Universal Polynomial Majorants on Convex Bodies

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Let $\mathbf{K}$ be a convex body in $\mathbb{R}^d$ ($d \geq 2$), and denote by $B_n(\mathbf{K})$ the set of all polynomials $p_n$ in $\mathbb{R}^d$ of total degree $\leq n$ such that $|p_n| \leq 1$ on $\mathbf{K}$. In this paper we consider the following question: does there exist a $p^*_n \in B_n(\mathbf{K})$ which majorates every element of $B_n(\mathbf{K})$ outside of $\mathbf{K}$? In other words can we find a minimal $\gamma > 1$ and $p^*_n \in B_n(\mathbf{K})$ so that $|p_n(x)| \leq \gamma |p^*_n(x)|$ for every $p_n \in B_n(\mathbf{K})$ and $x \in \mathbb{R}^d \setminus \mathbf{K}$? We discuss the magnitude of $\gamma$ and construct the universal majorants $p^*_n$ for even $n$. It is shown that $\gamma$ can be 1 only on ellipsoids. Moreover, $\gamma = O(1)$ on polytopes and has at most polynomial growth with respect to $n$, in general, for every convex body $\mathbf{K}$.

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Let $\mathbf{K} \subset \mathbb{R}^d$, $d \geq 2$, be a convex body i.e., it is a convex compact set with nonempty interior in $\mathbb{R}^d$. Consider the space $P_n^d$ of polynomials on $\mathbb{R}^d$ of total degree $\leq n$, endowed with the usual supremum norm on $\mathbf{K}$. Then the unit ball in this space is given by

$$B_n(\mathbf{K}) := \{ p \in P_n^d : \| p \|_{C(\mathbf{K})} \leq 1 \}.$$ 

In this paper we address the following question: is there a "largest" polynomial in $B_n(\mathbf{K})$ which majorates all elements of $B_n(\mathbf{K})$ everywhere on $\mathbb{R}^d \setminus \mathbf{K}$? In other words does there exist a $\gamma > 1$ and $p^*_n \in B_n(\mathbf{K})$ such that

$$|p_n(x)| \leq \gamma |p^*_n(x)|, \quad \forall p_n \in B_n(\mathbf{K}), \quad \forall x \in \mathbb{R}^d \setminus \mathbf{K}? \quad (1)$$

Such a $p^*_n$ majorates all $p_n \in B_n(\mathbf{K})$ at every point outside $\mathbf{K}$ (with the constant $\gamma$). In this sense $p^*_n$ is a universal majorant for polynomials in $B_n(\mathbf{K})$. Naturally, we are interested in the smallest possible $\gamma > 1$ for which (1) holds with some $p^*_n \in B_n(\mathbf{K})$. Thus we set $\gamma_n(\mathbf{K}) := \inf \{ \gamma : \text{there exists a } p^*_n \in B_n(\mathbf{K}) \text{ so that (1) holds} \}$.

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The above definition is motivated by the classical inequality of Chebyshev (see [1, p. 235]) stating that when \( d = 1 \) and \( K = [-1, 1] \) we have

\[
|p_n(x)| \leq |T_n(x)|, \quad \forall p_n \in B_d([-1, 1]), \quad \forall |x| > 1,
\]

where \( T_n(x) = \cos n \pi x \) is the Chebyshev polynomial. This means in our terminology that \( \gamma_d([-1, 1]) = 1 \) for every \( n \in \mathbb{N} \), with \( \pm T_n \) being the universal majorants.

In this paper we shall study the magnitude of \( \gamma_d(K) \) when \( d > 1 \) and \( K \) is a convex body in \( \mathbb{R}^d \). First, it has to be noted that the above question is meaningful only for even \( n \in \mathbb{N} \), because \( \gamma_{2n+1}(K) = \infty \) whenever \( d > 1 \) and \( n \in \mathbb{N} \). Indeed, if \( \gamma_{2n+1}(K) < \infty \), i.e., a universal majorant \( p_{2n+1}^* \in B_{2n+1}(K) \) exists, then it follows from (1) that \( \deg p_{2n+1}^* = 2n+1 \) (and not less), and \( p_{2n+1}^* \neq 0 \) on \( \mathbb{R}^d \setminus K \). Since \( d > 1 \) we can easily find a line \( L = \{at + b : t \in \mathbb{R}^1\} \) in \( \mathbb{R}^d \) (\( a, b \in \mathbb{R}^d \)) so that \( L \cap K = \emptyset \) and the univariate polynomial \( p_{2n+1}^* (at + b) \) has degree \( 2n+1 \). This yields that \( p_{2n+1}^* (at_0 + b) = 0 \) for some \( t_0 \in \mathbb{R}^1 \) contradicting the above observation that \( p_{2n+1}^* \neq 0 \) on \( \mathbb{R}^d \setminus K \).

On the other hand for even \( n \) one can give a simple example of a universal majorant in \( \mathbb{R}^d, d > 1 \). In what follows \( |x| \) denotes the Euclidean norm in \( \mathbb{R}^d \) \((d \geq 1), \langle x, y \rangle \) stands for the inner product of \( x, y \in \mathbb{R}^d \), \( B_d \) and \( \text{Int} K \) are the boundary and interior of \( K \), respectively.

**Example 1.** Let \( K = \{x \in \mathbb{R}^d : |x| \leq 1\} \) be the Euclidean unit ball in \( \mathbb{R}^d \). Then \( \gamma_d(K) = 1 \) with \( p_{2n}^* (x) = T_{2n}(|x|) \in B_{2n}(K) \) being a universal majorant. This follows immediately from (2) since \( T_{2n}(t) \), \( t \in \mathbb{R}^1 \) is an even polynomial.

Using affine transformations of \( \mathbb{R}^d \) the above example can be easily extended to arbitrary ellipsoids which means that \( \gamma_d(K) = 1 \) for any ellipsoid \( K \). Our first result gives a converse to this showing that \( \gamma_d(K) \) can attain its minimal value 1 only on ellipsoids.

**Theorem 1.** Let \( K \in \mathbb{R}^d, d \geq 2, \) be a convex body; \( n \in \mathbb{N} \). Then \( \gamma_d(K) = 1 \) if and only if \( K \) is an ellipsoid, i.e., \( K = \{x \in \mathbb{R}^d : |Ax + b| \leq 1\} \) for some \( A \in \mathbb{R}^{d \times d} \) (\( \det A \neq 0 \)) and \( b \in \mathbb{R}^d \). Moreover, in this case \( p_{2n}^* = \pm T_{2n}(|Ax + b|) \) are the only universal majorants.

Thus apart from ellipsoids we always have \( \gamma_d(K) > 1 \). It turns out that \( \gamma_d(K) = O(1) \) with a constant independent of \( n \) whenever \( K \) is a polytope. For a polytope \( K \) we shall denote by \( f_j(K) \) the number of its \( j \)-dimensional faces, \( 0 \leq j \leq d - 1 \).
Theorem 2. Let $K$ be a convex polytope in $\mathbb{R}^d$, $d \geq 2$. Then for every $n \in \mathbb{N}$

$$\gamma_{2n}(K) \leq \sum_{j=1}^{d-2} f_j(K) f_{d-j-1}(K) + 2f_{d-1}(K). \quad (3)$$

Moreover, if $K$ is central symmetric then we have $\gamma_{2n}(K) \leq f_{d-1}(K)$.

Using the above theorem and some known results on degree of approximation of convex bodies by polytopes with prescribed number of vertices or faces we can verify that $\gamma_{2n}(K)$ has at most polynomial growth in $n$ for every convex body $K$. Namely we have the next

Theorem 3. Let $K$ be a convex body in $\mathbb{R}^d$, $d \geq 2$. Then for every $n \in \mathbb{N}$

$$\gamma_{2n}(K) \leq c(d, K) n^{d(d-1)}, \quad (4)$$

where $c(d, K) > 0$ depends only on $d$ and $K$.

Note that in general, polynomials bounded by 1 on $K$ can grow exponentially outside $K$. Thus the polynomial growth $\gamma_{2n}(K) = O(n^{d(d-1)})$ given by Theorem 3 is very small relative to the size of polynomials $p_n \in B_d(K)$ outside of $K$. The estimate (4) can be improved further if $K$ has a $C^2_\infty$-boundary, i.e., its second fundamental form exists on $\text{Bd } K$ and the Gauss curvature is a positive continuous function on $\text{Bd } K$.

Theorem 4. If $K$ is a convex body in $\mathbb{R}^d$ ($d \geq 2$) with a $C^2_\infty$-boundary then $\gamma_{2n}(K) = O(n^{2d-1})$.

Above estimates can be used in order to obtain results on approximation of convex surfaces by algebraic surfaces. (We call zero sets of $p_n \in P^d_n$ algebraic surfaces of order $n$.) Denote by $d(A, B)$ the Hausdorff distance between $A, B \subset \mathbb{R}^d$.

Theorem 5. For any convex body $K$ in $\mathbb{R}^d$ ($d \geq 2$) there exists an algebraic surface $\Omega_n$ of order $n$ such that $d(\text{Bd } K, \Omega_n) \leq c \left(\frac{\log n}{n}\right)^2$, where $c > 0$ depends only on $K$ and $d$.

This paper is organized as follows. Section 1 contains some material on the geometry of convex bodies needed for our considerations. In Section 2 the proofs of Theorem 1–5 will be given. Finally, we shall conclude the paper by a discussion of some open problems.
1. GEOMETRY

First we need to introduce a certain quantity $k(x)$ which measures the distance from a given $x \in \mathbb{R}^d$ to the boundary $BdK$ of a convex body $K \subset \mathbb{R}^d$. This quantity was used in [5] and [6] for the study of multivariate Chebyshev and Bernstein Inequalities.

For given $A, B \in \mathbb{R}^d$ and $u \in S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$ such that $\langle u, B - A \rangle > 0$ consider the corresponding "slab" given by $S_u(A, B) := \{x \in \mathbb{R}^d : \langle u, A \rangle \leq \langle u, x \rangle \leq \langle u, B \rangle\}$.

For a fixed $\varepsilon > 0$ the "$\varepsilon$-dilation" of this slab is defined by $S_u^\varepsilon(A, B) := \{x \in \mathbb{R}^d : \langle u, A \rangle - \delta_u \leq \langle u, x \rangle \leq \langle u, B \rangle + \delta_u\}$ where $\delta_u := \varepsilon \frac{|u|}{2}$.

Finally, set $K_u := \bigcap \{S_u^\varepsilon(A, B) : S_u(A, B) = K, A, B \in \mathbb{R}^d, u \in S^{d-1}\}$, $\zeta_k(x) := \inf\{z : x \in K_u\}$.

Clearly, $\zeta_k(x) > 1$ for $x \in \mathbb{R}^d \setminus K$, $\zeta_k(x) = 1$ on $BdK$, and $\zeta_k(x) < 1$ inside $K$. Also, it is easy to see that when $K$ is central symmetric about $0$ then $\zeta_k(x) = \inf\{\varepsilon > 0 : \frac{x}{\varepsilon} \in K\}$ is the usual Minkowski functional. It is proved in [6] that for every $x \in \mathbb{R}^d \setminus K$

$$\sup\{|p_n(x) : p_n \in B_d(K)\} = T_d(\zeta_k(x)).$$

(5)

We shall also need the following lemmas on parallel supporting hyperplanes which are proved in [5] and [6]. (A special case of Lemma 1 also appears in [7].)

**Lemma 1.** Let $x \in \mathbb{R}^d \setminus K$. Then there exists a line $L$ passing through $x$ with $K \cap L = [A, B]$, such that $K$ possesses parallel supporting hyperplanes at $A$ and $B$. Moreover, for any such line

$$\zeta_k(x) = \frac{|x - A + B|}{2} / \frac{|A - B|}{2}.$$  (6)

For the proof of the above statement see [6], Corollary 1 and the proof of Theorem 1A on p. 422. The next lemma provides a similar statement for inner points of $K$.

**Lemma 2.** Let $x \in Int K$. Then there exists a line $L$ passing through $x$ with $K \cap L = [A, B]$ such that (6) holds, and $K$ possesses parallel supporting hyperplanes at $A$ and $B$. 

Note a slight difference in the statements of Lemmas 1 and 2: when \( x \in \mathbb{R}^d \setminus K \) by Lemma 1 (6) holds for every \( L \) as above, while for \( x \in \text{Int} \ K \) by Lemma 2 (6) holds for some \( L \) as above.

The first statement of Lemma 2 asserting that (6) holds for a certain line as above is a consequence of Proposition 2 in [5]. The second statement concerning parallel supporting hyperplanes is Proposition 1 of [5].

2. PROOFS

**Proof of Theorem 1.** The sufficiency in Theorem 1 is straightforward, it follows by a change of variables \( y = Ax + b \) (\( x, y \in \mathbb{R}^d \)) and Example 1.

Assume now that \( K \subset \mathbb{R}^d \) is such that \( \gamma_{2n}(K) = 1 \), and \( p_{2n}^* \in B_{2n}(K) \) is a corresponding universal majorant, so that

\[
|p_{2n}(x)| \leq |p_{2n}^*(x)|, \quad p_{2n} \in B_{2n}(K), \quad x \in \mathbb{R}^d \setminus K.
\]

Then it easily follows from (5) that

\[
|p_{2n}^*(x)| \equiv T_{2n}(\sigma_K(x)), \quad x \in \mathbb{R}^d \setminus K.
\]

In particular, we have that for every \( x \in \mathbb{R}^d \setminus K \) either \( p_{2n}^*(x) \equiv T_{2n}(\sigma_K(x)) \), or \( p_{2n}^*(x) \equiv -T_{2n}(\sigma_K(x)) \). Thus we may assume that

\[
p_{2n}^*(x) \equiv T_{2n}(\sigma_K(x)), \quad x \in \mathbb{R}^d \setminus K. \tag{7}
\]

First we shall verify that equality (7) holds for \( x \in K \), as well. Choose any \( \hat{x} \in \text{Int} K \). Then by Lemma 2 there exists a line \( L \) through \( \hat{x} \) with \( L \cap K = [A, B] \) such that

\[
\sigma_K(\hat{x}) = \frac{\hat{x} - A + B}{2}, \tag{8}
\]

and \( K \) possesses parallel supporting hyperplanes at \( A \) and \( B \). Let

\[
\hat{x} = \frac{-1 - t}{2} A + \frac{1 + t}{2} B,
\]

where it can be assumed that \( 0 \leq t \leq 1 \). Then by (8), \( \sigma_K(\hat{x}) = \hat{t} \). Moreover, by Lemma 1 for every \( x \in L \setminus K \) equality (6) holds, i.e. setting \( x_i = \frac{1}{2} A + \frac{t}{2} B \) we have \( \sigma_K(x_i) = t, \ t > 1 \). This and (7) yield that

\[
p_{2n}^*(x_i) \equiv T_{2n}(t), \quad t > 1.
\]
But of course the above equality of univariate polynomials has to extend from \( \{ t \in \mathbb{R}^1 : t > 1 \} \) to the whole line, i.e.,

\[
p_\ast^2 \left( \frac{1-t}{2} A + \frac{1+t}{2} B \right) = T_{2a}(t), \quad t \in \mathbb{R}. \tag{9}
\]

In particular, setting in (9) \( t = \bar{t} \) we obtain \( p_{\ast}^2(\bar{x}) = T_{2a}(\bar{t}) = T_{2a}(\bar{x}) \). Thus and by (7)

\[
p_{\ast}^2(x) = T_{2a}(\bar{x}), \quad x \in \mathbb{R}^d. \tag{10}
\]

The next step is to verify that \( K \) is central symmetric. Set

\[
\sigma_0 := \inf_{x \in K} \bar{x}(x), \quad K_0 := \bigcap_{x > \sigma_0} K_x.
\]

Clearly, \( \sigma_0 \geq 0, K_0 \neq \emptyset \) and \( \text{Int } K_x \neq \emptyset, \quad \sigma > \sigma_0 \). Furthermore, for every \( x \in K_0 \) we have \( \bar{x}_K(x) \leq \sigma_0 \), i.e., by minimality of \( \sigma_0 \) it follows that \( \bar{x}_K(x) = \sigma_0 \) whenever \( x \in K_0 \). This last observation implies that \( K_0 \) must be a singleton. Indeed, if \( a^*, b^* \in K_0 \) \( (a^* \neq b^*) \) then \( [a^*, b^*] \subset K_0 \), and hence \( \bar{x}_K(x) = \sigma_0 \) for \( x \in [a^*, b^*] \). This and (10) yield that \( p_{\ast}^2 = T_{2a}(\sigma_0) \) on the line \( L^* \) through \( a^* \) and \( b^* \), in an obvious contradiction with (10). Thus \( K_0 = [a^*] \). Consider now a line \( L^* \) through \( a^* \) with \( \text{K} \cap L^* = [a^*, B^*] \) such that \( K \) possesses parallel supporting hyperplanes at \( A^* \) and \( B^* \) (Lemma 2). By (9) and (10) we have with \( x^* = \frac{1-t}{2} A^* + \frac{1+t}{2} B^* \)

\[
T_{2a}(t) = p_{\ast}^2(x^*) = T_{2a}(\sigma_K(x^*)), \quad t \in \mathbb{R}^1. \tag{11}
\]

As \( t \) increases from \(-1\) to \( 1 \) the continuous function \( \sigma_K(x^*) \) decreases from \( 1 \) to \( \sigma_0 \), and then increases from \( \sigma_0 \) to \( 1 \). Thus in view of (11) we must have \( \sigma_0 = 0 \), and \( a^* = x_0^* = (A^* + B^*)/2 \). (In particular, \( \text{Int } K_x \neq \emptyset \) for every \( x > 0 \).) Similarly for any \( A \in \text{Bd } K \) there exists a \( B \in \text{Bd } K \) such that \( K \) possesses parallel supporting hyperplanes at \( A \) and \( B \). Thus using again (9) and (10)

\[
T_{2a}(t) = T_{2a} \left( \bar{x}_K \left( \frac{1-t}{2} A + \frac{1+t}{2} B \right) \right), \quad t \in \mathbb{R}.
\]

Again, as \( t \) varies in \([-1, 1]\) \( \bar{x}_K(A) \) must decrease from \( 1 \) to \( 0 \) and then increase from \( 0 \) to \( 1 \). Hence \([A, B]\) must contain \( a^* \) (otherwise
\(\mathbf{a}^* = \frac{\mathbf{A} + \mathbf{B}}{2}\) can not attain 0), and, in addition, \(\mathbf{a}^* = \mathbf{A} + \mathbf{B}\). This means that \(\mathbf{K}\) is central symmetric about \(\mathbf{a}^*\).

We may assume now that \(\mathbf{a}^* = \mathbf{0}\) and \(\mathbf{K}\) is symmetric about the origin.

Then \(\mathbf{z}_\mathbf{K}(t\mathbf{x}) = t\mathbf{z}_\mathbf{K}(\mathbf{x})\) whenever \(\mathbf{x} \in \mathbb{R}^d\) and \(t > 0\). The polynomial \(p^*_\mathbf{K}\) can be written as \(p^*_\mathbf{K}(\mathbf{x}) = \sum_{j=0}^{2n} h_j(\mathbf{x})\), where \(h_j\) is its \(j\)th homogeneous part, \(0 \leq j \leq 2n\). Furthermore \(T_{2n}(t) = \sum_{j=0}^{n} c_j t^{2j}\), where \(c_j \in \mathbb{R},\ 0 \leq j \leq n\). Then for every \(\mathbf{u} \in \mathbb{S}^{d-1}\) and \(t > 0\)

\[
p^*_\mathbf{K}(t\mathbf{u}) = \sum_{j=0}^{2n} h_j(t\mathbf{u}) = \sum_{j=0}^{2n} t^j h_j(\mathbf{u}),
\]

\[
T_{2n}(\mathbf{z}_\mathbf{K}(t\mathbf{u})) = T_{2n}(t\mathbf{z}_\mathbf{K}(\mathbf{u})) = \sum_{j=0}^{n} c_j t^{2j}(\mathbf{u})^2.
\]

Hence using (10) we obtain

\[
\sum_{j=0}^{2n} h_j(\mathbf{u}) t^j = \sum_{j=0}^{n} c_j \mathbf{z}_\mathbf{K}^2(\mathbf{u}) t^{2j}, \quad \mathbf{u} \in \mathbb{S}^{d-1}, \quad t > 0.
\]

This means that \(h_j(\mathbf{u}) = c_j \mathbf{z}_\mathbf{K}^2(\mathbf{u})\) for every \(\mathbf{u} \in \mathbb{S}^{d-1}\). In particular

\[
\mathbf{z}_\mathbf{K}^2(\mathbf{u}) = \frac{1}{c_1} h_1(\mathbf{u}) := H_2(\mathbf{u}), \quad \mathbf{u} \in \mathbb{S}^{d-1}.
\]

Evidently, \(H_2\) is a positive definite quadratic form, i.e.

\[K = \{ \mathbf{x} \in \mathbb{R}^d : \mathbf{z}_\mathbf{K}(\mathbf{x}) \leq 1 \} = \{ \mathbf{x} \in \mathbb{R}^d : H_2(\mathbf{x}) \leq 1 \}
\]

is an ellipsoid. In addition, by (10) the only possible majorants are \(\pm T_{2n}(\mathbf{z}_\mathbf{K}(\mathbf{x}))\).

**Proof of Theorem 2.** Let \(\mathbf{K} \subset \mathbb{R}^d\) be a polytope. Consider \(\mathbf{A}, \mathbf{B} \in \text{Bd} \mathbf{K}\) such that \(\mathbf{K}\) possesses parallel supporting hyperplanes \(\mathbf{H}_\mathbf{A}, \mathbf{H}_\mathbf{B}\) at \(\mathbf{A}\) and \(\mathbf{B}\), and denote by \(\mathcal{H}_{\mathbf{AB}}\) the set of normal vectors to such pairs of hyperplanes. Since \(\mathbf{K}\) is a polytope it is easy to see that for some \(\mathbf{u} \in \mathcal{H}_{\mathbf{AB}}\) the corresponding pair of hyperplanes \(\mathbf{H}_\mathbf{A}, \mathbf{H}_\mathbf{B}\) has the property that the faces \(\mathbf{F}_\mathbf{A} = \mathbf{K} \cap \mathbf{H}_\mathbf{A}\) and \(\mathbf{F}_\mathbf{B} = \mathbf{K} \cap \mathbf{H}_\mathbf{B}\) of the polytope \(\mathbf{K}\) contain a total of \(d-1\) linearly independent vectors. Let \(\mathcal{H}(\mathbf{K}) := \{ \mathbf{u}_1, \ldots, \mathbf{u}_N \} \subset \mathbb{S}^{d-1} := \{ \mathbf{y} = (y_1, \ldots, y_d) \in \mathbb{S}^{d-1} : y_1 \geq 0 \}\) be the set of normal vectors to pairs of hyperplanes with the
above properties. Since every \( u_j \in \mathcal{U}(K) \), \( 1 \leq j \leq N \), is uniquely determined by the corresponding pair of faces of \( K \) specified above it follows that

\[
N \leq \frac{d}{2} \sum_{j=1}^{d-2} f_j(K) f_{d-j-1}(K) + f_{d-1}(K). \tag{12}
\]

Moreover, \( \mathcal{U}(K) \cap \mathcal{U}_{AB} \neq \emptyset \) whenever \( K \) possesses parallel supporting hyperplanes at \( A, B \in \text{Bd} \ K \). Furthermore, for every \( u_j \in \mathcal{U}(K) \) select some \( A_j, B_j \in \text{Bd} \ K \) such that \( u_j \in \mathcal{U}_{A_j B_j} \), \( 1 \leq j \leq N \).

Finally, consider the polynomial \( \tilde{T}_{2n}(t) = (T_{2n}(t) + 1)/2 \in P^1 \). Obviously \( \tilde{T}_{2n} \geq 0 \) on \( \mathbb{R}^1 \), \( \tilde{T}_{2n} \leq 1 \) on \( [-1, 1] \), and \( \tilde{T}_{2n} \equiv 2T_{2n} \) on \( \mathbb{R}^1 \setminus [-1, 1] \). Now we set

\[
p^*_2(x) = \frac{1}{N} \sum_{j=1}^{N} \tilde{T}_{2n} \left( \left\langle \frac{x - A_j + B_j}{2}, u_j \right\rangle \right). \tag{13}
\]

Clearly, \( p^*_2 \in P^d_{2n} \). Moreover, we claim that \( |p^*_2| \leq 1 \) on \( K \), i.e., \( p^*_2 \in B_{2n}(K) \). Indeed, since \( K \) possesses parallel supporting hyperplanes at \( A_j \) and \( B_j \) with normal \( u_j \) we have (assuming, for instance that \( \langle A_j, u_j \rangle < \langle B_j, u_j \rangle \)) \( \langle A_j, u_j \rangle \leq \langle x, u_j \rangle \leq \langle B_j, u_j \rangle \), \( x \in K \). This easily implies

\[
\left| \left\langle \frac{x - A_j + B_j}{2}, u_j \right\rangle \right| \leq \left| \left\langle \frac{A_j - B_j}{2}, u_j \right\rangle \right|, \quad x \in K.
\]

Since \( |\tilde{T}_{2n}| \leq 1 \) on \( [-1, 1] \) we obtain by (13) that \( p^*_2 \in B_{2n}(K) \). Now we need to show that \( p^*_2 \) satisfies (1) with a proper \( \gamma \). Consider an arbitrary \( p_2 \in B_{2n}(K) \) and \( x^* \in \mathbb{R}^d \setminus K \). By Lemma 1 there exists a line \( L \) passing through \( x^* \) with \( K \cap L = [A^*, B^*] \) such that \( K \) possesses parallel supporting hyperplanes at \( A^*, B^* \in \text{Bd} \ K \). As it was observed above we can choose this pair of hyperplanes \( H_{A^*}, H_{B^*} \) (keeping \( A^*, B^* \) fixed) so that some \( u_j \in \mathcal{U}(K) \), \( 1 \leq j \leq N \), is the normal to these hyperplanes. Then by (6) using that \( x^*, A^*, B^* \in L \)

\[
as_K(x^*) = \frac{|x^* - A^* + B^*|}{|A^* - B^*|} = \frac{\left| \left\langle \frac{x^* - A^* + B^*}{2}, u_j \right\rangle \right|}{\left| \left\langle \frac{A^* - B^*}{2}, u_j \right\rangle \right|}. \tag{14}
\]
Recall that earlier we have already chosen \( A_j, B_j \) from the pair of hyperplanes \( H_{A_j^*}, H_{B_j^*} \) (with normal \( u_j \)). Hence without loss of generality, \( A^*, A_j \in H_{A^*}, B^*, B_j \in H_{B^*} \), i.e., \( A^* - A_j \) and \( B^* - B_j \) are normal to \( u_j \). Thus using (5), (14) and (13) we have for \( p_{2n} \in B_{2n}(K) \)
\[
|p_{2n}(x^*)| \leq T_{2n}(\sigma_K(x^*)) \leq 2\tilde{T}_{2n}(\sigma_K(x^*))
= 2\tilde{T}_{2n}\left(\frac{x^* - A_j + B_j}{2}, u_j\right) \leq 2Np_{2n}(x^*).
\]

Finally by (12) we arrive at estimate (3).

If remains to verify the sharper bound \( \gamma_{2n}(K) \leq f_{d-1}(K) \) in case when \( K \) is a central symmetric polytope. Assume that \( 0 \) is the center of symmetry of \( K \). Clearly, \( K \) has \( M := f_{d-1}(K)/2 \) pairs of parallel \((d - 1)\)-dimensional faces. Denote by \( o_j \), \( 1 \leq j \leq M \), the normals to these pairs of hyperplanes, and select any segments \([-A_j, A_j]\), \( 1 \leq j \leq M \) with endpoints in these pairs of faces. Finally, set
\[
\hat{\rho}_{2n}(x) = \frac{1}{M} \sum_{j=1}^{M} \tilde{T}_{2n} \left( \frac{\langle x, o_j \rangle}{\langle A_j, o_j \rangle} \right).
\]

As above, it follow that \( \hat{\rho}_{2n} \in B_{2n}(K) \). Now, for any \( x^* \in \mathbb{R}^d \setminus K \) the line \( L := \{tx^* : t \in \mathbb{R}\} \) intersects \( \partial K \) at some points \( \pm B \) which belong to a pair of parallel \((d - 1)\)-dimensional faces of \( K \) with normal \( o_k \) for some \( 1 \leq k \leq M \). Then \( B - A_k \perp o_k \) and proceeding as above we can show that for any \( p_{2n} \in B_{2n}(K) \)
\[
|p_{2n}(x^*)| \leq f_{d-1}(K) \hat{\rho}_{2n}(x^*), \text{ i.e., } \gamma_{2n}(K) \leq f_{d-1}(K).
\]

Proofs of Theorems 3 and 4. Now we proceed to proving Theorems 3 and 4. Their proofs are based on the "polytopal" estimate (3) for \( \gamma_{2n}(K) \) on one side, and some known results on the rate of approximation of convex bodies by polytopes. One such result proved in [3] (see also [4]) asserts that for any convex body \( K \in \mathbb{R}^d \) \((d \geq 2)\) and \( N \in \mathbb{N} \) there exists a polytope \( D \) with \( f_0(D) = N \) vertices so that
\[
\phi(K, D) \leq \frac{c}{N^{d/(d-1)}}, \tag{15}
\]
with an absolute constant \( c > 0 \). (Here as above \( \phi(K, D) \) stands for the Hausdorff distance between corresponding sets.) The approximating polytope \( D \) is constructed in [3] to be circumscribed to \( K \), it can be modified in an obvious way to be inscribed into \( K \). Moreover it is shown
in [2] that if $K$ is $C^2$ then for any $M \in \mathbb{N}$ there exists an inscribed polytope $D$ with $\max_{0 \leq j \leq d-1} f_j(D) \leq M$ such that

$$g(K, D) \leq \frac{c_1}{M^{2d-1}} \tag{16}$$

with some $c_1 > 0$ depending on $K$. In principle, (16) provides a stronger bound than (15) since it is known (see e.g. [8, p. 257]) that for any polytope $D$

$$f_j(D) \leq \epsilon(D) f_0(D)^{\frac{d-2}{2}} \quad 1 \leq j \leq d-1, \tag{17}$$

with some $\epsilon(d)$ depending only on $d$.

We shall also need the following well known corollary of Chebyshev Inequality (2): if $p_n \in P_n^j$ is a univariate polynomial and $|p_n| \leq 1$ on $[-1, 1]$ then

$$|p_n(t)| \leq e^{\epsilon \sqrt{\delta}}, \quad |t| \leq 1+\delta \quad (0 < \delta < 1) \tag{18}$$

with some absolute constant $c_0 > 0$.

After these preliminaries we turn to the proof of Theorem 3. Consider an arbitrary convex body $K$ in $\mathbb{R}^d$ ($d \geq 2$), and let $D \subseteq K$ be an inscribed polytope with $f_0(D) = N$ vertices so that (15) holds.

By estimate (3) of Theorem 2 and (17) we have

$$\#2_n(D) \leq c_1(d) N^{d-1} \tag{19}$$

Since $|p^*_n| \leq 1$ on $D \subseteq K$ it follows by (15) and (18) that

$$\|p^*_n\|_{C(K)} \leq \exp[c_2 n^{N/(1-d)}] \tag{20}$$

with some $c_2 > 0$ depending on $d$ and $K$. Hence setting $N := \lceil n^{d-1} \rceil + 1$ and $\tilde{p}_{2n} := e^{-c_2} p^*_n$ we obtain by (20) that $|\tilde{p}_{2n}| \leq 1$ on $K$, i.e., $\tilde{p}_{2n} \in B_{2n}(K)$. Moreover, using (19) we have for every $p_{2n} \in B_{2n}(K) \subseteq B_{2n}(D)$ and $x \in (\mathbb{R}^d \setminus K) \subseteq (\mathbb{R}^d \setminus D)$

$$|p_{2n}(x)| \leq c_1(d) e^{c_2 N^{d-1}} |\tilde{p}_{2n}(x)| \leq c_3 n^{d-1} |\tilde{p}_{2n}(x)|.$$ 

This verifies the upper bound (4) of Theorem 3.

The proof of Theorem 4 follows similarly by using estimate (16) with $M = \max_{0 \leq j \leq d-1} f_j(D)$ instead of (15). This together with (3) yields the bound $\gamma_{2n}(D) = O(M^2)$. Finally, setting $M := \lceil n^{d-1} \rceil + 1$ we arrive at $\gamma_{2n}(K) = O(n^{2d-1})$. This completes the proof of Theorems 3 and 4. 

\[\Box\]
Remark. It can be shown that when $K$ is central symmetric the approximating polytopes satisfying (15) and (16) can also be chosen to be central symmetric. Moreover, for central symmetric polytopes $D$ by Theorem 2 the sharper estimate $\gamma_2(D) \leq f_{\rho_{d-1}}(D)$ holds. This bound leads to an improvement of the above estimates for $\gamma_2(K)$. Indeed similarly to the proofs of Theorems 3 and 4 we can verify that in this case $\gamma_2(K) = O(n^{d-1}/2)$, and $\gamma_2(K) = O(n^{d-1})$ if, in addition, $K$ is also $C^2$.

Proof of Theorem 5. Consider an arbitrary point $x^*$ on the boundary of convex body $K$. Let $p_{2n}^* \in B_{2d}(K)$ be a universal majorant in $B_{2d}(K)$. Then by Theorem 3

$$|p_{2n}(x)| \leq cn^{d(d-1)} |p_{2n}^*(x)|, \quad p_{2n} \in B_{2d}(K), \quad x \in \mathbb{R}^d \setminus K. \quad (21)$$

We claim that there exists a point $\xi \in \mathbb{R}^d$ with $|x^* - \xi| = O((\frac{1}{n^{d-1}})^2)$ such that $|p_{2n}^*(\xi)| = 1$. In order to show this assume that $|p_{2n}^*| \leq 1$ in some ball $B_d(x^*)$ with center at $x^*$ and radius $\delta > 0$. Our claim will follow if we verify that such a $\delta$ must satisfy $\delta \leq c(\log n/n)^2$ for some $c > 0$ independent of $n$. There exists $y^* \in B_d K$ such that $K$ possesses parallel supporting hyperplanes at $x^*$ and $y^*$ with a normal $u^* \in S^{d-1}$. Let $L$ be the line through $x^*$ and $y^*$. We may assume that $|x^* - y^*| = 2$. (Clearly, $|x^* - y^*| \geq o_d(K)$, where $o_d(K)$ is the minimal distance between parallel supporting hyperplanes to $K$. Moreover $|x^* - y^*| \leq d(K) := \max \{|x - y| : x, y \in K\}$.) Set now $x_j := (1 + j\delta/2) x^* - j\delta y^* / 2, \quad j = 1, 2$. Evidently, $x_1, x_2 \in L \setminus K$, $|x_1 - x^*| = \delta$, and $|x_2 - x^*| = 2\delta$.

Consider the polynomial

$$p_{2n}(x) := T_{2n} \left( \begin{array}{c} x \cdot \frac{x^* + y^*}{2}, u^* \end{array} \right).$$

As in the proof of Theorem 2 it can be shown that $|p_{2n}| \leq 1$ on $K$, i.e., $p_{2n} \in B_{2d}(K)$. Then by (21) for $x_j \in \mathbb{R}^d \setminus K$

$$|p_{2n}^*(x_j)| \geq \frac{|p_{2n}(x_j)|}{cn^{d(d-1)}} = \frac{T_{2n}(1 + 2\delta)}{cn^{d(d-1)}}. \quad (22)$$

On the other hand since $|x^* - x_j| = \delta$ and $|p_{2n}^*| \leq 1$ on $B_d(x^*) \cup K$, we obtain, in particular, that $|p_{2n}^*| \leq 1$ on $[y^*, x_1]$, where $|y^* - x_1| = (1 + \frac{\delta}{2}) |x^* - y^*| = 2 + \delta$. Recall, that $y^*, x_1, x_2 \in L$ where $|(y^* + x_1)/2 - x_2| = 1 + 3\delta/2$. Now, applying (2) to the univariate polynomial
\[
p^*_n(t(x_1 - y^*) + x_1) \in B_d([y^*, x_1]), \quad |x_1 - y^*| = 2 + \delta,
\]
at the point \( x_2 \) with \(|(x_1 + y^*)/2 - x_2| = 1 + 3\delta/2 \) yields
\[
|p^*_n(x_2)| \leq T_{2d}(1 + \delta).
\]
This together with (22) implies
\[
ct^{d-1}T_{2d}(1 + \delta) \geq T_{2d}(1 + 2\delta).
\]  \hfill (23)
Furthermore, it is well known (see [1, p. 30]) that
\[
\frac{1}{2}(t + \sqrt{t^2 - 1})^n \leq T_{2d}(t) \leq (t + \sqrt{t^2 - 1})^n, \quad t > 1.
\]
This and (23) yield for \( 0 < \delta \leq \delta_0 \)
\[
ct^{d-1}(1 + \sqrt{3\delta})^n \geq \frac{1}{2}(1 + 2\sqrt{\delta})^n.
\]
Hence
\[
1 + c_d \frac{\log n}{n} \geq (2ct^{d-1})^{1/2n} \geq \frac{1 + 2\sqrt{\delta}}{1 + \sqrt{3\delta}} \geq 1 + c_0 \sqrt{\delta},
\]
i.e. we obtain that \( \delta = O((\log n)^2) \). Thus since \( |p^*_n(x^*)| \leq 1 \) there exists \( \tilde{x} \) such that \( |x^* - \tilde{x}| = O((\log n/n)^2) \) and \( |p^*_n(\tilde{x})| = 1 \). Consider now the polynomial \( g_{4n} = (p^*_n)^2 - 1 \in P_{4n}^d \). As we have shown above for every \( x^* \in \text{Bd} K \) there exists an \( \tilde{x} \) such that \( g_{4n}(\tilde{x}) = 0 \) and \( |x^* - \tilde{x}| \leq c(\log n/n)^2 \). This concludes the proof.

SOME OPEN PROBLEMS

The results proved above provide some insight on the magnitude of \( \gamma_{2d}(K) \), but a number of questions remains open. Namely it would be interesting to determine for what convex bodies \( K \)
\[
\sup_{n \in \mathbb{N}} \gamma_{2d}(K) < \infty.
\]  \hfill (24)
We have seen above that (24) holds for ellipsoids and polytopes. Using similar methods we can verify that (24) is true for finite intersections of central-symmetric polytopes and ellipsoids having the same center. This means that (24) holds not only for ellipsoids and polytopes. Is (24) true for
every convex body $K \subset \mathbb{R}^d$. Another open problem consists in characterizing those compact sets $K \subset \mathbb{R}^d$ for which $\gamma_{2n}(K)$ has subexponential growth, i.e.,

$$\limsup_{n \to \infty} \gamma_{2n}(K)^{1/n} = 1.$$  \hspace{1cm} (25)

Theorem 3 implies, in particular, that (25) holds for every convex body $K \subset \mathbb{R}^d$.

REFERENCES

2. K. Boróczky, Jr., Polytopal approximation bounding the number of $k$-faces, manuscript.