Sinc collocation method with boundary treatment for two-point boundary value problems

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Abstract

Sinc collocation method is proven to provide an exponential convergence rate in solving linear differential equations, even in the presence of singularities. But in order to treat the derivatives on boundaries, people often relied on the finite difference method, which would be expected to limit the accuracy. The present paper develops a Sinc collocation method with boundary treatment for two-point boundary value problems. Numerical results show that the method can directly and efficiently handle the boundary derivatives.

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1. Introduction

During the last three decades a variety of numerical methods based on the Sinc approximation have been developed, which are now collectively referred to as Sinc methods [6,1]. Sinc methods provide procedures for function approximation, solving initial and boundary value ordinary differential equation (ODE) problems, the approximate solution of PDEs and so on [6,7,10]. The books [6,1] provide excellent overviews of the existing Sinc methods for solving ODEs and PDEs. With the thought of solving PDE on general domains, a natural first step is the investigation of Sinc methods on the following two-point boundary value problem:

\[
\begin{align*}
-w''(x) + p(x)w'(x) + q(x)w(x) &= g(x), \quad x \in (a, b), \\
a_0w(a) + a_1w'(a) &= a_2, \\
b_0w(b) + b_1w'(b) &= b_2.
\end{align*}
\]  

Both Sinc Galerkin and Sinc collocation methods are useful in the treatment of problem (1), even in the presence of singularities. For problems with constant coefficients, the Sinc Galerkin method might well be the method of choice. For problems with variable coefficients, the Sinc collocation method is especially convenient, because the coefficients are more efficiently handled.

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By comparison to the finite difference, finite element and boundary element methods, the Sinc collocation approach has been shown to be more suitable in handling singularities, boundary layers and semi-infinite domains [10]. Furthermore, the residual error entailed in the Sinc collocation method is known to exhibit an exponential convergence rate. In [6,1], the Sinc collocation method with $2N + 1$ collocation points converges at the rate of $O(\exp(-k\sqrt{N}))$ with some $k > 0$. Recently, it has turned out that the Sinc collocation method can achieve convergence rate of $O(\exp(-k'(N/\log N)))$ with some $k' > 0$ when incorporated with the double exponential transformation and there convergence rates are best possible [7–9].

But there are still some difficulties in treating the derivatives at the boundaries, since the derivative of the Sinc function at the boundaries is undefined, which can lead to a numerical overflow near the boundaries [5]. In [5], Sinc collocation method along with the finite difference method for calculating derivatives near boundaries is used for solving the incompressible Navier–Stokes equations. Sinc collocation method for domain decomposition is discussed in [2–4]. Finite difference approximations are also used to match the boundary derivatives in the patching domain decomposition method. In addition, the use of the finite difference approximations would be expected to limit the accuracy achieved by the Sinc approximations. A more convenient technique to handle derivatives has yet to be developed.

The present paper provides a more convenient Sinc collocation method with boundary treatment (SCMBT) for problem (1), which can efficiently treat boundary derivatives. The paper is organized as follows: Section 1 is the introduction; in Section 2 we describe the existing Sinc collocation method for the two-point boundary problem with the zero Dirichlet boundary condition; in Section 3 we develop the Sinc collocation method with boundary treatment (SCMBT) for problem (1); in Section 4 we take some examples to illustrate the application of SCMBT; and Section 5 gives the brief comments.

2. Sinc collocation method

In this section, we describe the Sinc collocation method which has been widely used for the two-point boundary value problems.

2.1. Sinc collocation method on the entire interval ($-\infty, \infty$)

We first consider the Sinc collocation method for the following two-point boundary value problem with the zero Dirichlet boundary condition on $(-\infty, \infty)$:

\[
\begin{cases}
-v''(t) + p(t)v'(t) + q(t)v(t) = f(t), & t \in (-\infty, \infty), \\
\lim_{t \to \pm \infty} v(t) = 0.
\end{cases}
\]

(2)

The approximate solution to (2) is given by

\[v_N(t) \equiv \sum_{k=-N}^{N} v_k S(k, h)(t)\]  

(3)

and its first and second order derivatives are

\[
\frac{d}{dt} v_N(t) \equiv \sum_{k=-N}^{N} v_k \frac{d}{dt} S(k, h)(t),
\]

(4)

\[
\frac{d^2}{dt^2} v_N(t) \equiv \sum_{k=-N}^{N} v_k \frac{d^2}{dt^2} S(k, h)(t),
\]

(5)

where $S(k, h)(t)$ is the so-called Sinc function defined by

\[S(k, h)(t) = \frac{\sin[(\pi/h)(t - kh)]}{(\pi/h)(t - kh)},\]

and the step size $h$ is suitably chosen for a given positive integer $N$. 
We define
\[
\delta_{jk}^{(0)} \equiv S(j, h)(kh) = \begin{cases} 
1 & \text{if } j = k, \\
0 & \text{if } j \neq k,
\end{cases}
\]
\[
\delta_{jk}^{(1)} \equiv hS'(j, h)(kh) = \begin{cases} 
0 & \text{if } j = k, \\
\frac{(-1)^{k-j}}{k-j} & \text{if } j \neq k,
\end{cases}
\]
\[
\delta_{jk}^{(2)} \equiv h^2S''(j, h)(kh) = \begin{cases} 
-\frac{\pi^2}{3} & \text{if } j = k, \\
-\frac{2(-1)^{k-j}}{(k-j)^2} & \text{if } j \neq k,
\end{cases}
\]
and the Toeplitz matrices
\[
I^{(r)} = [\delta_{jk}^{(r)}], \quad r = 0, 1, 2,
\]
where \(\delta_{jk}^{(r)}\) denotes the \((j, k)\)th element of the matrix \(I^{(r)}\). In (3)–(5), setting \(t = jh\), for \(-N < j < N\), we can get
\[
v_j = \sum_{k=-N}^{N} \delta_{kj}^{(0)} v_k,
\]
\[
v'_j = \sum_{k=-N}^{N} \delta_{kj}^{(1)} \frac{h}{j} v_k,
\]
\[
v''_j = \sum_{k=-N}^{N} \delta_{kj}^{(2)} \frac{h^2}{j^2} v_k.
\]
Now, since \(\delta_{jk}^{(0)} = \delta_{kj}^{(0)}\), \(\delta_{jk}^{(1)} = -\delta_{kj}^{(1)}\), and \(\delta_{jk}^{(2)} = \delta_{kj}^{(2)}\), we can rewrite these equations in matrix form as
\[
v = Qv, \quad (6)
\]
\[
v' = Av, \quad (7)
\]
\[
v'' = Bv, \quad (8)
\]
where
\[
Q = I^{(0)}, \quad A = -\frac{I^{(1)}}{h}, \quad B = \frac{I^{(2)}}{h^2},
\]
and \(v = [v_{-N}, \ldots, v_N]^T, v' = [v'_{-N}, \ldots, v'_N]^T, v'' = [v''_{-N}, \ldots, v''_N]^T\). Note that \(v\) is the exact solution of problem (2) evaluated at the notes \(t = jh\). Applying the Sinc collocation formulas (6)–(8) to problem (2), we can finally get the system of equations
\[
Cv = f \quad (9)
\]
with
\[
C = -B + D(p)A + D(q)Q,
\]
\[
D(p) = \text{diag}(p(-Nh), \ldots, p(Nh)), \quad D(q) = \text{diag}(q(-Nh), \ldots, q(Nh)),
\]
and \(f = [f(-Nh), \ldots, f(Nh)]^T\). By solving the above system of equations (9), we can obtain the approximate solution \(v_N(t)\).
2.2. Convergence analysis

**Definition 1.** Let $H^1(D_d)$ denote the family of all functions analytic in $D_d$, with
\[ D_d = \{ z \in \mathbb{C} : |\Im z| < d \}, \]
such that if $D_d(\varepsilon)$ is defined for $0 < \varepsilon < 1$ by
\[ D_d(\varepsilon) = \{ z \in \mathbb{C} : |\Re z| < 1/\varepsilon, |\Im z| < d(1 - \varepsilon) \}, \]
then $N_1(f, D_d) < \infty$, with
\[ N_1(f, D_d) = \lim_{\varepsilon \to 0} \left( \int_{D_d(\varepsilon)} |f(z)| \, |dz| \right). \]

The following theorem shows that the Sinc collocation method converges at the rate of $\exp(-k'N/\log N)$, if the solution of problem (2) decays double exponentially.

**Theorem 2** (Sugihara [8]). Assume that problem (2) has a unique solution $v(t)$, and that solution $v(t)$ is analytic on the real line. Furthermore, assume, with positive constants $A, B, \alpha, \beta, \gamma$ and $d$, that

1. $p, p'$ and $q$ are analytic in the strip region $D_d$, and their absolute values on the real line are bounded from above as follows:
   \[ |p(t)|, |p'(t)|, |q(t)| \leq A \exp(B|t|) \quad \text{for all } t \in \mathbb{R}; \]
2. $pv, p'v$ and $qv$ belong to $H^1(D_d)$;
3. $p$ takes real values on the real line;
4. $\Re(2q(t) - p'(t)) \leq 0$ for all $t \in \mathbb{R}$;
5. $f$ belongs to $H^1(D_d)$ and decays double exponentially on the real line, that is,
   \[ |f(t)| \leq \alpha \exp(-\beta \exp(\gamma|t|)) \quad \text{for all } t \in \mathbb{R}; \]
6. $v$ belongs to $H^1(D_d)$ and decays double exponentially on the real line, that is,
   \[ |v(t)| \leq \alpha \exp(-\beta \exp(\gamma|t|)) \quad \text{for all } t \in \mathbb{R}. \]

Then we have
\[ \sup_{-\infty < t < \infty} |v(t) - v_N(t)| \leq C(\log N)N^{B/\gamma + 3/2} \exp \left[ -\frac{\pi d^2 \gamma N}{\log(\pi d^2 \gamma N/\beta)} \right] \]
for some $C$, where the mesh size $h$ in the Sinc collocation method is taken as
\[ h = \frac{\log(\pi d^2 \gamma N/\beta)}{\gamma N}. \]

2.3. Sinc collocation method on interval $[a, b]$

Here, we consider the Sinc collocation method for the following simple two-point boundary problem
\[ \begin{align*}
- u''(x) + p(x)u'(x) + q(x)u(x) &= f(x), \quad x \in [a, b], \\
u(a) &= u(b) = 0.
\end{align*} \tag{10} \]

The Sinc collocation method for this model problem is discussed in detail in [1,2,5–7].
Definition 3. Let $D$ be a simply connected domain having boundary $\partial D$. Let $a$ and $b$ denote two distinct points of $\partial D$, and let $\phi$ denote the conformal map of $D$ onto $D_d$ such that $\phi(a) = -\infty$ and $\phi(b) = \infty$, where $D_d$ is defined in Definition 1. Let $\psi = \phi^{-1}$ denote the inverse map, and let the arc $\Gamma$ be given by $\Gamma = \psi^{-1}(-\infty, \infty)$. For $h > 0$, let the points $x_k$ on $\Gamma$ be given by $x_k = \psi(kh), k \in \mathbb{Z}$.

First, we consider the following model problems:

$$u(x) = f_0(x), \quad (11)$$
$$\frac{du(x)}{dx} = f_1(x), \quad (12)$$
$$\frac{d^2u(x)}{dx^2} = f_2(x). \quad (13)$$

For $l$ a nonnegative real number, define the transformation of variable

$$v(t) = ((\phi')^l u) \circ \psi(t) = (\phi'(\psi(t)))^l u(\psi(t)). \quad (14)$$

A calculation by using the chain rule and $x = \psi(t)$ leads to the equalities

$$u(x) = \frac{1}{(\phi'(x))^l} v(t),$$
$$\frac{du(x)}{dx} = (\phi'(x))^{1-l} \frac{dv}{dt} + ((\phi'(x))^{-1})' v(t),$$
$$\frac{d^2u(x)}{dx^2} = (\phi'(x))^{2-l} \frac{d^2v}{dt^2} + (1 - 2l)(\phi'(x))^{-l} \phi''(x) \frac{dv}{dt} + (\phi'(x))^{-l}'' v(t).$$

Take Eq. (13) for example, the transformed equation is given by

$$(\phi'(x))^{2-l} \frac{d^2v}{dt^2} + (1 - 2l)(\phi'(x))^{-l} \phi''(x) \frac{dv}{dt} + (\phi'(x))^{-l}'' v(t) = f_2(x). \quad (15)$$

In addition, $v(t)$ satisfies the following conditions:

$$\lim_{t \to \pm\infty} v(t) = 0,$$

which follow from $\psi(-\infty) = 0, \psi(\infty) = 1$, and the homogeneous boundary conditions for $u$ in (10). We multiply both sides of (15) by $(\phi'(x))^{l-2}$ and take the Sinc points

$$t = jh, \quad \text{for} \quad -N < j < N.$$ 

Note that $x = \psi(t)$, thus $x = x_j = \psi(jh)$, as $t = jh$. Then applying the Sinc collocation method described in Section 2.1, we can obtain the following results:

$$B_1 v = D((\phi'(x))^{l-2}) f_2, \quad (16)$$

where

$$B_1 = \frac{I^{(2)}}{h^2} + (2l - 1)D((\phi'(x))^{-2} \phi''(x)) \frac{I^{(1)}}{h} + D((\phi'(x))^{l-2}(\phi'(x))^{-l}'') I^{(0)},$$

and $f_2 = [f_2(x_{-N}), \ldots, f_2(x_N)]$. Let $u = [u_{-N}, \ldots, u_N]^T$, $u_k$ is the approximation of the exact solution $u(x)$ of problem (13) at the nodes $x = x_j$. We now make the transformation

$$v = D((\phi')^l) u.$$
to arrive at the Sinc collocation results
\[ B\mathbf{u} = D((\phi')^{l-2})\mathbf{f}, \quad (17) \]
where
\[ B = B_1 D((\phi')^l). \]
In a similar way, we can get the Sinc collocation formulas for (11) and (12) as follows:
\[ Q\mathbf{u} = D((\phi')^{l-2})\mathbf{f}_0, \quad (18) \]
\[ A\mathbf{u} = D((\phi')^{l-2})\mathbf{f}_1, \quad (19) \]
where
\[ Q = D\left( \frac{1}{(\phi'(x))^2} \right) I^{0} D((\phi')^l), \]
\[ A = \left[ -D\left( \frac{1}{\phi^2} \right) \frac{I^{(1)}}{h} + D((\phi')^{l-2}(\phi')^{-l})I^{(0)} \right] D((\phi')^l). \]

Now we apply the Sinc collocation method to Eq. (10). Multiplying Eq. (10) by \((\phi'(x))^{l-2}\), we can get the system of equations
\[ C\mathbf{u} = D((\phi')^{l-2})\mathbf{f}, \quad (20) \]
where \(C = -B + D(p)A + D(q)Q\) and \(\mathbf{f} = [f(x_{-N}), \ldots, f(x_N)]^T\). Solve the above system of equations for \(\mathbf{u}\), then we get the approximate solution of (10) given by
\[ u_N(x) = \sum_{k=-N}^{N} u_k S(k, h) \circ \phi(x). \quad (21) \]

**Remark 4.** Generally, we choose \(l = 0, 1\), for they do have particular merit, though the development is valid for any \(l\). In addition, \(l = 1\) may be the choice when considering the boundary derivatives. The derivative of \(u(x)\) can be expressed as
\[ \frac{du(x)}{dx} = (\phi'(x))^{1-l} \frac{dv}{dr} + ((\phi'(x))^{-l})' v(t). \]
The fact is that \(\phi'(x)\) may be unbounded on \(\Gamma\), especially at endpoints \(a\) and \(b\). For example, let \(\phi(x) = \ln((x-a)/(b-x))\). Note that
\[ \phi'(x) = \frac{b - a}{(x - a)(b - x)}, \]
then we have \(\lim_{x \to a} \phi'(x) = -\infty\) and \(\lim_{x \to b} \phi'(x) = \infty\). Taking the double exponential transformation \(\phi(x) = \arcsinh(2/\pi)\arctanh((2/(b-a)x - (a + b)/(b - a)))\), we can get the similar results. So when \(l < 1\), \(du/dx\) may be unbounded near the endpoints. Thus in this situation the above Sinc collocation formula may not be useful for approximating the derivatives at boundaries.

### 3. The Sinc collocation method with boundary treatment (SCMBT)

#### 3.1. Description of the SCMBT

In this section, we consider the two-point boundary value problem (1) on \([0,1]\), that is \(a = 0\), \(b = 1\). The boundary conditions can be Dirichlet, Neumann or mixed boundary conditions.
Firstly, we consider the following model problem:
\[
\frac{d^2 w(x)}{dx^2} = g_2(x),
\]
and \(w(x)\) satisfies the following boundary conditions:
\[
\begin{align*}
& a_0 w(0) + a_1 w'(0) = a_2, \\
& b_0 w(1) + b_1 w'(1) = b_2.
\end{align*}
\]
We assume \(w'(0) < \infty, w'(1) < \infty\) and
\[
\begin{align*}
& c_{-N-1} \approx w(0), \\
& c_{-N-2} \approx w'(0), \\
& c_{N+1} \approx w(1), \\
& c_{N+2} \approx w'(1).
\end{align*}
\]
Then let
\[
\begin{align*}
u(x) &= w(x) - c_{-N-2} \varphi_{01}(x) - c_{-N-1} \varphi_{00}(x) - c_{N+1} \varphi_{10}(x) - c_{N+2} \varphi_{11}(x),
\end{align*}
\]
where \(\varphi_{00}(x), \varphi_{01}(x), \varphi_{10}(x)\) and \(\varphi_{11}(x)\) consist of the cardinal functions for univariate cubic Hermite interpolation:
\[
\begin{align*}
& \varphi_{00}(x) = (2x + 1)(1 - x)^2, \\
& \varphi_{01}(x) = x(1 - x)^2, \\
& \varphi_{10}(x) = x^2(3 - 2x), \\
& \varphi_{11}(x) = x^2(x - 1),
\end{align*}
\]
via the identities
\[
\begin{align*}
& \varphi_{00}(0) = 1, \quad \varphi_{00}(1) = \varphi_{00}'(0) = \varphi_{00}'(1) = 0, \\
& \varphi_{10}(0) = 0, \quad \varphi_{10}(1) = 1, \quad \varphi_{10}'(0) = \varphi_{10}'(1) = 0, \\
& \varphi_{01}(0) = \varphi_{01}(1) = 0, \quad \varphi_{01}'(0) = 1, \quad \varphi_{01}'(1) = 0, \\
& \varphi_{11}(0) = \varphi_{11}(1) = \varphi_{11}'(0) = 0, \quad \varphi_{11}'(1) = 1.
\end{align*}
\]
Then problem (22) can be transformed as
\[
\frac{d^2 u(x)}{dx^2} = \hat{g}_2(x),
\]
where \(\hat{g}(x) = g(x) - c_{-N-2} \varphi_{01}''(x) - c_{-N-1} \varphi_{00}''(x) - c_{N+1} \varphi_{10}''(x) - c_{N+2} \varphi_{11}''(x)\).

From (25) and (26), we know that \(u(x)\) satisfies the following boundary conditions:
\[
\begin{align*}
u(0) &= u(1) = 0 \\
u'(0) &= u'(1) = 0.
\end{align*}
\]

We assume the approximate solution of problem (27) is given by
\[
u_N(x) = \sum_{k=-N}^{N} u_k S(k, h) \circ \phi(x).
\]
We make the similar procedure as described in Section 2.3. In the SCMBT we choose \(l=1\) in the variable transformation (14), i.e.,
\[
v(t) = (\phi' u) \circ \psi(t) = \phi'(\psi(t)) u(\psi(t)).
\]
Then we evaluate Eq. (27) at Sinc points
\[
S = \{ x_j \mid j = -N-1, -N, \ldots, N, N+1 \}.\]
Note that we choose \( n + 2(n = 2N + 1) \) points here. Thus we can get the Sinc collocation results

\[
\tilde{B} \mathbf{u} = \tilde{Q} \hat{\mathbf{g}}_2
\]

with

\[
\tilde{B} = \left[ \frac{I^{(2)}}{h^2} + D \left( \frac{\phi''}{(\phi')^2} \right) \frac{I^{(1)}}{h} + D \left( \frac{1}{\phi'} \right) I^{(0)} \right] D(\phi'), \quad \tilde{Q} = D \left( \frac{1}{\phi'} \right),
\]

and \( \mathbf{u} = [u_{-N}, \ldots, u_N]^T \). The matrix \( \tilde{B} \) is \((n+2) \times n\) analogue of the matrix \( B \) in Section 2.3. The matrices \( I^{(r)} \), \( r = 0, 1, 2 \), are \((n+2) \times n\) and the diagonal matrices \( D(1/\phi') \), \( D(\phi''/(\phi')^2) \) and \( D((1/\phi')(1/\phi'')) \) are \((n+2) \times (n+2)\). The matrix \( D(\phi') \) is \( n \times n\). The above results follow from the fact that \( k = -N, \ldots, N \) in (28), \( j = -N - 1, -N, \ldots, N, N + 1 \) in \( S \) and \( I^{(r)} = [\delta_j^{(r)}] \). Let \( \hat{\mathbf{g}}_2 = [g_2(x_{-N-1}), g_2(x_{-N}), \ldots, g_2(x_N), g_2(x_{N+1})]^T \) and \( \mathbf{w} = [w_{-N}, \ldots, w_N]^T \).

Then from (26), we know that

\[
\mathbf{u} = \mathbf{w} - c_{-N-2} \tilde{\Phi}_{01} - c_{-N-1} \tilde{\Phi}_{00} - c_{N+1} \tilde{\Phi}_{10} - c_{N+2} \tilde{\Phi}_{11}
\]

and

\[
\hat{\mathbf{g}}_2 = g_2 - c_{-N-2} \Phi''_{01} - c_{-N-1} \Phi''_{00} - c_{N+1} \Phi''_{10} - c_{N+2} \Phi''_{11},
\]

where \( \tilde{\Phi}_{ij} = [\phi_{ij}(x_N), \ldots, \phi_{ij}(x_N)]^T \), \( \Phi''_{ij} = [\phi''_{ij}(x_{N-1}), \phi''_{ij}(x_{-N}), \ldots, \phi''_{ij}(x_N), \phi''_{ij}(x_{N+1})]^T \) for \( i = 0, 1, j = 0, 1 \). Then substituting the above expression of \( \mathbf{u} \) and \( \hat{\mathbf{g}}_2 \) into (29), we get

\[
\tilde{B}(\mathbf{w} - c_{-N-2} \tilde{\Phi}_{01} - c_{-N-1} \tilde{\Phi}_{00} - c_{N+1} \tilde{\Phi}_{10} - c_{N+2} \tilde{\Phi}_{11}) = \tilde{Q}(g_2 - c_{-N-2} \Phi''_{01} - c_{-N-1} \Phi''_{00} - c_{N+1} \Phi''_{10} - c_{N+2} \Phi''_{11}).
\]

By taking \( \mathbf{w} = [c_{-N-1}, c_{-N-2}, \mathbf{w}^T, c_{N+2}, c_{N+1}]^T = [w(0), w'(0), \mathbf{w}^T, w'(1), w(1)]^T \), we can arrive at the following Sinc collocation results:

\[
\tilde{B} \mathbf{w} = \tilde{Q} \mathbf{g}_2,
\]

where

\[
\tilde{B} = [B_1, B_2, \tilde{B}, B_3, B_4],
\]

\[
B_1 = \tilde{Q} \Phi''_{00} - \tilde{B} \tilde{\Phi}_{00}, \quad B_2 = \tilde{Q} \Phi''_{01} - \tilde{B} \tilde{\Phi}_{01},
\]

\[
B_3 = \tilde{Q} \Phi''_{11} - \tilde{B} \tilde{\Phi}_{11}, \quad B_4 = \tilde{Q} \Phi''_{10} - \tilde{B} \tilde{\Phi}_{10}.
\]

The matrix \( \tilde{B} \) is \((n+2) \times (n+4)\) and \( B_i (i = 1, 2, 3, 4) \) are \((n+2) \times 1\) column vectors. Eqs. (23), (24) and (30) make up a system of equations with \( n + 4 \) equations and \( n + 4 \) unknowns. Similarly, to solve the following model problems:

\[
w(x) = g_0(x),
\]

\[
\frac{dw(x)}{dx} = g_1(x),
\]

we can get the Sinc collocation formulas:

\[
\tilde{Q} \mathbf{c} = \tilde{Q} \mathbf{g}_0,
\]

\[
\tilde{A} \mathbf{c} = \tilde{Q} \mathbf{g}_1.
\]
As in Section 3.1, and with its inverse is given by

\[ \widetilde{Q} = D \left( \frac{1}{(\phi'(x))^2} \right) I^{(0)} D((\phi')) , \]

\[ Q_1 = \widetilde{Q} \Phi_{00} - \widetilde{\Phi}_{00}, \quad Q_2 = \widetilde{Q} \Phi_{01} - \widetilde{\Phi}_{01}, \quad Q_3 = \widetilde{Q} \Phi_{11} - \widetilde{\Phi}_{11}, \quad Q_4 = \widetilde{Q} \Phi_{10} - \widetilde{\Phi}_{10} \]

and

\[ A = [A_1, A_2, A_3, A_4], \quad \tilde{A} = \left[ -D \left( \frac{1}{\phi'} \right) \frac{I^{(1)}}{h} + D \left( \frac{1}{\phi'} \left( \left( \frac{1}{\phi'} \right)' \right) \right) I^{(0)} \right] D((\phi')) , \]

\[ A_1 = \widetilde{Q} \Phi'_{00} - \widetilde{\Phi}'_{00}, \quad A_2 = \widetilde{Q} \Phi'_{01} - \widetilde{\Phi}'_{01}, \quad A_3 = \widetilde{Q} \Phi'_{11} - \widetilde{\Phi}'_{11}, \quad A_4 = \widetilde{Q} \Phi'_{10} - \widetilde{\Phi}'_{10} . \]

Now, we apply the new Sinc collocation results to problem (1). Multiply both sides of the results by $1/\phi'$ and evaluate the results at Sinc points $S = [x_{-N-1}, \ldots, x_{N+1}]^T$. Then we can get

\[ M \mathbf{w} = \overline{Q} \mathbf{g}, \quad \text{(33)} \]

where

\[ M = \mathcal{B} + D(p) \mathcal{A} + D(q) \overline{Q} , \]

\[ D(p) = \text{diag}(p(x_{-N-1}), \ldots, p(x_{N+1})) , \quad D(q) = \text{diag}(q(x_{-N-1}), \ldots, q(x_{N+1})) , \]

and $\mathbf{g} = [g(x_{-N-1}), \ldots, g(x_{N+1})]^T$.

### 3.2. Analysis

Theorem 2 shows that the Sinc collocation method converges at the rate of $O(-kN/\log N)$ for $k > 0$ in the case $v(t)(t \in (-\infty, +\infty))$ decays double exponentially. Generally, this condition must be checked via a knowledge of the true solution $v(t)$ of the equation. We can find in SCMBT, as a consequence of variable transformation, $v(t)$ decays double exponentially without the knowledge of the exact solution.

We choose the variable transformation

\[ x = \psi(t) = \frac{1}{2} \tanh \left( \frac{\pi}{2} \sinh t \right) + \frac{1}{2} \]

and its inverse is given by

\[ \phi(x) = \text{arcsinh} \left( \frac{2}{\pi} \text{arctanh}(2x - 1) \right) . \]

As in Section 3.1, $u(0) = u(1) = u'(0) = u'(1) = 0$, we can know that $u(x)$ satisfies

\[ u(x) = \begin{cases} O(x^{1+\lambda_1}) & \text{as } x \to 0, \\ O((1-x)^{1+\lambda_2}) & \text{as } x \to 1, \end{cases} \]

for some $\lambda_1 > 0$ and $\lambda_2 > 0$. Note that

\[ \psi'(t) = \frac{(\pi/2) \cosh t}{\cosh^2((\pi/2) \sinh t)} \]

and

\[ \phi'(\psi(t)) = \frac{1}{\psi'(t)} = \frac{\cosh^2((\pi/2) \sinh t)}{(\pi/2) \cosh t} \sim O \left( \exp \left( -\frac{\pi}{2} (e - 1) \exp |t| \right) \right) \quad \text{as } t \to \infty \]
Table 1
The errors in Example 5

<table>
<thead>
<tr>
<th>N</th>
<th>4</th>
<th>8</th>
<th>12</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td>1.4168e-004</td>
<td>1.7727e-005</td>
<td>3.6856e-006</td>
<td>9.5069e-007</td>
</tr>
<tr>
<td>E^0_d</td>
<td>1.3295e-008</td>
<td>8.4241e-011</td>
<td>1.5491e-012</td>
<td>4.9738e-014</td>
</tr>
<tr>
<td>E^1_d</td>
<td>4.3166e-004</td>
<td>5.3245e-005</td>
<td>1.1059e-005</td>
<td>2.8522e-006</td>
</tr>
</tbody>
</table>

hold with arbitrary $0 < \varepsilon < 1$ which we need because of the factor $\cosh t$ in the denominator. Then we have

$$v(t) = (\phi' u) \circ (\psi(t)) = \frac{u(\psi(t))}{\psi'(t)} = O\left(\exp\left(-\frac{\pi}{2} (\varepsilon + \lambda) \exp(|t|)\right)\right) \quad \text{as } t \to \pm \infty$$

with $\lambda = \min(\lambda_1, \lambda_2)$. Since $\lambda, \varepsilon > 0$, so $v(t)$ decays double exponentially. Note that since $v(t) = (\phi' u)$ and $1/\phi' \leq \pi/2$ for all $x \in [0, 1]$, if $v(t)$ satisfies assumptions (1)–(5) of Theorem 2 with $\beta = (\pi/2)(\varepsilon + \lambda)$ and some $\alpha, \gamma$, we have

$$\sup_{0 < x < 1} |u(x) - u_N(x)| \leq C'(\log N) N^{B/\gamma + 3/2} \exp\left[\frac{-\pi d_\gamma N}{\log(\pi d_\gamma N / \beta)}\right].$$

Then at the Sinc nodes $S = \{x_j | j = -N, \ldots, N\}$, we have

$$\max_{-N \leq j \leq N} |w(x_j) - w_j| \leq C'(\log N) N^{B/\gamma + 3/2} \exp\left[\frac{-\pi d_\gamma N}{\log(\pi d_\gamma N / \beta)}\right].$$

Without the knowledge of the exact solution, we may be unable to determine the exact $\beta$, but one does not need optimal parameters for a good approximation. As we can see in the following section, in both examples we choose $\beta = \pi/2$ and the results perform like $\exp(-kN/\log N)$ for $k > 0$.

4. Numerical examples

We take two examples on the interval $[0, 1]$ to illustrate the performance of the SCMBT. In both examples, we take the double exponential transformation

$$x = \psi(t) = \frac{1}{2} \tanh\left(\frac{\pi}{2} \sinh t\right) + \frac{1}{2}.$$  

Let $E = \max_{-N \leq j \leq N} |w(x_j) - w_j|$ be the Sinc collocation absolute error on the Sinc points $S = \{x_j | j = -N, \ldots, N\}$. Let $E^0_d = |w'(0) - c_{-N-2}|$ and $E^1_d = |w'(1) - c_{N+2}|$.

Example 5. We consider the differential equation (see [1])

$$\begin{cases}
-w''(x) + \frac{1}{x} w'(x) + \frac{1}{x^2} w(x) = f(x), \\
w(0) - 2w'(0) = 1, \\
3w(1) + w'(1) = 9,
\end{cases} \quad (34)$$

where $f(x) = (\sqrt{x}/4)(-41x^2 + 34x - 1) - 2x + 1/x^2$ has the solution given by $w(x) = x^{5/2}(1 - x)^2 + x^3 + 1$. We apply SCMBT to the problem where the parameters $\beta, \gamma$ and $d$ are chosen as $\beta = \pi/2, \gamma = 2, d = \pi/4$. The errors are shown in Table 1. The results show that SCMBT can achieve high accuracy with little computational efforts. It can be observed that not only $E$ but also $E^0_d$ and $E^1_d$ converge to zero like $\exp(-kN/\log N)$.
Example 6. Consider the following homogeneous Dirichlet problem (see [6]):

\[ \varepsilon^2 w'' - w = -1, \quad 0 < x < 1, \]

\[ w(0) = w(1) = 0, \quad (35) \]

whose solution has a boundary layer in the neighborhoods of each of the points \( x = 0 \) and \( x = 1 \). The exact solution is given by

\[ w(x) = \frac{2 \sinh(x/2\varepsilon) \sinh((1 - x)/2\varepsilon)}{\cosh(1/2\varepsilon)}. \]

At first, by using \( \beta = \pi/2, \gamma = 2, d = \pi/4 \), which lead to \( h = \log(N\pi)/2N \), we apply SCMBT to the problem. The numerical errors for approximate first order derivatives on endpoints are given in Table 2 when \( \varepsilon = 0.01 \). Figs. 1 and 2 show \( \log_{10}(E) \) corresponding to \( N \) with \( \varepsilon = 0.01 \) and 0.001, which is denoted by SCMBT. Then we apply the Sinc collocation method described in Section 2.3 with \( l = 0 \) to the problem. In this method, we also choose \( \beta = \pi/2, \gamma = 2, d = \pi/4 \), but we define \( N_0 \equiv N + 1 \), which lead to \( h = \log(N_0\pi)/2N_0 = \log((N + 1)\pi)/2(N + 1) \). To show the performance of this method, \( \log_{10}(E) \) corresponding to \( N \) with \( \varepsilon = 0.01 \) and 0.001 is denoted by SCM in Figs. 1 and 2. We can observe that SCMBT performs better than SCM.

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### Table 2
The errors for approximate derivatives on boundaries in Example 6, with \( \varepsilon = 0.01 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E_d^0 )</td>
<td>0.0075</td>
<td>5.4590e-004</td>
<td>5.7995e-005</td>
</tr>
<tr>
<td>( E_d^1 )</td>
<td>0.0075</td>
<td>5.4590e-004</td>
<td>5.7995e-005</td>
</tr>
</tbody>
</table>

Fig. 1. \( \log_{10}(E) \) in Example 6 with \( \varepsilon = 0.01 \).
We want to point out that the condition number of the matrices in SCMBT may be large. In the examples, the equations are solved by using MATLAB with double precision.

5. Comments

In this paper, Sinc collocation method with boundary treatment was developed for two-point boundary value problems. The advantages of SCMBT are that the solution and its derivatives on boundaries can be easily treated and the method can solve problems with different boundary conditions.

SCMBT can directly treat the boundary derivatives, so the method can be easily applied to the patching domain decomposition method. We also expect SCMBT can be extended to boundary value problems for partial differential equations. This is left for another paper.

References