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Parallel Summation, Maxwell's Principle and the Infimum of Projections*

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In this paper we apply Maxwell's principle to give simple proofs of the properties of $R : S$, the *parallel sum* of two positive semi-definite linear operators. The parallel sum has been studied by Anderson, Ando, Duffin, Fillmore, Mitra, Williams, and others. In particular we give a short, elementary, and geometric proof of the result of Anderson and Duffin that gives the infimum of two orthogonal projections as twice their parallel sum.

1. ELECTRICAL MOTIVATION

To fix ideas, let us consider the network shown in Fig. 1. This connection of two resistors is called the *parallel connection*. We are given a real number c which denotes the current through the battery. We wish to determine v the voltage across the resistors. Letting x be the current through the first resistor and y be the current through the second, Kirchhoff's current law [10] gives

$$c = x + y. \tag{K}$$

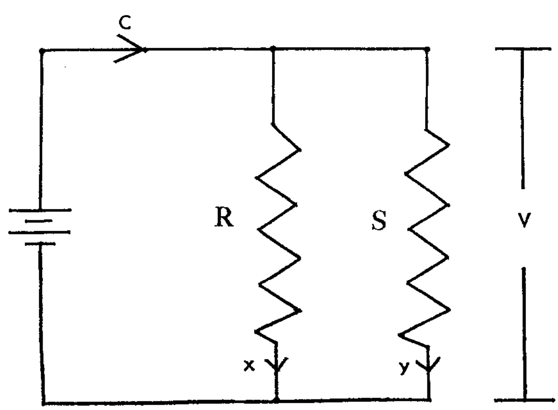

FIG. 1. The parallel connection.

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On the other hand by Kirchhoff's voltage drop law and Ohm's law [13] we have

$$v = Rx = Sy \tag{K'}$$

Here the symbols R and S are the resistance values of the corresponding resistors.

Equations (K) and (K′) can be easily solved to give

$$v = \frac{1}{1/R + 1/S}\, c. \tag{P}$$

This equation holds whenever R and S are strictly positive real numbers. It shows that the two resistors together act as if they were a single resistor whose resistance is given by $((R)^{-1} + (S)^{-1})^{-1}$. This real number is called the *joint resistance* of the network. If we rewrite (P) then it will continue to hold for nonnegative values of R and S,

$$v = R(R + S)^{\dagger}\, Sc, \tag{P$_1$}$$

where $a^{\dagger} = 1/a$ if $a \neq 0$, and $0^{\dagger} = 0$. Let us call the quantity $R(R + S)^{\dagger}\, S$ the *parallel sum* of the real numbers R and S, since it is derived as the joint resistence of two resistors with resistance values R and S. For a shorthand notation we write $R\colon S$ as an abreviation of $R(R + S)^{\dagger}\, S$.

James Clerk Maxwell in his famous treatise [11] showed how to replace one of Kirchhoff's laws by a variational principle. Applying this to replace Kirchhoff's voltage drop law in the parallel connection of Fig. 1 we get

$$R\colon S = \min_{x+y=1} Rx^2 + Sy^2. \tag{P$_2$}$$

The verification that this formula is the same as the formula of (P_1) is an elementary argument of calculus. (Set the derivative equal to zero!) The usefulness of (P_2) is shown, for instance, in that it gives a very easy proof of Lehman's inequality [5],

$$R\colon T + S\colon U \leqslant (R + S)\colon (T + U)$$

which holds for nonnegative real numbers R, S, T, and U.

It is well known that a possible generalization of nonnegative real numbers are the positive semi-definite matrices (or linear operators). With this in mind, Anderson and Duffin [1] *defined* the parallel sum of two n by n positive semi-definite matricies by the formula of (P_1)

$$R\colon S = R(R + S)^{\dagger}\, S,$$

where the dagger now denotes the Moore–Penrose pseudoinverse [14].

The motivation of Anderson and Duffin was based on the electrical interconnection of linear time-invariant resistive n-ports. See [5] for a simple explanation of these terms.

As noticed by Ando [3] and others [4, 7], the variational formulation gives an equivalent definition of the joint resistance (and thus the parallel sum) which simplifies many proofs.

So letting R and S be positive semi-definite linear operators on some (finite or infinite dimensional) Hilbert space let us define

$$(R\colon Sc, c) = \inf_{x+y=c} (Rx, x) + (Sy, y).$$

Here (,) denotes the inner product. It is not immediately clear that the above is in fact a definition. This will be shown in Section 4. However, assuming that it does indeed uniquely define $R\colon S$, we proceed to prove some of the properties of the parallel sum in Sections 2 and 3.

2. The Parallel Sum

Let U be a (finite or infinite dimensional) complex Hilbert space. Let $(\cdot, \cdot)$ denote the inner product of U. A (bounded) linear operator $R\colon U \to U$ is ***positive semi-definite*** if $(Rx, x) \geqslant 0$. Let us ***define*** the parallel sum, $R\colon S$ of two positive semi-definite linear operators R and S by the formula,

$$(R\colon Sc, c) = \inf_{x+y=c} (Rx, x) + (Sy, y).$$

If indeed the above defines a linear operator then it must be unique since a classical formula determines the values of $R\colon S$ from the quadratic form. The existence of the linear operator $R\colon S$ will be given in Section 4. Let us now prove, assuming the existence of $R\colon S$, some of the properties of $R\colon S$. Recall that $R \leqslant S$ means that $S - R$ is positive semi-definite. This partial order is equivalent to the condition that $(Rx, x) \leqslant (Sx, x)$ for all $x \in U$.

Theorem 1. *Let R and S be positive semi-definite linear operators on a complex Hilbert space U. Then*

(a) *$R\colon S$, the parallel sum of R and S is also positive semi-definite.*

(b) *$R\colon S = S\colon R$.*

(c) *If T is also a positive semi-definite linear operator on U, then*

$$(R\colon S)\colon T = R\colon (S\colon T).$$

(d) Series-parallel inequality. *Given R, S, T, U all positive semi-definite, then*

$$(R: T) + (S: U) \leqslant (R + S): (T + U).$$

(e) Transformer inequality. *If P is a linear operator taking W to U then*

$$P^*(R: S)\, P \leqslant (P^*RP): (P^*SP).$$

(f) Parallel-inner product inequality. $(R: Sc, c) \leqslant (Rc, c): (Sc, c)$.

(g) $\| R: S \| \leqslant \| R \| : \| S \|$.

Proof. Part (a) is trivial.
Parts (b) and (c) are almost as easy,

$$(R: (S: T)\, c, c) = ((R: S): Tc, c) = \inf_{x+y+z=c} (Rx, x) + (Sy, y) + (Tu, u).$$

For part (d) we see that to prove $A \leqslant B$ we must show that $(Ac, c) \leqslant (Bc, c)$ for all $c \in U$. So fix c and consider

$$\begin{aligned}
&(((R: T) + (S: U))\, c, c) \\
&\quad = (R: Tc, c) + (S: Uc, c) \\
&\quad = \inf_{x+y=c} (Rx, x) + (Ty, y) + \inf_{v+w=c} (Sv, v) + (Uw, w) \\
&\quad = \inf\{(Rx, x) + (Sv, v) + (Ty, y) + (Uw, w) \mid x + y = c;\, v + w = c\} \\
&\quad \leqslant \inf_{x+y=c} (Rx, x) + (Sx, x) + (Ty, y) + (Uy, y) \\
&\quad = ((R + S): (T + U)\, c, c).
\end{aligned}$$

For part (e)

$$\begin{aligned}
(P^*(R: S)\, Pc, c) &= ((R: S)\, Pc, Pc) \\
&= \inf_{x+y=Pc} (Rx, x) + (Sy, y) \\
&\leqslant \inf_{\substack{x+y=Pc \\ x=Pw \\ y=Pv \\ v+w=c}} (Rx, x) + (Sy, y) \\
&= \inf_{v+w=c} (RPx, Px) + (SPy, Py) \\
&= \inf_{v+w=c} (P^*RPv, v) + (P^*SPu, u) \\
&= ((P^*RP): (P^*SP)\, c, c).
\end{aligned}$$

Part (f) follows from (e) by taking $W = \mathbb{C}$.
Part (g) follows easily from the fact that $\| R \| = \sup\{(Rx, x) \mid \| x \| = 1\}$. ∎

COROLLARY 1. *Let S_{ij}, $i = 1,\ldots, m$, $j = 1,\ldots, n$ all be positive semi-definite operators on U, then*

$$\sum_{i=1}^{m} \prod_{j=1}^{n} : S_{ij} \leqslant \prod_{i=1}^{m} : \sum_{j=1}^{n} S_{ij},$$

where

$$\prod_{i=1}^{n} : A_i = A_1 : A_2 : \cdots : A_n .$$

Proof. This follows from 1(d) by induction. ∎

COROLLARY 2. *Let P_i be linear operators taking W to U. Set $\phi(A) = \sum_{i=1}^{n} P_i^* A P_i$, then*

$$\phi(R: S) \leqslant \phi(R): \phi(S).$$

Proof. Follows immediately from Corollary 1 and Theorem 1(e). ∎

Corollary 2 implies in particular that $\phi(R: S) \leqslant \phi(R): \phi(S)$ when U is finite dimensional and ϕ is *completely positive.* This result is due to Ando [3]. Our proof is a simple modification of a theorem of R. J. Duffin and the author [7] and is simpler than that in [3].

Theorem 1 and Corollary 1 are contained in [1] in the case U is finite dimensional. For U infinite dimensional Fillmore and Williams [8] gave an equivalent definition of $R: S$ and showed that $R: S = S: R$.

Anderson and Trapp in [2] gave an another equivalent definition of $R: S$ and showed Theorem 1. Our proofs are simpler.

Let $\| A \|_p$ denote the Schatten p-norm of the compact selfadjoint linear operator A, $p \geqslant 1$,

$$\| A \|_p = \left(\sum | \lambda_i |^p\right)^{1/p},$$

where $\{\lambda_i\}$ are the eigenvalues of A with multiplicities. Then if R is positive semi-definite, then set $n_p(R) = \sup\{\mathrm{tr}[S^{1/2}RS^{1/2}] \mid S$ is a compact positive semi-definite operator with $\| S \|_q \leqslant 1$, where $1/p + 1/q = 1\}$ then one has $n_p(R) = +\infty$ if R is not compact and $n_p(R) = \| R \|_p$ if R is compact. Here $S^{1/2}$ denotes the unique positive semi-definite operator whose square is S, see [15]. These results are trivial in finite dimensional space, and follow from the spectral theorem for self-adjoint operators in infinite dimensions. This leads us to *define* $\| R \|_p = +\infty$ for non-compact positive semi-definite R. With this convention we have the following

THEOREM 2. *Let R and S be positive semi-definite then $\| R: S \|_p \leqslant \| R \|_p : \| S \|_p$, with the convention that $\alpha: +\infty = \alpha$ for $\alpha \geqslant 0$.*

Proof. By the above characterization of $\| R \|_p$ it suffices to show that

$$\operatorname{tr}[R\colon S] \leqslant \operatorname{tr}[R]\colon \operatorname{tr}[S].$$

But letting $\{e_i\}_{i=1}^{\infty}$ be a basis of U, then

$$\begin{aligned} \operatorname{tr}[R\colon S] &= \sum_{i=1}^{\infty} (R\colon Se_i\,, e_i) \\ &\leqslant \sum_{i=1}^{\infty} (Re_i\,, e_i)\colon (Se_i\,, e_i). \end{aligned}$$

The last inequality follows from Theorem 1(f).

But

$$\sum_{i=1}^{\infty} (Re_i\,, e_i)\colon (Se_i\,, e_i) \leqslant \left(\sum_{i=1}^{\infty} Re_i\,, e_i\right)\colon \left(\sum_{i=1}^{\infty} Se_i\,, e_i\right) = \operatorname{tr}[R]\colon \operatorname{tr}[S].$$

This follows from an infinite version of Corollary 2. There is no problem in the limits because the terms are all positive. ∎

Theorem 2 is due to Ando [3] in finite dimensions. His proof is based on an alternate proof of $\operatorname{tr}[R\colon S] \leqslant \operatorname{tr}[R]\colon \operatorname{tr}[S]$. The result that $\operatorname{tr}[R\colon S] \leqslant \operatorname{tr}[R]\colon \operatorname{tr}[S]$ is due to Anderson and Duffin in the finite dimensional case.

3. The Infimum of Orthogonal Projections

Let P and Q be orthogonal projections onto range (P) and range (Q) respectively. Halmos in a problem of [9] asks for a formula for $P \wedge Q$, the infimum of the two projections. This is defined as the orthogonal projection onto $S = \operatorname{range}(P) \cap \operatorname{range}(Q)$. The remarkable formula of the following theorem is due to Anderson and Duffin [1] in the case that U is finite dimensional. Filmore and Williams latter extended this result to the case where U is infinite dimensional, see [8].

Our proof is geometric and completely elementary. Note that if P is an orthogonal projection then P is positive semi-definite.

Theorem 3. *Let P and Q denote orthogonal projections onto range (P) and range (Q), respectively, then*

$$P \wedge Q = 2P\colon Q,$$

where $P \wedge Q$ is the orthogonal projection onto $\operatorname{range}(P) \cap \operatorname{range}(Q)$.

Proof. Let $c \in S = \text{range}(P) \cap \text{range}(Q)$ then

$$(P{:}\,Qc, c) = \inf_{x+y=c} (Px, x) + (Qy, y)$$
$$= \left(P\frac{c}{2}, \frac{c}{2}\right) + \left(Q\frac{c}{2}, \frac{c}{2}\right) = \frac{1}{2}\| c \|^2.$$

The penultimate equality follows from "setting the derivative equal to zero."

On the other hand if $c \in S^\perp = (\text{range}(P) \cap \text{range}(Q))^\perp = \text{range}(P)^\perp \oplus \text{range}(Q)^\perp = \ker(P) \oplus \ker(Q)$, (where $\ker(A)$ denotes the null space of A) then we can find $x_0 \in \ker(P)$, $y_0 \in \ker(Q)$ such that $x_0 + y_0 = c$. Thus we have

$$(P{:}\,Qc, c) = \inf_{x+y=c} (Px, x) + (Qy, y)$$
$$= (Px_0, x_0) + (Qy_0, y_0) = 0.$$

But now we are almost done. With respect to a suitable orthonormal basis we may write any $x \in U$ as $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ with $x_1 \in S$, and $x_2 \in S^\perp$; with respect to this basis we have shown

$$P{:}\,Q = \begin{bmatrix} \frac{1}{2}I & * \\ * & 0 \end{bmatrix}.$$

But since $P{:}\,Q$ is positive semi-definite, an elementary lemma on positive semi-definite operators will show that

$$P{:}\,Q = \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & 0 \end{bmatrix},$$

and we are done. ∎

4. The Existence of the Parallel Sum

In this section we wish to show that there is a unique linear operator $R{:}\, S$ such that

$$(R{:}\, Sc, c) = \inf_{x+y=c} (Rx, x) + (Sy, y) \qquad \text{(PV)}$$

whenever R and S are positive semi-definite linear operators on a complex Hilbert space. Since the quadratic form uniquely determines the linear operator in a complex inner product space it suffices to show that there is a linear operator $R{:}\, S$ such that $(R{:}\, Sc, c) = f(c)$, where

$$f(c) = \inf_{x+y=c} (Rx, x) + (Sy, y).$$

If $(R + S)$ is positive definite, and hence invertible, then by setting the derivative equal to zero one has

$$f(c) = (R(R + S)^{-1} Sc, c).$$

If $(R + S)$ is not positive definite, then a simple "$\epsilon - \delta$" proof, see, e.g., [6], will show

$$\begin{aligned} f(c) &= \lim_{\epsilon \downarrow 0} \inf_{x+y=c} (Rx, x) + (Sy, y) + \epsilon \| x \|^2 + \epsilon \| y \|^2 \\ &= \lim_{\epsilon \downarrow 0} \inf_{x+y=c} ((R + \epsilon I) x, x) + (S + \epsilon Iy, y) \\ &= \lim_{\epsilon \downarrow 0} ((R + \epsilon I) : (S + \epsilon I) c, c). \end{aligned}$$

The latter exists since $(R + \epsilon I) + (S + \epsilon I)$ is positive definite. Thus

$$f_\epsilon(c) = \inf_{x+y=c} (Rx, x) + (Sy, y) + \epsilon \| x \|^2 + \epsilon \| y \|^2$$

is given by a quadratic form,

$$f_\epsilon(c) = ((R + \epsilon I)(R + S + 2\epsilon I)^{-1} (S + \epsilon I) c, c).$$

Moreover, since $f(c) = \lim_{\epsilon \downarrow 0} f_\epsilon(c)$, $f(c)$ is a quadratic form. This follows since $f_\epsilon(c)$ decreases when ϵ decreases, see [15].

If $R + S$ is positive definite then $R: S = R(R + S)^{-1} S$. If U is finite dimensional then the general case can be shown to be

$$R: S = R(R + S)^{\dagger} S.$$

In the case of that U is infinite dimensional, then one obtains the formula of Fillmore and Williams

$$R: S = R^{1/2}((R + S)^{1/2\dagger} R^{1/2})^* ((R + S)^{1/2\dagger} S^{1/2}) S^{1/2}.$$

Either of the above formulas follow from applying continuity arguments to $R(R + S)^{-1} S$.

5. Generalizations

If U is finite dimensional then system (K)–(K′)

$$x + y = c \qquad \text{(K)}$$

$$Rx = Sy = v \qquad \text{(K')}$$

may have a solution if R and S are not positive semi-definite. Since (K)–(K′) are determined from the ***network*** interpretation of R: S let us call R and S ***parallel summable*** if given any $c \in U$ there is an x, y and a ***unique*** v that solve (K), (K′). It is simple to show that this definition is equivalent to that of [14]. If R and S are parallel summable let us define the ***parallel sum***, R: S, by the formual R: $Sc = v$ where c and v are given as in (K)–(K′). The structure of (K)–(K′) shows trivially that the parallel sum is commutative and associative when the parallel sum is defined. This simplifies the proofs of [14].

Analyzing the analogous network equations of the hybrid connection will give simple proofs of most of the results of [12].

References

1. W. N. Anderson, Jr and R. J. Duffin, Series and parallel addition of operators, *J. Math. Anal. Appl.* **26** (1969), 576–594.
2. W. N. Anderson, Jr. and G. E. Trapp, Shorted operators, II, *SIAM J. Appl. Math.* **28** (1975), 60–71.
3. T. Ando, Inequalities for joint resistance, to appear.
4. R. J. Duffin, Non-linear networks, IIa, *Bull. Amer. Math. Soc.* **53** (1947), 963–971.
5. R. J. Duffin, Network models, *in* "Mathematical Aspects of Electrical Network Analysis," S.I.A.M.–A.M.S. proceedings, Vol. III, Amer. Math. Soc., Providence, R. I., 1971.
6. R. J. Duffin, The Lagrange multiplier method for convex programs, *Proc. Nat. Acad. Sci. USA* **72** (1975), 1778–1781.
7. R. J. Duffin and T. D. Morley, Inequalities induced by electrical connections, *in* "Recent Applications of Generalized Inverses," M. Z. Nashed, Ed.), Pitman, New York, 1978.
8. P. A. Fillmore and J. P. Williams, On Operator Ranges, *Advances in Math.* **7** (1971), 254–281.
9. P. Halmos, "A Hilbert Space Problem Book," Springer–Verlag, New York, 1974.
10. G. Kirchhoff, Über die Auflösung der Gleichungen, auf welche man bei der Untersuchungen der Linearen Vertailung Galvanisher Ströme geführt wird, *Poggendorf Ann. Physik* **72** (1847), 497–508.
11. J. C. Maxwell, "A Treatise on Electricity and Magnetism," 3rd ed., Dover, New York 1953.
12. S. K. Mitra and G. E. Trapp, On hybrid addition of matrices, *Linear Algebra Appl.* **10** (1975), 19–35.
13. G. S. Ohm, "Die galvanische Kette, mathematisch bearbeitet von dr. G. S. Ohm," Eiemann, Berlin, 1827.
14. C. R. Rao and S. K. Mitra, "Generalized Inverse of Matrices and its Applications, Wiley, New York, 1971.
15. F. Riesz and B. Sz.-Nagy, "Functional Analysis," Ungar, New York, 1955.