A KAM theorem for the defocusing NLS equation ✤

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A B S T R A C T

In this paper we prove a KAM theorem for the defocusing NLS equation in one space dimension with periodic boundary conditions. The novelty of our result is that it is valid not only near the zero solution, but on the entire Sobolev space $H^N(T, \mathbb{C})$ with $N \in \mathbb{Z}_{>1}$. In particular, the invariant tori which persist under small Hamiltonian perturbations might be far away from the zero potential.

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1. Introduction

Consider the defocusing nonlinear Schrödinger equation (dNLS) in one space dimension

$$i \partial_t u = -\partial_x^2 u + 2|u|^2 u \quad (1.1)$$

on the Sobolev space $H^N_C \equiv H^N(T, \mathbb{C})$ of complex valued functions on $\mathbb{R}$ of period one,

$$H^N_C = \left\{ u(x) = \sum_{j \in \mathbb{Z}} \hat{u}_j e^{2\pi i j x}; \|u\|_N < \infty \right\},$$

where

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\[ \|u\|_N = \left( |\hat{u}_0|^2 + \sum_{j \neq 0} j^{2N}|\hat{u}_j|^2 \right)^{\frac{1}{2}}, \]

and \( \hat{u}_j, j \in \mathbb{Z}, \) denote the Fourier coefficients of \( u. \) It is an integrable PDE and according to [12], admits global Birkhoff coordinates. Indeed, recall from [12] that the (complex) NLS equation can be viewed as a Hamiltonian system with phase space \( H^N_C = H^N_{C \times H} \) and Poisson bracket

\[ \{F, G\}(\phi_1, \phi_2) = -i \int_0^1 (\partial_{\phi_1} F \partial_{\phi_2} G - \partial_{\phi_2} F \partial_{\phi_1} G) \, dx \]

where \( \partial_{\phi_i} F \) denotes the \( L^2 \)-gradient of \( F \) with respect to \( \phi_i \) \( (i = 1, 2). \) The Hamiltonian equations of motion are given by

\[ \partial_t \phi_1 = -i \partial_{\phi_2} H_{NLS}, \quad \partial_t \phi_2 = i \partial_{\phi_1} H_{NLS} \]

where

\[ H_{NLS}(\phi_1, \phi_2) = \int_0^1 (\partial_\phi \phi_1 \partial_\phi \phi_2 + \phi_1^2 \phi_2^2) \, dx. \]

The defocusing NLS equation \((1.1)\) is then obtained by restricting the complex NLS equation to the invariant subspace \( H^N_{r} = \{ \phi \in H^N_{C} | \phi_2 = \bar{\phi}_1 \}. \) Note that \( H^N_{r} \) is a real subspace of \( H^N_{C}. \) To describe the Birkhoff coordinates introduce the model space

\[ h^N_r = \{ (q, p) = (q_j, p_j)_{j \in \mathbb{Z}}; q_j, p_j \in \mathbb{R}; \|q\|_N + \|p\|_N < \infty \} \]

where

\[ \|p\|_N = \left( p_0^2 + \sum_{j \neq 0} j^{2N}|p_j|^2 \right)^{\frac{1}{2}}. \]

The corresponding complex Hilbert space is denoted by \( h^N_C. \) The space \( h^N_r \) is endowed with the Poisson structure induced by the standard symplectic form \( \sum_{j \in \mathbb{Z}} dq_j \wedge dp_j. \) In [12] one finds a detailed proof of the following result on Birkhoff coordinates for \((1.1).\)

**Theorem 1.1.** There exists a real analytic map \( \Phi : h^0_r \to h^0_r \) with the following properties

(B1) \( \Phi \) is canonical, i.e. for any \( C^1 \)-functions \( F, G \) on \( h^0_r, \) \( \{ F \circ \Phi, G \circ \Phi \} = \{ F, G \} \circ \Phi. \)

(B2) For any \( N \in \mathbb{Z}_{\geq 0}, \) the restriction of \( \Phi, \Phi |_{H^N_r} : H^N_r \to h^N_r \), is a real analytic diffeomorphism.

(B3) \( \Phi \) defines global Birkhoff coordinates for NLS on \( H^1_r. \) That is, on \( h^1_r, \) the transformed NLS Hamiltonian \( H_{NLS} \circ \Phi^{-1} \) is a real analytic function of the actions \( I_j = \frac{1}{2}(p_j^2 + q_j^2) \) \( (j \in \mathbb{Z}). \)

(B4) The differential of \( \Phi \) at \( \phi = 0, d_0 \Phi, \) is the Fourier transform.

To state our KAM theorem, we need first to introduce some more notations. Let us denote by \( T_\tau, \tau \in \mathbb{R}, \) the flow of translation on \( L^2_C, \) i.e. for any \( \phi \in L^2_C, T_\tau \phi(x) = \phi(x + \tau). \) Note that \( \tau \to T_\tau(\phi) \) solves the linear PDE \( \partial_\tau \phi = \partial_\phi. \) Actually the latter is a Hamiltonian PDE

\[ \partial_\tau \phi_1 = -i \partial_{\phi_2}(iH_2), \quad \partial_\tau \phi_2 = i \partial_{\phi_1}(iH_2) \]
where \( H_2 \) is the Hamiltonian

\[
H_2(\phi_1, \phi_2) = \int_0^1 \phi_2 \partial_x \phi_1 \, dx
\]

which is the second Hamiltonian in the NLS-hierarchy – see e.g. [12, Section 4]. In particular, \( H_2 \) Poisson commutes with \( H_{\text{NLS}} \).

\[
\{ H_2, H_{\text{NLS}} \} = 0.
\]

Actually, a large class of Hamiltonians Poisson commutes with \( H_2 \). Indeed, consider a Hamiltonian of the form

\[
P(\phi) = \int_0^1 F(x, \phi_1(x), \phi_2(x)) \, dx
\]

where \( F = F(x, \zeta, \eta) \) is a polynomial in two complex variables \( \zeta, \eta \). As \( H^1_c \to C^0(\mathbb{T}, \mathbb{C}^2) \) by the Sobolev embedding theorem, for any \( N \geq 1 \), the functional \( P \) is defined on \( H^N_c \). Note that for \( i = 1, 2 \), \( \partial_{\phi_i} P = f_i(x, \phi_1(x), \phi_2(x)) \) with \( f_1 = \partial_x F, \ f_2 = \partial_\eta F \) and that \( (\partial_{\phi_1} P, \partial_{\phi_2} P) \in H^N_c \) for any \( \phi \) in \( H^N_c \). By a straightforward computation,

\[
\{ P, H_2 \} = -i \int_0^1 (\partial_{\phi_1} P \cdot \partial_{\phi_2} H_2 - \partial_{\phi_2} P \cdot \partial_{\phi_1} H_2) \, dx
\]

\[
= -i \int_0^1 (\partial_{\phi_1} P \cdot \partial_x \phi_1 + \partial_{\phi_2} P \cdot \partial_x \phi_2) \, dx
\]

\[
= -i \int_0^1 \frac{d}{dx} F(x, \phi(x)) \, dx + i \int_0^1 (\partial_x F)(x, \phi(x)) \, dx.
\]

As \( F(x, \phi(x)) \) is 1-periodic, \( \int_0^1 \frac{d}{dx} F(x, \phi(x)) \, dx \) vanishes. Furthermore,

\[
\int_0^1 (\partial_x F)(x, \phi(x)) \, dx = \sum_{\text{finite}} \int_0^1 (\partial_x a_{ij})(x) \phi_1(x)^i \phi_2(x)^j \, dx.
\]

Hence \( \{ P, H_2 \} \) vanishes identically if all the coefficients \( a_{ij} \) of the polynomial \( F \) are constant. More generally, \( \{ P, H_2 \} \) vanishes identically for any Hamiltonian \( P \) on its domain of definition if it is of the form

\[
P(\phi) = \int_0^1 F(\phi_1(x), \phi_2(x)) \, dx
\]

where \( F(\zeta, \eta) \) is an arbitrary analytic function on some domain of \( \mathbb{C}^2 \).
Next we need to introduce notation to parametrize finite-dimensional tori invariant under the defocusing NLS. For \( I_A = \{ l_j \}_{j \in A} \in \mathbb{R}^A_{>0} \) with \( A \subseteq \mathbb{Z} \) finite and \( \mathbb{R}^A_{>0} = (\mathbb{R}_{>0})^A \), denote by \( \mathbb{T}_{I_A} \) the torus in \( \mathbb{h}^0 \) given by

\[
\mathbb{T}_{I_A} = \{ (q_j, p_j)_{j \in \mathbb{Z}} : q_j^2 + p_j^2 = 2l_j \quad \forall j \in A; \quad p_j = q_j = 0 \quad \forall j \notin A \}
\]

and by \( \mathcal{T}_{I_A} \) its image by \( \Phi^{-1} \), \( \mathcal{T}_{I_A} = \Phi^{-1}(\mathbb{T}_{I_A}) \). For \( \Pi \subseteq \mathbb{R}^A_{>0} \) a compact subset of positive Lebesgue measure, denote by \( \mathbb{T}_\Pi \) and \( \mathcal{T}_\Pi \) the sets

\[
\mathbb{T}_\Pi = \bigcup_{I_A \in \Pi} \mathbb{T}_{I_A} \quad \text{and} \quad \mathcal{T}_\Pi = \Phi^{-1}(\mathbb{T}_\Pi).
\]

We will consider Hamiltonian perturbations \( H_{\text{NLS}} + \epsilon K \) on \( H_r^N, N \in \mathbb{Z}_{\geq 1} \), with the following assumptions on \( K \):

- \((P_1)\) \( K \) is analytic on some open neighborhood \( U \equiv U_\Pi \) of \( \mathcal{T}_\Pi \) in \( H_r^N \) and real valued on \( U \cap H_r^N \);
- \((P_2)\) the \( L_2 \)-gradients \( \partial_{\phi_1} K, \partial_{\phi_2} K \) are bounded as functions from \( U \) to \( H_r^N \) and verify the normalization condition

\[
\sup \{ \| \partial_{\phi_1} K \|_N + \| \partial_{\phi_2} K \|_N : \phi \in U \} \leq 1;
\]

- \((P_3)\) \( \{ K, H_2 \} = 0 \).

Examples of Hamiltonians satisfying conditions \((P_1)\)–\((P_3)\) are polynomials in \( \phi_1, \phi_2 \) of the form

\[
\sum_{\text{finite}}^{1} a_{ij} \phi_1(x)^i \phi_2(x)^j \, dx
\]

where the complex coefficients \( a_{ij} \) are constant and satisfy \( a_{ij} = \bar{a}_{ji} \).

Our KAM theorem states that for any \( A \subseteq \mathbb{Z} \) finite and for any \( \epsilon > 0 \) sufficiently small, many of the NLS-invariant tori \( \mathcal{T}_{I_A} \) persist under perturbation of the NLS Hamiltonian by \( \epsilon K \) with \( K \) satisfying \((P_1)\)–\((P_3)\). Moreover, these tori and their linear flows are only slightly deformed. Let us now state our KAM theorem in a more formal way. Denote by \( \mathbb{T}^A \) the \(|A|\)-dimensional torus \((\mathbb{R}/2\pi \mathbb{Z})^A \) and by \( \text{meas}(W) \) the Lebesgue measure of a Lebesgue measurable subset \( W \subseteq \mathbb{R}^A \).

**Theorem 1.2.** Let \( N \in \mathbb{Z}_{\geq 1} \) and let \( A \subseteq \mathbb{Z} \) be a finite index set. Furthermore let \( \Pi \subseteq \mathbb{R}^A_{>0} \) be a compact subset of positive Lebesgue measure. Then for any Hamiltonian \( K \) satisfying \((P_1)\)–\((P_3)\), there exists \( \epsilon_0 > 0 \) so that the following holds:

- \((KAM1)\) there exists a family of closed subsets \( \Pi_\epsilon \subseteq \Pi, |\epsilon| \leq \epsilon_0, \) with \( \lim_{\epsilon \to 0} \text{meas}(\Pi \setminus \Pi_\epsilon) = 0; \)
- \((KAM2)\) for any \( |\epsilon| \leq \epsilon_0 \), there exists a Lipschitz family of real analytic torus embeddings

\[
\Xi_\epsilon : \mathbb{T}^A \times \Pi_\epsilon \to U \cap H_r^N;
\]

- \((KAM3)\) for any \( |\epsilon| \leq \epsilon_0 \), there exists a Lipschitz map

\[
f_\epsilon : \Pi_\epsilon \to \mathbb{R}^A
\]

such that for any \( |\epsilon| \leq \epsilon_0 \), \( l_A \in \Pi_\epsilon \), and \( \theta_A \in \mathbb{T}^A \), the curve \( t \mapsto \Xi_\epsilon(\theta_A + t f_\epsilon(l_A), l_A) \) is a quasi-periodic solution of

\[
\partial_t \phi_1 = -i \partial_{\phi_2} H_{\text{NLS}} - i \epsilon \partial_{\phi_1} K, \quad \partial_t \phi_2 = i \partial_{\phi_1} H_{\text{NLS}} + i \epsilon \partial_{\phi_2} K.
\]
Related work. Theorem 1.2 confirms that the KAM type theorem of [7], when applied to dNLS, does not only hold near $\phi = 0$, but is actually valid on the entire phase space. In [8], Geng and You prove an abstract KAM result in spaces with exponential weights near an equilibrium solution of certain linear integrable PDEs for a special class of perturbations. They then apply their theorem, among other equations, to the beam equation and to a class of nonlinear Schrödinger equations in arbitrary space dimension. We note that the existence of quasi-periodic solutions of such equations was proved earlier in [2], by the C-W-B method. At the same time, Theorem 1.2 complements the KAM type theorem proved in [10] where instead of imposing condition (P3), dNLS is studied on various invariant subspaces of $H^N$, including the subspace of odd functions and the one of even functions of $H^N$. The perturbations considered in [10] are assumed to induce Hamiltonian vector fields which are tangent to the subspaces considered so that the perturbed equation evolves on these subspaces. In [8], Geng and You overcome the difficulties caused by the asymptotics of the NLS frequencies $(\omega_j)_{j \in \mathbb{Z}}$. In fact, for $j \in \mathbb{Z}$ large, $\omega_j \sim \omega_{-j}$, i.e., $\omega_j$ and $\omega_{-j}$ are in ‘near resonance’. In earlier work (see [14, 10]), NLS-invariant subspaces of $H^N$ were considered so that the near resonances mentioned above are no longer relevant when dNLS is restricted to these subspaces. In [8], Geng and You prove the difficulties caused by these near resonances by imposing a symmetry condition on the perturbations – cf. [8], condition (A4). Condition (P3), introduced above, is a coordinate-free way of formulating their condition (A4). In Section 2 we express condition (P3) in Birkhoff coordinates. It allows to apply a KAM theorem with symmetries, a version of a by now standard abstract KAM theorem of the type obtained in [18] (cf. also [7]), which we state in Section 4. Taking into account the properties of the frequencies of dNLS, discussed in Section 3, Theorem 1.2 is then proved in Section 4. In subsequent work we plan to apply the arguments used in the proof of Theorem 1.2 to other equations as well. In Section 6 we prove the KAM theorem with symmetries stated in Section 4.

2. $H_2$-symmetry

Let us consider a real analytic Hamiltonian $P$, defined on an open neighborhood $U \subseteq H^N_A$ of the form introduced in Section 1 with $\Pi \subseteq \mathbb{R}^A_{>0}$ where $A \subseteq \mathbb{Z}$ is finite. We want to compute the Poisson bracket $\{P_iH_2\}$ in Birkhoff coordinates $(q, p) (q_j, p_j)_{j \in \mathbb{Z}}$. For this purpose it is convenient to introduce action–angle coordinates $I_A = (I_j)_{j \in A}, \theta = (\theta_j)_{j \in A}$ and complex coordinates $w = (w_j)_{j \in B}, z = (z_j)_{j \in B}$, $B = \mathbb{Z} \setminus A$. Note that for $j \in A$, one has $I_j > 0$ and hence the angle variable $\theta_j$ is well defined $\text{mod } 2\pi$. The coordinates $q, p$ are related to $I_A, \theta_A, w, z$ as follows: for $j \in A$

\[(q_j, p_j) = \sqrt{2I_j}(\cos \theta_j, -\sin \theta_j),\]

where $I_j = (p_j^2 + q_j^2)/2$ whereas for $j \in B$,\n
\[w_j = \frac{1}{\sqrt{2}}(q_j - ip_j), \quad z_j = \frac{1}{\sqrt{2}}(q_j + ip_j).\]

Note that for any $j \in B$, $dw_j \wedge dz_j = idq_j \wedge dp_j$ and $w_jz_j = I_j$ whereas for $j \in A$ one has $d\theta_j \wedge dI_j = dq_j \wedge dp_j$. Assume that $P: U \to \mathbb{C}$ is a real analytic Hamiltonian. Then the Taylor expansion of $P \circ \Phi^{-1}$ at $I_A = \xi \in \Pi, w = 0, z = 0$ is of the form

\[\sum P_{k\ell m} e^{ik \cdot \theta} y^\ell w^m z^n \quad (2.1)\]
where $y = I_A - x$ and where $k, \ell, m, n$ are integer vectors, $k \in \mathbb{Z}_A^A$, $\ell \in \mathbb{Z}_{\geq 0}$, $m, n \in \mathbb{Z}_{>0}$ with $|m|, |n| < \infty$. Here $|m| = \sum_{j \in B} m_j$ and in (2.1) we have used the multi-index notation

$$y^\ell = \prod_{j \in A} y_j^{\ell_j}, \quad k \cdot \theta = \sum_{j \in A} k_j \theta_j, \quad w^m = \prod_{j \in B} w_j^{m_j}.$$ 

Further introduce the sequence $v = (v_j)_{j \in \mathbb{Z}}$ where $v_j = j$, for any $j \in \mathbb{Z}$. With the notation $v_A = (v_j)_{j \in A}$ and $v_B = (v_j)_{j \in B}$ one then has

$$k \cdot v_A = \sum_{j \in A} j k_j \quad \text{and} \quad m \cdot v_B = \sum_{j \in B} j m_j.$$ 

By Theorem 1.1, there exists a neighborhood $W$ of $H_0^0$ in $H_0^0$ so that the Birkhoff map $\Phi$ is defined on $W$ and has range $V := \Phi(W) \subseteq h_0^0$ so that for any $N \geq 0$,

$$\Phi : W \cap H_0^N \rightarrow V \cap h_0^N \quad (2.2)$$

is a bi-analytic diffeomorphism.

**Proposition 2.1.**

(i) On $h_0^1 \cap V$, $iH_2 \circ \Phi^{-1}(q, p) = \sum_{j \in \mathbb{Z}} 2\pi j I_j$. In particular, for $I_A = \xi + y$ one has $iH_2 \circ \Phi^{-1}(q, p) = 2\pi (c + \sum_{j \in \mathbb{Z}} j y_j + \sum_{j \in B} j w_j z_j)$, where $c = \sum_{j \in A} j \xi_j$.

(ii) Let $P : U \rightarrow \mathbb{C}$ be given as above. Then, at any point $I_A = \xi \in \Pi$, $w = 0$, $z = 0$, the function $\{P \circ \Phi^{-1}, iH_2 \circ \Phi^{-1}\}$ admits a Taylor expansion in $y = I_A - \xi$, $w, z$ of the form

$$\{P \circ \Phi^{-1}, iH_2 \circ \Phi^{-1}\} = 2\pi i \sum_{k, \ell, m, n} (k \cdot v_A + (n - m) \cdot v_B) P_{k \ell m n} e^{ik \cdot \theta} y^\ell w^m z^n.$$ 

**Proof.** (i) follows from [11], Proposition 3.4 and the remark following it and (ii) results from a straightforward computation, taking into account that the Birkhoff coordinates are canonical. 

As an immediate consequence of Proposition 2.1 one has the following

**Corollary 2.1.** For $P : U \rightarrow \mathbb{C}$ with $\{P, H_2\} \equiv 0$, the coefficients of the Taylor expansion (2.1) of $P \circ \Phi^{-1}$ at $I_A = \xi, w = 0, z = 0$ satisfy for any $k \in \mathbb{Z}_A^A, \ell \in \mathbb{Z}_{\geq 0}, m, n \in \mathbb{Z}_{>0}$

$$\text{if } P_{k \ell m n} \neq 0 \text{ then } k \cdot v_A + (n - m) \cdot v_B = 0. \quad (2.3)$$

**Proof.** As $\Phi$ and hence $\Phi^{-1}$ are canonical one has $0 = \{P, H_2\} \circ \Phi^{-1} = \{P \circ \Phi^{-1}, H_2 \circ \Phi^{-1}\}$. The claimed statement then follows from item (ii) of Proposition 2.1. 

As an illustration of implications of (2.3), consider $P$ in Corollary 2.1 with the property that $P \circ \Phi^{-1}$ admits an expansion of the form

$$\sum_{|k| \leq K, j \in B} p_{kj} e^{ik \cdot \theta} w_j z^{-j}. \quad (2.4)$$

It then follows from (2.3) that $p_{kj} = 0$ for any $j \in B$ with $2|j| > K \max_{i \in A} |i|$. In particular, the sum in (2.4) is finite.
3. NLS frequencies

Let \( W \) and \( V \) be the open neighborhoods introduced in Section 2 – see (2.2). Note that \( H_{\text{NLS}} \circ \Phi^{-1} \) is well-defined on \( V \cap h^1_c \) and analytic there. By Theorem 1.1, \( H_{\text{NLS}} \circ \Phi^{-1} \) only depends on the action variables \( I_j, j \in \mathbb{Z} \), and it then follows that \( H_{\text{NLS}} \circ \Phi^{-1} \) is a real analytic function of \( I_j, j \in \mathbb{Z} \). For any \( j \in \mathbb{Z} \),

\[
\omega_j := \partial_{I_j} H_{\text{NLS}} \circ \Phi^{-1}
\]

is called the \( j \)th NLS frequency of the (defocusing) NLS. We note that due to Theorem 1.1, the frequencies are analytic functions on \( V^N \) for any \( N \in \mathbb{Z}_{\geq 1} \) where \( V^N \subseteq \ell^{1,2N}(\mathbb{Z}, \mathbb{C}) \) denotes the open neighborhood of \( \ell^{1,2N}(\mathbb{Z}, \mathbb{R}) \) given by

\[
V^N := \left\{ I = \left( \frac{p_j^2 + q_j^2}{2} \right)_{j \in \mathbb{Z}} : (q_j, p_j)_{j \in \mathbb{Z}} \in V \cap h^N_c \right\}. \tag{3.1}
\]

Here \( \ell^{1,\alpha}_c \equiv \ell^{1,\alpha}(\mathbb{Z}, \mathbb{C}) \) denotes the Banach space consisting of all complex sequences \( v = (v_j)_{j \in \mathbb{Z}} \) with

\[
\|v\|_{\ell^{1,\alpha}} = |v_0| + \sum_{j \neq 0} |j|^\alpha |v_j| < \infty.
\]

The expansion of \( H_{\text{NLS}} \circ \Phi^{-1} \) at \( I = 0 \) is calculated in [14]. It leads to the following asymptotic expansion of the frequencies in a neighborhood of \( I = 0 \) in \( \ell^{1,2}(\mathbb{Z}, \mathbb{C}) \) (see [10, Corollary 3.2])

\[
\omega_j = 4\pi^2 j^2 + 4 \sum_{i \neq j} I_i + 2I_j + O(I^2)
\]

and of their partial derivatives

\[
\partial_I \omega_j = 4 - 2\delta_{ij} + O(I). \tag{3.2}
\]

As an application one obtains the following results (cf. [10]).

**Proposition 3.1.** For any \( \emptyset \neq A \subseteq \mathbb{Z} \) with \( |A| < \infty \), the following functions, when restricted to \( \mathbb{R}^A_{\geq 0} \), satisfy

(i) \( \det((\partial_{I_j} \omega_j)_{i,j \in A}) \big|_{I=0} \neq 0 \); in particular \( \det((\partial_{I_j} \omega_j)_{i,j \in A}) \neq 0 \);

(ii) for any \( k \in \mathbb{Z}^A \) and \( a, b \in B \),

(M1) \( k \cdot \omega_A \pm \omega_a \neq 0 \);

(M2) \( k \cdot \omega_A \pm (\omega_a + \omega_b) \neq 0 \);

(M3) if in addition \( a \neq b \) then \( k \cdot \omega_A + \omega_a - \omega_b \neq 0 \).

**Proof.** (i) It follows from (3.2) that

\[
\det((\partial_{I_j} \omega_j)_{i,j \in A}) \big|_{I=0} = -(-2)^{|A|}(2|A| - 1) \neq 0.
\]

(ii) Let \( A' := A \cup \{a\} \) and \( k^\pm \in \mathbb{Z}^A \) with \( k_j^\pm = k_j \) for \( j \in A \) and \( k_A^\pm = \pm 1 \). In particular, \( k^\pm \neq 0 \). As by (i), \( \det((\partial_{I_j} \omega_j)_{i,j \in A'}) \) doesn’t vanish identically on \( \mathbb{R}^{A'}_{\geq 0} \), it follows that there exists \( j \in A' \) so that \( \partial_{I_j}(\sum_{i \in A} k_i \omega_i \pm \omega_a) \) doesn’t vanish identically. This proves (M1). The statements (M2) and (M3) are proved in a similar way. \( \square \)
Remark 3.1. Consider the case (M2) with $a = b$. Let $A' = A \cup \{a\}$, $k^\pm \in A'$ with $k^\pm_j = k_j$ for any $j \in A$ and $k^\pm_0 = \pm 2$. Then $k^\pm \neq 0$. Hence again, $\partial_{ij}(\sum_{i \in A} k_i \omega_i \pm 2 \omega_0)$ cannot vanish identically for all $j \in A'$ at the same time.

Proposition 3.1 will allow us to prove Kolmogorov’s and Melnikov’s conditions for NLS on the entire phase space – see Section 5 for details. Finally we state the asymptotics of the frequencies derived in [10]. There, they are stated for potentials of real type, $\phi \in H^1_r$. The proof of Theorem 5.10 in [10] shows that the asymptotics actually hold on $W \cap H^1_c$.

Proposition 3.2. For $\phi \in W \cap H^1_c$ or equivalently, for $I$ in $V^1_I$,

$$\omega_j = 4\pi^2 j^2 + O(1)$$

locally uniformly on $W \cap H^1_c$. Hence by [13], Theorem A.3, and by Theorem 1.1,

$$V^1_I \to \ell^\infty(\mathbb{Z}, \mathbb{C}), \quad I \mapsto (\omega_j - 4j^2 \pi^2)_{j \in \mathbb{Z}}$$

is real analytic.

Note that the asymptotics (3.3) imply that

$$\omega_j - \omega_{-j} = O(1).$$

It means that the frequencies $\omega_j$ and $\omega_{-j}$ are not well separated as $|j| \to \infty$. This causes the additional difficulties, alluded to in the introduction, when estimating the measure of the set of good parameters in the proof of Theorem 1.2.

4. An infinite-dimensional KAM theorem with symmetries

Theorem 1.2 is derived from an abstract KAM Theorem with parameters in infinite dimension, first obtained by Kuksin [15] and then further developed by Pöschel [18], cf. also [13]. We need a version of this result taking into account the occurrence of near resonance (3.4). Following the exposition in [13] and [18], consider small perturbations of a family of infinite-dimensional integrable Hamiltonians $H = H(y, u, v; \xi)$ with parameter $\xi \in \Pi \subset \mathbb{R}^A$ and $\Pi$ is a compact subset.
of $\mathbb{R}^A$ of positive Lebesgue measure. The symplectic form on $\mathcal{M}^N$ is the standard one given by $\sum_{j \in A} dx_j \wedge dy_j + \sum_{j \in B} du_j \wedge dv_j$. The Hamiltonian equations of motion of $H$ are therefore
\[ \dot{x} = \omega(\xi), \quad \dot{y} = 0, \quad \dot{u} = \Omega(\xi)v, \quad \dot{v} = -\Omega(\xi)u, \]
where for any $j \in B$, $(\Omega(\xi)u)_j = \Omega_j(\xi)u_j$. Hence, for any parameter $\xi \in \Pi$, on the $|A|$-dimensional invariant torus,
\[ T_0 = T^A \times \{0\} \times \{0\} \times \{0\}, \]
the flow is rotational with internal frequencies $\omega(\xi) = (\omega_j(\xi))_{j \in A}$. In the normal space, described by the $(u, v)$ coordinates, we have an elliptic equilibrium at the origin, whose frequencies are $\Omega(\xi) = (\Omega_j(\xi))_{j \in B}$. Hence, for any $\xi \in \Pi$, $T_0$ is an invariant, rotational, linearly stable torus for the Hamiltonian $H$. Our aim is to prove the persistence of this torus under small perturbations $H + P$ of the integrable Hamiltonian $H$ for a large Cantor set of parameter values $\xi$. To this end we make assumptions on the frequencies of the unperturbed Hamiltonian $H$ and on the perturbation $P$.

**Assumption $A$ (Frequencies).**

(A1) The map $\xi \mapsto \omega(\xi)$ between $\Pi$ and its image $\omega(\Pi)$ is a homeomorphism which, together with its inverse, is Lipschitz continuous.

(A2) There exists a real sequence $(\overline{\Omega}_j)_{j \in B}$, independent of $\xi \in \Pi$, of the form
\[ \overline{\Omega}_j = |j|^d + a_1|j|^d_1 + \cdots + a_D|j|^d_D \quad (4.2) \]
where $d = d_0 > d_1 > \cdots > d_D \geq 0$ with $D \in \mathbb{Z}_{\geq 0}, d > 1$, and $a_1, \ldots, a_D \in \mathbb{R}$, so that $\xi \mapsto (\Omega_j - \overline{\Omega}_j)_{j \in B}$ is a Lipschitz continuous map on $\Pi$ with values in $\ell^\infty_{-\delta} = \ell^\infty_{-\delta}(B, \mathbb{R})$ for some $0 \leq \delta < 1 \wedge (d - 1)$.

(A3) For any $(k, e)$ in $Z := \{(k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B \setminus (0, 0) : |e| \leq 2; k \cdot \nu_A + e \cdot \nu_B = 0\}$ with $e \neq 0$
\[ \text{meas}\{\xi \in \Pi : k \cdot \omega(\xi) + e \cdot \Omega(\xi) = 0\} = 0. \quad (4.3) \]
Recall that for integer vectors such as $e$, the norm $|e|$ is given by $|e| = \sum_{j \in B} |e_j|$. Furthermore, we note that Assumption (A1) implies that (4.3) holds for $e = 0$.

The second set of assumptions concerns the perturbing Hamiltonian $P$ and its vector field, $X_P = (\partial_y P, -\partial_x P, \partial_x P, -\partial_y P)$. We use the notation $i_\xi X_P$ for $X_P$ evaluated at $\xi$. Finally, we denote by $\mathcal{M}^N_C$ the complexification of the phase space $\mathcal{M}^N$, $\mathcal{M}^N_C = (\mathbb{C}/2\pi \mathbb{Z})^A \times \mathbb{C}^A \times \mathbb{C}^{2N} \times \mathbb{C}^{2N}$. Note that at each point of $\mathcal{M}^N_C$, the tangent space is given by
\[ T^N_C := \mathbb{C}^A \times \mathbb{C}^A \times \mathbb{C}^{2N} \times \mathbb{C}^{2N}. \]

**Assumption $B$ (Perturbation).**

(B1) There exists a neighborhood $V$ of $T_0$ in $\mathcal{M}^N_C$ such that $P$ is a function on $V \times \Pi$ and its Hamiltonian vector field defines a map
\[ X_P : V \times \Pi \rightarrow T^N_C. \quad (4.4) \]
Moreover, $i_\xi X_P$ is real analytic on $V$ for each $\xi \in \Pi$, and $i_w X_P$ is uniformly Lipschitz on $\Pi$ for each $w \in V$. (Here $i_\xi X_P$ denotes the vector field $X_P$, evaluated at the parameter value $\xi$; $i_w X_P$ is defined similarly.)
\( (B2) \ \{P, S\} = 0 \) where

\[
S = a + b \sum_{j \in A} jy_j + c \sum_{j \in B} j(u_j^2 + v_j^2) / 2
\]

(4.5)

with \( a \in \mathbb{R} \) and \( b, c \in \mathbb{R} \setminus \{0\} \).

To state the KAM theorem we need to introduce some domains and norms. For \( s > 0 \) and \( r > 0 \) we introduce the complex \( T_0 \)-neighborhoods

\[
D(s, r) = \left\{ |\Im x| < s \right\} \times \left\{ |y| < r^2 \right\} \times \left\{ \|u\|_N + \|v\|_N < r \right\} \subset \mathcal{M}_C^N.
\]

Here, for \( z \) in \( \mathbb{R} \) or \( \mathbb{C} \), \( |z| = \max_{j \in A} |z_j| \).

For a vector field \( Y \) in \( \mathcal{P}_C^N \) with components \( (Y_x, Y_y, Y_u, Y_v) \) introduce the weighted norm

\[
\|Y\|_{r, N} = |Y_x| + \frac{1}{r^2} |Y_y| + \frac{1}{r} \|Y_u\|_N + \frac{1}{r} \|Y_v\|_N.
\]

Such weights are convenient when estimating the components of a Hamiltonian vector field \( XP = (\partial_y P, -\partial_x P, \partial_u P, -\partial_v P) \) on \( D(s, r) \) in terms of \( r \). For a vector field \( Y : V \times \Pi \rightarrow \mathcal{P}_C^N \) we then define the norms

\[
\|Y\|_{r, N; V \times \Pi}^{\sup} = \sup_{(w, \xi) \in V \times \Pi} \|Y(w, \xi)\|_{r, N},
\]

\[
\|Y\|_{r, N; V \times \Pi}^{lip} = \sup_{\xi, \xi' \in \Pi, \xi \neq \xi'} \frac{\|\Delta_{\xi, \xi'} Y\|_{r, N; V}}{|\xi - \xi'|},
\]

where \( \Delta_{\xi, \xi'} Y = i_{\xi'} Y - i_\xi Y \), and

\[
\|i_\xi Y\|_{r, N; V}^{\sup} = \sup_{w \in V} \|Y(w, \xi)\|_{r, N}.
\]

In a completely analogous way, the Lipschitz semi-norm of the map \( F : \Pi \rightarrow \ell^{\infty,-\delta} \) is defined as

\[
|F|_{\Pi, \ell^{\infty,-\delta}}^{lip} = \sup_{\xi, \xi' \in \Pi, \xi \neq \xi'} \frac{\|\Delta_{\xi, \xi'} F\|_{\ell^{\infty,-\delta}}}{|\xi - \xi'|}.
\]

Finally, let \( 1 \leq M < \infty \) be a constant satisfying

\[
|\omega|_{\Pi}^{lip} + |\Omega|_{\Pi, \ell^{\infty,-\delta}}^{lip} \leq M.
\]

(4.6)

Note that if Assumption \( A \) and Assumption \( B \) hold such an \( M \) exists.

**Theorem 4.1.** Suppose \( H \) is a family of Hamiltonians of the form (4.1) defined on the phase space \( \mathcal{M}_C^N \), \( N \in \mathbb{Z}_{\geq 1} \), and depending on parameters in \( \Pi \) so that Assumption \( A \) is satisfied with \( d \) and \( \delta \). Furthermore, assume that \( s > 0 \). Then there exist a positive constant \( \gamma \) depending on the finite subset \( A \subset \mathbb{Z} \) of (4.1), \( d, \delta, \) the frequencies \( \omega \) and \( \Omega \) of \( H \), and \( s \) such that for any perturbed Hamiltonian \( H + P \) with \( P \) satisfying Assumption \( B \) on a neighborhood \( V \) of \( T_0 \) in \( \mathcal{M}_C^N \), with \( D(s, r) \subset V \) for some \( r > 0 \), and the smallness condition
\[\epsilon := \|X_p\|_{\sup_{r,N:D(s,r)\times \Pi}}^\star + \frac{\alpha}{M}\|X_p\|_{\lip{r,N:D(s,r)\times \Pi}} \leq \alpha \gamma \] (4.7)

for some \(0 < \alpha < 1\), the following holds. There exist

1. a closed subset \(\Pi_\ast \subset \Pi\), depending on the perturbation \(P\), with \(\text{meas}(\Pi \setminus \Pi_\ast) \to 0\) as \(\alpha \to 0\),
2. a Lipschitz family of real analytic torus embeddings \(\Psi : \mathbb{T}^A \times \Pi_\ast \to \mathcal{M}^N\),
3. a Lipschitz map \(f : \Pi_\ast \to \mathbb{R}^A\),

such that for any \(\xi \in \Pi_\ast\), \((\mathbb{T}^A \times \{\xi\})\) is an invariant torus of the perturbed Hamiltonian \(H + P\) at \(\xi\) and the flow of \(H + P\) on this torus is given by

\[\mathbb{T}^A \times \mathbb{R} \to \mathcal{M}^N, \quad (x, t) \mapsto \Psi(x + tf(\xi), \xi).\]

Thus for any \(x \in \mathbb{T}^A\) and \(\xi \in \Pi_\ast\), the curve \(t \mapsto \Psi(x + tf(\xi), \xi)\) is a quasi-periodic solution for the Hamiltonian \(i_\xi(H + P)\). Moreover, for any \(\xi \in \Pi_\ast\), the embedding \(\Psi(\cdot, \xi) : \mathbb{T}^A \to \mathcal{M}^N\) is real analytic on \(D(s/2) = \{|x| < s/2\}\), and

\[
\|\Psi - \Psi_0\|_{\sup_{r,N:D(s/2)\times \Pi_\ast}} + \frac{\alpha}{M}\|\Psi - \Psi_0\|_{\lip{r,N:D(s/2)\times \Pi_\ast}} \leq \frac{c\epsilon}{\alpha},
\]

\[|f - \omega|_{\Pi_\ast} + \frac{\alpha}{M}|f - \omega|_{\Pi_\ast} \leq c\epsilon,
\]

where

\[\Psi_0 : \mathbb{T}^A \times \Pi \to \mathbb{T}_0, \quad (x, \xi) \mapsto (x, 0, 0, 0)\]
is the trivial embedding, and \(c\) is a positive constant which depends on the same parameters as \(\gamma\).

Remark 4.1.

1. Note that (4.2) implies that for any \(j \in B\) with \(-j \in B\), one has \(\Omega_{-j} = \Omega_j\). Theorem 4.1 continues to hold under a weaker version of (4.2) where the coefficients for \(j > 0\) and \(j < 0\) might take different values, \(a_{\frac{j}{2}}, \ldots, a_{\frac{-j}{2}}\). However for the applications we have in mind, condition (A2) as stated suffices. Furthermore, it is straightforward to verify that Theorem 4.1 also continues to hold if \(\delta\) and/or some of the exponents in (4.2) are negative. We add the condition \(\delta \geq 0\) and \(\delta D \geq 0\) for convenience.
2. Theorem 4.1 remains true if \(S\) in Assumption (B2) is replaced by \(\sum_{j \in \mathbb{A}} \rho(j)y_j + \sum_{j \in \mathbb{B}} \rho(j)(u_j^2 + v_j^2)/2\) where \((\rho(j))_{j \in \mathbb{Z}}\) is a real sequence, satisfying for some constants \(\kappa_0 > 0\), \(\kappa_1 > 0\) and \(C_\rho > 0\),

\[|\rho(j) - \rho(-j)| \geq C_\rho |j|^\kappa_1 > 0, \quad \forall |j| \geq \kappa_0 > 0.
\]

It turns out that Theorem 4.1 can be shown by adapting the proofs of Theorem A and Corollary C in [18], taking into account the symmetry condition (B2). The latter condition is used in an essential way to obtain the claimed measure estimate of Theorem 4.1 — see Section 6.4.

We conclude this section with a brief outline of the KAM proof in the presence of symmetries. As in the case without symmetries, it employs the rapidly converging iteration scheme of Newton type, involving an infinite sequence of coordinate transformations. At the \(v\)th step of the scheme, a Hamiltonian \(H_P + P_v\) is considered where \(H_P\) is a Hamiltonian of the form (4.1), and \(P_v\) is a small perturbation satisfying the symmetry condition \(|\{P_v, S\}| = 0\). In the case considered, the Hamiltonian \(S\) is in normal form, given by the expression (4.5). One then constructs a canonical transformation \(\Psi_v\) with the property that \((H_P + P_v) \circ \Psi_v\) takes the form \(H_{v+1} + P_{v+1}\) where \(H_{v+1}\) is again of the
form (4.1) and $P_{v+1}$ is a much smaller error term than $P_v$, satisfying in addition $\{P_{v+1}, S\} = 0$. The composition of the infinite sequence of coordinate changes $\psi_0, \psi_1, \ldots$ transforms the initial Hamiltonian $H + P$ – at least formally – into a normal form $H_\infty$. For the construction of these coordinate transformations a set of parameters $\xi$ has to be excluded. The measure of this set is then estimated, using that $\{P_v, S\} = 0$ for any $v$. Let us now describe the construction of the transformation $\psi_v$ in more detail. For brevity, we drop the index $v$ in $H_v, P_v, R_v$ and write

$$H + P = H + R + (P - R),$$

where $R$ is obtained from $P$ by truncating its Fourier and Taylor series expansion. From $\{P, S\} = 0$ one deduces that $\{R, S\} = 0$ as well. The canonical transformation $\psi_v$ is constructed as the time-1-map of the flow $X^1_F$ of a Hamiltonian vector field $X_F, \psi_v = X^1_F|_{t=1}$, where the Hamiltonian $F$ satisfies $\{F, S\} = 0$. To find such a Hamiltonian $F$, one expands $(H + P) \circ X^1_F$ with respect to $t$ at $t = 0$. Recall that for any Hamiltonian $G$,

$$\frac{d}{dt} G \circ X^1_F = \{G, F\} \circ X^1_F.$$

Hence

$$R \circ X^1_F = R + \int_0^1 \{R, F\} \circ X^1_F \, dt$$

and

$$H \circ X^1_F = H + \{H, F\} + \int_0^1 (1 - t) \{\{H, F\}, F\} \circ X^1_F \, dt.$$

Altogether, one thus has

$$(H + R) \circ \psi_v = H + R + \{H, F\} + \int_0^1 (1 - t) \{\{H, F\} + R, F\} \circ X^1_F \, dt.$$

The latter integral is of quadratic order in $R$ and $F$ and will be part of the new error term. The aim is to determine $F$ in such a way that $H_+ := H + R + \{H, F\}$ is again of the form (4.1) and $\{F, S\} = 0$. Setting $\hat{H} := H_+ - H$, this amounts to solve the system of linear equations

$$\{F, H\} + \hat{H} = R \quad \text{and} \quad \{F, S\} = 0 \quad (4.8)$$

for $F$ and $\hat{H}$ with $\hat{H}$ being of the form (4.1), and $R$ given as above. We will explicitly construct a solution $F, \hat{H}$ of (4.8). It then follows that

$$(1 - t)\{H, F\} + R = (1 - t)\hat{H} + tR,$$

and hence

$$(H + P) \circ \psi_v = H_+ + Q + (P - R) \circ \psi_v \quad (4.9)$$

with
Then \( H_\perp \) is the new normal form \( H_{v+1} \) and \( Q + (P - R) \circ \Psi_v \), the new perturbative term \( P_{v+1} \). Note that the term \( Q \) is of quadratic order in \( R, F, \hat{H} \). Furthermore, one has \( S \circ X^t_F = S \).

\[
Q = \int_0^1 \left\{(1 - t)\hat{H} + tR, F\right\} \circ X^t_F \, dt. \tag{4.10}
\]

Hence

\[
\left\{(P - R) \circ \Psi_v, S\right\} = \left\{(P - R) \circ \Psi_v, S \circ \Psi_v\right\} = \left\{P - R, S\right\} \circ \Psi_v = 0
\]

and, with \( \tilde{G}(t) := (1 - t)\hat{H} + tR \),

\[
\{Q, S\} = \int_0^1 \left\{\{G(t), F\} \circ X^t_F, S \circ X^t_F\right\} dt
\]

\[
= \int_0^1 \left\{\{G(t), F\}, S\right\} \circ X^t_F \, dt.
\]

As \( \hat{H} \) and \( S \) are both in normal form one has \( \left\{\hat{H}, S\right\} = 0 \). Together with the already established identities \( \{F, S\} = 0 \) and \( \{R, S\} = 0 \), one then concludes by the Jacobi identity that \( \{Q, S\} = 0 \). Altogether it follows that \( \{P_{v+1}, S\} = 0 \). In Section 6, we complete the proof of Theorem 4.1.

5. Proof of Theorem 1.2

In this section we show how Theorem 1.2 can be deduced from Theorem 4.1, using similar arguments as in [13] – see also [10]. Recall the set-up of Theorem 1.2. The subset \( A \subseteq \mathbb{Z} \) is of finite cardinality, \( \Pi \subseteq \mathbb{R}_{>0}^A \) is compact and of positive Lebesgue measure, \( T_{\Pi} \) is a union of \( A \)-tori in \( h_\perp^0 \) indexed by \( \xi \in \Pi \), and \( T_{\Pi} = \Phi^{-1}(\Sigma_{\Pi}) \subseteq \bigcap_{N \geq 1} H_{N_r}^N \). Consider the perturbed NLS Hamiltonian \( H_\epsilon = H_{\text{NLS}} + \epsilon K \), where \( K \) is a real analytic map, \( K : U \to \mathbb{C} \), with \( U \equiv U_{\Pi} \) a complex neighborhood of \( T_{\Pi} \) in \( H_c^N \) for some \( N \in \mathbb{Z}_{>1} \), so that properties (P1)–(P3) of Theorem 1.2 hold.

As a first step we apply the Birkhoff map \( \Phi^{-1} \) of Theorem 1.1,

\[
\Phi^{-1} : h_r^N \to H_r^N.
\]

Since \( \Phi^{-1} \) is real analytic, there is a complex neighborhood \( V \) of \( T_{\Pi} \) in the complexification of \( h_r^N \), which is mapped bi-analytically onto the neighborhood \( U \) of \( T_{\Pi} \). If necessary, we choose \( U \) and/or \( V \) smaller. Hence we have the following diagram where each arrow represents a bi-analytic diffeomorphism given by an approximate restriction of \( \Phi^{-1} \):

\[
\begin{array}{c}
\mathbb{T} \subset \mathbb{T}_{\Pi} \subset V \subset h_c^N \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathbb{T}_I \subset \mathbb{T}_{\Pi} \subset U \subset H_c^N.
\end{array}
\]

Now we consider the transformed Hamiltonian \( H_\epsilon \circ \Phi^{-1} \). Define \( \tilde{H}_{\text{NLS}} := H_{\text{NLS}} \circ \Phi^{-1} \) and \( \tilde{K} := K \circ \Phi^{-1} \big|_V \) so that

\[
H_\epsilon \circ \Phi^{-1} = \tilde{H}_{\text{NLS}} + \epsilon \tilde{K}.
\]
Then $H_e \circ \Phi^{-1}$ is real analytic on $V \supset \mathbb{T}_U$. Let us first look at the integrable Hamiltonian $\hat{H}_{\text{NLS}}$. By Theorem 1.1, $\hat{H}_{\text{NLS}}$ depends on $(q, p)$ only through the actions, $I_j = (q_j^2 + p_j^2)/2$, $j \in \mathbb{Z}$. As in Section 3, we view $\hat{H}_{\text{NLS}}$ as a real analytic function of the $l = (l_j)_{j \in \mathbb{Z}}$ defined on $V_1^N$ where $V_1^N$ has been introduced in (3.1). Using Taylor’s formula and the definition of the frequencies, $\omega_j(l) := \delta_j \hat{H}_{\text{NLS}}(l)$, we obtain

$$\hat{H}_{\text{NLS}}(I^0 + J) = \hat{H}_{\text{NLS}}(I^0) + \sum_{j \in \mathbb{Z}} \omega_j(I^0) J_j + Q \quad (5.1)$$

where $Q := \sum_{i, j \in \mathbb{Z}} Q_{ij}(I^0, J) J_i J_j$ and

$$Q_{ij}(I^0, J) := \int_0^1 (1 - t) \partial_l \omega_j(I^0 + t J) \, dt.$$ 

Note that $\partial_l \omega_j = \partial_l \partial_j \hat{H}_{\text{NLS}}(l)$ and hence the $Q_{ij}$ are symmetric in $i$ and $j$. Using the asymptotics of $\omega_j$ and the analyticity properties of $(\omega_j - 4\pi^2 j^2)_{j \in \mathbb{Z}}$ of Proposition 3.2 it follows from Cauchy’s estimate (see e.g. [13, Lemma A.2]) that

$$\sup_{j \in \mathbb{Z}} \left| \sum_{i \in \mathbb{Z}} Q_{ij}(I^0, J) J_i \right| \leq C \| J \|_{\ell^1,2N}$$

and hence

$$|Q| = \left| \sum_{i, j} Q_{ij}(I^0, J) J_i J_j \right| \leq C \| J \|_{\ell^1,2N}^2 \quad (5.2)$$

uniformly in $I^0$ on some complex neighborhood of $\mathbb{T}_U$ and $\| J \|_{\ell^1,2N}$ sufficiently small. Furthermore, by assumption (P2), the Hamiltonian vector field $X_K$ of $K$, given by $X_K = -i(\partial_{\phi_2} K, -\partial_{\phi_1} K)$, is defined on $U$ and of order 1, $\| X_K \|_N = O(1)$. The Hamiltonian vector field of the transformed Hamiltonian $\tilde{K} = K \circ \Phi^{-1}$.

$$X_{\tilde{K}} = (\Phi^{-1})^* X_K = d\Phi \cdot X_K \circ \Phi^{-1},$$

is then defined on $V$. In view of Theorem 1.1, we may shrink $V$, if necessary, so that $d\Phi \circ \Phi^{-1}$ is uniformly bounded on $V$. Hence

$$\| X_{\tilde{K}} \|_N = O(1) \quad (5.3)$$

uniformly on $V$.

As a second step we introduce symplectic polar coordinates near the tori in the family $\mathbb{T}_U$. For each $\xi = (\xi_j)_{j \in A} \in \mathbb{I}$ we then introduce new coordinates by setting for $j \in A$

$$\sqrt{2(\xi_j + y_j)e^{-i\xi_j}} := q_j + ip_j, \quad \sqrt{2(\xi_j + y_j)e^{i\xi_j}} := q_j - ip_j$$

whereas for $j \in B$, the Birkhoff coordinates $q_j, p_j$ play the role of $u_j, v_j$ of Section 4,

$$u_j := q_j, \quad v_j := p_j.$$
For each $\xi \in \Pi$, this transformation is real analytic and symplectic on $D(s, r) \subseteq V$ for all $s > 0$ and $r > 0$ sufficiently small. In the following we fix such an $s$, while we keep the freedom of choosing $r$ smaller later in the proof. Using the expansion of $H_{NLS}$ in (5.1) and setting $I^0$ to be element with components $\xi_j$ for $j \in A$ and $0$ for $j \in B$, the integrable Hamiltonian $H_{NLS}$ in the new coordinates is, up to a constant depending only on $\xi$, given by $H + Q$ with

$$H = H(y, u, v; \xi) = \sum_{j \in A} \omega_j(\xi) y_j + \sum_{j \in B} \Omega_j(\xi)(u_j^2 + v_j^2)/2,$$

(5.4)

where $\Omega_j(\xi) := \omega_j(\xi)$ for $j \in B$, and, according to (5.1), $Q \equiv Q(y, u, v; \xi)$ is given by

$$Q = \sum_{i, j} Q_{ij}(\xi, J)(J_i J_j) \quad \text{with} \quad J_j = y_j (j \in A) \quad \text{and} \quad J_j = (u_j^2 + v_j^2)/2 (j \in B),$$

(5.5)

where we have identified $I^0$ with $\xi$. We want to apply Theorem 4.1 for $H$, defined by (5.4), $P := Q + \epsilon \tilde{K}$, and $S := iH_2 \circ \Phi^{-1}$. We now verify Assumptions (A1)–(A3) and (B1)–(B2). Concerning (A1), recall that by Proposition 3.1, $\det((\partial_{y_j} \Omega_i)_{i, j \in A}) \neq 0$ on $\Pi$. Since this determinant is a real analytic function, it is nonzero almost everywhere on $\Pi$. In particular, for any given $\eta > 0$ we may excise from $\Pi$ a relatively open subset $\Pi_\eta$ with $\text{meas}(\Pi_\eta) < \eta$ such that on $\Pi \setminus \Pi_\eta$ the above determinant is uniformly bounded away from zero. Moreover, we may cover $\Pi \setminus \Pi_\eta$ by finitely many closed subsets $\Pi_i$, so that on each subset the map $\xi \rightarrow \omega(\xi)$ is a bianalytic homeomorphism onto its image in $\mathbb{R}^A$. Henceforth it suffices to consider each such parameter set $\Pi_i$ one at a time.

Next let us verify (A2). The external frequencies $\Omega_j$, $j \in B$, may be written as $\Omega_j(\xi) = \widehat{\Omega}_j + \tilde{\Omega}_j(\xi)$ with $\widehat{\Omega}_j = 4\pi^2 j^2$ and

$$\tilde{\Omega}_j(\xi) := \Omega_j - \widehat{\Omega}_j = \partial_{y_j} H_{NLS}(\xi) - 4\pi^2 j^2.$$

By Proposition 3.2, $\tilde{\Omega} : \xi \mapsto (\tilde{\Omega}_j(\xi))_{j \in B}$ maps $\Pi$ into $\ell^\infty(B; \mathbb{R})$ and is analytic on a complex neighborhood of $\Pi$ with values in $\ell^\infty(B, \mathbb{C})$. Hence $\tilde{\Omega}$ is also Lipschitz by Cauchy’s estimate. In summary, Assumption (A2) is satisfied with $d = 2$ and $\delta = 0$.

To see that Assumption (A3) holds note that by Proposition 3.1, $k \cdot \omega(\xi) + e \cdot \Omega(\xi) \neq 0$ for every $k \in \mathbb{Z}^A$ and $e \in \mathbb{Z}^B$ with $1 \leq |e| \leq 2$. Since each such expression is real analytic in $\xi$, its zero set is a set of measure zero and (A3) follows.

Toward Assumption (B2), first note that by Proposition 2.1(i), $iH_2 \circ \Phi^{-1}$ is of the form $S$, described in (B2). As $\Phi^{-1}$ is canonical and $Q$, given by (5.5), is in normal form, it follows that $\{Q, iH_2 \circ \Phi^{-1}\} = 0$. Furthermore, in view of Assumption (P3), $\{K \circ \Phi^{-1}, iH_2 \circ \Phi^{-1}\} = \{K, iH_2\} \circ \Phi^{-1} = 0$. Altogether we have shown that

$$\{ P, iH_2 \circ \Phi^{-1}\} = \{ Q + \epsilon \tilde{K}, iH_2 \circ \Phi^{-1}\} = 0$$

and Assumption (B2) follows.

It remains to check Assumption (B1). As already mentioned, the perturbation $P$ consists of two parts

$$P = Q + \epsilon \tilde{K}. $$

In view of the definition (5.5), the Hamiltonian vector field of $Q$ is given by

$$X_Q = (\partial_y Q, 0, \partial_v Q, -\partial_u Q).$$

To estimate the size of $X_Q$ we apply Cauchy’s estimate to each of its components. From the estimate (5.2) together with the bounds $|y| < r^2$ and $\|u\|N + \|v\|N < r$ one then gets that $\|X_Q\|_{\sup_{\sup_{r, r; D(s, r) \times \Pi_i}}} \leq \epsilon \tilde{K},$ 0. T. Kappeler, Z. Liang / J. Differential Equations 252 (2012) 4068–4113
cr^2$. As $Q$ analytically extends to some complex neighborhood of $\Pi$, again by Cauchy’s estimate, one obtains a similar bound for the Lipschitz semi-norm of $X_Q$,

$$\|X_Q\|_{lip}^{r,N;D(s,r)\times\Pi_t} \leq cr^2.$$

Taking the weight factors in the norm $\|\cdot\|_{r,N}$ into account and using (5.3), one gets the following estimate for the second term in $P$, $\|X^\theta\|_{r,N;D(s,r)\times\Pi_t} \leq \frac{c}{r^2}$. Arguing as for $Q$, one obtains a bound of the same form for the Lipschitz semi-norm, $\|X^\theta\|_{lip}^{r,N;D(s,r)\times\Pi_t} \leq \frac{c}{r^2}$. Altogether, we thus have shown that for any $0 < \alpha \leq M$ and $r > 0$ small enough,

$$\|X_Q + \epsilon \tilde{K}\|_{lip}^{r,N;D(s,r)\times\Pi_t} \leq \epsilon (r^2 + \frac{2}{r^2}).$$

(5.6)

In particular, we have verified Assumption (B1) with $V$ in (4.4) given by $D(s,r)$.

To meet the smallness condition (4.7) of Theorem 4.1 for $P = Q + \epsilon \tilde{K}$ choose $r$ and $\alpha$ as follows

$$r^2 = \sqrt{\epsilon}, \quad \alpha = \frac{2C}{\gamma} \sqrt{\epsilon},$$

(5.7)

with $\epsilon$ so small that $\alpha < 1$. Here, $C$ is taken from the preceding estimate, and $\gamma$ is taken from Theorem 4.1. We then obtain

$$C \left( r^2 + \frac{\epsilon}{r^2} \right) = 2C \sqrt{\epsilon} = \gamma \alpha.$$

The estimate (5.6) then implies that (4.7) holds. The conclusions of Theorem 1.2 now follow from the ones of Theorem 4.1. Let us only comment on the measure theoretic statement of Theorem 1.2. By Theorem 4.1 and the choice (5.7) of $\alpha$, for each $\Pi_t$ there exists $\Pi_t,\epsilon \subseteq \Pi_t$ so that

$$\text{meas}(\Pi_t \setminus \Pi_t,\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$$

Finitely many sets $\Pi_t$ cover the parameter domain $\Pi$ up to a set of measure $\eta$. By first choosing $\eta$ and then $\epsilon$ small enough we can assure that

$$\text{meas}\left(\Pi \setminus \bigcup_{i} \Pi_{i,\epsilon}\right) \to 0 \quad \text{as } \epsilon \to 0.$$

The proof of Theorem 1.2 is now complete. □

6. Proof of Theorem 4.1

The aim of this section is to prove Theorem 4.1. It is based on the proof of a KAM theorem without symmetries presented in [18].

6.1. Linearized equation

In this subsection we study the linear system (4.8)

$$\{F, H\} + \hat{H} = R \quad \text{and} \quad \{F, S\} = 0$$

(6.1)
where \( H, S, R \) are given Hamiltonians and \( F, \dot{H} \) are to be determined. It is convenient to introduce complex coordinates \( w = (w_j)_{j \in B}, z = (z_j)_{j \in B} \) defined by

\[
w = \frac{1}{\sqrt{2}}(u - iv) \quad \text{and} \quad z = \frac{1}{\sqrt{2}}(u + iv).
\]

In these complex coordinates, the Hamiltonians \( H \equiv H(y, w, z; \xi) \) and \( S = S(y, w, z) \) are given by

\[
H = \sum_{j \in A} \omega_j y_j + \sum_{j \in B} \Omega_j w_j z_j,
\]

\[
S = a + b \sum_{j \in A} y_j + c \sum_{j \in B} w_j z_j.
\]

Here \( \omega_j = \omega_j(\xi) \) and \( \Omega_j = \Omega_j(\xi) \) depend on the parameter \( \xi \) and \( a, b, c \) are real constants with \( b \neq 0, c \neq 0 \). In the sequel we will assume that the constants \( a, b, c \) are given by \( a = 0, b = c = 1 \) – the case where \( a \in \mathbb{R}, b, c \in \mathbb{R} \setminus \{0\} \) are arbitrary is proved in the same way. \( H \) is assumed to be regular on the domain \( D(s, r) \times \Pi \) in the sense that for each \( \xi \in \Pi \), \( i\xi H \equiv H(\cdot; \xi) \) is real analytic on \( D(s, r) \) and \( H(y, w, z; \cdot) \) is Lipschitz in \( \xi \), uniformly on \( D(s, r) \). The Hamiltonian \( R = R(x, y, w, z; \xi) \) is also assumed to be regular on \( D(s, r) \times \Pi \) and to be of the form

\[
R = \sum_{2||l| + |m + n| \leq 2} R_{klmn} e^{ikx} y^l w^m z^n. \tag{6.2}
\]

Here and in the sequel, a sum such as in (6.2) extends over all integer vectors \( k \in \mathbb{Z}^A \), \( l \in \mathbb{Z}^A_{\geq 0} \), and \( m, n \in \mathbb{Z}^B_{\geq 0} \). Hence \( R \) is a polynomial in \( y, w, z \) of degree two – the \( y_j, j \in A \), being variables of degree two – whose coefficients depend regularly on \( x \) and \( \xi \) in the sense above. Moreover, the Hamiltonian vector field \( X_R \equiv X_R(x, y, w, z; \xi) \) associated with \( R \) is assumed to be a regular map

\[
X_R : D(s, r) \times \Pi \to T^*_C
\]

and \( R \) is assumed to satisfy the symmetry conditions

\[
\{R, S\} = 0. \tag{6.4}
\]

The latter identity means that for any \( k \in \mathbb{Z}^A, l \in \mathbb{Z}^A_{\geq 0}, m, n \in \mathbb{Z}^B_{\geq 0} \) and \( \xi \in \Pi \)

\[
R_{klmn} \cdot (k \cdot v_A + (n - m) \cdot v_B) = 0. \tag{6.5}
\]

The mean value \( [R] \) of \( R \) is defined by

\[
[R] = \sum_{||l| + |m| = 1} R_{0lmm} y^l w^m z^n.
\]

Note that \( [R] \) is of the same form as \( H \). To shorten notation we drop the subscripts \( N \) and \( \Pi \) in \( \| \cdot \|_{r, N; D(s, r) \times \Pi} \) and write \( \| \cdot \|_{r, D(s, r)} \) instead of \( \| \cdot \|_{r, N; D(s, r) \times \Pi} \) as well as \( |\omega|_{lip}, |\Omega|_{lip} \) instead of \( |\omega|_{\Pi}, |\Omega|_{\Pi} \). In the sequel, we will always assume that \( \Omega \) satisfies condition (A2) of Section 4, i.e., \( \Omega = \overline{\Omega} + \hat{\Omega} \) where \( \overline{\Omega} \) is independent of \( \xi \) with

\[
\overline{\Omega} = |j|^d + \cdots \quad \text{for some } d > 1.
\]
and \( \hat{\Omega} = \Omega - \mathbb{Z}^B \) is a Lipschitz map, \( \hat{\Omega} : \Pi \to \ell^{\infty,-\delta}(B,\mathbb{R}) \) for some \( 0 \leq \delta < d - 1 \). Finally, for any \( e \in \mathbb{Z}^B \) with finite support we define

\[
|e|_{\delta} := \sum_{j \in B} (j)^{\delta} |e_j|.
\]

**Lemma 6.1.** Let \( \alpha > 0, s > 0, r > 0 \) and assume that \( H \) and \( R \) are regular on \( D(s, r) \times \Pi \) and that \( R \) satisfies (6.3) and (6.4). Moreover assume that for any \( \xi \in \Pi \) and any \( (k, e) \in \mathbb{Z} \)

\[
|k \cdot \omega(\xi) + e \cdot \Omega(\xi)| \geq \alpha A_k^{-1} \cdot 1 \vee |e|_{\delta}^{1/2}
\]

(6.6)

where the sequence \( (A_k)_{k \in \mathbb{Z}^A} \subseteq \mathbb{R} \) satisfies \( A_k \geq 1 \). Then the linear system (6.1) has a unique solution \( F, \hat{H} \) when normalized by \( [F] = 0, [\hat{H}] = \hat{H} \). The following estimates hold:

\[
\|X_{\hat{H}}^{\sup}_{r,D(s,r)} \| \leq \|X_R^{\sup}_{r,D(s,r)} \|, \quad \|X_{\hat{H}}^{lip}_{r,D(s,r)} \| \leq \|X_R^{lip}_{r,D(s,r)} \|
\]

and for any \( 0 < \sigma \leq s \)

\[
\|X_{\hat{F}}^{\sup}_{r,D(s-\sigma,r)} \| \leq \frac{16B_\sigma}{\alpha} \|X_R^{\sup}_{r,D(s,r)} \|
\]

\[
\|X_{\hat{F}}^{lip}_{r,D(s-\sigma,r)} \| \leq \frac{25B_\sigma}{\alpha} \left( \|X_R^{lip}_{r,D(s,r)} \| + \frac{M}{\alpha} \|X_R^{sup}_{r,D(s,r)} \| \right)
\]

where \( M \geq 1 \) satisfies \( M \geq |\omega|^{lip} + |\Omega|^{\infty,-1} \) and \( B_\sigma = (2^{4|A|} \sum_{k \in \mathbb{Z}^A} (k)^{4} A_k^{4} e^{-2|k|\sigma})^{1/2} \).

**Proof.** We are looking for solutions \( F \) and \( \hat{H} \) of (6.1) which admit expansions of the form

\[
F = \sum_{2||+|m+n|\leq 2} F_{klmn} e^{ik \cdot x} y^l w^m z^n
\]

and

\[
\hat{H} = \sum_{|k|+|m|=1} \hat{H}_{klmn} y^l w^m z^n.
\]

Use that \( \{x_j, y_j\} = 1 \) for any \( j \in A \) and \( \{w_j, z_j\} = 1 \) for any \( j \in B \) and that all other brackets between coordinate functions vanish to conclude

\[
\{e^{ik \cdot x} y^l w^m z^n, y_j\} = ik_j e^{ik \cdot x} y^l w^m z^n
\]

(6.7)

and

\[
\{e^{ik \cdot x} y^l w^m z^n, w_j z_j\} = i(n_j - m_j) e^{ik \cdot x} y^l w^m z^n.
\]

(6.8)

It then follows that

\[
\{F, H\} = \sum i F_{klmn} (k \cdot \omega + (n - m) \cdot \Omega) e^{ik \cdot x} y^l w^m z^n.
\]
One then finds by comparison of coefficients that the system (6.1) admits the solution $F$ and $\hat{H}$ given by

$$iF_{klmn} = \begin{cases} \frac{R_{klmn}}{k \cdot \omega + (n-m) \cdot \Omega}, & \text{if } R_{klmn} \neq 0 \text{ and } (k, n-m) \neq (0, 0), \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{H}_{0lmn} = R_{0lmn}. \tag{6.9}$$

By (6.5), one has $R_{klmn}(k \cdot \nu_A + (n-m) \cdot \nu_B) = 0$. Thus the small divisor conditions (6.6) guarantee that $F_{klmn}$ is well defined for any $k, l, m, n$. Furthermore $[F] = 0$ and $[\hat{H}] = \hat{H}$. When normalized in this way, $F$ and $H$ are uniquely determined. Clearly, one has $\{\hat{H}, S\} = 0$ and

$$\{F, S\} = \sum iF_{klmn}(k \cdot \nu_A + (n-m) \cdot \nu_B)e^{ik \cdot x}y^1w^mz^n$$

which by the definition of $F$ equals

$$\sum \frac{R_{klmn}(k \cdot \nu_A + (n-m) \cdot \nu_B)}{k \cdot \omega + (n-m) \cdot \Omega}e^{ik \cdot x}y^1w^mz^n.$$

In view of (6.5) it then follows that $\{F, S\} = 0$. To derive the claimed estimates we decompose $R = R^0 + R^1 + R^2$ and write

$$R^0 = R^{00} = R^{000} + R^{001}, \quad R^1 = R^{10} + R^{01}, \quad R^2 = R^{20} + R^{11} + R^{02}$$

where $R^a$ comprises all terms with $|m+n| = a$,

$$R^{000} = \sum R_{k000}e^{ik \cdot x}, \quad R^{001} = \sum_{j \in A} R_{j}^{001} y_j = \sum_{|l|=1} R_{k00l}e^{ik \cdot x}y^l$$

and $R^{ab}$ are given by

$$R^{10} = \sum_{j \in B} R_{j}^{10} w_j, \quad R^{01} = \sum_{j \in B} R_{j}^{01} z_j,$$

$$R^{20} = \sum_{i,j \in B} R_{ij}^{20} w_i w_j, \quad R^{11} = \sum_{i,j \in B} R_{ij}^{11} z_i z_j, \quad R^{02} = \sum_{i,j \in B} R_{ij}^{02} z_i z_j.$$

The coefficients $R_{j}^{ab}$ and $R_{ij}^{ab}$ are given by the corresponding derivatives of $R$ with respect to the 

components of $w$ and $z$ at $w = 0, z = 0$ and depend on $x$ and $\xi$ whereas the coefficients $R_{j}^{001}$ are given by $\partial_y R^{1}_{y=0}$ and also depend on $x$ and $\xi$. So e.g. for any $j \in B$,

$$R_{j}^{10} = \partial_{w_j} R|_{w=0, z=0} = \sum R_{k0m0}e^{ik \cdot x} \quad \text{and} \quad m^j = (\delta_{ji})_{i \in B}.$$

The functions $F$ and $\hat{H}$ are decomposed in a similar way. The linear system $\{F, H\} + \hat{H} = R$ then may be written as follows

$$\{F^{ab}, H\} = R^{ab} - [R^{ab}], \quad R^{ab} = [R^{ab}]$$

and it suffices to obtain the claimed estimates for each of the Hamiltonians $F^{ab}$, $\hat{H}^{ab}$ individually. By the definition of $\hat{H}^{ab}$, the claimed estimates for $\hat{H}$ hold trivially. Concerning the terms $F^{ab}$, they all
can be treated in a similar fashion. So we concentrate on $F^{10}$ and $F^{11}$ only. Let us begin with $F^{10}$. We want to estimate $X_{F^{10}} = (0, -\partial_x F^{10}, 0, i\partial_w F^{10})$ in terms of $X_R$. It is convenient to introduce the notation $\hat{R} = (\hat{R}_j)_{j \in B}$ with

$$
\hat{R}_j \equiv R_j^{10} = \sum_k \hat{R}_{kj} e^{ik \cdot x} = \partial_w R|_{w=0,z=0}.
$$

By the definition of the norm $\| \cdot \|_{r,D(s,r)}$ one has

$$
\| \hat{R} \|_{D(s)} \leq r \| X_R \|_{r,D(s,r)} \tag{6.10}
$$

where $D(s) := \{ x \in \mathbb{C}^A/2\pi \mathbb{Z}^A : |\Re x| < s \}$. By assumption, $\hat{R} : D(s) \to \ell^2 N \equiv \ell^2 N(B, \mathbb{C})$ is analytic and has a Fourier expansion with Fourier coefficients $(\hat{R}_{kj})_{j \in B}$, $k \in \mathbb{Z}^A$, satisfying the $L^2$-bound

$$
\sum_{k \in \mathbb{Z}^A} \| (\hat{R}_{kj})_{j \in B} \|_N e^{2|k|}\leq 2|\alpha| (\| \hat{R} \|_{D(s)})^2.
$$

Actually, due to the symmetry conditions (6.5), for each $k \in \mathbb{Z}^A$, $\hat{R}_{kj} = 0$, and hence $\hat{F}_{kj} = 0$, for any $j \in B$ except possibly for $j = j(k) = k \cdot \nu_A$. For any $k \in \mathbb{Z}^A$, $\hat{R}_{kj(k)} : \Pi \to \mathbb{C}$ is Lipschitz and the corresponding coefficient $\hat{F}_{kj(k)}$ of $\hat{F}$ is given by

$$
i \hat{F}_{kj(k)} = \frac{\hat{R}_{kj(k)}}{k \cdot \omega - \Omega_{j(k)}}.
$$

By the small divisors assumption (6.6), $|k \cdot \omega - \Omega_{j(k)}| \geq \alpha A_k^{-1}$ for any $k \in \mathbb{Z}^A$. Hence

$$
\| (\hat{F}_{kj})_{j \in B} \|_N \leq \frac{A_k}{\alpha} \| (\hat{R}_{kj})_{j \in B} \|_N \tag{6.11}
$$

and thus

$$
\| \hat{F} \|_{D(s-\sigma)} \leq \sum_k \| (\hat{F}_{kj})_{j \in B} \|_N e^{2|k|}\sigma} \leq \sum_k A_k^2 e^{-2|k|}\frac{1}{\alpha} \left( \sum_k \| (\hat{R}_{kj})_{j \in B} \|_N e^{2|k|}\sigma} \right)^{\frac{1}{2}} \leq \frac{B \sigma}{\alpha} \| \hat{R} \|_{D(s)}.
$$

or, as $\hat{F} = \partial_w F^{10}$,

$$
\frac{1}{r} \| \partial_w F^{10} \|_{D(s-\sigma)} \leq \frac{B \sigma}{\alpha} \| X_R \|_{r,D(s,r)}.
$$

The other nonzero component of $X_{F^{10}}$ is given by $\partial_x F^{10} = \sum_k ik(\sum_{j \in B} \hat{F}_{kj} w_j) e^{ik \cdot x}$. As by (6.11),

$$
\sum_{j \in B} \hat{F}_{kj} w_j \leq \| (\hat{F}_{kj})_{j \in B} \|_N \| w \|_N \leq \frac{A_k}{\alpha} \| (\hat{R}_{kj})_{j \in B} \|_N \| w \|_N.
$$
one gets
\[ \frac{1}{r} \| \partial_r F^{10} \|_{D(s-\sigma, r)}^{\sup} \leq \sum_k \frac{A_k}{\alpha} |k| \| (\hat{R}_{kj}) \|_N e^{k|s-\sigma|} \leq \frac{B_\sigma}{\alpha} \| \hat{R} \|_{D(s)}^{\sup}. \]

It then follows from (6.10) that
\[ \frac{1}{r^2} \| \partial_r F^{10} \|_{D(s-\sigma, r)}^{\sup} \leq \frac{B_\sigma}{\alpha} \| X_R \|_{r, D(s, r)}^{\sup}. \]

Altogether, we have proved that
\[ \| X_{F^{10}} \|_{r, D(s-\sigma, r)}^{\sup} \leq \frac{1}{r^2} \| \partial_r F^{10} \|_{D(s-\sigma, r)}^{\sup} + \frac{1}{r} \| \partial_w F^{10} \|_{D(s-\sigma, r)}^{\sup} \leq \frac{2B_\sigma}{\alpha} \| X_R \|_{r, D(s, r)}^{\sup}. \]

Next we want to estimate \( \| X_{F^{10}} \|_{r, D(s-\sigma, r)}^{\lip} \). Let \( \alpha_k := k \cdot \omega - \Omega_{j(k)} \) and \( \Delta \equiv \Delta_{\xi \zeta} \) for \( \xi, \zeta \in \mathcal{P} \). Then, for any \( k \in \mathbb{Z}^A \) and with \( j \equiv j(k) \),

\[ i \Delta \hat{F}_{kj} = \Delta^{-1}(\hat{R}_{kj}) = \alpha_k^{-1} \Delta(\hat{R}_{kj}) + \hat{R}_{kj} \Delta(\alpha_k^{-1}) \]

and

\[ -\Delta(\alpha_k^{-1}) = \frac{\Delta \alpha_k}{\alpha_k(\xi) \cdot \alpha_k(\zeta)} = k \cdot \Delta \omega - \Delta \Omega_j. \]

By the small divisors assumption (6.6), \( |\alpha_k| \geq A_k^{-1}(j)^{\frac{1}{\delta}} \). Recall that \( \Omega_j = \Pi_j + \hat{\Omega}_j \), where \( \Delta \Omega_j = \Delta \hat{\Omega}_j = O(j^{\delta}) \). One then gets

\[ |\Delta \alpha_k^{-1}| \leq \frac{A_k^2}{\alpha^2} \left( |k| |\Delta \omega| + |\Delta \Omega_j| \right) \]

and thus

\[ \| (\Delta \hat{F}_{kj})_{j \in B} \|_N \leq \frac{A_k}{\alpha} \| (\Delta \hat{R}_{kj})_{j \in B} \|_N + \frac{A_k^2}{\alpha^2} (|k| |\Delta \omega| + |\Delta \Omega_j|_{E_{\infty, -\delta}}) \| (\hat{R}_{kj})_{j \in B} \|_N. \quad (6.12) \]

Summing up to the Fourier series as before we obtain

\[ \| \Delta \hat{F} \|_{D(s-\sigma)} \leq \frac{B_\sigma}{\alpha} \| \Delta \hat{R} \|_{D(s)}^{\sup} + \frac{B_\sigma}{\alpha^2} (|\Delta \omega| + |\Delta \Omega_j|_{E_{\infty, -\delta}}) \| \hat{R} \|_{D(s)}^{\sup}. \]

Dividing this inequality by \( |\xi - \zeta| \) and taking the supremum over \( \xi \neq \zeta \) in \( \mathcal{P} \) yields, with \( \hat{\omega} = \partial_w F^{10} \),

\[ \frac{1}{r} \| \partial_w F^{10} \|_{D(s-\sigma, r)}^{\lip} \leq \frac{B_\sigma}{\alpha} \left( \| X_R \|_{r, D(s, r)}^{\lip} + \frac{M}{\alpha} \| X_R \|_{r, D(s, r)}^{\sup} \right) \]

where we used that \( M \geq |\omega|^{\lip} + |\Omega_j|_{E_{\infty, -\delta}}^{\lip} \).
Now let us estimate the Lipschitz semi-norm of the other nonzero component $\partial_x F^{10}$ of $X_{F^{10}}$. Note that

$$\Delta \partial_x F^{10} = \sum_k i k \left( \sum_{j \in B} \Delta \tilde{\mathcal{F}}_{kj} w_j \right) e^{ikx}.$$ 

Hence by (6.12)

$$\frac{1}{r} \left\| \Delta \partial_x F^{10} \right\|_{D(s-\sigma,r)} \leq \sum_k |k| \left\| (\Delta \tilde{\mathcal{F}}_{kj})_{j \in B} \right\|_N e^{k(s-\sigma)}$$

$$\leq \sum_k |k| e^{k(s-\sigma)} \left( \frac{A_k}{\alpha} \sum_{\ell} \left\| \Delta \tilde{\mathcal{R}}_{kj} \right\|_{D(s)} + \frac{A_{2k}}{\alpha^2} \left( \sum_{\ell} \left\| \Delta \omega \right\|_{\ell_{\infty,\sigma}} \right)^2 \right)$$

and thus by the definition of $B_\sigma$

$$\frac{1}{r^2} \left\| \Delta \partial_x F^{10} \right\|_{D(s-\sigma,r)} \leq \frac{B_\sigma}{\alpha} \left( \sum_{\ell} \left\| \Delta \tilde{\mathcal{R}}_{kj} \right\|_{D(s)} + \frac{B_\sigma}{\alpha^2} \left( \sum_{\ell} \left\| \Delta \omega \right\|_{\ell_{\infty,\sigma}} \right)^2 \right)$$

leading as above to the estimate

$$\frac{1}{r^2} \left\| \partial_x F^{10} \right\|_{D(s-\sigma,r)} \leq \frac{B_\sigma}{\alpha} \left( \sum_{\ell} \left\| X_R \right\|_{D(s)} + \frac{M}{\alpha} \left\| X_R \right\|_{D(s)} \right).$$

Altogether we have shown

$$\left\| X_{F^{10}} \right\|_{D(s-\sigma,r)} \leq \frac{2B_\sigma}{\alpha} \left( \sum_{\ell} \left\| X_R \right\|_{D(s)} + \frac{M}{\alpha} \left\| X_R \right\|_{D(s)} \right).$$

Let us now turn our attention to the term $\tilde{F} = F^{11}$. We want to estimate

$$X_{F^{11}} = (0, -\partial_x F^{11}, -i\partial_z F^{11}, i\partial_w F^{11}).$$

Recall that $\tilde{R} \equiv R^{11} = \sum_{i,j} \tilde{R}^{11}_{ij} v_i w_j$. For convenience, let $\tilde{R}^{ij} := \tilde{R}^{11}_{ij}$ and denote the operator corresponding to $(\tilde{R}^{ij})_{i,j \in B}$ by $\tilde{R}$. Note that $\tilde{R}^{ij} = \partial_w / \partial_z R^{ij}_{w=0,z=0}$. Due to the special form of $R$, $\tilde{R}$ can be viewed as the Jacobian of $\partial_x R|_{w=0}$ with respect to $w$ at $w = 0$. In particular, it can be viewed as a linear operator on $\ell^2_{\infty} \equiv \ell^2_{\infty}(B, \mathbb{C})$. Hence by the Cauchy estimate for analytic maps between Banach spaces (cf. [18, Lemma A.3])

$$\left\| \tilde{R} \right\|_{D(s)} \leq \frac{1}{r} \left\| \partial_z R \right\|_{D(s)} \leq \left\| X_R \right\|_{D(s)}$$

where $\left\| \tilde{R} \right\|$ denotes the operator norm on $\ell^2_{\infty}(B, \mathbb{C})$. This is equivalent to the statement that $\tilde{R} = ((i)^{-N} R^{(j)N}_{ij})_{i,j \in E}$ is a bounded operator on $\ell^2 \equiv \ell^2(B, \mathbb{C})$. Expanding $\tilde{R}$ into its Fourier series with operator valued coefficients $\tilde{R}_k = (\tilde{R}^{(j)N}_{ij})_{i,j \in B}, k \in \mathbb{Z}^A$, one gets as before

$$\sum_{k \in \mathbb{Z}^A} \left\| \tilde{R}_k \right\|^2 e^{2|k|s} \leq 2^{|A|} \left( \left\| \tilde{R} \right\|_{D(s)} \right)^2.$$
where now \( \| \tilde{R}_k \| \) denotes the operator norm of \( \tilde{R}_k : \ell^2 \to \ell^2 \). The corresponding coefficient \( \tilde{F}_k = (\tilde{F}_{k,ij})_{i,j \in B} \) is given by (cf. (6.9))

\[
i \tilde{F}_{k,ij} = \begin{cases} \frac{\tilde{R}_{k,ij}}{k \omega + \Omega_i - \Omega_j}, & \text{if } \tilde{R}_{k,ij} \neq 0 \text{ and } |i| + |j| \neq 0, \\ 0, & \text{otherwise.} \end{cases}
\]

(6.14)

Note that by the symmetry conditions (6.5), \( \tilde{R}_{k,ij} \neq 0 \) implies that \( j = i + k \cdot \nu_A \). Hence for any \( k \in \mathbb{Z}^A \), in each row and in each column of the infinite matrix \( (\tilde{R}_{k,ij})_{i,j \in B} \) and thus also of the infinite matrix \( (\tilde{F}_{k,ij})_{i,j \in B} \) — there is at most one nonzero entry. Therefore the operator norm of \( \tilde{R}_k = (\tilde{R}_{k,ij})_{i,j \in B} \) can be computed to be

\[
\| \tilde{R}_k \| = \sup_{i,j \in B} |\tilde{R}_{k,ij}|.
\]

By (6.5)–(6.6), \( \tilde{R}_{k,ij} \neq 0 \) with \( |i| + |j| \neq 0 \) implies that \( |k \cdot \omega + \Omega_i - \Omega_j| \geq \alpha A_k^{-1} \). Hence \( \| \tilde{F}_k \| \leq A_k \| \tilde{R}_k \| \) uniformly on \( \Pi \). Summing up over \( k \) leads to

\[
\| \tilde{F} \|_{D(s-\sigma)} \leq \frac{B_\sigma}{\alpha} \| \tilde{R} \|_{D(s)}.
\]

(6.15)

Going back to the operator norm of linear operators on \( \ell^2 \) one gets, in view of (6.13),

\[
\frac{1}{r} \| \partial_\nu F^{11} \|_{D(s-\sigma),r} \leq \frac{B_\sigma}{\alpha} \| X_{F^{11}} \|_{r, D(s),r}.
\]

(6.16)

Similarly one has

\[
\frac{1}{r} \| \partial_w F^{11} \|_{D(s-\sigma),r} \leq \frac{B_\sigma}{\alpha} \| X_{F^{11}} \|_{r, D(s),r}.
\]

To estimate \( \frac{1}{r^2} \| \partial_x F^{11} \|_{D(s-\sigma),r} \) note that \( \partial_x F^{11} = i \sum_k k (\sum_{i,j} \tilde{F}_{ij} z_i w_j) e^{ik \cdot x} \). As \( |\sum_{i,j} \tilde{F}_{ij} z_i w_j| \leq \| \tilde{F} \| \| z \|_{N} \| w \|_{N} \), it follows from (6.15) and the definition of \( B_\sigma \)

\[
\frac{1}{r^2} \| \partial_x F^{11} \|_{D(s-\sigma),r} \leq \frac{B_\sigma}{\alpha} \| X_{F^{11}} \|_{r, D(s),r}.
\]

(6.17)

Altogether we thus have proved that

\[
\| X_{F^{11}} \|_{r, D(s-\sigma),r} \leq \frac{3B_\sigma}{\alpha} \| X_{F^{11}} \|_{r, D(s),r}.
\]

Next we want to estimate \( \| X_{F^{11}} \|_{r, D(s-\sigma),r} \). The Lipschitz estimate of \( \hat{F} \) is obtained in a similar fashion as the one of \( \hat{F} \). Indeed, let

\[
j \equiv j(i,k) := i + k \cdot \nu_A \quad \text{and} \quad \alpha_{k,i} := k \cdot \omega + \Omega_i - \Omega_j.
\]

Then

\[
i \Delta \hat{F}_{k,ij} = \alpha_{k,i}^{-1} \Delta \tilde{R}_{k,ij} + \tilde{R}_{k,ij} \Delta \alpha_{k,i}^{-1}.
\]
The small divisors assumption (6.6) then implies that
\[ |\alpha_{k,i}| \geq \alpha A_k^{-1}(\langle i \rangle^\delta + \langle j \rangle^\delta)^{1/2}. \]

Using that
\[ \frac{|\Delta (\Omega_i - \Omega_j)|}{\langle i \rangle^\delta + \langle j \rangle^\delta} \leq \frac{|\Delta \Omega_i|}{\langle i \rangle^\delta} + \frac{|\Delta \Omega_j|}{\langle j \rangle^\delta}, \]
one gets above
\[ |\Delta \alpha_{k,i}^{-1}| \leq \frac{A_k^2}{\alpha^2} \left( |k| |\Delta \omega| + \frac{|\Delta \Omega_i|}{\langle i \rangle^\delta} + \frac{|\Delta \Omega_j|}{\langle j \rangle^\delta} \right) \leq \frac{A_k^2}{\alpha^2} (|k| |\Delta \omega| + 2|\Delta \Omega|_{\ell^{\infty,-\delta}}) \]
and therefore, uniformly on \( \Pi \),
\[ \|\Delta \tilde{F}_k\| \leq \frac{A_k}{\alpha} \|\Delta \tilde{R}_k\| + \frac{A_k^2}{\alpha^2} (|\Delta \omega| + 2|\Delta \Omega|_{\ell^{\infty,-\delta}}) \|\tilde{R}_k\|. \]

Summing up over \( k \) this leads to
\[ \|\Delta \tilde{F}\|_{\sup D(s-\sigma)} \leq \frac{B_\sigma}{\alpha} \|\Delta \tilde{R}\|_{\sup D(s)} + \frac{B_\sigma}{\alpha^2} (|\Delta \omega| + 2|\Delta \Omega|_{\ell^{\infty,-\delta}}) \|\tilde{R}\|_{\sup D(s)}. \]

Going back to the operator norm of linear operators on \( \ell^{2,N} \) one gets
\[ \|\Delta \tilde{F}\|_{\sup D(s-\sigma)} \leq \frac{B_\sigma}{\alpha} \|\Delta \tilde{R}\|_{\sup D(s)} + \frac{B_\sigma}{\alpha^2} (|\Delta \omega| + 2|\Delta \Omega|_{\ell^{\infty,-\delta}}) \|\tilde{R}\|_{\sup D(s)}. \quad (6.16) \]

Dividing this inequality by \( |\xi - \zeta| \) and taking the supremum over \( \xi \neq \zeta \) in \( \Pi \) yields
\[ \|\tilde{F}\|_{\ell^p D(s-\sigma)} \leq \frac{2B_\sigma}{\alpha} \left( \|\tilde{R}\|_{\ell^p D(s)} + \frac{M}{\alpha} \|\tilde{R}\|_{\sup D(s)} \right). \]

Finally arguing as in (6.15) one concludes that
\[ \frac{1}{r} \|\partial_z F^{11}\|_{\ell^p D(s-\sigma,r)} = \sup_{\|w\|_{\ell^p} < r} \frac{1}{r} \|\tilde{F}w\|_{\ell^p D(s-\sigma,r)} \leq \frac{2B_\sigma}{\alpha} \left( \|X_R\|_{\ell^{p},D(s,r)} + \frac{M}{\alpha} \|X_R\|_{\sup D(s,r)} \right). \quad (6.17) \]

Similarly one has
\[ \frac{1}{r^l} \|\partial_w F^{11}\|_{\ell^p D(s-\sigma,r)} \leq \frac{2B_\sigma}{\alpha} \left( \|X_R\|_{\ell^{p},D(s,r)} + \frac{M}{\alpha} \|X_R\|_{\sup D(s,r)} \right). \]

To estimate \( \frac{1}{r^l} \|\partial \Delta F^{11}\|_{\ell^p D(s-\sigma,r)} \) note that
\[ -i\partial_k \Delta F^{11} = \sum_k \left( \sum_{i,j} \Delta \tilde{F}_{ij} z_i w_j \right) e^{ikx}. \]

As \( \sum_{i,j} |\Delta \tilde{F}_{ij} z_i w_j| \leq \|\Delta \tilde{F}\|_{\ell^p} \|z\|_N \|w\|_N \) it follows from (6.16) and the definition of \( B_\sigma \)
\[
\frac{1}{r^2} \| \partial_x \Delta F^{11} \|_{D(s-\sigma, r)}^{\sup} \leq \sum_k |k| \| \Delta \hat{F}_k \|_{e^{k(s-\sigma)}} \\
\leq \frac{B_\sigma}{\alpha} \| \Delta \hat{R} \|_{D(s)}^{\sup} + \frac{B_\sigma}{\alpha^2} (|\Delta \omega| + 2|\Delta \Omega|) \| \hat{R} \|_{D(s)}^{\sup}.
\]

With the same arguments which lead to (6.17) one then concludes that
\[
\frac{1}{r^2} \| \partial_x F^{11} \|_{D(s-\sigma, r)}^{\lip} \leq \frac{2B_\sigma}{\alpha} \left( \| X_R \|_{l,D(s,r)}^{\lip} + \frac{M}{\alpha} \| X_R \|_{r,D(s,r)}^{\sup} \right).
\]

Altogether we thus have proved that
\[
\| X_F \|_{r,D(s-\sigma, r)}^{\sup} \leq \frac{6B_\sigma}{\alpha} \left( \| X_R \|_{r,D(s,r)}^{\lip} + \frac{M}{\alpha} \| X_R \|_{r,D(s,r)}^{\sup} \right).
\]

All the other components $F^{ab}$ admit the same type of estimates. More precisely, $\| X_{F^{a_0}} \|_{r,D(s-\sigma, r)}^{\sup}$, $\| X_{F^{a_1}} \|_{r,D(s-\sigma, r)}^{\sup}$ and $\| X_{F^{a_2}} \|_{r,D(s-\sigma, r)}^{\sup}$ are each bounded by $\frac{3B_\sigma}{\alpha} \| X_R \|_{r,D(s,r)}^{\sup}$ whereas $\| X_{F^{a_0}} \|_{r,D(s-\sigma, r)}^{\sup}$ and $\| X_{F^{a_1}} \|_{r,D(s-\sigma, r)}^{\sup}$ are bounded by $\frac{2B_\sigma}{\alpha} \| X_R \|_{r,D(s,r)}^{\sup}$. Altogether, one gets
\[
\| X_F \|_{r,D(s-\sigma, r)}^{\sup} \leq \frac{16B_\sigma}{\alpha} \| X_R \|_{r,D(s,r)}^{\sup}.
\]

Similarly, by the estimates above, one obtains
\[
\| X_F \|_{r,D(s-\sigma, r)}^{\lip} \leq \frac{25B_\sigma}{\alpha} \left( \| X_R \|_{r,D(s,r)}^{\lip} + \frac{M}{\alpha} \| X_R \|_{r,D(s,r)}^{\sup} \right)
\]
as claimed. \( \square \)

Following [18], the estimates may be compactly written as follows. For $\lambda \geq 0$, define for a parameter dependent vector field $Y : D(s, r) \times \Pi \to \mathcal{P}_C^n$ with components $(Y_x, Y_y, Y_u, Y_v)$ and parameter $\xi \in \Pi$,
\[
\| Y \|_{r,D(s,r)}^{\lambda} : = \| Y \|_{r,D(s,r)}^{\sup} + \lambda \| Y \|_{r,D(s,r)}^{\lip}.
\]

Furthermore, let $\| Y \|_{r,D(s,r)}^a$ stand for either $\| Y \|_{r,D(s,r)}^{\sup}$ or $\| Y \|_{r,D(s,r)}^{\lip}$.

**Corollary 6.1.** Under the assumptions of Lemma 6.1, one has for $0 < \sigma \leq s$ and $0 \leq \lambda \leq \frac{\alpha}{M}$
\[
\| X_{\hat{H}} \|_{r,D(s,r)}^a \leq \| X_R \|_{r,D(s,r)}^a,
\]
and
\[
\| X_F \|_{r,D(s-\sigma, r)}^{\lambda} \leq \frac{41B_\sigma}{\alpha} \| X_R \|_{r,D(s,r)}^{\lambda}.
\]

Moreover, if $A_k = \langle k \rangle^\tau$, then
\[
B_\sigma \leq b \cdot \sigma^{-2(\tau + |A| + 2)}
\]
(6.18)

with some constant $b \geq 1$ depending only on $A$ and $\tau$. 

6.2. KAM step

At the \( v \)th step of the iteration scheme we are given a Hamiltonian \( H_v + P_v \) where \( H_v \) is in normal form and \( P_v \) is a small perturbation satisfying \( \{ P_v, S \} = 0 \). More precisely, \( H_v \) and \( P_v \) are assumed to be regular on \( D(s_v, r_v) \times \Pi_v \) with \( 0 < s_v \leq s_{v-1} \) and \( 0 < r_v \leq r_{v-1} \) in the sense defined at the beginning of Section 6.1. Furthermore, \( \Pi_v \subseteq \Pi \) is a compact subset and \( H_v \) is of the form

\[
H_v = \omega^v \cdot y + \Omega^v w \cdot z,
\]

with \( \omega^v = (\omega_j^v)_{j \in A} \) and \( \Omega^v w = (\Omega_j^v w_j)_{j \in B} \) satisfying \( |\omega^v|^{lip} + |\Omega^v|^{lip} \leq M_v \) and the small divisors condition on \( \Pi_v \)

\[
|k \cdot \omega^v + e \cdot \Omega^v| \geq \alpha_v A_k^{-1} \cdot 1 \vee |e|^2 \beta
\]

for any \( (k, e) \in \mathcal{Z} \) where \( A_k = (k)^{\tau} \). The perturbation \( P_v \) satisfies in addition the symmetry condition \( \{ P_v, S \} = 0 \). In this subsection we now drop the index \( v \) and write ‘+’ for ‘+1’ to simplify notation. Thus \( P = P_v \) and \( P_+ = P_{v+1} \) and so on. In the following, \( C \) stands for a constant which depends only on \( A \) and \( \tau \) – actually the dependence on \( \tau \) only enters through the constant \( b \) in (6.18). Furthermore we assume that the perturbation is so small that we can choose \( 0 < \eta < \frac{1}{15} \) and \( 0 < \sigma < \frac{1}{2} \) with \( \sigma \leq 1 \), such that

\[
\|X_P\|^{sup}_{r, D(s, r)} + \frac{\alpha}{M} \|X_P\|^{lip}_{r, D(s, r)} \leq \frac{\alpha \sigma^2 \eta^2}{c_0}
\]

where \( \kappa = 2 \tau + |A| + 3 \) and \( c_0 \geq 1 \) is a sufficiently large constant depending only on \( A \) and \( \tau \), which will be specified later and will enter the smallness condition of the perturbation \( P \) in Theorem 4.1, encoded in \( \gamma \).

Approximation of \( P \)

We now approximate \( P \) by its Taylor polynomial \( R \) of degree two in \( y, w, \) and \( z \) of the form (6.2). This leads to corresponding approximations of the partial derivatives \( \partial_\xi P, \partial_y P, \partial_w P, \) and \( \partial_\eta P \) which constitute the Hamiltonian vector field \( X_P \). As in the proof of Lemma 6.1, we represent \( R \) in the form \( \sum_{0 \leq i+j \leq 2} R^{ij} \). The components of the Hamiltonian vector fields \( X_R \) can then be expressed in terms of the derivatives up to order 2 of components of \( X_P \) evaluated at \( y = 0, w = 0, z = 0 \). Since \( P(\cdot; \xi) \) is analytic, Cauchy's estimate then leads to the estimate

\[
\|X_R\|^n_{r, D(s, r)} \leq C \|X_P\|^n_{r, D(s, r)},
\]

where we recall that \( C \) stands for a constant which depends only on \( A \) and \( \tau \). Next we need to estimate how accurate \( X_R \) approximates \( X_P \). We claim that

\[
\|X_P - X_R\|^n_{\eta r, D(s, 4\eta r)} \leq C \eta \|X_P\|^n_{r, D(s, r)},
\]

To prove this inequality note that

\[
X_P - X_R = (\partial_y (P - R), -\partial_\xi (P - R), -i\partial_z (P - R), i\partial_w (P - R)).
\]

Let us begin by estimating \( \partial_y P - \partial_y R \). As \( \partial_y R = \partial_y P |_{y=0, w=0, z=0} \) one has
\[
\frac{\partial_y P - \partial_y R}{\sup_{D(\eta,t)}} = \frac{1}{\eta} \int_0^1 \left( \frac{y \cdot \partial_y (\partial_y P)(x, ty, tw, tz)}{\sup_{D(\eta,t)}} \right) dt \\
\leq C_{\eta} \frac{(4\eta)^2}{((1-4\eta)^r)^2} \sup_{D(\eta,t)} \| \partial_y P \|_{\sup_{D(\eta,t)}} \\
\leq C_{\eta} \sup_{D(\eta,t)} \| \partial_y P \|_{\sup_{D(\eta,t)}}.
\]

Similarly one gets
\[
\frac{\sup_{D(\eta,t)}}{\sup_{D(\eta,t)}} (w \cdot \partial_w) (\partial_y P) (z \cdot \partial_z (\partial_y P)) \leq C_{\eta} \sup_{D(\eta,t)} \| \partial_y P \|_{\sup_{D(\eta,t)}}.
\]

As \( \| \partial_y P \|_{\sup_{D(\eta,t)}} \leq \| X_P \|_{r,D(\eta,t)} \), it then follows that
\[
\| \partial_y P - \partial_y R \|_{\sup_{D(\eta,t)}} \leq C_{\eta} \| X_P \|_{r,D(\eta,t)}.
\]

In a similar way one shows that
\[
\| \partial_y P - \partial_y R \|_{\sup_{D(\eta,t)}} \leq C_{\eta} \| X_P \|_{r,D(\eta,t)}.
\]

Next let us estimate the component \( \partial_w P - \partial_w R \). Note that
\[
\partial_w P(x, y, w, z) = \partial_w P(x, 0, w, z) + \int_0^1 y \cdot \partial_y (\partial_w P)(x, ty, w, z) dt.
\]

The error term \( \int_0^1 y \cdot \partial_y (\partial_w P)(x, ty, w, z) dt \) is not part of \( X_R \) and Cauchy's estimate leads to
\[
\frac{1}{\eta} \left\| \int_0^1 y \cdot \partial_y (\partial_w P)(x, ty, w, z) dt \right\|_{\sup_{D(\eta,t)}} \leq C_{\eta} \frac{(4\eta)^2}{((1-4\eta)^r)^2} \sup_{D(\eta,t)} \| \partial_w P \|_{\sup_{D(\eta,t)}} \\
\leq C_{\eta} \frac{1}{r} \| \partial_w P \|_{\sup_{D(\eta,t)}} \leq C_{\eta} \| X_P \|_{r,D(\eta,t)}.
\]

Now expand \( \partial_w P(x, 0, w, z) \).
\[
\partial_w P(x, 0, w, z) - \partial_w P(x, 0, 0, 0) = \int_0^1 \frac{d}{dt} \partial_w P(x, 0, tw, tz) dt =: I.
\]

As \( \frac{d}{dt} \partial_w P(x, 0, tw, tz) = w \cdot \partial_w (\partial_w P)(x, 0, tw, tz) + z \cdot \partial_z (\partial_w P)(x, 0, tw, tz) \) we get
\[
I = w \cdot \partial_w (\partial_w P)(x, 0, 0, 0) + z \cdot \partial_z (\partial_w P)(x, 0, 0, 0) + I + II + III + IV.
\]
where

\[ II = \int_0^1 (1 - s)(w \cdot \partial_w) \cdot (\hat{w} \cdot \partial_w)\partial_w P(x, 0, s, w, z)\, ds \big|_{\hat{w} = w}, \]

\[ III = \int_0^1 (1 - s)(z \cdot \partial_z) \cdot (\hat{z} \cdot \partial_z)\partial_w P(x, 0, s, w, z)\, ds \big|_{\hat{z} = z}, \]

\[ IV = 2 \int_0^1 (1 - s)(z \cdot \partial_z)(w \cdot \partial_w)\partial_w P(x, 0, s, w, z)\, ds. \]

The error terms II, III, IV are not part of \( X_R \) and by Cauchy's estimate for second derivatives one gets

\[ \frac{1}{\eta r} \| II \|_{sup}^2 \leq \frac{1}{\eta r} (1 - 4\eta r) \| \partial_w P \|_{sup}^2 \leq C \eta \| X_P \|_{sup}^2 \]

For III and IV similar estimates are obtained. Altogether we then get

\[ \frac{1}{\eta r} \| \partial_w P - \partial_w R \|_{sup}^2 \leq C \eta \| X_P \|_{sup}^2. \]

In a similar way one shows that

\[ \frac{1}{\eta r} \| \partial_z P - \partial_z R \|_{lip}^2 \leq C \eta \| X_P \|_{lip}^2. \]

By the same arguments one also has

\[ \frac{1}{\eta r} \| \partial_z P - \partial_z R \|_{lip}^2 \leq C \eta \| X_P \|_{lip}^2. \]

Finally, we need to consider \( \partial_x P - \partial_x R \). First expand \( \partial_x P \) with respect to \( y \),

\[ \partial_x P(x, y, w, z) = \partial_x P(x, 0, w, z) + (y \cdot \partial_y)(\partial_x P)(x, 0, w, z) + V \]

where

\[ V := \int_0^1 (1 - t)(y \cdot \partial_y)(\hat{y} \cdot \partial_y)\partial_x P(x, s, y, w, z)\, ds \big|_{\hat{y} = y} \]

is not part of \( \partial_x R \). By Cauchy's estimate one gets

\[ \frac{1}{\eta r} \| V \|_{sup}^2 \leq \frac{C}{\eta r} (1 - 4\eta r) \| \partial_x P \|_{lip}^2 \leq C \eta^2 \frac{1}{r^2} \| \partial_x P \|_{lip}^2 \leq C \eta^2 \| X_P \|_{lip}^2. \]

As \( R \) is an affine function of \( y \) it follows that the term VI in the expansion

\[ (y \cdot \partial_y)(\partial_x P)(x, 0, w, z) = (y \cdot \partial_y)(\partial_x P)(x, 0, 0, 0) + VI \]
is not part of $\partial_x R$ where

$$V I = \int_0^1 (w \cdot \partial_w)(y \cdot \partial_y)(\partial_x P)(x, 0, tw, tz) \, dt + \int_0^1 (z \cdot \partial_z)(y \cdot \partial_y)(\partial_x P)(x, 0, tw, tz) \, dt.$$  

Arguing as above one has

$$\frac{1}{(\eta r)^2} \|V I\|_{D(s,4\eta r)}^{\sup} \leq \frac{C \eta}{(\eta r)^2} \frac{(\eta r)^3}{((1 - 4\eta r)^3} \|\partial_x P\|_{D(s,r)}^{\sup} \leq C \eta \|X_P\|_{r,D(s,r)}^{\sup}.$$  

The remaining term $\partial_t P(x, 0, w, z)$ has to be expanded in $w$ and $z$ up to order 2. The remainder term $V I I$ can then be written in terms of integrals and Cauchy’s estimate can be applied to show that

$$\frac{1}{(\eta r)^2} \|V I I\|_{D(s,4\eta r)}^{\sup} \leq C \eta \|X_P\|_{r,D(s,r)}^{\sup}.$$  

Altogether we thus have proved that

$$\|X_P - X_R\|_{\eta r, D(s,4\eta r)}^{\sup} \leq C \eta \|X_P\|_{r,D(s,r)}^{\sup}.$$  

In a similar way one shows that

$$\|X_P - X_R\|_{lip}\eta r, D(s,4\eta r)} \leq C \eta \|X_P\|_{r,D(s,r)}^{lip}$$  

and (6.22) is established.

Solution of linearized equation

Since the small divisors assumption (6.19) are supposed to hold, we can solve the linear system

$$\{F, H\} + \hat{H} = R, \quad \{F, S\} = 0$$

with the help of Lemma 6.1. By Corollary 6.1 and the estimates (6.21) we obtain

$$\|X_H\|_{r,D(s,r)}^{\sup} \leq C \|X_P\|_{r,D(s,r)}^{\sup}$$

(6.23)

and, for any $0 \leq \lambda \leq \frac{\alpha}{M}$,

$$\|X_F\|_{r,D(s-\sigma,r)}^\lambda \leq C \alpha^{-1} \sigma^{-1} \lambda \|X_P\|_{r,D(s,r)}^{\lambda}$$

(6.24)

where we recall that $\kappa = 2\tau + |A| + 3$. By the construction of $F$ and the estimates of $X_F$ of Lemma 6.1 it follows that $X_F$ is a real analytic map $X_F : D(s - \sigma, r) \to \mathcal{P}_C^N$ where $\mathcal{P}_C^N = (\mathcal{P}_C^N, \|\cdot, N\|$). At each point $\tau = (x, y, w, z) \in D(s - \sigma, r)$, the differential $dX_F$ defines a bounded linear operator on $\mathcal{P}_C^N$. Note that the $\|\cdot, N$-distance in $\mathcal{P}_C^N$ between $D(s - 2\sigma, \frac{r}{2})$ and the boundary of $D(s - \sigma, r)$ can be estimated from below by $\sigma \lambda \frac{1}{\lambda} \geq \frac{\sigma}{2}$. Hence by Cauchy’s estimate

$$\|dX_F\|_{r,D(s-2\sigma, \frac{r}{2})}^{\sup} \leq C \sigma^{-1} \|X_F\|_{r,D(s-\sigma, r)}^{\sup}$$

(6.25)
where for any $r \in D(s - 2\sigma, \frac{\xi}{2})$, $\|d_r X_F\|$ denotes the operator norm on $P^N_C$,

$$\|d_r X_F\| = \sup_{\|Y\|_{r,N} \leq 1} \|d_r X_F \cdot Y\|_{r,N}.$$ 

Similarly, one sees that

$$\|d X_F\|_{r,D(s-2\sigma, \frac{\xi}{2})}^{lip} \leq C\sigma^{-1}\|X_F\|_{r,D(s-\sigma, r)}^{lip}. \quad (6.26)$$

Canonical transformation

The preceding estimates together with (6.20) and (6.24) imply that for any $0 \leq \lambda \leq \frac{\eta}{M}$

$$\frac{1}{\sigma} \|X_F\|_{r,D(s, r, \xi)}^{lip}, \|d X_F\|_{r,D(s-2\sigma, \frac{\xi}{2})}^{lip} \leq Cc_0^{-1} \eta^2. \quad (6.27)$$

Note that the $\|\cdot\|_{r,N}$-distance of $D(s-3\sigma, \frac{\xi}{2})$ to the boundary of $D(s-2\sigma, \frac{\xi}{2})$ is at least $\sigma \wedge \frac{1}{27} \geq \frac{\eta}{M}$. Now choose $c_0$ in (6.20) sufficiently large to insure that for any $|t| \leq 1$ the flow $X^t_F$ exists on $D(s-3\sigma, \frac{\xi}{2})$ and maps $D(s-3\sigma, \frac{\xi}{2})$ into $D(s-2\sigma, \frac{\xi}{2})$. Similarly, the flow $X^t_F$ maps $D(s-4\sigma, \frac{\xi}{2})$ into $D(s-3\sigma, \frac{\xi}{2})$. By [14, Lemma A.4], together with the estimate (6.27) above we have

$$\|X^t_F - Id\|_{r,D(s-3\sigma, \frac{\xi}{2})}^* \leq C\|X^t_F\|_{r,D(s-\sigma, r)}^*. \quad (6.28)$$

Since the $\|\cdot\|_{r,N}$ distance of $D(s-4\sigma, \frac{\xi}{2})$ to the boundary of $D(s-3\sigma, \frac{\xi}{2})$ is at least $\sigma \wedge \frac{1}{27} \geq \sigma/32$, it then follows from Cauchy's estimate that

$$\|d X^t_F - Id\|_{r,D(s-4\sigma, \frac{\xi}{2})}^* \leq C\sigma^{-1}\|X^t_F\|_{r,D(s-\sigma, r)}^*. \quad (6.29)$$

In particular, we notice that for any $-1 \leq t \leq 1$, $X^t_F : D(s-3\sigma, \frac{\xi}{2}) \times \Pi \to D(s-2\sigma, \frac{\xi}{2})$ is regular and for any $\xi \in \Pi$, $X^t_F(\cdot, \xi)$ defines a canonical coordinate transformation.

New Hamiltonian

Taking the pull back of $H + P$ by the canonical transformation $\Phi = X^t_F|_{t=1}$ one obtains the Hamiltonian $H_+ + P_+$, defined on $D(s-3\sigma, \frac{\xi}{2})$, where $H_+ = H + \hat{H}$ and, by (4.9)-(4.10)

$$P_+ = (P - R) \circ X^1_F + \int_0^1 (1 - t)\hat{H} + tR, F) \circ X^t_F \, dt.$$ 

We have already verified at the end of Section 4 that $S \circ X^1_F = S$ and $\{P_+, S\} = 0$. We now want to estimate the $\|\cdot\|_{r,N}$-norm of the vector field $X_{P_+}$ in terms of the size of $X_P$. First note that $X_{P_+}$ is given by

$$X_{P_+} = (X^1_F)^*(X_P - X_R) + \int_0^1 (X^t_F)^*[X_{(1-t)\hat{H} + tR} , X_F] \, dt.$$ 

It is shown in [14, pp. 130–132], that for any $0 \leq t \leq 1$, $0 < \eta < \frac{\alpha}{16}$, $0 \leq \lambda \leq \frac{\eta}{M}$ and any vector field $Y : D(s-2\sigma, 4\eta r) \to P^N_C$, 

$$\|Y\|_{r,N} \leq C\eta^2.$$ 

Thus, since $\|X_{P_+}\|_{r,N} \leq \|X_P\|_{r,N}$ the vector field $X_{P_+}$ is regular and $\Phi = X^t_F|_{t=1}$ is a canonical coordinate transformation.
\[
\| (X_F^t)^* Y \|_{H^r, D(s-5\sigma, \eta r)} \leq C \| Y \|_{H^r, D(s-2\sigma, 4\eta r)} ^{\lambda}.
\] (6.30)

By (6.22), one has \( \| X_P - X_R \|_{H^r, D(s, 4\eta r)} ^{\lambda} \leq C \eta \| X_P \|_{r, D(s, r)} ^{\lambda} \) and hence in view of (6.30),
\[
\| (X_F^t)^* (X_P - X_R) \|_{H^r, D(s-5\sigma, \eta r)} ^{\lambda} \leq C \eta \| X_P \|_{r, D(s, r)} ^{\lambda}.
\] (6.31)

It remains to consider the commutator \([X_G, X_F]\) where \( G(t) = (1-t) \dot{H} + t R \). Note that \([X_G, X_F] = dX_F \cdot X_G - dX_G \cdot X_F\). Hence at each point \( r \in D(s - 2\sigma, \frac{r}{2}) \), \( \| d_t X_F \| \) denotes the operator norm of \( d_t X_F : \mathcal{P}_C^N \to \mathcal{P}_C^N \) with respect to the norm \( \| \cdot \|_{r, N} \). By (6.24)–(6.26),
\[
\frac{1}{\sigma} \| X_F \|_{r, D(s-\sigma, r)} ^{\lambda}, \quad \| dX_F \|_{r, D(s-2\sigma, \frac{r}{2})} ^{\lambda} \leq C \alpha^{-1} \sigma^{-\kappa} \| X_P \|_{r, D(s, r)} ^{\lambda}
\]
whereas by (6.21), \( \| X_G \|_{r, D(s, r)} ^{\lambda} \leq C \| X_P \|_{r, D(s, r)} ^{\lambda} \). By Cauchy’s estimate one then also has
\[
\| dX_G \|_{r, D(s-\sigma, \frac{r}{2})} ^{\lambda} \leq C \sigma^{-1} \| X_P \|_{r, D(s, r)} ^{\lambda}.
\]
Combining the above estimates yields
\[
\| [X_G, X_F] \|_{r, D(s-2\sigma, \frac{r}{2})} ^{\sup} \leq C \alpha^{-1} \sigma^{-\kappa} (\| X_P \|_{r, D(s, r)} ^{\sup}) ^{2}.
\] (6.32)

Furthermore, as
\[
\| [X_G, X_F] \|_{r, D(s-2\sigma, \frac{r}{2})} ^{lip} \leq \| dX_F \|_{r, D(s-2\sigma, \frac{r}{2})} ^{lip} \| X_G \|_{r, D(s-2\sigma, \frac{r}{2})} ^{\sup} \\
+ \| dX_F \|_{r, D(s-2\sigma, \frac{r}{2})} ^{\sup} \| X_G \|_{r, D(s-2\sigma, \frac{r}{2})} ^{lip} + \| dX_G \|_{r, D(s-2\sigma, \frac{r}{2})} ^{lip} \| X_F \|_{r, D(s-2\sigma, \frac{r}{2})} ^{\sup} \\
+ \| dX_G \|_{r, D(s-2\sigma, \frac{r}{2})} ^{\sup} \| X_F \|_{r, D(s-2\sigma, \frac{r}{2})} ^{lip}
\]
one also concludes that
\[
\| [X_G, X_F] \|_{r, D(s-2\sigma, \frac{r}{2})} ^{lip} \leq C \alpha^{-1} \sigma^{-\kappa} \| X_P \|_{r, D(s, r)} ^{lip} \cdot \| X_P \|_{r, D(s, r)} ^{\sup} + C \alpha^{-1} \sigma^{-\kappa} \cdot M \alpha^{-1} \cdot (\| X_P \|_{r, D(s, r)} ^{\sup}) ^{2}.
\] (6.33)

Using that for any vector \( Y \in \mathcal{P}_C^N \), \( \| Y \|_{H^r, N} ^{\alpha} \leq \| Y \|_{r, N} ^{\alpha} \) it then follows from (6.30), (6.32), and (6.33) and the fact that \( 4\eta r < \frac{r}{2} \)
\[
\| (X_F^t)^* [X_G, X_F] \|_{H^r, D(s-\lambda, \eta r)} ^{\lambda} \leq C \eta^{-2} \alpha^{-1} \sigma^{-\kappa} \| X_P \|_{r, D(s, r)} ^{\sup} \| X_P \|_{r, D(s, r)} ^{\lambda} \\
\leq C \eta^{-2} \alpha^{-1} \sigma^{-\kappa} (\| X_P \|_{r, D(s, r)} ^{\lambda}) ^{2}
\]
for any \( 0 \leq \lambda \leq \frac{\alpha}{M} \) and any \( 0 \leq t \leq 1 \). Combined with (6.31) it leads to the following estimate of the new error term \( X_{P+} \).
\[
\| X_{P+} \|_{H^r, D(s-\lambda, \eta r)} ^{\lambda} \leq C \eta^{-2} \alpha^{-1} \sigma^{-\kappa} (\| X_P \|_{r, D(s, r)} ^{\lambda}) ^{2} + C \eta \| X_P \|_{r, D(s, r)} ^{\lambda}.
\] (6.34)
New normal form

We already have seen that $H = H + \hat{H}$ where by (6.23), $\|X_H\|_{r,D(s,r)}^* \leq C\|X_P\|_{r,D(s,r)}^*$. Note that $\hat{H}$ is of the form

$$\hat{H}(y, w, z; \xi) = \hat{\omega}(\xi) \cdot y + \hat{\Omega}(\xi)w \cdot z$$

and hence

$$|\hat{\omega}|^* \leq C\|X_P\|_{r,D(s,r)}^*.$$  \hfill (6.35)

Taking into account that $\sup_{\|w\|_N} \|\hat{\Omega}(\xi)w\|_N = |\hat{\Omega}(\xi)|\ell_\infty$, it also follows that

$$|\hat{\Omega}|^* \ell_\infty \leq C\|X_P\|_{r,D(s,r)}^*.$$  \hfill (6.36)

In order to bound the small divisors for the new frequencies $\omega_+ = \omega + \hat{\omega}$ and $\Omega_+ = \Omega + \hat{\Omega}$ for $k \in \mathbb{Z}^A$ with $|k| \leq K$ with $K$ to be chosen later in the proof. Observe that for any $(k, e) \in \mathbb{Z}$ with $|k| \leq K$, using that $|e| \leq 2 |k| \cdot \hat{\omega} + e \cdot \hat{\Omega} \sup \leq |k| \cdot \hat{\omega} \sup + |e| \cdot \hat{\Omega} \sup \leq (K + 2)C\|X_P\|_{r,D(s,r)}^\sup \leq \hat{\alpha} A_k^{-1}$

where $\hat{\alpha}$ satisfies $\hat{\alpha} > C\|X_P\|_{r,D(s,r)}^\sup \cdot (K + 2) \max_{|k| \leq K} |A_k|$. It turns out that one can choose $\hat{\alpha}$ so that $\alpha_+ := \alpha - \hat{\alpha} > 0$ -- see Lemma 6.3. With the small divisors assumption (6.19) it then follows that for any $(k, e) \in \mathbb{Z}$ with $|k| \leq K$, $\omega_+$, $\Omega_+$ satisfy on $\Pi_v$

$$|k \cdot \omega_+ + e \cdot \Omega_+| \geq \alpha_+ A_k^{-1} \cdot 1 \vee |e|_\delta^2.$$  \hfill (6.37)

6.3. Iteration and proof of Theorem 4.1

To iterate the KAM step infinitely many times we now choose sequences for all the relevant parameters. Following [18], we choose a geometric sequence for $\sigma$, choose the $\eta$'s to minimize the error estimate (6.34) and change $\alpha$ and $M$ only slightly.

Let $c_1$ be twice the maximum of all those constants $C$ obtained during the KAM step which depend only on $A \subset \mathbb{Z}$ and $\tau$. For any $\nu \in \mathbb{Z}_{\geq 0}$ set

$$\alpha_\nu = \frac{\alpha_0}{2} \left(1 + 2^{-\nu}\right), \quad M_\nu = M_0(2 - 2^{-\nu}), \quad \lambda_\nu = \frac{\alpha_\nu}{M_\nu}$$

with $0 < \alpha_0 < 1$ and $M_0 \geq 1$ satisfying $M_0 \geq |\omega|_{lip} + |\Omega|_{lip, \infty \cdot \delta}$. Then $(\alpha_\nu)_{\nu \geq 0}$ is decreasing and $(M_\nu)_{\nu \geq 0}$ increasing. Hence $(\lambda_\nu)_{\nu \geq 0}$ is decreasing as well and

$$\lambda_0/4 \leq \lambda_\nu \leq \lambda_0.$$  \hfill (6.38)

Furthermore, with $\kappa = 2\tau + |A| + 3$,

$$\sigma_{\nu+1} = \frac{\sigma_\nu}{2}, \quad \epsilon_{\nu+1} = \frac{c_1\epsilon_\nu^{4/3}}{(\alpha_\nu \sigma_\nu^2)^{1/3}}, \quad \eta_\nu^3 = \frac{\epsilon_\nu}{\alpha_\nu \sigma_\nu^2}$$
perturbation is expressed by the inequality

\[ \gamma \leq (c_0 + 2^{k+3}c_1)^{-3} \]  \hspace{1cm} (6.39)

where \( c_0 \) appears in (6.20). Finally let \( K_v = K_0 2^v \) and \( K_0^{+1} = \frac{1}{c_1 c_0} \). The smallness condition of the perturbation is expressed by the inequality

\[ \epsilon = \epsilon_0 \leq \gamma_0 \alpha_0 \sigma_0^k. \]  \hspace{1cm} (6.40)

Then one has the following bounds for the sequence \((\epsilon_v)_v \geq 0\).

Lemma 6.2. For any \( v \geq 0 \),

(i) \( \epsilon_v \leq \gamma_0 \alpha_0 \sigma_v^k 2^{-v} \);

(ii) \( \epsilon_{v+1} \leq 2^{-k-3} \epsilon_v \) and \( \sum_0^\infty \epsilon_v \leq 2 \epsilon_0 \);

(iii) \( \alpha_v^{-1} \sigma_v^{1-k} \epsilon_v \leq 2^{-1} \sigma_0^{-1} \sigma_0^k \epsilon_0 2^{-v} \).

Proof. (i) The claimed estimate is proved by induction. For \( v = 0 \), the estimate holds by assumption (6.40). To prove the induction step, note that by definition, \( \epsilon_{v+1} = c_1 \epsilon_v (\alpha_v^{-1} \sigma_v^{-k} \epsilon_v)^{1/3} \). By the induction hypotheses, \( (\alpha_v^{-1} \sigma_v^{-k} \epsilon_v)^{1/3} \leq (\gamma_0 2^{-v})^{1/3} \) and by the smallness condition of \( \gamma_0 \), one has

\[ c_1 (\alpha_v^{-1} \sigma_v^{-k} \epsilon_v)^{1/3} \leq c_1 (\gamma_0 2^{-v})^{1/3} \leq c_1 \frac{1}{c_1 2^{k+3}} \leq 2^{-k-3} \]  \hspace{1cm} (6.41)

which together with the induction hypothesis implies

\[ \epsilon_{v+1} \leq 2^{-k-3} \cdot \gamma_0 \alpha_v \sigma_v^k 2^{-v} = \gamma_0 \cdot 2^{-2} \alpha_v \cdot (2^{-1} \sigma_v)^k \cdot 2^{-v-1} \leq \gamma_0 \alpha_{v+1} \sigma_{v+1}^k 2^{-v-1}. \]

(ii) By the definition of \( \epsilon_{v+1} \), \( \epsilon_{v+1}/\epsilon_v = c_1 (\alpha_v^{-1} \sigma_v^{-k} \epsilon_v)^{1/3} \). As by (6.41) \( \epsilon_{v+1}/\epsilon_v \leq 2^{-k-3} \) item (ii) follows.

(iii) The claimed estimate clearly holds in the case \( v = 0 \). To prove the induction step first note that by (ii), \( \epsilon_{v+1}/\epsilon_v \leq 2^{-k-3} \).

Hence

\[ \frac{\epsilon_{v+1} \alpha_{v+1}^{-1} \sigma_{v+1}^{1-k}}{\epsilon_v \alpha_v^{-1} \sigma_v^{1-k}} = \frac{\epsilon_{v+1}}{\epsilon_v} \cdot \frac{\alpha_v}{\alpha_{v+1}} \cdot \left( \frac{\sigma_{v+1}}{\sigma_v} \right)^{1-k} \leq 2^{-3} \]  \hspace{1cm} (6.42)

and the claimed estimate for \( \epsilon_{v+1} \) then follows from the induction hypothesis. \( \Box \)

In [18], a version of the following Iterative Lemma is proved. It can be proved in the same way as in [18] and hence we omit its proof.

Lemma 6.3. Suppose that \( H_v + P_v \) is regular on \( D_v \times \Pi_v \) in the sense defined at the beginning of Section 6.1 where \( H_v = \omega^v(\xi) \cdot y + \Omega^v(\xi) w \cdot z \) is a regular Hamiltonian on \( D_v \times \Pi_v \) in normal form satisfying \( |\omega_v|_{H_v} + |\Omega^v|_{H_v} \leq M_v \) and
\[ |k \cdot \omega^v(\xi) + e \cdot \Omega^v(\xi)| \geq \alpha_v A_k^{-1} \cdot 1 \vee |e|^{1/2}, \quad \forall \xi \in \Pi_v, \forall (k, e) \in \mathcal{Z}, \quad (6.43) \]

and where \( P_v \) satisfies \( \{ P_v, S \} = 0 \) and

\[ \|X_{P_v}\|_{r_v, D_v}^{\lambda_v} \leq \epsilon_v. \quad (6.44) \]

Then there exist a regular map \( \Phi_{v+1}: D_{v+1} \times \Pi_v \to D_v \) with \( \Phi_{v+1}(\cdot, \xi) \), being a real analytic symplectic coordinate transformation on \( D_{v+1} \) for any \( \xi \in \Pi_v \), and a closed subset \( \Pi_{v+1} \) of \( \Pi_v \), \( \Pi_{v+1} = \Pi_v \\setminus \bigcup_{|k| > K_\nu, (k, e) \in \mathcal{Z}} \mathcal{R}^v_{ke}(\alpha_{v+1}) \), where

\[ \mathcal{R}^v_{ke}(\alpha_{v+1}) = \{ \xi \in \Pi_v: |k \cdot \omega^{v+1}(\xi) + e \cdot \Omega^{v+1}(\xi)| < \alpha_{v+1} A_k^{-1} \cdot 1 \vee |e|^{1/2} \} \]

such that \( (H_v + P_v) \circ \Phi_{v+1} = H_{v+1} + P_{v+1} \) satisfies the same assumptions as \( H_v + P_v \), but with \( v + 1 \) in place of \( v \).

**Remark 6.1.** We point out that the dependence of the set \( \mathcal{R}^v_{ke}(\alpha_{v+1}) \) on the perturbation \( P \) is not indicated in the notation. We will see in Section 6.4 that the measure of this set can be bounded in terms of \( \alpha_{v+1} \) independently of the perturbation.

By \( (6.28)-(6.29) \) together with \( (6.24) \) and the assumption \( (6.44) \) we obtain the following estimates.

\[ \frac{1}{\sigma_v} \|\Phi_{v+1} - id\|_{r_v, D_{v+1}}^{\lambda_v} \leq c_1 \alpha_v^{-1} \sigma_v^{-\kappa} \epsilon_v, \quad (6.45) \]

whereas by \( (6.35)-(6.36) \) together with assumption \( (6.44) \) one gets

\[ |\omega^{v+1} - \omega^v|_{\Pi_v}^{\lambda_v} \leq c_1 \epsilon_v. \quad (6.46) \]

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Given the assumptions of Theorem 4.1, we want to apply Lemma 6.3 (Iterative Lemma) with \( v = 0 \). Set

\[ s_0 := s; \quad r_0 := r; \quad \alpha_0 := \alpha; \quad M_0 := M; \quad \epsilon_0 := \epsilon \]

and

\[ D_0 = D(s, r); \quad H_0 := H; \quad P_0 := P \]

with \( s, \alpha, M, \epsilon, H, \) and \( P \) given as in Theorem 4.1. As in the Iterative Lemma, choose \( \sigma_0 = s_0/40, \lambda_0 = \alpha_0/M_0, \) and \( \gamma_0 \) and assume that \( \epsilon_0 := \epsilon = \|X_{P_0}\|_{r_0, D_0}^{\lambda_0} \leq \gamma_0 \alpha_0 \sigma_0^{\kappa} \). Setting \( \gamma := \gamma_0 \sigma_0^{\kappa} \) one then gets

\[ \epsilon_0 = \|X_{P_0}\|_{r_0, D(s, r)}^{\lambda_0} \leq \gamma_0 \alpha_0 \sigma_0^{\kappa} = \alpha \gamma. \]

In particular, inequality \( (4.7) \) of Theorem 4.1 with \( \gamma \) chosen as above is satisfied. By Assumption \( (B2), \{P_0, S\} = \{P, S\} = 0 \). Furthermore for \( v = 0 \), the small divisors condition \( (6.43) \) holds on

\[ \Pi_0 := \Pi \setminus \bigcup_{(k, e) \in \mathcal{Z}} \mathcal{R}^0_{ke}(\alpha_0), \]

where

\[ R_{ke}^0(\alpha_0) = \{ \xi \in \Pi : |\kappa \cdot \omega(\xi) + e \cdot \Omega(\xi)| < \alpha_0 \psi^{-1} \cdot 1 \lor |e|^{1/2} \} \].

Thus the Iterative Lemma applies and we obtain a decreasing sequence of domains \( D_v \times \Pi_v \) and regular maps \( \Phi_v : D_v \times \Pi_v \to D_{v-1} \), \( v \geq 1 \), with the properties listed in Lemma 6.3. Set \( \Phi^v := \Phi_1 \circ \cdots \circ \Phi_v : D_v \times \Pi_v \to D_0 \). In particular, \((H_0 + P_0) \circ \Phi^v = H_v + P_v\) and the estimates (6.45)–(6.46) hold. To prove the convergence of the sequence \( \Phi^v \), note that the sequence \( (r_v)_{v \geq 0} \) is decreasing. Thus for any \( Y \in \mathcal{P}^N_C \) one has \( \|Y\|_{r_v,N} \leq \|Y\|_{r_{v+1},N} \). Hence for a linear operator \( T : (\mathcal{P}^N_C, \| \cdot \|_{r_v,N}) \to (\mathcal{P}^N_C, \| \cdot \|_{r_{v+1},N}) \)

\[ \|T\|_{r_{v+1},r_v} = \sup_{\|f\|_{r_{v+1},N} \leq 1} \|Tf\|_{r_v,N} \leq \sup_{\|f\|_{r_v,N} \leq 1} \|Tf\|_{r_v,N} = \|T\|_{r_v,N}. \]

By the mean value theorem one has

\[ \| \Phi^{v+1} - \Phi^v \|_{\sup \, D_{v+1}} \leq \| d\Phi^v \|_{r_v,0,0} \cdot \| \Phi^{v+1} \|_{r_v,0,0} - \| \Phi^v \|_{r_v,0,0} \]

where for any \( x \in D_{v+1} \), \( d_x \Phi^v \) is viewed as a linear map \( (\mathcal{P}^N_C, \| \cdot \|_{r_v,N}) \to (\mathcal{P}^N_C, \| \cdot \|_{r_0,N}) \). Viewed in the chain rule \( d\Phi^v = d\Phi_1 \circ \cdots \circ d\Phi_v \) and thus by the considerations above,

\[ \| d\Phi^v \|_{r_v,0,0} \leq \prod_{\mu=1}^v \| d\Phi_{\mu} \|_{r_{\mu-1},0,0} \leq \prod_{\mu=0}^{v-1} (1 + 2^{-\mu - 2}) \leq 2 \]

where we used that by (6.45), for any \( \mu \geq 1 \),

\[ \| d\Phi_{\mu} \|_{r_{\mu-1},0,0} \leq 1 + \| d\Phi_{\mu} - \| \|_{r_{\mu-1},0,0} \leq 1 + c_1 \alpha_{\mu-1}^{-1} \sigma_{\mu-1}^{-\kappa} \epsilon_{\mu-1} \]

and by Lemma 6.2,

\[ c_1 \alpha_{\mu-1}^{-1} \sigma_{\mu-1}^{-\kappa} \epsilon_{\mu-1} \leq c_1 \gamma_0 2^{-\mu - 1} \leq c_1 \frac{1}{(2x + 3 \gamma_1)^3} 2^{-\mu + 1} \leq 2^{-\mu - 2}. \]

Similarly, one argues for the Lipschitz semi-norm,

\[ \| \Phi^{v+1} - \Phi^v \|_{lip \, D_{v+1}} \leq \| d\Phi^v \|_{lip \, D_{v+1}} \| \Phi^{v+1} - \| \|_{lip \, D_{v+1}} + \| d\Phi^v \|_{lip \, D_{v+1}} \| \Phi^{v+1} - \| \|_{lip \, D_{v+1}} \]

and shows as for \( \| d\Phi^v \|_{lip \, D_{v+1}} \) that \( \| d\Phi^v \|_{lip \, D_{v+1}} \) is uniformly bounded. As already pointed out at the beginning of this subsection, one has \( \alpha_{M_0} \leq \kappa_v \leq \kappa_0 \) and hence

\[ \| \Phi^{v+1} - \Phi^v \|_{lip \, D_{v+1}} \leq C \| \Phi^{v+1} - \| \|_{lip \, D_{v+1}} \].

Combined with (6.45) this leads to

\[ \| \Phi^{v+1} - \Phi^v \|_{lip \, D_{v+1}} \leq C c_1 \epsilon_v \alpha_v^{-1} \sigma_v^{1-\kappa}. \] (6.47)

Therefore, \( (\Phi^v)_{v \geq 1} \) converges uniformly on \( \bigcap_{v \geq 0} (D_v \times \Pi_v) = D(s/2,0) \times \Pi_* \) to a Lipschitz continuous family of real analytic torus embeddings \( \Psi : \mathbb{T}^A \times \Pi_* \to \mathcal{M}^N \). Here \( \Pi_* = \bigcap_{v \geq 0} \Pi_v \) and
\[D_a = D(s/2, 0) = D(s/2) \times \{0\} \times \{0\} \subseteq \mathcal{M}^N.\]

Recall from the statement of Theorem 4.1 that \(\Phi^0 \equiv \Psi_0\) denotes the trivial torus embedding \(\mathbb{T}^A \times \Pi_a \rightarrow \mathbb{T}_0\). Then by (6.47)

\[
\|\Psi - \Psi_0\|_{r_0, D_a}^{\lambda_0} \leq \sum_0^\infty \|\Phi^{v+1} - \Phi^v\|_{r_0, D_a}^{\lambda_0} \leq C_1 \sum_0^\infty \alpha_1^{-1} \sigma_1^{1-k} \epsilon_v.
\]

By Lemma 6.2, \(\sum_0^\infty \alpha_1^{-1} \sigma_1^{1-k} \epsilon_v \leq 2 \alpha_1^{-1} (\sum_0^\infty \alpha_1^{1-k}) \epsilon_0\) and hence \(\|\Psi - \Psi_0\|_{r_0, D_a}^{\lambda_0} \leq c \epsilon / \alpha\) as claimed in Theorem 4.1. Taking into account (6.40) and the estimate (6.46) one sees that the frequencies \(\omega^v(\xi) \in \mathbb{R}^A\) and \(\Omega^v(\xi) \in \ell^{\infty,-\delta}\) converge uniformly on \(\Pi_a\) to Lipschitz continuous functions \(f : \Pi_a \rightarrow \mathbb{R}^A\) respectively \(\Omega^* : \Pi_a \rightarrow \ell^{\infty,-\delta}\). Furthermore, letting \(\omega^0\) denote the frequency vector \(\omega\) of the unperturbed Hamiltonian \(H\), it follows that \(f(\xi) - \omega(\xi) = \sum_0^\infty (\omega^{v+1}(\xi) - \omega^v(\xi))\) can be estimated as

\[
|f - \omega|_{\Pi_a}^{\lambda_0} \leq C \sum_0^\infty |\omega^{v+1} - \omega^v|_{\Pi_a}^{\lambda_0} \leq C \sum_0^\infty \epsilon_v.
\]

By Lemma 6.2(ii), \(\sum_0^\infty \epsilon_v \leq 2 \epsilon_0\). As \(\epsilon_0 = \epsilon\) we thus have shown that \(|f - \omega|_{\Pi_a}^{\lambda_0} \leq C \epsilon\) as claimed in Theorem 4.1. On the embedded tori, the flow of the perturbed Hamiltonian \(H + P\) can be computed as follows. First note that

\[
\|X_{H+P} \circ \Phi^v - d\Phi^v \cdot X_{H^v}\|_{r_0, D_v, \times \Pi_v}^{\sup} \leq \|d\Phi^v\|_{r_v, r_0, D_v, \times \Pi_v} \|\Phi^v\|_{r_0, r_v, D_v, \times \Pi_v} \leq C \||X_{H^v}\|_{r_0, r_v, D_v, \times \Pi_v}^{\sup}.
\]

In the limit, one thus obtains that \(X_{H+P} \circ \Psi = d\Psi \cdot X_{H^v}\) on \(D(s/2, 0)\) where

\[H_a(y, w, z; \xi) := f(\xi) \cdot y + \Omega^*(\xi)w \cdot z.\]

It thus follows that for any \(x \in \mathbb{T}^A\) and \(\xi \in \Pi_a\)

\[X_{H+P}^t(\Psi(x; \xi)) = \Psi(x + tf(\xi); \xi)\]

as claimed in Theorem 4.1. It remains to show the claims of item (i) of Theorem 4.1, concerning the set \(\Pi \setminus \Pi_a\). This will be done in the subsequent Section 6.4. \(\square\)

6.4. Set of excluded parameters

The aim of this subsection is to prove item (i) of Theorem 4.1. While we again follow the line of arguments used in [13] and [18], there are notable differences due to the near resonances of the frequencies of the unperturbed Hamiltonian which we will point out in the course of the proof.

The KAM iteration leads to a decreasing sequence \((\Pi_v)_{v \geq 0}\) of closed subsets of the parameter space \(\Pi\). Recall that \(\Pi \setminus \Pi_a = \Pi \setminus (\bigcap_{v \geq 0} \Pi_v)\) where

\[\Pi_0 = \bigcup_{k \in \mathbb{Z}^A} \mathcal{R}_{kk}^0(\alpha_0) \quad \text{and} \quad \Pi_v = \bigcup_{|k| > K_v} \mathcal{R}_{kk}^v(\alpha_v) \quad \text{for} \quad v \geq 1.
\]

Recall that \(\mathcal{Z} \subseteq \mathbb{Z}^A \times \mathbb{Z}^B\) is given by

\[\mathcal{Z} = \{(k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B \setminus \{(0, 0)\} : |e| \leq 2; k \cdot v_A + e \cdot v_B = 0\}.\]
$K_{v}$ is given by $K_{v} = (c_{1}g_{0})^{-1}2^{v}$, and for $(k,e) \in \mathbb{Z}^{A} \times \mathbb{Z}^{B}$,

$$R_{ke}^{v}(\alpha_{v}) = \{|k \cdot \omega^{v}(\xi) + e \cdot \Omega^{v}(\xi)| < \alpha_{v}2^{v-1} \cdot 1 \vee |e|^{\frac{1}{2}}\}$$

with $\Pi_{-1} = \Pi$. Here $\omega^{v} = (\omega_{j}^{v})_{j \in A}$ and $\Omega^{v} = (\Omega_{j}^{v})_{j \in B}$ are the frequencies obtained in the KAM iteration with $\omega^{0} = \omega$, $\Omega^{0} = \Omega$ denoting the ones of the unperturbed Hamiltonian $H$. \textcolor{red}{We will prove that by choosing $\gamma_{0}$ sufficiently small – and hence $K_{0}$ sufficiently large – measures intersected by $\Pi_{0}$ can be estimated as claimed. Note that the set $\Pi_{0}$ is defined in terms of the frequencies of $H$ whereas for $v \geq 1$ the set $\Pi_{v}$ depends on the perturbation $P.$ To estimate measures intersected by $\Pi_{v}$ we need to make some preparations. It is convenient to extend the frequencies $\omega^{v}$, $\Omega^{v}$, defined on $\Pi_{v-1}$, to all of $\Pi.$ Indeed, each component of $\omega^{v+1} - \omega^{v} : \Pi_{v} \rightarrow \mathbb{R}^{A}$ and of $\Omega^{v+1} - \Omega^{v} : \Pi_{v} \rightarrow \ell^{\infty}$ has a Lipschitz continuous extension from $\Pi_{v}$ to $\Pi$ which preserves its minimum, maximum, and Lipschitz semi-norm – see e.g. [13, Lemma M.5]. Since we use the sup norm for $\omega^{v+1} - \omega^{v}$ and $\Omega^{v+1} - \Omega^{v}$ we obtain in this way extensions $(\omega^{v+1} - \omega^{v})^{v}$ of $\omega^{v+1} - \omega^{v}$ and $(\Omega^{v+1} - \Omega^{v})^{v}$ of $\Omega^{v+1} - \Omega^{v}$ to all of $\Pi$ satisfying

$$\left|\left(\omega^{v+1} - \omega^{v}\right)^{v}\right|_{\Pi}^{\lambda_{v}} = \left|\omega^{v+1} - \omega^{v}\right|_{\Pi_{v}}^{\lambda_{v}}, \quad \left|\left(\Omega^{v+1} - \Omega^{v}\right)^{v}\right|_{\Pi,\ell^{\infty}}^{\lambda_{v}} = \left|\Omega^{v+1} - \Omega^{v}\right|_{\Pi_{v},\ell^{\infty}}^{\lambda_{v}}. \quad (6.48)$$

Now define $\tilde{\omega}^{v+1}$, $\tilde{\Omega}^{v+1}$ by telescoping sums

$$\tilde{\omega}^{v+1} = \omega + \sum_{\mu=0}^{v} (\omega^{\mu+1} - \omega^{\mu}) \quad \text{and} \quad \tilde{\Omega}^{v+1} = \Omega + \sum_{\mu=0}^{v} (\Omega^{\mu+1} - \Omega^{\mu}).$$

Then $\tilde{\omega}^{v} : \Pi \rightarrow \mathbb{R}^{A}$ and $\tilde{\Omega}^{v} : \Pi \rightarrow \ell^{\infty}$ are Lipschitz continuous extensions of $\omega^{v}$ respectively $\Omega^{v}$. Moreover, by Lemma 6.2(ii), $\sum_{0}^{\infty} \epsilon_{\mu} \leq 2c_{0}$ and hence it follows from (6.46) and (6.48) that for any $v \geq 0$,

$$\left|\tilde{\omega}^{v+1} - \omega\right|_{\Pi}^{\lambda_{v}} \leq c_{1} \sum_{0}^{v} \epsilon_{\mu} \leq 2c_{1} \epsilon_{0}. \quad (6.49)$$

As by (6.38), $\lambda_{0}/4 \leq \lambda_{v} \leq \lambda_{0}$, one then has

$$\left|\tilde{\omega}^{v} - \omega\right|_{\Pi}^{\lambda_{0}/4} \leq 2c_{1} \epsilon_{0}. \quad (6.50)$$

In particular,

$$\left|\tilde{\omega}^{v} - \omega\right|_{\Pi}^{lip} \leq 8c_{1} \epsilon_{0} \leq \frac{8c_{1} \epsilon_{0}}{\lambda_{0}}. \quad (6.51)$$

Recall that $\lambda_{0} = \alpha_{0}/M_{0}$, $\epsilon_{0} \leq \gamma_{0}\alpha_{0}\sigma_{0}^{K}$, $\sigma_{0} \leq 1/4$, and $\gamma_{0} \leq (2^{k+3}c_{1})^{-3}$. Thus

$$2c_{1} \epsilon_{0} \leq \alpha_{0}/2 \quad \text{and} \quad \frac{8c_{1} \epsilon_{0}}{\lambda_{0}} \leq 8c_{1} \sigma_{0}^{K} M_{0} \gamma_{0}. \quad (6.52)$$

By Assumption (A1), there exists a constant $1 \leq L < \infty$ satisfying

$$L \geq \left|\omega^{-1}\right|_{lip(\Pi)}. \quad (6.49)$$

Now require that $\gamma_{0}$ is chosen so small that in addition to (6.39), one has

$$8c_{1} \sigma_{0}^{K} M_{0} \gamma_{0} \leq 1/2L. \quad (6.50)$$
Then, for any $v \geq 0$,
\[ |\tilde{\omega}^v - \omega|^\sup_{\Pi}, |\tilde{\Omega}^v - \Omega|^\sup_{\Pi, l^\infty} \leq \alpha/2 \quad \text{and} \quad |\tilde{\omega}^v - \omega|^\text{lip}_{\Pi}, |\tilde{\Omega}^v - \Omega|^\text{lip}_{\Pi, l^\infty} \leq 1/2L \]
and for any $(k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B$, $R^{\nu}_{ke}(\alpha_v)$ is contained in
\[ \tilde{R}_{ke}(\alpha_v) := \{ \xi \in \Pi : |k \cdot \tilde{\omega}^v(\xi) + e \cdot \tilde{\Omega}^v(\xi)| < \alpha_v A_k^{-1} \cdot 1 \vee |e|_2^\frac{1}{2} \} . \]
In addition, we assume that $M = M_0 \geq 1$ bounds the frequencies,
\[ |\omega|^\sup_{\Pi} + |\Omega - \Omega(\xi)| \leq M \quad \text{and} \quad |\omega|^\text{lip}_{\Pi} + |\Omega|^\text{lip}_{\Pi, l^\infty} \leq M. \tag{6.51} \]
It turns out that we need not to distinguish between the different values of $v$ in $\tilde{\omega}^v$ and $\tilde{\Omega}^v$. In the sequel we only use the fact that $\tilde{\omega}^v$ and $\tilde{\Omega}^v$ are Lipschitz maps $\omega'$ and $\Omega'$, defined on $\Pi$, which satisfy the following inequalities
\[ |\omega' - \omega|^\sup_{\Pi} + |\Omega' - \Omega|^\sup_{\Pi, l^\infty} \leq \alpha/2 \quad \text{and} \quad |\omega' - \omega|^\text{lip}_{\Pi} + |\Omega' - \Omega|^\text{lip}_{\Pi, l^\infty} \leq 1/2L. \tag{6.52} \]
Henceforth we consider functions $\omega'$, $\Omega'$ which satisfy these estimates – they may even depend on $k$ and $e$ – and for any $(k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B$ define
\[ R'_{ke}(\alpha) = \{ \xi \in \Pi : |k \cdot \omega'(\xi) + e \cdot \Omega'(\xi)| < \alpha A_k^{-1} \cdot 1 \vee |e|_2^\frac{1}{2} \} . \]
First we derive the following estimate for meas($R'_{ke}(\alpha)$).

**Lemma 6.4.** For any $(k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B \setminus \{(0, 0)\}$ with $|k| \geq 6LM|e|_k$
\[ \text{meas}(R'_{ke}(\alpha)) \leq 12L(2M^\frac{1}{2})^{|A|} \alpha |k|^{-\frac{1}{2}} A_k^{-1} \]
where $\rho = \text{diam}(\Pi)$ denotes the diameter of $\Pi$.

**Proof.** Taking into account Assumption (A1) we introduce the unperturbed frequencies $\zeta = \omega(\xi)$ as new parameters with domain $\bar{\Pi} = \omega(\Pi)$ and consider the resonance zones $\tilde{R}_{ke} = \omega(\mathcal{R}'_{ke})$ in $\bar{\Pi}$. Writing $\hat{\omega}$ and $\hat{\Omega}$ for the pull back of $\omega'$ and $\Omega'$ by $\omega^{-1}$, we then have by (6.49), (6.52)
\[ |\hat{\omega} - \text{id}|_{\bar{\Pi}}^{\text{lip}} \leq |\omega - \omega|_{\Pi}^{\text{lip}} \cdot |\omega^{-1}|_{\Pi}^{\text{lip}} \leq 1/2. \]
In view of (6.49), (6.51), (6.52) and using that $L \geq 1$, $M \geq 1$, the Lipschitz semi-norm of $\hat{\omega}$ can be bounded as follows
\[ |\hat{\Omega}|_{\bar{\Pi}, l^\infty} \leq \left( |\Omega' - \Omega|^\text{lip}_{\Pi, l^\infty} + |\Omega'|^{\text{lip}}_{\bar{\Pi}, l^\infty} \right) |\omega^{-1}|_{\Pi}^{\text{lip}} \leq \left( \frac{1}{2L} + M \right) L \leq 2LM. \tag{6.53} \]
To estimate $\text{meas}(\tilde{R}_{ke}(\alpha))$, let $g(\zeta) := k \cdot \hat{\omega}(\zeta) + e \cdot \hat{\Omega}(\zeta)$. Choose a vector $v \in \{-1, 1\}^A$ such that $k \cdot v = |k|$ and write any vector in $\mathbb{R}^A$ as a linear combination of $v$ and an element $w$ in the orthogonal complement $v^\perp$ of $v$. Introduce the following affine function of the real variable $r$, $\zeta = \zeta(r) := rv + w$. For any $t > s$ with $\xi(t), \xi(s)$ in $\tilde{\Pi}$ one has
\[ k \cdot \hat{\omega}(\zeta)|_{t}^{s} = k \cdot \xi_{s}^{t} + k \cdot (\hat{\omega}(\zeta) - \xi_{s}^{t}) \geq |k|(t - s) - \frac{1}{2} |k|(t - s) = \frac{1}{2} |k|(t - s). \tag{6.54} \]
Moreover by (6.53) and the assumption $6LM|e|_b \ll |k|$, 

$$|e \cdot \hat{\omega}(\zeta)|^2_b = |e \cdot (\hat{\omega}(\zeta(t)) - \hat{\omega}(\zeta(s)))| \leq |e|_b|\hat{\omega}|_{\text{lip}} \rho \ll (t - s) \leq 2LM|e|_b(t - s) \leq \frac{1}{3}|k|(t - s). \quad (6.55)$$

Altogether we have shown that uniformly for $w \in v^\perp$ with $rv + w|_{r=t,s} \in \tilde{\Pi}$,

$$g(rv + w)|^2_b \geq \frac{1}{6}|k|(t - s).$$

It follows that for each point $w \in v^\perp$ so that $rv + w \in \tilde{\Pi}$ for some $r \in \mathbb{R}$, the set

$$\{ r \in \mathbb{R}: rv + w \in \tilde{\Pi}; \, |g(rv + w)| < \eta \}$$

is contained in an interval $I_w$ of length $\text{meas}(I_w) \leq 12\eta |k|^{-1}$. With $\eta = \alpha A^{-1}_k \cdot 1 \vee |e|_b^{\frac{1}{2}}$ and Fubini's theorem one then concludes that

$$\text{meas}(\hat{\mathcal{R}}_{ke}(\alpha)) \leq \frac{12\alpha A^{-1}_k}{|k|} \cdot 1 \vee |e|_b^{\frac{1}{2}} \cdot (\text{diam}(\tilde{\Pi}))^{\frac{1}{2}}. \quad (\alpha \in \mathbb{R})$$

As $6LM|e|_b \ll |k|$ and $LM \gg 1$ one then gets for any $e \in \mathbb{Z}^B$

$$\text{meas}(\hat{\mathcal{R}}_{ke}(\alpha)) \leq \frac{12\alpha |k|^{-\frac{1}{2}} A^{-1}_k}{|k|} \cdot (\text{diam}(\tilde{\Pi}))^{\frac{1}{2}}. \quad (\alpha \in \mathbb{R})$$

Going back to the original parameter domain $\Pi$ by the inverse $\omega^{-1}$ of the frequency map and noting that $\text{diam}(\tilde{\Pi}) \ll |\omega|_{\tilde{\Pi}}^{\text{lip}} \rho$ with $\rho$ denoting the diameter $\text{diam}(\tilde{\Pi})$ of $\tilde{\Pi}$ it then follows that

$$\text{meas}(\hat{\mathcal{R}}_{ke}(\alpha)) \leq (|\omega^{-1}|_{\tilde{\Pi}}^{\text{lip}})^{|A|} \text{meas}(\hat{\mathcal{R}}_{ke}(\alpha)) \leq 12L^{|A|}(M \rho)^{|A|^{-1}} \alpha |k|^{-\frac{1}{2}} A^{-1}_k$$

as claimed. \hfill \Box

It is convenient to introduce $\Lambda := \{ e \in \mathbb{Z}^B: 1 \leq |e| \leq 2 \}$ and

$$\Lambda_r := \{ e \in \Lambda: e = (\delta_{aj})_{j \in \mathbb{Z}} - (\delta_{-aj})_{j \in \mathbb{Z}}, \, a \in \mathbb{Z} \setminus \{ 0 \} \}.$$ 

By a slight abuse of terminology we refer to $\Lambda_r$ as the subset of resonant sites of $\Lambda$. It turns out that the estimates involving resonant sites have to be dealt with separately. Recall that the unperturbed frequencies satisfy $|\Omega - \Omega|_{\tilde{\Pi}, \text{lip}, \infty, \delta}^{\text{sup}} \leq M$ where for $j \in B$,

$$\overline{\Omega}_j = |j|^d + a_1 |j|^{d_1} + \cdots + a_D |j|^{d_D}$$

with $d \equiv d_0 > d_1 > \cdots > d_D \geq 0$, $a_1, \ldots, a_D \in \mathbb{R}$, and $d > 1$, $0 \leq \delta < 1 \land (d - 1)$. 

Lemma 6.5. There exists $E \geq 1$ depending only on $M$ and the approximation $\Omega$ of the unperturbed frequencies $\Omega$ so that for any $e \in \Lambda \setminus \Lambda_r$ with $|e|_\ast := \max_{j \in B}|j|: e_j \neq 0| \geq E$

\[
|e \cdot \Omega'(\xi)| \geq \frac{1}{8}|e|_{d-1} \quad \forall \xi \in \Pi.
\]

Proof. We only prove the claimed estimate for $e \in \Lambda \setminus \Lambda_r$ with $e \cdot \Omega'(\xi)$ of the form $\Omega'_i - \Omega'_j$ for some $i, j \in B$ with $j \neq -i$ which is the most subtle case. Write $\Omega'_i = \Omega + (\Omega'_i - \Omega_i)$ and $\Omega_i = \Omega_i + (\Omega_i - \Omega_i)$ and use that $|\Omega' - \Omega|_{\sup, \ell} \leq \alpha/2$ and $|\Omega - \Omega|_{\sup, \ell} \leq M$ to conclude that

\[
|e \cdot \Omega'| \geq |\Omega_i - \Omega_j| - M(i^\delta + j^\delta) \cdot \alpha.
\]

Without loss of generality assume that $i = |i| > j = |j|$. As $0 < \alpha < 1 \leq M$ and $i \geq 1$ it then follows that

\[
|e \cdot \Omega'| \geq |\Omega_i - \Omega_j| - 3Mi^\delta \geq i^d - j^d - \sum_{l=1}^D |a_l|(i^{d_l} - j^{d_l}) - 3Mi^\delta.
\]

For $j = 0$ we get, with $C = \sum_{l=1}^D |a_l|$,\n
\[
|e \cdot \Omega'| \geq i^d - Ci^1 - 3Mi^\delta.
\]

Choosing $E \geq 1$ sufficiently large and using that in this case $|e|_{d-1} = |i|^{d-1} + 1$ and $d_1, \delta < d$ it then follows that for any $e$ with $|e|_\ast \geq E$

\[
|e \cdot \Omega'| \geq \frac{1}{8}|e|_{d-1} \quad \forall \xi \in \Pi.
\]

If $j \geq 1$ note that $i^x - j^x$ is monotone increasing in $x \geq 0$ and hence

\[
|e \cdot \Omega'| \geq i^d - j^d - C(i^{d_1} - j^{d_1}) - 3Mi^\delta.
\]

Using that $i^d - j^d = (i - j)(i^{d-1} + j^{d-1}) + jj^{d-1} - ij^{d-1}$ and as $d \geq 1$,

\[
i^d - j^d + jj^{d-1} - ij^{d-1} = i(i^{d-1} - j^{d-1}) + j(i^{d-1} - j^{d-1}) \geq 0
\]

it then follows that

\[
2(i^d - j^d) \geq (i - j)(i^{d-1} + j^{d-1}) \geq (i - j)i^{d-1}.
\]

On the other hand

\[
i^{d_1} - j^{d_1} = \int_i^j d_1 x^{d_1 - 1} dx \leq d_1 \cdot (i - j) \cdot 1 \lor i^{d_1 - 1}.
\]

Altogether we then get
\[ |e \cdot \Omega'| \geq \frac{1}{4} (i-j)^{d-1} \left( 1 - 4Cd_1 \frac{1 \vee id_{d-1}}{id_{d-1}} \right) + \frac{1}{4} i^{d-1} \left( 1 - 12M_\delta^{d_1-d+1} \right) \]
\[ \geq \frac{1}{8} (i^{d-1} + j^{d-1}) \left( 2 - 4Cd_1 \frac{1 \vee id_{d-1}}{id_{d-1}} - 12M_\delta^{d_1-d+1} \right). \]

Choosing \( E \geq 1 \) larger, if necessary, it follows also in this case that for any \( e \) with \( |e|_\delta \geq E \)
\[ |e \cdot \Omega'(\xi)| \geq \frac{1}{8} |e|_{d-1} \ \forall \xi \in \Pi \]
as claimed. \( \Box \)

**Lemma 6.6.** For any \( e \in A \setminus A_r \) with \( |e|_\delta \geq E \) and \( E \) given as in Lemma 6.5 and for any \( k \in \mathbb{Z}^A \) and \( 0 < \alpha < \frac{1}{16} \) with \( \mathcal{R}'_{ke}(\alpha) \neq \emptyset \) one has
\[ |k| \geq \frac{1}{16} (1 + M)^{-1} |e|_{d-1}. \]

**Proof.** Again we only prove the claimed estimate for \( e \in A \setminus A_r \) with \( e \cdot \Omega'(\xi) \) of the form
\[ \Omega_i'(\xi) - \Omega_j'(\xi) \] for some \( i, j \in B \) with \( j \neq -i \) which is again the most subtle case. Since by assumption \( \mathcal{R}'_{ke}(\alpha) \neq \emptyset \) there exists \( \xi \in \Pi \) such that
\[ |k \cdot \omega'(\xi) + \Omega_i'(\xi) - \Omega_j'(\xi)| < \alpha A_k^{-1} |e|_\delta^{1/2}. \]

By Lemma 6.5 one then concludes that for any \( e \in A \setminus A_r \) with \( |e|_\delta \geq E \)
\[ |k| |\omega'|_{\Pi} \geq |\Omega_i' - \Omega_j'| - |k \cdot \omega' + \Omega_i' - \Omega_j'| \geq \frac{1}{8} |e|_{d-1} - \alpha |e|_\delta^{1/2}. \]

As \( \alpha < \frac{1}{16} \) and, by assumption, \( d-1 > \delta \), it then follows that \( |k| |\omega'|_{\Pi} \geq \frac{1}{16} |e|_{d-1} \). On the other hand, by (6.51) and (6.52),
\[ |\omega'|_{\Pi} \leq |\omega' - \omega|_{\Pi} + |\omega|_{\Pi} \leq \frac{\alpha}{2} + M \leq 1 + M \]
yielding \( |k| \geq \frac{1}{16} (1 + M)^{-1} |e|_{d-1} \) as claimed. \( \Box \)

Introduce
\[ E_{nr} := (2E^{d-1-\delta}) \vee (6 \cdot 48 \cdot LM(1 + M)) \] and \( K_{nr} := 6LM \max_{|e|_{d-1-\delta} \leq E_{nr} \delta} |e|_\delta \)
where the subscript index \( nr \) stands for 'nonresonant'.

**Lemma 6.7.** For any \( 0 < \alpha < \frac{1}{16} \) and \( (k, e) \in \mathbb{Z}^A \times \mathbb{Z}^B \) with \( e \in A \setminus A_r \) and either \( |k| \geq K_{nr} \) or \( |e|_{d-1-\delta} \geq E_{nr} \),
\[ \text{meas}(\mathcal{R}'_{ke}(\alpha)) \leq 12L(LM\rho)^{|A|^{-1}} \alpha |k|^{-1/2} A_k^{-1}. \]
Lemma 6.8. Introduce $\text{Enr}$ and either $|k| \geq K_r$ or $|e|_{d-1-\delta} \geq E_r$. Note that $|e|_{d-1-\delta}|e|_{\delta} \leq 3|e|_{d-1}$. Together with the assumption $|e|_{d-1-\delta} \geq E_{nr} \geq 6 \cdot 48 \cdot LM(1 + M)$ it then follows that

$$|k| \geq \frac{1}{16} (1 + M)^{-1} \left( \frac{1}{3} \right) \cdot 6 \cdot 48 \cdot LM(1 + M)|e|_{\delta} \geq 6LM|e|_{\delta}.$$  

If $|e|_{d-1-\delta} < E_{nr}$, then $|k| \geq K_{nr} \geq 6LM|e|_{\delta}$. So in both cases, Lemma 6.4 applies, yielding the claimed estimate.

Next we treat the case of a resonant site, $e \in \Lambda_r$. Let $C_A = 1 \vee \max_{e \in A} |i|$ and introduce

$$E_r := 2(6LMCA)^{(d-1-\delta)/(1-\delta)} \quad \text{and} \quad K_r := 6LM \max_{e \in \Lambda_r} \max_{|e|_{d-1-\delta} \leq E_r} |e|_{\delta}$$

where the subscript index $r$ stands for ‘resonant’.

Lemma 6.8. For any $(k, e) \in \mathcal{Z}$ with $e \in \Lambda_r$ and either $|k| \geq K_r$ or $|e|_{d-1-\delta} \geq E_r$

$$\text{meas}(\mathcal{R}_{ke}^r(\alpha)) \leq 12L(LM\rho)^{|A|^{-1}} \alpha |k|^{-\frac{3}{2}} A_{e}^{-1}.$$  

We remark that in the proof of Lemma 6.8 the assumption $\delta < 1$ of Assumption (A2) is used in an essential way.

Proof. Note that for $e \in \Lambda_r$, $|e|_{d-1-\delta} = 2|i|^{d-1-\delta}$. Using that $0 \leq \delta < 1$ it then follows that $|e|_{d-1-\delta} \geq E_r$ implies $|i|^{1-\delta} \geq 6LMCA$. On the other hand, as $(k, e) \in \mathcal{Z}$, it follows that $2|i| = |k \cdot v_A| \leq |k|C_A$ which then leads to

$$|k| \geq C_A^{-1}2|i| = C_A^{-1}2|i|^{1-\delta} \geq 6LM|e|_{\delta}.$$  

If $|e|_{d-1-\delta} < E_r$, then by assumption $|k| \geq K_r$ and hence $|k| \geq 6LM|e|_{\delta}$ as well. Thus in both cases we can again apply Lemma 6.4 to get the claimed estimate.

It is convenient to combine the statements of Lemma 6.4 for $e = 0$, Lemma 6.7, and Lemma 6.8. Introduce

$$E_* = E_r \vee E_{nr} \quad \text{and} \quad K_* = K_r \vee K_{nr}. \quad \text{(6.56)}$$

Corollary 6.2. For any $(k, e) \in \mathcal{Z} \setminus \mathcal{Z}_* \quad \text{with} \quad e \in \Lambda \quad \text{and} \quad \text{for any} \quad (k, 0) \in \mathcal{Z},$$

$$\text{meas}(\mathcal{R}_{ke}^r(\alpha)) \leq 12L(LM\rho)^{|A|^{-1}} \alpha |k|^{-\frac{3}{2}} A_{e}^{-1},$$

where

$$\mathcal{Z}_* := \{ (k, e) \in \mathcal{Z}: \ 0 \leq |k| < K_*; \ 0 \leq |e|_{d-1-\delta} < E_* \}.$$  

To continue, introduce for any $k \in \mathcal{Z}^A$ the resonance sets

$$\mathcal{R}’_{ke}(\alpha) := \bigcup_{(k, e) \in \mathcal{Z} \setminus \mathcal{Z}_*} \mathcal{R}_{ke}^r(\alpha).$$
Remark 6.1. Note that for any $0 < \alpha < 1/16$, $R'_0(\alpha) = \emptyset$. Indeed, let $(0, e) \in Z \setminus Z_s$. If $e \in A_r$, then $0 = k \cdot v_A + e \cdot v_B = 2i$ for some $i \in Z$ with $i, -i \in B$. But $i = 0$ contradicts that $e \in A_r$. It remains to treat the case $e \in A \setminus A_r$. The assumption $(0, e) \notin Z_s$ implies that $|e|_{d-1-\delta} \geq E_s$. As $E_s \geq E_{nr} \geq 2^{d-1-\delta}$ it follows that $|e|_s \geq E$. By Lemma 6.6 it then follows that $R'_{0e}(\alpha) = \emptyset$.

The case $k \neq 0$ is treated in the following lemma. Recall that $\rho$ denotes the diameter of $\Pi$.

Lemma 6.9. Assume that $0 < \alpha < 1/16$. Then, for any $k \in Z^A \setminus \{0\}$,

$$\text{meas}(R'_k(\alpha)) \leq C \rho |A|^{-1} \alpha |k|^{-\frac{1}{2} + 1 + \sqrt{2(d-1)^{-1}}} A_k^{-1},$$

where $C$ is a constant depending on $L, M, A, d, \delta$ and the coefficients in the expansion of $\Omega$.

Proof. First note that

$$R'_k(\alpha) = R'^{0}_k(\alpha) \cup R'^{tr}_k(\alpha) \cup R'^{nr}_k(\alpha)$$

where

$$R'^{0}_k(\alpha) = \begin{cases} R'_{k0}(\alpha) & \text{if } (k, 0) \in Z \setminus Z_s, \\ \emptyset & \text{if } (k, 0) \in Z_s, \end{cases}$$

and

$$R'^{tr}_k(\alpha) = \bigcup_{e \in A_r} R'_{ke}(\alpha); \quad R'^{nr}_k(\alpha) = \bigcup_{e \notin A \setminus A_r} R'_{ke}(\alpha).$$

By Corollary 6.2,

$$\text{meas}(R'_k(\alpha)) \leq 12L(M \rho)^{|A|^{-1}} \alpha |k|^{-\frac{1}{2}} A_k^{-1}. \quad (6.57)$$

Toward $R'^{0}_k(\alpha)$ note that for $(k, e) \in Z \setminus Z_s$ with $e \in A_r$ it follows that

$$0 < 2|i| = |k \cdot v_A| \leq C_A |k|.$$

Hence

$$|\{ (k, e) \in Z \setminus Z_s: e \in A_r \}| \leq C_A |k| \quad (6.58)$$

and thus again by Corollary 6.2,

$$\text{meas}(R'^{tr}_k(\alpha)) \leq 12C_A L(M \rho)^{|A|^{-1}} \alpha |k| \cdot |k|^{-\frac{1}{2}} A_k^{-1}. \quad (6.59)$$

To estimate $\text{meas}(R'^{nr}_k(\alpha))$ we argue as follows. Consider $(k, e) \in Z \setminus Z_s$ with $R'_{ke}(\alpha) \neq \emptyset$ and $e \in A \setminus A_r$. If $|e|_{d-1-\delta} \geq E_s$, then $|e|_s \geq E$ and hence by Lemma 6.6,

$$|k| \geq \frac{1}{16} (1 + M)^{-1} |e|_{d-1}$$

and thus
\[
\begin{align*}
\sharp \{ e \in A \setminus A_\tau : (k, e) \in \mathcal{Z} : |e|_{d-1} & \geq E_*; \mathcal{R}_{ke}^i(\alpha) \neq \emptyset \} \\
& \leq \sharp \{ e \in A \setminus A_\tau : |e|_{d-1} \leq 16(1 + M)|k| \} \\
& \leq 3 \cdot 9 \cdot (16(1 + M))^{2(d-1)-1} |k|^{2(d-1)-1}.
\end{align*}
\] (6.60)

Finally
\[
\sharp \{ e \in A \setminus A_\tau : (k, e) \in \mathcal{Z} \setminus \mathcal{Z}_* : 1 \leq |e|_{d-1} \leq E_* \} \leq 3 \cdot (2E_*^{(d-1)-1} + 1)^2.
\]

Altogether, we then get again from Corollary 6.2 that \(\text{meas}(\mathcal{R}_{ke}^{\text{mr}}(\alpha))\) is bounded by
\[
3(9(16(1 + M))^{2(d-1)-1} |k|^{2(d-1)-1} + (2E_*^{(d-1)-1} + 1)^2) \cdot 12L \mu(\mathcal{R}_\rho)^{|A|-1} \alpha |k|^{-\frac{1}{2}} A_k^{-1}.
\] (6.61)

Combining (6.57), (6.59), (6.61) leads to the claimed estimate for \(\text{meas}(\mathcal{R}_{ke}^{i}(\alpha))\). \(\square\)

**Proof of Theorem 4.1(i).** First we need to choose the parameters \(K_0, \tau, \) and \(\alpha\). Recall that \(K_0\) is given by \(K_0 = (c_1 \gamma_0)^{-r-1}\) where \(\gamma_0\) satisfies the smallness condition (6.39) and (6.50). In view of the definition, \(A_k = (k)^r\), and of Lemma 6.9, choose \(\tau \geq |A| + \frac{1}{2} + 1 \vee 2(d-1)^{-1}\). Furthermore, if necessary, choose \(0 < \gamma_0\) smaller so that \(K_0 \geq K_*\) where \(K_*\) is given by (6.56). Finally let \(0 < \alpha < 1/16\). With these choices we now estimate \(\text{meas}(\mathcal{P} \setminus \mathcal{P}_n)\). Write \(\mathcal{P} \setminus \mathcal{P}_n = \bigcup_{i=1}^4 \mathcal{X}_\alpha^i\) where
\[
\begin{align*}
\mathcal{X}_\alpha^1 &= \bigcup_{|k| < K_*} \mathcal{R}_{k\alpha}^0(\alpha_0), \\
\mathcal{X}_\alpha^2 &= \bigcup_{|k| < K_*} \mathcal{R}_{k\alpha}^0(\alpha_0), \\
\mathcal{X}_\alpha^3 &= \bigcup_{|k| \geq K_*} \mathcal{R}_{k\alpha}^0(\alpha_0), \\
\mathcal{X}_\alpha^4 &= \bigcup_{|k| \geq K_*} \mathcal{R}_{k\alpha}^0(\alpha_0).
\end{align*}
\]

and
\[
\begin{align*}
\mathcal{X}_\alpha^5 &= \bigcup_{|k| \geq K_*} \mathcal{R}_{k\alpha}^0(\alpha_0), \\
\mathcal{X}_\alpha^6 &= \bigcup_{|k| \geq K_*} \mathcal{R}_{k\alpha}^0(\alpha_0).
\end{align*}
\]

We will estimate \(\text{meas}(\mathcal{X}_\alpha^1), 1 \leq i \leq 4,\) separately. First note that in view of (6.58) and (6.60), for each \(0 \leq |k| < K_*\), the set \(\{ e \in A : (k, e) \in \mathcal{Z} ; \mathcal{R}_{k\alpha}^0(\alpha) \neq \emptyset \}\) is finite. Hence \(\mathcal{X}_\alpha^1\) is a finite union of resonance sets \(\mathcal{R}_{k\alpha}^0(\alpha_0)\). By its definition, \(\mathcal{R}_{k\alpha}^0(\alpha_0)\) is a closed subset of the compact set \(\mathcal{P} \subseteq \mathbb{R}^4\) and monotone increasing with respect to \(\alpha\). Furthermore, by Assumption (A3), \(\text{meas}(\mathcal{R}_{k\alpha}^0(0)) = 0\). Hence it follows that

\[
\lim_{\alpha \to 0} \text{meas}(\mathcal{X}_\alpha^1) = 0.
\]

By Corollary 6.2
\[
\text{meas}(\mathcal{X}_\alpha^2) \leq \sum_{k \neq 0} 12L \mu(\mathcal{R}_\rho)^{|A|-1} \alpha |k|^{-\frac{1}{2}} A_k^{-1} \leq C \rho^{|A|-1} \alpha \sum_{k \neq 0} |k|^{-\frac{1}{2}} A_k^{-1}.
\] (6.62)

and by Lemma 6.9,
\[
\text{meas}(\mathcal{X}_\alpha^2) \leq \text{meas} \left( \sum_{|k| \geq K_*} \mathcal{R}_{k\alpha}^0(\alpha) \right) \leq \sum_{|k| \geq K_*} C \rho^{|A|-1} \alpha |k|^{-\frac{1}{2} + 1 \vee 2(d-1)^{-1}} A_k^{-1}.
\] (6.63)
By the choice of $\tau \geq |A| + \frac{1}{2} + 1 \vee 2(d - 1)^{-1}$, one then gets

$$\text{meas}(\mathcal{Z}_\alpha^2) + \text{meas}(\mathcal{Z}_\alpha^4) \leq C\rho |A|^{-1}\alpha \left( \sum_{k \neq 0} |k|^{-\frac{1}{2}} A_k^{-1} + \sum_{|k| \geq K_\nu} |k|^{-\frac{1}{2} + 1 \vee 2(d - 1)^{-1}} A_k^{-1} \right)$$

$$\leq 2C\rho |A|^{-1}\alpha \sum_{k \neq 0} \frac{1}{|k|^{1+|A|}} \leq CC'\rho |A|^{-1}\alpha \quad (6.64)$$

where $C'$ is a constant only depending on $|A|$. Toward $\text{meas}(\mathcal{Z}_\alpha^4)$, recall that $\mathcal{R}_{ke}^v(\alpha_v) \subseteq \mathcal{R}_{ke}^v(\alpha)$ as $\alpha_v \leq \alpha$, for any $v \geq 1$. As $K_\nu \leq K_0 < K_\nu$ for any $v \geq 1$ one concludes that for any nonempty resonance set $\mathcal{R}_{ke}^v(\alpha_v)$ in $\mathcal{Z}_\alpha^4$ one has $(k, e) \in \mathcal{Z} \setminus \mathcal{Z}_\alpha$. Lemma 6.9 then implies that

$$\text{meas}(\mathcal{Z}_\alpha^4) \leq \sum_{v \geq 1} \sum_{|k| > K_\nu} C\rho |A|^{-1}\alpha |k|^{-\frac{1}{2} + 1 \vee 2(d - 1)^{-1}} A_k^{-1} \leq CC'\rho |A|^{-1}\alpha \sum_{v \geq 1} \frac{1}{K_\nu}. \quad (6.65)$$

By the choice of $K_\nu$, $K_\nu = K_0 2^v$, one then gets

$$\text{meas}(\mathcal{Z}_\alpha^4) \leq CC'\rho |A|^{-1}\alpha \sum_{v \geq 1} K_0^{-1} / 2^v \leq CC'K_0^{-1} \rho |A|^{-1}\alpha.$$

In particular it follows that

$$\sum_{i=2}^{4} \text{meas}(\mathcal{Z}_\alpha^i) = O(\alpha), \quad \alpha \to 0.$$

This finishes the proof of item (i) of Theorem 4.1. □

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**References**


