ s_\upsilon=\alpha+\varepsilon_\upsilon-1/2(p+q)+1+2\kappa}} = \frac{1}{\Gamma\left(\pm\frac{1}{2}(\varepsilon_\sigma-\varepsilon_\upsilon)\right)}.$$

These factors tend to 0 if $\varepsilon \to 0$. Of course, it is necessary to check the absence of poles of the numerator in the dangerous domain.

Therefore a nonzero contribution to the limit (4.23) can be given only by the terms

$$2\sum_{\sigma=1}^{p}\int_{i\mathbb{R}^p} H_\sigma(\alpha,\varepsilon,s)\,ds_1\cdots ds_{\sigma-1}\,ds_{\sigma+1}\cdots ds_p. \tag{4.24}$$

Our expression is symmetric under permutations of s_j and hence all summands of (4.18) give the same contribution to the limit. Thus, deleting ε in (4.24) we obtain the formula

$$\begin{aligned}
&\mathfrak{F}_{\kappa+1}(\alpha)-\mathfrak{F}_\kappa(\alpha)\\
&\quad=\frac{2\pi i\cdot 2p\cdot 2^{p\alpha}}{\prod\Gamma(\alpha-j+1)}\int_{\mathbb{R}^{p-1}}\left[\frac{\Gamma\left(\frac{1}{2}\left(\alpha-\frac{1}{2}(p+q)+1-s_1\right)\right)}{s_1-\alpha+\frac{1}{2}(p+q)-1-2\kappa}\,\Gamma\right.\\
&\qquad\times\left(\frac{1}{2}\left(\alpha-\frac{1}{2}(p+q)+1+s_1\right)\right)\\
&\qquad\times\prod_{2\leqslant k\leqslant p,\ \pm}\Gamma\left(\frac{1}{2}\left(\alpha-\frac{1}{2}(p+q)+1\pm s_k\right)\right)\\
&\qquad\cdot\prod_{2\leqslant k\leqslant p,\ \pm}\frac{\Gamma\left(\frac{1}{2}(q-p)\pm s_k\right)}{\Gamma(\pm s_k)}\prod_{2\leqslant k<l\leqslant p,\ \pm}\\
&\qquad\left.\left.\times\frac{\Gamma\left(\frac{1}{2}(1+s_k\pm s_l)\right)\Gamma\left(\frac{1}{2}(1-s_k\pm s_l)\right)}{\Gamma\left(\frac{1}{2}(s_k\pm s_l)\right)\Gamma\left(\frac{1}{2}(-s_k\pm s_l)\right)}\,\Phi_s(t)\right]\right|_{s_1=\alpha-1/2(p+q)+1+2\kappa}\\
&\qquad\times ds_2\cdots ds_p,
\end{aligned}$$

where $\alpha\in\Pi_{\kappa+1}\cap\mathcal{O}_\delta$.

4.5. *Calculations*

Lemma 4.3.b is an obvious corollary of the last formula. Nevertheless we present some elements of the calculation, since this is essential for understanding Section 4.7.

$$(1)\quad \left[\frac{\Gamma\left(\frac{1}{2}\left(\alpha+\frac{1}{2}(p+q)-1\pm s_k\right)\right)}{\Gamma\left(\frac{1}{2}(s_1\pm s_k)\right)\Gamma\left(\frac{1}{2}(-s_1\pm s_k)\right)}\right]\Bigg|_{s_1=\alpha-1/2(p+q)+2+2\kappa}$$

$$=\frac{\frac{1}{2}\left(-\alpha+\frac{1}{2}(p+q)-1\pm s_k\right)-\kappa}{\Gamma\left(\frac{1}{2}\left(-\alpha+\frac{1}{2}(p+q)+1\pm s_k\right)\right)}.$$

We observe that the factor $Y(\alpha; s)$ (see (4.2) and (2.2)) is canceled. This factor was the origin of singularities in our integral (2.1)–(2.5).

$$(2)\quad \Gamma(\tfrac{1}{2}(1+s_1\pm s_k))|_{s_1=\alpha-1/2(p+q)+1+2\kappa}$$
$$=\Gamma(\tfrac{1}{2}(\alpha-\tfrac{1}{2}(p+q)+1+2\kappa\pm s_k)).$$

We observe appearance of the factor (4.8), which is very similar to the factor $Y(\alpha; s)$. Later it will be an origin of new singularities.

$$(3)\quad \left[\frac{\Gamma\left(\frac{1}{2}\left(\alpha-\frac{1}{2}(p+q)+1+s_1\right)\right)}{\Gamma(s_1)\,\Gamma(-s_1)}\right]\Bigg|_{s_1=\alpha-1/2(p+q)+1+2\kappa}$$

$$=\frac{\left(-\alpha+\frac{1}{2}(p+q)-2\kappa-1\right)}{\Gamma\left(-\alpha+\frac{1}{2}(p+q)-\kappa\right)}.$$

This gives formulas (4.6)–(4.12) and completes the proof of Lemma 4.3.

4.6. *Analytic Continuation through the Line* $\operatorname{Re}\alpha<\frac{1}{2}(p+q)-1-2\kappa$

Lemma 4.3 gives the analytic continuation of $\mathfrak{F}_\kappa$ to $\mathcal{O}_\delta\cap\Pi_{\kappa+1}$. Evidently, the expression for the analytic continuation is analytic in the strip

$$-2\kappa-1<\alpha-\tfrac{1}{2}(p+q)+1<-2(\kappa-1) \qquad (4.25)$$

and hence we obtain the analytic continuation of $\mathfrak{F}_\kappa$ to the whole strip (4.25).

4.7. *Proof of Theorem* 2.2

The Plancherel formula (2.1)–(2.5) is correct if $\operatorname{Re} \alpha > \frac{1}{2}(p+q)-1$. We want to construct the analytic continuation of its right part to the domain $\operatorname{Re} \alpha < \frac{1}{2}(p+q)-1$. Let us move α to the left side.

First we pass across the line $\alpha = \frac{1}{2}(p+q)-1$. Then we obtain the additional summand $\mathfrak{F}^0(\alpha) := \mathfrak{F}_1(\alpha) - \mathfrak{F}_0(\alpha)$ given by formulas (4.6)–(4.12) for $\kappa = 0$. This is the summand of the Plancherel formula corresponding to $m=1$, $u_1 = 0$.

Let us compare the formulas (4.6)–(4.12) for $\mathfrak{F}^0(\alpha)$ and (2.1)–(2.5). First, we have in (4.6)–(4.12) the additional factor (4.9). This factor has singularities, but all these singularities lie in the domain $\alpha > \frac{1}{2}(p+q)-1$. The factors (2.2) and (4.8) are very similar (α is changed to $\alpha+1$). The factors (2.14)–(2.15) and (4.10)–(4.11) also are very similar. In fact, (4.10)–(4.11) is the Gindikin–Karpelevich density for $\mathrm{O}(p-1, q-1)$.

Hence, we can construct the analytic continuation of $\mathfrak{F}^0(\alpha)$ in the same way as above. The first singularity of $\mathfrak{F}^0(\alpha)$ on our course is the line $\operatorname{Re} \alpha = \frac{1}{2}(p+q)-2$. After passing across the line we obtain one more summand $\mathfrak{F}^{00}(\alpha)$ corresponding to $m=2$, $u_1 = u_2 = 0$.

The line $\operatorname{Re} \alpha = \frac{1}{2}(p+q)-3$ contains singularities of the integral $\mathfrak{F}(\alpha)$ and also singularities of $\mathfrak{F}^{00}(\alpha)$. Hence we obtain two additional summands $\mathfrak{F}^1(\alpha)$ and $\mathfrak{F}^{000}(\alpha)$ corresponding to $m=1$, $u_1 = 1$ and $m=3$, $u_1 = u_2 = u_3 = 0$, etc., etc., etc.

Formally, we must give a complete description of the inductive step but it literally repeats the arguments of Sections 4.1–4.6.

5. POSITIVE DEFINITENESS OF SPHERICAL FUNCTIONS

Thus, we obtain the expansion of $\mathscr{B}_\alpha(s)$ in spherical functions having the form

$$\mathscr{B}_\alpha(z) = \int_{\mathbb{C}^p} \Phi_s(z)\, d\mu_{our}(s), \tag{5.1}$$

where the positive D_p-invariant measure $\mu_{our}(z)$ is described in Theorem 2.2. Our purpose is to prove positive definiteness of all spherical functions $\Phi_s(z)$ that are contained in the support of the measure μ_{our}.

By the abstract Plancherel theorem, there exists the unique expansion

$$\mathscr{B}_\alpha(z) = \int_{\mathbb{C}^p} \Phi_s(z)\, d\mu_{truth}(s), \tag{5.2}$$

where μ_{truth} is a positive D_p-invariant measure on $\mathbb{C}^n$ supported by the space $\hat{G}_{sph}$ of positive definite spherical functions.

Substitute $z=0$ to (5.2). Then $\mathscr{B}_\alpha(0)=1$, $\Phi_s(0)=1$ and hence

$$\int_{\mathbb{C}^p} d\mu_{truth} = 1. \tag{5.3}$$

We denote by $\operatorname{supp} \mu_{truth}$ and $\operatorname{supp} \mu_{our}$ the supports of the measures μ_{truth} and μ_{our}.

5.1. *Preliminary Remarks on the Supports of the Measures*

Consider the bounded polyhedron $Q \subset \mathbb{R}^p$ described in Theorem 1.14. Consider the tube $\tilde{Q} \subset \mathbb{C}^p$ defined by the condition

$$s \in \tilde{Q} \qquad \text{iff} \qquad \operatorname{Re} s \in Q.$$

Then

$$\operatorname{supp} \mu_{truth} \subset \tilde{Q}; \qquad \operatorname{supp} \mu_{our} \subset \tilde{Q} \tag{5.4}$$

(the first is a corollary of Theorem 1.14; the second is a corollary of Theorem 2.2).

Denote by $\mathbb{R} \cup i\mathbb{R}$ the union of the real and imaginary axes in $\mathbb{C}$. Then

$$\begin{aligned} \operatorname{supp} \mu_{our} &\subset (\mathbb{R} \cup i\mathbb{R}) \times \cdots \times (\mathbb{R} \cup i\mathbb{R}); \\ \operatorname{supp} \mu_{truth} &\subset (\mathbb{R} \cup i\mathbb{R}) \times \cdots \times (\mathbb{R} \cup i\mathbb{R}) \end{aligned} \tag{5.5}$$

(the first is a corollary of Theorem 2.2 and the second is a corollary of Lemma 1.13).

5.2. *Heat Kernel*

Let $\Delta_1, \ldots, \Delta_p$ be Laplace operators (see [17, Section 2.5]) on the symmetric space G/K. The operator Δ_j is some G-invariant partial differential operator of order $2j$ on $G/K = \mathrm{B}_{p,q}$ with rational coefficients. The operator Δ_1 is the usual Laplace–Beltrami operator on G/K (see [17, Section 2.2.4]).

The spherical functions are joint eigenfunctions of the operators Δ_j (see [17, Section 4.2]). We have equalities

$$\Delta_j \Phi_s(z) = a_j(s)\, \Phi_s(z)$$

where a_j are some polynomials invariant with respect to the Weyl group D_p. If $\Phi_s \neq \Phi_{s'}$, then $a_j(s) \neq a_j(s')$ for some j.

In particular,

$$\Delta_1 \Phi_s(z) = (\lambda + s^2)\, \Phi_s(z),$$

where λ is a constant and

$$s^2 := s_1^2 + \cdots + s_p^2.$$

By conditions (5.4)–(5.5), *the eigenvalues* $(\lambda + s^2)$ *are real and they are uniformly bounded above on the supports of the measures* μ_{our}, μ_{truth}.

Consider the Cauchy problem for the *heat equation*

$$\left(\frac{\partial}{\partial \tau} - \Delta_1\right) F(z, \tau) = 0, \qquad F(z, 0) = f(z)$$

on G/K. Let $R_\tau(z, u)$ be the heat kernel. This means that the solution of the Cauchy problem for the heat equation is given by the formula

$$F(z, \tau) = A_\tau f(z) := \int_{G/K} R_\tau(z, u)\, f(u)\, d\lambda(u),$$

where λ is the G-invariant measure on G/K.

LEMMA 5.1. *For each* $\tau > 0$ *and* N *there exists a constant* $C(\tau, N)$ *independent of* z, u *such that*

$$R_\tau(z, u) \leqslant C(\tau, N)(1 + \operatorname{dist}(z, u))^{-N},$$

where $\operatorname{dist}(\cdot, \cdot)$ *is the distance in* G/K *associated with Riemannian metric.*

Proof. Since the heat kernel is G-invariant, we can assume $u = 0$. Then

$$R_\tau(z, 0) = \int_{s \in i\mathbb{R}} \exp\{\tau(\lambda + s^2)\}\, \Phi_s(z)\, ds.$$

By the integral formula (1.38) for spherical functions,

$$R_\tau(z, 0) = \int_{s \in i\mathbb{R}} \int_{\mathbf{k} \in K} \exp\{\tau(\lambda + s^2)\}\, \Psi_s(z^{[\mathbf{k}]})\, d\mathbf{k}\, ds.$$

Rapid decrease of the last expression is more or less obvious (behavior of heat kernels is investigated in detail in [60]). ∎

Similar estimates are valid for partial derivatives of $R_\tau(z, u)$ of any order. For spherical functions we have the equality

$$A_\tau \Phi_s(z) = \exp\{\tau(\lambda + s^2)\}\ \Phi_s(z).$$

LEMMA 5.2. *Let μ_{our}, μ_{truth} be the same as above. Then*

$$\text{(a)}\quad A_\tau \mathscr{B}_\alpha(z) = \int_{G/K} \exp\{\tau(\lambda + s^2)\}\ \Phi_s(z)\ d\mu_{our}(s) \tag{5.6}$$

$$\text{(b)}\quad A_\tau \mathscr{B}_\alpha(z) = \int_{G/K} \exp\{\tau(\lambda + s^2)\}\ \Phi_s(z)\ d\mu_{truth}(s). \tag{5.7}$$

Proof. We must prove the possibility of changing the order of the integration. It is sufficient to show absolute convergence of the integrals

$$\begin{gathered} \int_{G/K} \int_{\mathbb{C}^p} R_\tau(z, u)\ \Phi_s(u)\ d\mu_{our}(s)\ d\lambda(u); \\ \int_{G/K} \int_{\mathbb{C}^p} R_\tau(z, u)\ \Phi_s(u)\ d\mu_{truth}(s)\ d\lambda(u). \end{gathered} \tag{5.8}$$

We notice that the heat kernel rapidly decreases in u for fixed z, spherical functions $\Phi_{\sigma_j}(u)$ are bounded (see Theorem 1.14). In the first case the density of $\mu_{our}(s)$ exponentially decreases if $|s| \to \infty$. In the second case we have (5.3).

LEMMA 5.3. *For each polynomial $r(x_1, ..., x_p)$ we have*

$$r(\Delta_1, ..., \Delta_p)\ A_\tau \mathscr{B}_\alpha(z) = \int_{G/K} r(a_1(s), ..., a_p(s)) \exp\{\tau(\lambda + s^2)\}\ \Phi_s(z)\ d\mu_{our}(s)$$

$$r(\Delta_1, ..., \Delta_p)\ A_\tau \mathscr{B}_\alpha(z) = \int_{G/K} r(a_1(s), ..., a_p(s)) \exp\{\tau(\lambda + s^2)\}\ \Phi_s(z)\ d\mu_{truth}(s).$$

Proof. It is sufficient to prove that all partial derivatives by z of integrals (5.6), (5.7) absolutely converge. It is obvious by the following reasons.

1. The integrand rapidly decreases in the variable s.

2. For a given z partial derivatives of the heat kernel by u rapidly decrease.

3. Spherical functions are bounded. ∎

5.3. *Proof of Positive Definiteness*

Consider $\sigma \in \operatorname{supp} \mu_{our}$. Consider the function

$$\eta(s) = \exp\{\tau(\lambda + s^2)\} \sum_{j=1}^{p} (a_j(s) - a_j(\sigma))^2$$

Let M be the maximum of η on the support of μ_{our} (call to mind that $\operatorname{supp} \mu_{our} \subset Q$). Then the function

$$\zeta(s) = M - \eta(s)$$

satisfies conditions

$$\zeta(\sigma) = M; \qquad \zeta(s) < M \qquad \text{if} \quad s \neq w\sigma \qquad \text{for all} \quad w \in D_p.$$

Consider the sequence of functions

$$\xi_k(s) = C_k(M - \eta(s))^k \exp\{\tau(\lambda + s^2)\},$$

where C_k is determined by the condition $\int \xi_k(s)\, d\mu_{our} = 1$. Obviously, the sequence $\xi_k(s)$ converges to distribution $\sum_{w \in D_p} \delta(s - w\sigma)$.

The function $\xi_k(s)$ is a polynomial expression in $a_1(s), ..., a_p(s)$, $\exp\{\tau(\lambda + s^2)\}$:

$$\xi_k(s) = P_k(a_1(s), ..., a_p(s), \exp\{\tau(\lambda + s^2)\}).$$

Consider the operator

$$\Xi_k := P_k(\Delta_1, ..., \Delta_p, A_\tau).$$

By Lemma 5.3, we have

$$\Xi_k \mathscr{B}_\alpha(z) = \int_{G/K} \xi_k(s)\, \Phi_s(z)\, d\mu_{our}(s) \tag{5.9}$$

$$\Xi_k \mathscr{B}_\alpha(z) = \int_{G/K} \xi_k(s)\, \Phi_s(z)\, d\mu_{truth}(s). \tag{5.10}$$

We have $\xi_k(s) \geqslant 0$. Hence, by Lemma 1.5(d) and (5.10), the function $\Xi_k \mathscr{B}_\alpha(s)$ is positive definite. By (5.9), the sequence $\Xi_k \mathscr{B}_\alpha(s)$ converges to $\Phi_\sigma(z)$. Thus, $\Phi_\sigma(z)$ is a pointwise limit of positive definite functions and hence it is positive definite.

6. OTHER SERIES

6.1. *Hermitizations*

Below we present the list of hermitizations (see Section 0.1).[26]

Each Riemannian noncompact classical symmetric space G/K can be realized as a matrix ball.[27] A *matrix ball* is a space of all matrices of a given size over $\mathbb{K}=\mathbb{R}, \mathbb{C}, \mathbb{H}$ with norm <1 satisfying (or not satisfying) some symmetry condition. The list of matrix balls is given in Table 1.

The last column contains the hermitization $\widetilde{G}/\widetilde{K}$ of G/K. The embedding of matrix balls $G/K \to \widetilde{G}/\widetilde{K}$ in all cases is obvious.

Remark. Some spaces of small dimension are present in the left column two times (for instance the Lobachevskii plane $O(2,1)/O(2)\times O(1)$). Two associated hermitizations are different.

Table 2 contains hermitizations related to future tubes.

Exceptional hermitizations[28, 29] are given in the Table 3.

6.2. *Kernel Representations*

A *kernel representation*[30] ρ of G is a restriction of a highest weight representation $\tilde{\rho}$ of $\widetilde{G}$ to G. The constructive description of the scalar-valued kernel representations of $O(p,q)$ given in Sections 1.11 and 1.12 is valid for all series 1–10.

Below we discuss only scalar-valued kernel-representations.

6.3. *Plancherel Formula for Large* α

This formula was obtained in [35][31] for series 1–10.

[26] The classical part of the list is contained in Jaffee [22]. Olshanskii in [41, 42] observed that all cases (including exceptional cases) can be easily reduced to Nagano [29]. The list 1–18 is in one-to-one correspondence with the list of compressive semigroups of symmetric spaces and with the list of causal symmetric spaces; see [41, 42]. The list 1–10 is in one-to-one correspondence with the list of real classical categories; see [32, Addendum A].

[27] This observation is present in [30].

[28] EIII and EIV are real form of E_6, EVII is a real forms of E_7, and FII is a real form of F_4; see [54].

[29] Here we observe one of the cases where a phenomenon existing for classical groups does not exist for all exceptional groups.

[30] His definition was proposed in [33].

[31] The case of hermitian spaces 3, 7, 10, 11 was considered by Berezin [4] for $\alpha=\beta$; see the notations in Section 0.2 (the proof is published in [52]). For the hermitian case $\alpha\neq\beta$ the Plancherel formula independent of [35] was obtained by Zhang [56]. The future tube case is a simple exercise. The hermitian cases 13, 14 are covered by [52]; the case 15 is reduced to one of the Gindikin integrals [11]. Probably only for the exceptional cases 16, 17, 18 is the formula not known. Hence, the possibility of obtaining the solution in the "general case" (i.e., 16–18) is yet preserved.

TABLE 1

	G/K	$\mathbb{K}$	Size	Condition	$\widetilde{G}/\widetilde{K}$
1.	$\mathrm{GL}(n, \mathbb{R})/(\mathrm{O}(n, \mathbb{R})$	$\mathbb{R}$	$n \times n$	$z = z^t$	$\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$
2.	$\mathrm{O}(p, q)/\mathrm{O}(p) \times \mathrm{O}(q)$	$\mathbb{R}$	$p \times q$		$\mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)$
3.	$\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$	$\mathbb{C}$	$n \times n$	$z = z^t$	$\mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$ $\times \mathrm{Sp}(2n, \mathbb{R})/\mathrm{U}(n)$
4.	$\mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$	$\mathbb{C}$	$n \times n$	$z = z^*$	$U(n, n)/\mathrm{U}(n) \times \mathrm{U}(n)$
5.	$\mathrm{O}(n, \mathbb{C})/\mathrm{O}(n)$	$\mathbb{R}$	$n \times n$	$z = z^t$	$\mathrm{SO}^*(2n)/U(n)$
6.	$\mathrm{Sp}(2n, \mathbb{C})/\mathrm{Sp}(n)$	$\mathbb{H}$	$n \times n$	$z = -z^*$	$\mathrm{Sp}(4n, \mathbb{R})/\mathrm{U}(2n)$
7.	$\mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)$	$\mathbb{H}$	$p \times q$		$[\mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)]$ $\times [\mathrm{U}(p, q)/\mathrm{U}(p) \times \mathrm{U}(q)]$
8.	$\mathrm{GL}(n, \mathbb{H})/\mathrm{Sp}(n)$	$\mathbb{H}$	$n \times n$	$z = z^*$	$\mathrm{SO}^*(2n)/U(n)$
9.	$\mathrm{Sp}(p, q)/\mathrm{Sp}(p)$ $\times \mathrm{Sp}(q)$	$\mathbb{C}$	$p \times q$		$\mathrm{U}(2p, 2q)/\mathrm{U}(2p) \times \mathrm{U}(2q)$
10.	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$	$\mathbb{C}$	$n \times n$	$z = z^t$	$\mathrm{SO}^*(2n)/\mathrm{U}(n) \times \mathrm{SO}^*(2n)/\mathrm{U}(n)$

Consider for simplicity the nonhermitian case or hermitian case $\alpha = \beta$ (see the notations of Section 0.2. In all these cases, the Plancherel formula has the form

$$\prod_{k=1}^{p} \cosh^{-\alpha} t_k = E(\alpha) \int_{1\mathbb{R}^p} \prod_{k \leqslant p,\, \pm} \Gamma(\tfrac{1}{2}(\alpha - h + s_k))\, \mathfrak{R}(s)\, \Phi_s(t)\, ds, \tag{6.1}$$

where m is the rank of G, $\mathfrak{R}(s)$ is the Gindikin–Karpelevich density, $E(\alpha)$ is a meromorphic, factor and h is a constant. In fact, h is the last point of square integrability. This means that $\int_{G/K} |\mathscr{B}_\alpha(z)|^2\, dz$ is finite for $\alpha > h$ and infinite for $\alpha = h$.

6.4. *The Analytic Continuation of the Plancherel Formula*

Our arguments from Sections 4 and 5 do not depend on series and they are valid for all series 1–10.

In fact, the considerations of Section 4 prove that the following formal procedure gives the correct result. We fix $m = 0, 1, \ldots, p := \operatorname{rank} G$ and the collection of numbers $u_1 \leqslant u_2 \leqslant \cdots \leqslant u_m$ such that $\alpha + 2u_m + (m-1) \dim \mathbb{K} < h$.

TABLE 2

	G/K	$\tilde{G}/\tilde{K}$
11.	$\mathrm{SO}(2,n)/\mathrm{O}(n)\times\mathrm{O}(2)$	$[\mathrm{SO}(2,n)/\mathrm{O}(n)\times\mathrm{O}(2)]\times[\mathrm{SO}(2,n)/\mathrm{O}(n)\times\mathrm{O}(2)]$
12.	$\mathrm{SO}(1,p)\times\mathrm{O}(1,q)$	$\mathrm{SO}(2,p+q)$

Let

$$\begin{aligned} Q_0^{m;\,u}(s) &= \prod_{k\leqslant p,\ \pm} \Gamma\left(\frac{1}{2}(\alpha-h+s_k)\right)\cdot \mathfrak{R}(s)\,\Phi_s(t) \\ Q_k^{m;\,u}(s) &= \left.\frac{Q_{k-1}^{m;\,u}(s)}{s_k-\alpha+h-2u_k-k\dim\mathbb{K}}\right|_{s_k=\alpha-h+2u_k+k\dim\mathbb{K}}. \end{aligned} \tag{6.2}$$

Then the analytic continuation of (6.1) has the form

$$\prod_{k=1}^{p}\cosh^{-\alpha} t_k=\sum_{m;\,u_1,\ldots,u_m} E(\alpha)\,\frac{(2\pi)^m\,p!}{(p-m)!}\int_{i\,\mathbb{R}^{p-m}} Q_m^{m;\,u}(s)\,ds_{m+1}\cdots ds_m. \tag{6.3}$$

In the cases $\mathbb{K}=\mathbb{C}$, $\mathbb{H}$ (see Table 1) this formula can be considered as a final result in a closed form.

In the case $\mathbb{K}=\mathbb{R}$ ($G=\mathrm{O}(p,q)$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{R})$) the substitutions (6.2) are impossible without cancellations and in this case formula (6.3) gives an algorithmic procedure of calculation of Plancherel measure.

Remark. For the groups $\mathrm{U}(p,q)$ it is easy to obtain the Plancherel formula (see the explicit final expression in [33]) using Berezin–Karpelevich formula for spherical functions (see [5, 20]) and Molev unitarizability results [28]; partially this idea was also realized in [18].

6.5. *Kernel Representations of Compact Groups*

The hermitization procedure is also valid for compact Riemannian symmetric spaces. To obtain the list of hermitizations, we must replace the

TABLE 3

13.	$E\mathrm{III}/\mathrm{SO}(10)\times\mathrm{SO}(2)$	$[E\mathrm{III}/\mathrm{SO}(10)\times\mathrm{SO}(2)]\times[E\mathrm{III}/\mathrm{SO}(10)\times\mathrm{SO}(2)]$
14.	$E\mathrm{VII}/E\mathrm{III}\times\mathrm{SO}(2)$	$[E\mathrm{VII}/E\mathrm{III}\times\mathrm{SO}(2)]\times[E\mathrm{VII}/E\mathrm{III}\times\mathrm{SO}(2)]$
15.	$F\mathrm{II}/\mathrm{Spin}(9)$	$E\mathrm{III}/\mathrm{SO}(10)\times\mathrm{SO}(2)$
16.	$\mathrm{Sp}(2,2)/\mathrm{Sp}(2)\times\mathrm{Sp}(2)$	$E\mathrm{III}/\mathrm{SO}(10)\times\mathrm{SO}(2)$
17.	$\mathrm{GL}(4,\mathbb{H})/\mathrm{Sp}(2)$	$E\mathrm{VII}/E\mathrm{III}\times\mathrm{SO}(2)$
18.	$E\mathrm{IV}\times\mathbb{R}/F_4$	$E\mathrm{VII}/E\mathrm{III}\times\mathrm{SO}(2)$

TABLE 1′

	G/K	$\mathbb{K}$	Size	Condition	$\tilde{G}/\tilde{K}$
1′.	$\mathrm{U}(n)/\mathrm{O}(n, \mathbb{R})$	$\mathbb{R}$	$n\times n$	$z=z^t$	$\mathrm{Sp}(n)/\mathrm{U}(n)$
2′.	$\mathrm{O}(p+q)/\mathrm{O}(p)\times\mathrm{O}(q)$	$\mathbb{R}$	$p\times q$		$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$
3′.	$\mathrm{Sp}(n)/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^t$	$\mathrm{Sp}(n)/\mathrm{U}(n)\times\mathrm{Sp}(n)/\mathrm{U}(n)$
4′.	$\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^*$	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$
5′.	$\mathrm{O}(n)\times\mathrm{O}(n)/\mathrm{O}(n)$	$\mathbb{R}$	$n\times n$	$z=-z^t$	$\mathrm{O}(2n)/\mathrm{U}(n)$
6′.	$\mathrm{Sp}(n)\times\mathrm{Sp}(n)/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$z=z^*$	$\mathrm{Sp}(2n)/\mathrm{U}(2n)$
7′.	$\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)$	$\mathbb{H}$	$p\times q$		$[\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)]$ $\times[\mathrm{U}(p+q)/\mathrm{U}(p)\times\mathrm{U}(q)]$
8′.	$\mathrm{U}(2n)/\mathrm{Sp}(n)$	$\mathbb{H}$	$n\times n$	$z=z^*$	$\mathrm{O}(2n)/\mathrm{U}(n)$
9′.	$\mathrm{Sp}(p+q)/\mathrm{Sp}(p)=\mathrm{Sp}(q)$	$\mathbb{C}$	$p\times q$		$\mathrm{U}(2p+2q)/\mathrm{U}(2p)\times\mathrm{U}(2q)$
10′.	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathbb{C}$	$n\times n$	$z=z^t$	$\mathrm{O}(2n)/\mathrm{U}(n)\times\mathrm{O}(2n)/\mathrm{U}(n)$

groups G, $\tilde{G}$ in Tables 1–3 by their compact forms. For instance, the Table 1 transforms to Table 1′.

The construction of the kernel representation of $\mathrm{O}(p+q)$ given in Section 1.28 can be literally translated to all series 1–10. For this purpose we must replace the space $\mathrm{Mat}_{p,q}(\mathbb{R})$ by the space Mat of all matrices over $\mathbb{K}$ (see the second column) having the size given in the third column and satisfying the condition in the forth column. The group G acts on Mat by fractional linear transformations.[32]

Integrals evaluated in [35] easily give Plancherel formulas for all kernel representations in cases 1–10.

The case of hermitian symmetric spaces was earlier considered by Zhang [56].

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[32] More details on these models of Riemannian compact symmetric spaces are contained in [37].

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