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Hyers-Ulam Stability of Linear Differential Equations of First Order

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Abstract—We prove the Hyers-Ulam stability of linear differential equations of first order,

$$\varphi(t)y'(t) = y(t).$$

Indeed, this paper deals with a generalization of a paper by Alsina and Ger [1] or of papers by Miura, Takahasi and Choda [2] and by Miura [3].

Keywords—Hyers-Ulam stability, Differential equation.

1. INTRODUCTION

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems [4]. Among those was the question concerning the stability of homomorphisms:

> Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

In the following year, Hyers affirmatively answered the question of Ulam for the case where G_1 and G_2 are Banach spaces. Furthermore, the result of Hyers has been generalized by Rassias [5,6]. Since then, the stability problems of various functional equations have been investigated by many authors (see [7–10]).

Throughout this paper, let $I = (a, b)$ be an open real interval, where we assume that a and b satisfy $-\infty \le a < b \le +\infty$. Assume further that $\varphi : I \to \mathbb{R}$ is a given function for which the integral $\int_a^t d\tau/\varphi(\tau)$ exists for any $t \in I$.

Alsina and Ger were the first authors who investigated the Hyers-Ulam stability of a differential equation (see [1]). In fact, they proved that if a differentiable function $y : I \to \mathbb{R}$ satisfies $|y'(t) - y(t)| \le \varepsilon$ for all $t \in I$, then there exists a differentiable function $g : I \to \mathbb{R}$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|y(t) - g(t)| \le 3\varepsilon$ for every $t \in I$.

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The above result of Alsina and Ger has recently been generalized by Miura, Takahasi and Choda [2], by Miura [3], and also by Takahasi, Miura and Miyajima [11]. Indeed, they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(t)$, while Alsina and Ger investigated the differential equation $y'(t) = y(t)$.

The aim of this paper is to investigate the Hyers-Ulam stability of the following linear differential equation of first order:

$$\varphi(t)y'(t) = y(t). \tag{1}$$

More precisely, we prove that if either $\varphi(t) > 0$ holds for all $t \in I$ or $\varphi(t) < 0$ holds for all $t \in I$, and further if a differentiable function $y : I \to \mathbb{R}$ satisfies the inequality $|\varphi(t)y'(t) - y(t)| \le \varepsilon$ for all $t \in I$, then there exists a real number c such that

$$\left| y(t) - c\exp\left\{ \int_a^t \frac{d\tau}{\varphi(\tau)} \right\} \right| \le \varepsilon,$$

for any $t \in I$.

2. PRELIMINARIES

Following an idea of Alsina and Ger [1] (see also [2,3,11]), we will first prove a preliminary lemma.

LEMMA 1. *Assume that a differentiable function $z : I \to \mathbb{R}$ is given.*

(a) *The inequality $z(t) \le \varphi(t)z'(t)$ is true for all $t \in I$ if and only if there exists a differentiable function $\alpha : I \to \mathbb{R}$ such that $\alpha'(t)\varphi(t) \ge 0$ and*

$$z(t) = \alpha(t)\exp\left\{ \int_a^t \frac{d\tau}{\varphi(\tau)} \right\},$$

for all $t \in I$.

(b) *The inequality $z(t) \ge \varphi(t)z'(t)$ holds true for any $t \in I$ if and only if there exists a differentiable function $\beta : I \to \mathbb{R}$ such that $\beta'(t)\varphi(t) \le 0$ and*

$$z(t) = \beta(t)\exp\left\{ \int_a^t \frac{d\tau}{\varphi(\tau)} \right\},$$

for all $t \in I$.

PROOF. (a) Assume that the inequality $z(t) \le \varphi(t)z'(t)$ holds for all $t \in I$. Let us define a function $\alpha : I \to \mathbb{R}$ to be

$$\alpha(t) = \exp\left\{ -\int_a^t \frac{d\tau}{\varphi(\tau)} \right\} z(t).$$

Obviously, α is differentiable on I. Differentiation of α with respect to t yields

$$\alpha'(t) = \frac{1}{\varphi(t)} \exp\left\{ -\int_a^t \frac{d\tau}{\varphi(\tau)} \right\} (\varphi(t)z'(t) - z(t)),$$

i.e., our hypothesis implies that $\alpha'(t)\varphi(t) \ge 0$ is true for all $t \in I$.

Conversely, assume that there exists a differentiable function $\alpha : I \to \mathbb{R}$ such that $\alpha'(t)\varphi(t) \geq 0$ for each $t \in I$. Let us define a function $z : I \to \mathbb{R}$ by

$$z(t) = \alpha(t) \exp\left\{ \int_a^t \frac{d\tau}{\varphi(\tau)} \right\}.$$

Then, z is differentiable on I. By differentiation of z, we obtain

$$\varphi(t)z'(t) = \alpha'(t)\varphi(t) \exp\left\{ \int_a^t \frac{d\tau}{\varphi(\tau)} \right\} + z(t) \geq z(t),$$

for all $t \in I$, which completes the proof of (a).

For the proof of part (b), we only need to replace $z(t)$ by $-z(t)$ and apply part (a). ∎

The following theorem provides us with an explicit form of a differentiable function which satisfies inequality (2) below.

THEOREM 2. *Given an $\varepsilon > 0$, a differentiable function $y : I \to \mathbb{R}$ is a solution of the following inequality:*

$$|\varphi(t)y'(t) - y(t)| \leq \varepsilon, \tag{2}$$

for all $t \in I$, if and only if there exists a differentiable function $\alpha : I \to \mathbb{R}$ such that

$$y(t) = \varepsilon + \alpha(t) \exp\left\{ \int_a^t \frac{d\tau}{\varphi(\tau)} \right\} \tag{3}$$

and

$$0 \leq \alpha'(t)\varphi(t) \leq 2\varepsilon \exp\left\{ -\int_a^t \frac{d\tau}{\varphi(\tau)} \right\}, \tag{4}$$

for any $t \in I$.

PROOF. First, assume that a differentiable function $y : I \to \mathbb{R}$ is a solution of inequality (2). Equivalently, y satisfies

$$y(t) - \varepsilon \leq \varphi(t)y'(t) \leq y(t) + \varepsilon, \tag{5}$$

for each $t \in I$.

Define $z(t) = y(t) - \varepsilon$. It then follows from the inequality on the left side of (5) that $z(t) \leq \varphi(t)z'(t)$ holds for every $t \in I$. According to Lemma 1(a), there exists a differentiable function $\alpha : I \to \mathbb{R}$ such that the expression in (3) holds for all $t \in I$, where α additionally satisfies

$$\alpha'(t)\varphi(t) \geq 0, \tag{6}$$

for any $t \in I$.

Analogously, define $z(t) = y(t) + \varepsilon$. For this case, the inequality on the right side of (5) implies that $z(t) \geq \varphi(t)z'(t)$ holds for any $t \in I$. According to Lemma 1(b), there exists a differentiable function $\beta : I \to \mathbb{R}$ such that

$$y(t) + \varepsilon = \beta(t) \exp\left\{ \int_a^t \frac{d\tau}{\varphi(\tau)} \right\} \tag{7}$$

and

$$\beta'(t)\varphi(t) \leq 0, \tag{8}$$

for all $t \in I$.

Differentiate both expressions in (3) and (7) with respect to t and then equate both resulting equalities. Then, we have

$$\begin{aligned} y'(t) &= \alpha'(t) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} + \frac{\alpha(t)}{\varphi(t)} \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} \\ &= \beta'(t) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} + \frac{1}{\varphi(t)} \left(\alpha(t) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} + 2\varepsilon\right), \end{aligned}$$

where the last equality follows from a repeated application of (3) and (7).

Consequently, we have

$$\beta'(t) = \alpha'(t) - \frac{2\varepsilon}{\varphi(t)} \exp\left\{-\int_a^t \frac{d\tau}{\varphi(\tau)}\right\},$$

which together with (6) and (8) imply the validity of (4) for any $t \in I$.

Now, assume that a function $y : I \to \mathbb{R}$ is given by (3) for all $t \in I$, where $\alpha : I \to \mathbb{R}$ is a differentiable function and satisfies the inequalities in (4) for any $t \in I$. Differentiate $y(t)$ with respect to t and multiply by $\varphi(t)$ the resulting equation and then subtract $y(t)$ from the resulting equation. Then, we get

$$\varphi(t)y'(t) - y(t) = \alpha'(t)\varphi(t) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} - \varepsilon.$$

By (4) and the last equation, we conclude that $-\varepsilon \leq \varphi(t)y'(t) - y(t) \leq \varepsilon$ for all $t \in I$, which is equivalent to the inequality of (2). ∎

3. HYERS-ULAM STABILITY OF DIFFERENTIAL EQUATION (1)

In the following theorem, we prove the Hyers-Ulam stability of the differential equation (1) which obviously improves a result of Alsina and Ger (see [1, Remark]) as well as a result of Miura, Takahasi and Choda (see [2, Theorem 2.3]; see also [3,11]).

THEOREM 3. *If either $\varphi(t) > 0$ holds for all $t \in I$ or $\varphi(t) < 0$ holds for all $t \in I$, and if a differentiable function $y : I \to \mathbb{R}$ satisfies inequality (2) for all $t \in I$, then there exists a real number c such that*

$$\left| y(t) - c \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\}\right| \leq \varepsilon, \tag{9}$$

for any $t \in I$.

PROOF. First, we assume that $\varphi(t) > 0$ holds true for all $t \in I$ and that a differentiable function $y : I \to \mathbb{R}$ satisfies inequality (2) for all $t \in I$.

Define $c := \lim_{t \to b-} \alpha(t)$, where $\alpha : I \to \mathbb{R}$ is given in Theorem 2. Since $\varphi(t) > 0$ for all $t \in I$, we can divide the inequalities in (4) by $-\varphi(t)$ and integrate them from t to b:

$$\begin{aligned} 0 \geq \alpha(t) - c = -\int_t^b \alpha'(s)\, ds &\geq 2\varepsilon \int_t^b \left(-\frac{1}{\varphi(s)}\right) \exp\left\{-\int_a^s \frac{d\tau}{\varphi(\tau)}\right\} ds \\ &= 2\varepsilon \exp\left\{-\int_a^b \frac{d\tau}{\varphi(\tau)}\right\} - 2\varepsilon \exp\left\{-\int_a^t \frac{d\tau}{\varphi(\tau)}\right\}, \end{aligned}$$

for all $t \in I$. From the above inequalities, we further get

$$\varepsilon \geq \varepsilon + (\alpha(t) - c) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} \geq 2\varepsilon \exp\left\{-\int_t^b \frac{d\tau}{\varphi(\tau)}\right\} - \varepsilon \geq -\varepsilon,$$

for every $t \in I$. These inequalities, together with (3), prove the validity of inequality (9) for any $t \in I$.

We now assume that $\varphi(t) < 0$ holds true for all $t \in I$ and that a differentiable function $y : I \to \mathbb{R}$ is a solution of inequality (2). If we define $c := \lim_{t \to a+} \alpha(t)$ and apply a similar argument to this case, we obtain the desired result. ∎

Here, we notice that $\gamma \exp\{\int_a^t d\tau / \varphi(\tau)\}$ is the general solution of the differential equation (1), where γ is an arbitrary real constant.

Theorem 3 states that each solution of inequality (2) can be approximated by a solution of the differential equation (1) within a distance ε. Unfortunately, there is no efficient way to find out the constant c occurring in inequality (9), even though we can get some information on the behavior of $y(t)$ from Theorem 2.

However, if an initial condition, say $y(a)$, is known, then the following corollary may help to evaluate the lower and upper bounds for $y(t)$.

COROLLARY 4. *Assume that a is a real number and that $\varphi(t) > 0$ holds for all $t \in I$. Let a function $y : I \cup \{a\} \to \mathbb{R}$ be differentiable on I and continuous at a on the right. If y satisfies inequality (2) for all $t \in I$ and for some $\varepsilon > 0$, then*

$$(y(a) - \varepsilon) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} + \varepsilon \le y(t) \le (y(a) + \varepsilon) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} - \varepsilon, \tag{10}$$

for any $t \in I$.

PROOF. If y satisfies inequality (2) for all $t \in I$, Theorem 2 implies that there exists a differentiable function $\alpha : I \to \mathbb{R}$ satisfying (3) and (4) for any $t \in I$.

Since $\varphi(t) > 0$ for $t \in I$, we can divide the inequalities in (4) by $\varphi(t)$ and integrate them from a to t so that

$$0 \le \alpha(t) - c \le 2\varepsilon - 2\varepsilon \exp\left\{-\int_a^t \frac{d\tau}{\varphi(\tau)}\right\},$$

where we define $c := \lim_{t \to a+} \alpha(t)$, and so

$$0 \le \alpha(t) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} - c \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} \le 2\varepsilon \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} - 2\varepsilon.$$

From these inequalities, and considering (3), we conclude that (10) is true because if we let $t \to a+$ in (3), then we obtain $c = y(a) - \varepsilon$. ∎

If $\varphi(t) < 0$ is assumed for each $t \in I$, we can prove the following corollary by using an analogous argument. Hence, we omit the proof.

COROLLARY 5. *Assume that a is a real number and that $\varphi(t) < 0$ is true for any $t \in I$. Moreover, assume that $y : I \cup \{a\} \to \mathbb{R}$ is a function which is differentiable on I and continuous at a on the right. If y satisfies inequality (2) for all $t \in I$ and for some $\varepsilon > 0$, then*

$$(y(a) + \varepsilon) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} - \varepsilon \le y(t) \le (y(a) - \varepsilon) \exp\left\{\int_a^t \frac{d\tau}{\varphi(\tau)}\right\} + \varepsilon,$$

for each $t \in I$.

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