Stochastic Processes and their Applications 27 (1988) 43-56
North-Holland

ON THE LARGE VALUES OF THE WIENER PROCESS

Endre CSÁKI

Mathematical Institute of the Hungarian Academy of Sciences, Realtanoda u. 13-15, 1053 Budapest, Hungary

Karl GRILL

Institut für Statistik und Wahrscheinlichkeitstheorie, TU Wien, Wiedner Hauptstrasse 8-10. 1040 Wien, Austria

Received 5 March 1987
Revised 15 July 1987

Let $(W(t), t \geq 0)$, be a standard Wiener process and define

$$M^+(t) = \max\{W(u): u \leq t\},$$

$$M^-(t) = \max\{-W(u): u \leq t\},$$

$$Z(t) = \max\{u \leq t: W(u) = 0\}.$$

We investigate the asymptotic behaviour of $Z(t)$ and $M^-(t)$ under the condition that $M^+(t)$ (or, equivalently, $W(t)$) gets very large, i.e. as large as indicated by the law of iterated logarithm.

AMS 1980 Subject Classifications: Primary 60F15; Secondary 60J65, 60G17.

Wiener process * strong laws * law of iterated logarithm * strong approximation * partial sum process

1. Introduction

Let $(W(t), t \geq 0)$ be a standard Wiener process defined on some probability space $(\Omega, F, \mathbf{P})$. For $t \geq 0$, define

$$M^+(t) = \max\{W(u): u \leq t\},$$

$$M^-(t) = \max\{-W(u): u \leq t\},$$

$$M(t) = \max\{|W(u)|: u \leq t\} = \max(M^+(t), M^-(t)),$$

$$Z(t) = \max\{u \leq t: W(u) = 0\}.$$

In order to present the results of our paper in a pleasant form, it is worthwhile to recall some definitions; see, e.g. Révész [6, 7].

Research Supported by Hungarian National Foundation for Scientific Research grant No. 1808.

Let $X(t)$ be a stochastic process. Then we formulate:

Definition 1. The function $f_1(t)$ belongs to the upper-upper class of $X(t)$ ($f_1 \in \mathrm{UUC}(X(t))$) if $X(t) \leqslant f_1(t)$ a.s. for all t large enough.

Definition 2. The function $f_2(t)$ belongs to the upper-lower class of $X(t)$ ($f_2 \in \mathrm{ULC}(X(t))$) if $X(t) > f_2(t)$ a.s. i.o.

Definition 3. The function $f_3(t)$ belongs to the lower-upper class of $X(t)$ ($f_3 \in \mathrm{LUC}(X(t))$) if $X(t) < f_3(t)$ a.s. i.o.

Definition 4. The function $f_4(t)$ belongs to the lower-lower class of $X(t)$ ($f_4 \in \mathrm{LLC}(X(t))$) if $X(t) \geqslant f_4(t)$ a.s. for all t large enough.

For each of the variables we consider, a precise description of the four classes is known. Let us cite two theorems of that kind which will be important in the sequel.

Theorem A (Kolmogorov-Erdös-Feller-Petrowski). *Let* $f(t) = \psi(t)t^{1/2}$ *with* $\psi(t) \uparrow \infty$. *Furthermore, let*

$$I_0(\psi) = \int_1^\infty t^{-1}\psi(t)\exp(-\psi^2(t)/2)\,\mathrm{d}t.$$

Then $f(t) \in \mathrm{UUC}(W(t),\ t \geqslant 0)$ *iff* $I_0(\psi) < \infty$.

This remains true if we replace W by M^+, M^-, or M.

Theorem B (Chung-Erdös). *Let* $g(x) \uparrow \infty$ *and*

$$\hat{I}(g) = \int_1^\infty x^{-1}g^{-1/2}(x)\,\mathrm{d}x.$$

Then $tg^{-1}(t) \in \mathrm{LLC}(Z(t),\ t \geqslant 0)$ *iff* $\hat{I}(g) < \infty$.

We shall investigate how the asymptotics of some of these quantities change if we know that, simultaneously, another one gets extremely large or small.

Some similar work has been done by Csáki [3] who asks how small $M^+(t)$ and $M^-(t)$ can simultaneously get, and by Csáki, Földes and Révész [4] who investigate $M(t)$ and its location $v(t)$ which is the largest $u \leqslant t$ such that $M(t) = |W(u)|$.

2. Formulation of the theorems

In the sequel, $f(t)$ will be a fixed function satisfying the conditions of Theorem A for ULC($W(t)$, $t \geq 0$), i.e.

$$f(t) = t^{1/2}\psi(t)$$

with

$$\psi(t) \uparrow \infty$$

and

$$I_0(\psi) = \infty.$$

Define $T_1 = \{t \geq 0\colon W(t) \geq f(t)\}$.

Furthermore, let $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$ be real functions satisfying the following conditions:

$$\alpha(t), \gamma(t) \text{ monotone,} \tag{1}$$

$$0 < \alpha(t) < 1, \tag{2}$$

$$\beta(t), \delta(t) \downarrow 0 \tag{3}$$

$$t^{1/2}\gamma(t) \uparrow \infty, \quad t^{1/2}\delta(t) \uparrow \infty, \tag{4}$$

$$t\alpha(t) \uparrow \infty, \quad t\beta(t) \uparrow \infty, \tag{5}$$

Then we have:

Theorem 1

$$t\alpha(t) \in \text{UUC}(Z(t), t \in T_1) \quad \textit{iff} \quad I_1(\alpha, \psi) < \infty,$$

where

$$I_1(\alpha, \psi) = \int_1^\infty \frac{1}{\sqrt{\alpha(t)(1-\alpha(t))}} \exp\left(-\frac{\psi^2(t)}{2(1-\alpha(t))}\right) \frac{\mathrm{d}t}{t}.$$

Theorem 2

$$t\beta(t) \in \text{LLC}(Z(t), t \in T_1) \quad \textit{iff} \quad I_2(\beta, \psi) < \infty,$$

where

$$I_2(\alpha, \psi) = \int_1^\infty \theta_1^\infty\, \psi^2(t)\beta^{1/2}(t) \exp\left(-\frac{\psi^2(t)}{2}\right) \frac{\mathrm{d}t}{t}.$$

Theorem 3

$$g(t) = t^{1/2}\gamma(t) \in \text{UUC}(M^-(t), t \in T_1) \quad \textit{iff} \quad f(t) + 2g(t) \in \text{UUC}(W(t), t \geq 0).$$

Theorem 4

$$t^{1/2}\delta(t) \in \mathrm{LLC}(M^-(t), t \in T_1 \quad \textit{iff} \quad I_3(\delta, \psi) < \infty,$$

where

$$I_3(\delta, \psi) = \int_1^\infty \delta(t)\psi^2(t) \exp\left(-\frac{\psi^2(t)}{2}\right) \frac{\mathrm{d}t}{t}.$$

Remarks. (1) In these theorems, T_1 can be replaced by $T_2 = \{t \geq 0\colon M^+(t) \geq f(t)\}$, but in Theorems 1 and 3 we have to assure that $\psi(t)$ is not too small. In fact, if $2f(t) \in \mathrm{ULC}(W(t),\ t \geq 0)$, then one easily verifies that the Wiener process can go up as far as $f(t)$ and still return to zero up to time t, so Theorem 1 breaks down altogether. As for Theorem 3, there is a similar situation, but it can be restated as follows:

Theorem 3*

$$g(t) \in \mathrm{UUC}(M^-)(t), t \in T_2)$$
$$\textit{iff} \quad \min(f(t) + 2g(t), g(t) + 2f(t)) \in \mathrm{UUC}(W(t), t \geq 0).$$

In case $3f(t) \in \mathrm{UUC}(W(t),\ t \geq 0)$, this reverts back to our original Theorem 3.

(2) One can turn Theorems 1 to 4 around in order to get theorems about how big $W(t)$ can get if $Z(t)$ or $M^-(t)$ is small or large, and, again, $W(t)$ can be replaced by $M^+(t)$ in these theorems. There is, however, again the problem that for small $\psi(t)$ this replacement is not possible. Also, the results obtained may not be very surprising (consider, e.g., the case $Z(t) = t$), so we shall not go into this.

In order to give some impression of the meaning of our theorems, let us put $\psi(t) = \sqrt{(2-\varepsilon) \log\log t}$. Then Theorems 1 to 4 imply that if t varies over the set of those points where $W(t) \geq \sqrt{(2-\varepsilon) t \log\log t}$, then the following conclusions hold with probability one:

1. $\limsup_{t \in T_1}(Z(t)/t) = \varepsilon/2$, which could also be obtained from Strassen's law of the iterated logarithm.
2. $Z(t) \leq t \log^{-\eta} t$ infinitely often iff $\eta \leq \varepsilon$.
3. $\limsup_{t \in T_1}(M^-(t)/\sqrt{2t \log\log t}) = (1 - \sqrt{1 - \varepsilon/2})/2$, which again follows from Strassen's law,
4. $M^-(t) \leq t^{1/2} \log^{-\eta/2} t$ infinitely often iff $\eta \leq \varepsilon$.

Note that 2 and 4 are somewhat surprising since we know that large values of $M(t)$ go with small values of $Z(t)$ (and $M^-(t)$). Hence intuition might suggest that if $t^{1/2}\psi(t) \in \mathrm{ULC}(W(t),\ t \geq 0)$ and $t\beta(t) \in \mathrm{LUC}(Z(t), t \geq 0)$ then the events $\{W(t) \geq t^{1/2}\psi(t)\}$ and $\{Z(t) \leq t\beta(t)\}$ can occur infinitely often with probability 1 simultaneously as $t \to \infty$. Our results, however, show that this is not the case.

A particular instance of Theorem 3 that is of particular interest is the following

Consequence 1. *Let* $V(t)=\min(M^{+}(t), M^{-}(t))$. *Then* $f(t)\in \mathrm{UUC}(V(t),\ t\geq 0)$ *iff* $3f(t)\in \mathrm{UUC}(W(t), t\geq 0)$.

Finally, let us state one more consequence of our theorems. To this end, let $X_1, X_2, \ldots$ be a sequence of i.i.d.r.v.'s with $\mathbf{E}X_1=0$, $\mathbf{E}X_1^2=1$, and $\mathbf{E}X_1^{2+\delta}<\infty$. In this case, it follows from Skorohod's strong embedding scheme that we can define a version of the sequence X_1, $X_2, \ldots$ together with a Wiener process such that

$$|W(n)-S_n|=\mathrm{o}(n^{1/(2+\delta)}+\varepsilon)$$

and

$$|Z(n)-T_n|=\mathrm{o}(n^{2/(2+\delta)}+\varepsilon),$$

where

$$S_n=X_1+X_2+\cdots+X_n,, \quad S_0=0,$$

and

$$T_n=\max\{k<n\colon S_kS_{k+1}\leq 0\}.$$

This implies that we have the following

Consequence 2. *Theorems* 1 *to* 4 *remain valid if in their statements* $W(t)$ *and* $Z(t)$ *are replaced by* S_n *and* T_n, *respectively.*

3. The proofs

Only Theorem 1 will be proved in detail; as to the others, we shall only give the basic estimates, since the further calculations are largely the same as those we make in proving Theorem 1, in fact somewhat simpler. So we think we can forgo boring the reader by going through them.

In order to simplify notation, in the sequel C (with or without index) will always denote an absolute constant whose value may vary from one occurrence to the other, such that, e.g., notations like $C=C+1$ are possible. We shall use the notation $f(x)\equiv g(x)$ with the meaning that a relation $C_1g(x)\leq f(x)\leq C_2g(x)$ with positive constants C_1, C_2 holds.

Lemma 1. *Let* $0\leq t_1<t$ *and* x *be real numbers, and* $A\subset[t_1, t]$ *and* b *be Borel sets. Then*

$$\mathbf{P}(W(t)\in B, Z(t)\in A|\, W(t_1)=x)=\int_{A\times B} f(u, v|t_1, x)\,\mathrm{d}u\,\mathrm{d}v \tag{6}$$

with

$$f(u, v|t_1, x)=\frac{|v|}{2\pi\sqrt{(u-t_1)(t-u)^3}}\exp\left(-\tfrac{1}{2}\left(\frac{x^2}{u-t_1}+\frac{v^2}{t-u}\right)\right). \tag{7}$$

Remark. Clearly, the unconditional distribution of $W(t)$ and $Z(t)$ is obtained by setting $t_1 = x = 0$ in equations (6) and (7).

Proof. We shall make use of the technique used by Billingsley [1, pages 80-83] to obtain the distribution of $Z(t)$. Namely, we calculate the distribution of related quantities for the sum of Bernoulli $(-1, 1)$ random variables, and by normalizing these and passing to the limit, the distribution we seek is obtained.

So let $(X_n, n \in \mathbb{N})$ be a sequence of i.i.d.r.v.'s with

$$\mathbf{P}(X_n = 1) = \mathbf{P}(X_n = -1) = \tfrac{1}{2}$$

and let

$$S_n = \sum_{k=1}^{n} X_k,$$

where we interpret the empty sum as zero. Furthermore, let

$$Y_n = \max\{k \leq n \colon S_k = 0\}.$$

To avoid parity problems in the sequel, let us assume that n, m, n_1, k, l are all even; for all other combinations that yield positive probabilities, similar calculations can be carried out. So, let us now calculate

$$\begin{aligned}
&\mathbf{P}(S_n = m, Y_n = l \mid S_{n_1} = k) \\
&\quad = \mathbf{P}(S_n = m, S_l = 0, \forall j > l \colon S_j > 0 \mid S_{n_1} = k) \\
&\quad = \mathbf{P}(S_l = 0 \mid S_{n_1} = k)\mathbf{P}(S_n = m \mid S_1 = 0)\mathbf{P}(\forall j > l \colon S_j > 0 \mid S_l = 0, S_n = m) \\
&\quad = \binom{l - n_1}{\frac{l - n_1 + k}{2}} 2^{n_1 - l} \binom{n - l}{\frac{n - l + m}{2}} 2^{l - n} \frac{|m|}{n - l}.
\end{aligned}$$

Now putting

$$m = v\sqrt{N},$$

$$k = x\sqrt{N}, \qquad l = uN, \qquad n = tN, \qquad n_1 = t_1 N,$$

for $N \to \infty$, the latter probability is asymptotically equal to $4N^{-3/2} f(u, v)$, so an application of Donsker's Theorem [1, page 68] finally proves Lemma 1.

Corollary 1. *If $x \geq \sqrt{t_1}$ and $A \subset [t_1, t]$, then*

$$\mathbf{P}(W(t) \in B, Z(t) \in A \mid W(t_1) = x) \leq \mathbf{P}(W(t) \in B, Z(t) \in A).$$

Proof. It follows from the Remark to Lemma 1 that it suffices to prove that

$$f(u, v \mid t_1, x) \leq f(u, v \mid 0, 0).$$

This is equivalent to

$$\frac{1}{\sqrt{u-t_1}}\exp\left(-\frac{x^2}{2(u-t_1)}\right)\leqslant\frac{1}{\sqrt{u}},$$

which in turn is equivalent to

$$\frac{u}{u-t_1}\leqslant\exp\left(\frac{x^2}{u-t_1}\right).$$

For $x\geqslant\sqrt{t_1}$, the last inequality is a consequence of the elementary $e^x\geqslant 1+x$.

Lemma 2. *Let $x>0$ and $0<y<1$ be chosen in such a way that $x^2y/1-y$ is bounded away from zero. Then*

$$\mathbf{P}(Z(t)\geqslant ty,\ W(t)\geqslant x\sqrt{t})\asymp\frac{(1-y)^{3/2}}{x^2y^{1/2}}\exp\left(-\frac{x^2}{2(1-y)}\right). \tag{8}$$

Proof. It follows by elementary calculations from Lemma 1 that

$$\mathbf{P}(Z(t)\geqslant ty,\ W(t)\geqslant x\sqrt{t})=\frac{\exp(-x^2/2)}{\pi}\int_{\sqrt{y/(1-y)}}^{\infty}\frac{\mathrm{d}v}{1+v^2}\exp\left(-\frac{x^2v^2}{2}\right).$$

The integral on the right-hand side can be estimated above by multiplying both numerator and denominator of the integrand by v, then replacing v in the denominator by $\sqrt{y/(1-y)}$. The lower estimate is obtained by replacing the upper limit of integration by $\sqrt{y/(1-y)+1/x^2}$. For v in this range the integrator is still $\geqslant C(1-y)\exp(-x^2y/(2(1-y))$, so Lemma 2 is proved by multiplying this expression by the length of the interval of integration, which is of order $(1/x^2)\sqrt{(1-y)/y}$.

Lemma 3. *If x is bounded away from zero and x^2y is bounded above then*

$$\mathbf{P}(Z(t)\geqslant ty,\ W(t)\geqslant\geqslant x\sqrt{t})\asymp\sqrt{y}\exp\left(-\frac{x^2}{2}\right).$$

For M^-, simple reflection arguments yield the following two lemmas:

Lemma 4. *If x and y are both bounded away from zero, then*

$$\mathbf{P}(M^-(t)\geqslant x\sqrt{t},\ W(t)\geqslant y\sqrt{t})=1-\Phi(2x+y)\asymp\frac{1}{2x+y}\exp\left(-\frac{(2x+y)^2}{2}\right).$$

Lemma 5. *If xy is bounded away from infinity, then*

$$\mathbf{P}(M^-(t)\leqslant x\sqrt{t},\ W(t)\geqslant y\sqrt{t})=\Phi(2x+y)-\Phi(y)\asymp x\exp\left(-\frac{y^2}{2}\right).$$

Remark. Lemmas 3 to 5 are given only for reference. They are needed in the proofs of Theorems 2 to 4, respectively.

For the remainder of this section, let us define

$$\rho(t)=\frac{\psi(t)}{1-\alpha(t)}. \tag{9}$$

For the time being, we shall demand the additional regularity condition

$$\rho(t)t^{-1/4}\downarrow 0. \tag{10}$$

We shall show later that this condition does not entail any significant loss of generality. Yet, it will help to make some of our arguments easier.

Now, let us define a sequence (t_k) in the following way: Fix $t_0>0$ and let, for all k,

$$t_{k+1}=t_k\left(1+\frac{1}{\rho^2(t_k)}\right).$$

The following lemma will be used on various occasions.

Lemma 6. *There are positive constants a and b such that for every nonincreasing nonnegative function f we have*

$$a\int_{t_n}^{\infty}\frac{\rho^2(t)}{t}f(t)\,\mathrm{d}t\leq\sum_{k=n}^{\infty}f(t_k)\leq b\int_{t_{n+1}}^{\infty}\frac{\rho^2(t)}{t}f(t)\,\mathrm{d}t.$$

The proof is elementary and will be omitted. In order to prove Theorem 1, let us now assume that $I_1(\alpha,\psi)<\infty$. Define the events

$$A_k=\{t_{k-1}\alpha(t_{k-1})\leq Z(t_k)\leq t_{k-1},\ \max_{t_{k-1}\leq t\leq t_k}W(t)\geq\psi(t_{k-1})\sqrt{t_{k-1}}\}$$

and

$$B_k=\{t_{k-1}\leq Z(t_k),\ \max_{t_{k-1}\leq t\leq t_k}W(t)\geq\psi(t_{k-1})\sqrt{t_{k-1}}\}.$$

By Lemma 2 and the reflection principle it follows that

$$\mathbf{P}(A_k)\leq C\frac{(1-\alpha(t_{k-1}))^{3/2}}{\psi^2(t_{k-1})\alpha^{1/2}(t_{k-1})}\exp\left(-\frac{\psi^2(t_{k-1})}{2(1-\alpha(t_{k-1}))}\right).$$

Also, by the reflection principle,

$$\begin{aligned}\mathbf{P}(B_k)&\leq C\left(1-\Phi\left(\frac{\psi(t_{k-1})\sqrt{t_{k-1}}}{\sqrt{t_k-t_{k-1}}}\right)\right)\leq C(1-\Phi(\psi(t_{k-1})\rho(t_{k-1})))\\&=\mathrm{o}(\mathbf{P}(A_k)).\end{aligned}$$

So, it follows from Lemma 6 that

$$\sum_{k=1}^{\infty}\mathbf{P}(A_k\cup B_k)\leq\sum_{k=1}^{\infty}(\mathbf{P}(A_k)+\mathbf{P}(B_k))<\infty.$$

The Borel-Cantelli lemma now implies that only finitely many of the events A_k and B_k can occur with probability one, which is clearly equivalent to $t\alpha(t) \in \mathrm{UUC}(Z(t), t \in T_1)$. Thus one part of Theorem 1 is proved.

For the second part, now let $I_1(\alpha, \psi) = \infty$. We define the events

$$D_k = \left\{ t_k\left(\alpha(t_k) + \frac{1}{\rho^2(t_k)} \right) \leq Z(t_k) \leq t_k(\alpha(t_k) + \frac{2}{\rho^2(t_k)}), \; W(t_k) \geq \psi(t_k)\sqrt{t_k} \right\}.$$

We shall prove that with probability one infinitely many of the events D_k occur. To this end, we shall make use of the Borel-Cantelli lemma in the following form:

Lemma 7. *Let $(A_k, k \in \mathbb{N})$ be a sequence of events satisfying the following conditions*

(i) $\sum_{k=1}^{\infty} \mathbf{P}(A_k) = \infty$,

(ii) $\lim_{n\to\infty} \frac{\sum_{k=1}^n \sum_{r=1}^n \mathbf{P}(A_k A_r)}{(\sum_{k=1}^n \mathbf{P}(A_k))^2} \leq 1.$

Then

$$\mathbf{P}(A_k \text{ i.o.}) = 1.$$

Concerning the condition (i) of Lemma 7, Lemma 1 implies that

$$\mathbf{P}(D_k) \geq C \frac{(1-\alpha(t_k))^{3/2}}{\psi^2(t_k)\alpha^{1/2}(t_k)} \exp\left(-\frac{\psi^2(t_k)}{2(1-\alpha(t_k))} \right).$$

Applying Lemma 6, again we obtain

$$\sum_{k=1}^{\infty} \mathbf{P}(D_k) = \infty.$$

Turning to the second condition, let $k < r$ and consider

$$P(D_k D_r) = \mathbf{P}(D_k D_r \{Z(t_r) > t_k\}) + \mathbf{P}(D_k D_r \{Z(t_r) \leq t_k\}).$$

The Corollary to Lemma 1 implies that

$$\mathbf{P}(D_k D_r \{Z(t_r) > t_k\}) \leq \mathbf{P}(D_k)\mathbf{P}(D_r). \tag{11}$$

It remains to estimate

$$\mathbf{P}(D_k D_r \{Z(t_r) \leq t_k\}).$$

By assumption (10) this probability is zero if k is large enough and $t_r > 4t_k$, because the definition of D_k demands that, on one hand,

$$Z(t_r) = Z(t_k) \geq t_r\left(\alpha(t_r) + \frac{1}{\rho^2(t_r)} \right),$$

and, on the other hand,

$$Z(t_k) \leqslant t_k\left(\alpha(t_k) + \frac{2}{\rho^2(t_k)}\right).$$

Now

$$t_k\alpha(t_k) \leqslant t_r\alpha(t_r)$$

by monotonicity of $t\alpha(t)$, and by assumption (10),

$$\frac{t_r}{\rho^2(t_r)} > \frac{2t_k}{\rho^2(t_k)},$$

so we even have

$$D_kD_r\{Z(t_r) \leqslant t_k\} = \emptyset.$$

For $t_r \leqslant 4t_k$, we distinguish the two cases that α is nondecreasing or nonincreasing, respectively. In the first case, there is an $\alpha_0 > 0$ such that $\alpha(t) \geqslant \alpha_0$ for all t. Furthermore,

$$t_r \geqslant t_k\left(1 + \frac{r-k}{\rho^2(t_r)}\right) \quad \text{and} \quad \rho(t_r) \leqslant \sqrt{2}\rho(t_k).$$

So, finally,

$$t_r\left(\alpha(t_r) + \frac{1}{\rho^2(t_r)}\right) \geqslant t_k\left(\alpha(t_k) + \frac{1 + (r-k)\alpha_0}{2\rho^2(t_k)}\right).$$

If $r > k + 4/\alpha_0$, then this implies that again $D_kD_r\{Z(t_k) \leqslant t_k\} = \emptyset$, so we finally obtain

$$\sum_{r>k} \mathbf{P}(D_kD_r\{Z(t_r) \leqslant t_k\}) \leqslant \frac{4}{\alpha_0}\mathbf{P}(D_k). \tag{12}$$

Now, assume that α is nonincreasing. In this case, we have

$$\begin{aligned}
&\mathbf{P}(D_kD_r\{Z(t_r) \leqslant t_k\}) \\
&\quad \leqslant \mathbf{P}(D_k\{W(t_r) \geqslant \psi(t_r)\sqrt{t_r}\})\}) \leqslant \mathbf{P}(\mathrm{D}_k\{\mathrm{W}(t_r) \geqslant \psi(t_k)\sqrt{t_r}\}) \\
&\quad \leqslant \int_{\psi/(t_k)}^{\infty} \exp(-\xi^2/2)\,\mathbf{P}(t_k\alpha(t_k) \leqslant Z(t_k) \mid W(t_k) = \xi\sqrt{t_k}) \\
&\qquad \times \mathbf{P}(W(t_r) \geqslant \psi(t_k)\sqrt{t_k} \mid W(t_k) = \xi\sqrt{t_k})\,\mathrm{d}\xi.
\end{aligned}$$

From Lemma 1, it follows that

$$\begin{aligned}
\mathbf{P}(Z(t_k) \geqslant \alpha(t_k)t_k \mid W(t_k) = \xi\sqrt{t_k}) &= 2\left(1 - \Phi\left(\frac{\xi\sqrt{\alpha(t_k)}}{\sqrt{1-\alpha(t_k)}}\right)\right) \\
&\leqslant 2\left(1 - \Phi\left(\frac{\psi(t_k)\sqrt{\alpha(t_k)}}{\sqrt{1-\alpha(t_k)}}\right)\right),
\end{aligned}$$

so

$$\mathbf{P}(D_k\{W(t_r)\geq \psi(t_k)\sqrt{t_r}\})\leq \sqrt{2/\pi}\left(1-\Phi\left(\psi(t_k)\frac{\sqrt{\alpha(t_k)}}{\sqrt{1-\alpha(t_k)}}\right)\right)$$
$$\times\int_{\psi(t_k)}^{\infty} \mathrm{e}^{-\xi^2/2}\,\mathrm{d}\xi\left(1-\Phi\left(\frac{\psi(t_k)\sqrt{t_r}-\xi\sqrt{t_k}}{\sqrt{t_r-t_k}}\right)\right). \quad (13)$$

The last integral can be estimated in the following way:

$$\int_{\psi(t_k)}^{\infty} \mathrm{e}^{-\xi^2/2}\,\mathrm{d}\xi\left(1-\Phi\left(\frac{\psi(t_k)\sqrt{t_r}-\xi\sqrt{t_k}}{\sqrt{t_r-t_k}}\right)\right)$$
$$\leq \frac{1}{\psi(t_k)}\int_{\psi(t_k)}^{\infty} \mathrm{e}^{-\xi^2/2}\xi\,\mathrm{d}\xi\left(1-\Phi\left(\frac{\psi(t_k)\sqrt{t_r}-\xi\sqrt{t_k}}{\sqrt{t_r-t_k}}\right)\right)$$
$$=\frac{1}{\psi(t_k)}\exp\left(-\frac{\psi^2(t_k)}{2}\left(1+\sqrt{\frac{t_k}{t_r}}\right)\left(1-\Phi\left(\psi(t_k)\frac{\sqrt{t_r}-\sqrt{t_k}}{\sqrt{t_r-t_k}}\right)\right)\right). \quad (14)$$

Now, (13) and (14) together imply that

$$\mathbf{P}(D_kD_r\{Z(t_r)\leq t_k\})\leq C\mathbf{P}(D_k)\left(1-\Phi\left(\psi(t_k)\frac{\sqrt{t_r}-\sqrt{t_k}}{\sqrt{t_r-t_k}}\right)\right).$$

Summing over τ, we obtain:

$$\sum_{r>k}\mathbf{P}(D_kD_r\{Z(t_r)\leq t_k\}\leq C\mathbf{P}(D_k)\sum_{t_k<t_r\leq 4t_k}\left(1-\Phi\left(\psi(t_k)\frac{\sqrt{t_r}-\sqrt{t_k}}{\sqrt{t_r-t_k}}\right)\right)$$

We estimate this sum by an application of Lemma 6:

$$\sum_{t_k<t_r\leq 4t_k}\left(1-\Phi\left(\psi(t_k)\frac{\sqrt{t_r}-\sqrt{t_k}}{\sqrt{t_r-t_k}}\right)\right)$$
$$\leq \int_{t_k}^{4t_k}\left(1-\Phi\left(\psi(t_k)\frac{\sqrt{t_r}-\sqrt{t}}{\sqrt{t_r-t}}\right)\right)\frac{\rho^2(t)\,\mathrm{d}t}{t}.$$

As α is nonincreasing, it is in particular bounded away from one, and together with (10) this means that for $t_k\leq t\leq 4t_k$ we have

$$\rho(t)\leq C\rho(t_k)\leq C\psi(t_k).$$

Inserting this in the last integral and making the substitution

$$u=\psi(t_k)\frac{\sqrt{t}-\sqrt{t_k}}{\sqrt{t-t_k}},$$

we finally obtain that this integral is bounded, which implies that

$$\sum_{r>k}\mathbf{P}(D_kD_r\{Z(t_r)\leq t_k\})\leq C\mathbf{P}(D_k) \quad (15)$$

in this case too.

For nonincreasing α, (11) and (15) together imply that

$$\sum_{k=1}^{N}\sum_{r=1}^{N}\mathbf{P}(D_kD_r)\leq\sum_{k=1}^{N}\sum_{r=1}^{N}\mathbf{P}(D_k)\mathbf{P}(D_r)+C\sum_{k=1}^{N}\mathbf{P}(D_k).$$

For nondecreasing α the same follows from (11) and (12). Thus in both case the hypothesis of Lemma 7 is fulfilled, and we can conclude that with probability one infinitely many events D_n occur, which clearly implies that infinitely often $Z(t)\geq t\alpha(t)$ and $W(t)\geq\psi(t)\sqrt{t}$ hold simultaneously, which means nothing but $\alpha\in \mathrm{ULC}(Z(t),\ t\in T_1)$.

We have now completely proved Theorem 1 under the additional regularity assumption (10). It remains to show that the general case can be reduced to this special case.

First assume that

$$\liminf_{t\to\infty}\frac{\psi^2(t)\alpha(t)}{1-\alpha(t)}<1.$$

This can, of course, only happen if $\alpha\to 0$. In this case we can find arbitrarily large t such that

$$\frac{\psi^2(t)\alpha(t)}{1-\alpha(t)}\leq 1.$$

If we have such a t, then for all u between $t/2$ and t, we have

$$\frac{\psi^2(u)\alpha(u)}{1-\alpha(u)}\leq\frac{1}{1-\alpha_0}, \tag{16}$$

where $\alpha_0=\max_{t\geq 0}\alpha(t)$. We define a sequence θ_j in the following way: θ_0 is chosen so that

$$\frac{\psi^2(\theta_0)\alpha(\theta_0)}{1-\alpha(\theta_0)}\leq 1,$$

and, for $k>0$, θ_k is the least $t>2\theta_{k-1}$ for that

$$\frac{\psi^2(t)\alpha(t)}{1-\alpha(t)}\leq 1.$$

Now, let T_0 be the union of all intervals $[\theta_k/2,\ \theta_k]$. If, in addition, the integral

$$\int_{T_0}\frac{\psi(t)}{t}\exp\left(-\frac{\psi^2(t)}{2}\right)\mathrm{d}t$$

diverges, then a modification of Theorem A yields that for infinitely many $t\in T_0$ we have $W(t)\geq\psi(t)\sqrt{t}$. For these t, however, (16) implies that the conditional probability of $Z(t)\geq t\alpha(t)$, given that $W(t)>\psi(t)\sqrt{t}$, is greater than some positive constant. A simple argument based on the law of large numbers then yields that infinitely often both $W(t) \geqslant \psi(t)\sqrt{t}$ and $Z(t) \geqslant t\alpha(t)$ simultaneously. On the other hand, one readily checks that $I_1(\psi, \alpha) = \infty$, so the statement of Theorem 1 holds in this case too.

If the integral

$$\int_{T_0} \frac{\psi(t)}{t} \exp\left(-\frac{\psi^2(t)}{2}\right) dt$$

is finite, then one easily finds that removing T_0 from the range of integration in the definition of $I_1(\alpha, \psi)$ does not change its convergence behaviour, nor does the removal of those k for which $t_k \in T_0$ change the convergence behaviour of the sums that figure in the proof of Theorem 1. Also, from Theorem A we obtain that for $t \in T_0$ we have $W(t) < \psi(t)\sqrt{t}$ eventually with probability one. This means that in the sequel we can restrict our attention to $t \notin T_0$, or, in other words, we may assume that

$$\liminf_{t\to\infty} \frac{\psi^2(t)\alpha(t)}{1-\alpha(t)} \geqslant 1.$$

Define now the set

$$K_0 = \{k: \rho(t_{k+1}) > 2\rho(t_k)\} = \{k(n), n \in \mathbb{N}\},$$

where the $k(n)$ are in increasing order.

Clearly,

$$\rho(t_{k(n)}) > 2^n \rho(t_{k(0)}),$$

and

$$\mathbf{P}(A_{k(n)}) \equiv \frac{1}{\rho(t_{k(n)})} \frac{\sqrt{1-\alpha(t_{k(n)})}}{\psi(t_{k(n)})\sqrt{\alpha(t_{k(n)}}} \exp\left(-\frac{\psi(t_{k(n)})\rho(t_{k(n)})}{2}\right)$$
$$\leqslant \frac{C}{\rho(t_{k(n)})} \leqslant C2^{-n},$$

so

$$\sum_{k\in K_0} \mathbf{P}(A_k) < \infty.$$

Similar calculations can be carried out for the events B_k and D_k. This, however, implies that there is no loss of generality if in the proof of Theorem 1 we carry out all summations only over $k \notin K_0$. This implies in particular that in the following calculations the use of Lemma 6 is justified.

Now, define

$$K_1 = \{k \notin K_0: \exists r < k: r \notin K_0, t_k^{-1/4}\rho(t_k) \geqslant t_r^{-1/4}\rho(t_r)\}$$
$$= \{k'(n), n \in \mathbb{N}\}.$$

$$\mathbf{P}(A_{k'(n)}) \equiv \frac{1}{\rho(t_{k'(n)})} \frac{\sqrt{1-\alpha(t_{k'(n)})}}{\psi(t^{k'(n)})\sqrt{\alpha(t_{k'(n)})}} \exp\left(-\frac{\psi(t_{k'(m)})\rho(t_{k'(n)})}{2}\right)$$
$$\leqslant \exp(-C\rho(t_{k'(n)})) \leqslant \exp(-Cn^{1/4}),$$

and again

$$\sum_{k \in K_1} \mathbf{P}(A_k) < \infty.$$

Here, too, the same holds for B_k and D_k. As before, this means that we can restrict all summations in the proof of Theorem 1 to $k \notin K_1$. One also readily verifies that this restriction does not affect the convergence of the integrals figuring in our proof. Thus Theorem 1 is completely proved.

Finally, let us say a few words about the proofs of the other theorems. The outline of these is the same as that of the one given here for Theorem 1. Of course, the basic probability estimates will be taken from Lemma 3, 4, or 5, respectively. One more point that may present some difficulty is the definition of the sequence t_k. In this definition, ρ is to be replaced by $\psi + 2\gamma$ in the case of Theorem 3, and by ψ for Theorems 2 and 4. In the condition (10) we have to replace ρ by these quantities, too. By the same method that we used for Theorem 1 in the preceding paragraphs, these regularity conditions can be shown to entail no loss of generality.

References

[1] P. Billingsley, Convergence of Probability Measures (Wiley, New York, 1968).

[2] K.L. Chung and P. Erdös, On the application of the Borel-Cantelli lemma, Trans. Am. Math. Soc. 72 (1952) 179-186.

[3] E. Csáki, On the lower limits of maxima and minima of Wiener process and partial sums, Z. Wahrsch. Verw. Geb. 43 (1978) 205-222.

[4] E. Csáki, A. Földes and P. Révész, On the maximum of a Wiener process and its location, Z. Wahrsch. Verw. Geb. (to appear).

[5] W. Feller, An Introduction to Probability Theory and its Application, Vol. II (Wiley, New York, 1966).

[6] P. Révész, A note to the Chung-Erdös-Sirao theorem, in: L.M. Chakravarty, Asymptomic Theory of Statistical Tests and Estimation (Academic, New York, 1980); 147-158.

[7] P. Révesz, On the increments of Wiener and related processes, Ann. Probab. 10 (1982) 613-622.