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Self-maps of the product of two spheres fixing the diagonal

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ABSTRACT

We compute the monoid of essential self-maps of $S^n \times S^n$ fixing the diagonal. More generally, we consider products $S \times S$, where S is a suspension. Essential self-maps of $S \times S$ demonstrate the interplay between the pinching action for a mapping cone and the fundamental action on homotopy classes under a space. We compute examples with non-trivial fundamental actions.

1. Introduction

This paper investigates self-maps of $S^n \times S^n$ fixing the diagonal $\Delta : D = S^n \to S^n \times S^n$. More precisely, we consider maps $f : S^n \times S^n \to S^n \times S^n$ with $f\Delta = \Delta$, and the set, $[S^n \times S^n, S^n \times S^n]^{\Delta}$, of homotopy classes of such maps, where we only admit the homotopy $H : f \simeq g$ if, for each t, $0 \leqslant t \leqslant 1$, the map H_t also fixes the diagonal. The function

$$\varphi_n : \left[S^n \times S^n, S^n \times S^n\right]^{\Delta} \to \left[S^n \times S^n, S^n \times S^n\right]^{*},$$

takes the homotopy class of a map relative D to the homotopy class of the same map relative the base point $* \in D$. There is a fundamental action of $F = \pi_{n+1}(S^n) \oplus \pi_{n+1}(S^n)$ on $[S^n \times S^n, S^n \times S^n]^{\Delta}$ such that $\varphi_n(f) = \varphi_n(g)$ if and only if there is an $\alpha \in F$ such that $f + \alpha = g$, see Section 2.

Theorem 1.1. *The function φ_n is injective, that is, the fundamental action is trivial for $n \geqslant 1$.*

In other words, given a self-map F of $S^n \times S^n$ with $F\Delta \simeq \Delta$, there is a homotopy $F \simeq G$ with $G\Delta = \Delta$ and the homotopy class of G is uniquely determined by F.

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More generally, given a suspension S of a co-H-group with diagonal $\Delta : D = S \subset S \times S$ and a map $v : D = S \to U$, we consider the function

$$\varphi : [S \times S, U]^v \to [S \times S, U]^*,$$

where $[S \times S, U]^v$ is the set of homotopy classes of maps under D. In Section 4 we compute both sets in terms of Whitehead products and obtain a criterion for φ to be injective.

The function φ is not injective in general, that is, there are spaces U and maps $v : S \to U$, for $S = S^2$, such that the fundamental action on $[S \times S, U]^v$ is non-trivial.

We compute the orbits of the fundamental action for $S = S^n \times S^n$ in Sections 4 and 5, apparently providing the first example in the literature where such orbits are computed for subspaces other than points. As a special case of Theorem 4.2 we obtain

Theorem 1.2. *Take a map $v : S^n \to U$. Then*

$$\left[S^n \times S^n, U\right]^v = \bigcup_u \pi_{2n}(U)/I_u,$$

where u ranges over the set $\mathcal{I}$ of all $u = (u', u'') \in \pi_n(U) \times \pi_n(U)$ with $[u', u''] = 0$ and $u' + u'' = v$. Let $w = u'' + (-1)^{n-1}u'$, so that $v = w + (1 + (-1)^n)u'$. Then

$$I_u = \left\{[\alpha, w] \,\middle|\, \alpha \in \pi_{n+1}(U)\right\}$$

and

$$J_u = \left\{[\alpha, w] + \left[\gamma, u'\right] \,\middle|\, \alpha, \gamma \in \pi_{n+1}(U)\right\}.$$

The orbits of the fundamental action are given by the quotient groups J_u/I_u acting on $\pi_{2n}(U)/I_u$. Thus φ is injective if and only if $I_u = J_u$ for all $u \in \mathcal{I}$.

To determine the monoid $[S^n \times S^n, S^n \times S^n]^\Delta$, consider the monoid $\mathcal{N}$ of (2×2)-matrices over $\mathbb{Z}$ given by

$$\mathcal{N} = \left\{\begin{bmatrix} a' & a'' \\ b' & b'' \end{bmatrix} \,\middle|\, a' + a'' = 1,\ b' + b'' = 1\right\} \subset \mathrm{End}(\mathbb{Z} \oplus \mathbb{Z}).$$

Let $\mathcal{M}$ be the submonoid of matrices with $a', a'', b', b'' \in \{0, 1\}$. There are four canonical self-maps of $S^n \times S^n$ which fix the diagonal, namely the identity, I, the interchange map, T, $P' = \Delta \circ \mathrm{pr}_1$ and $P'' = \Delta \circ \mathrm{pr}_2$, where $\mathrm{pr}_i : S^n \times S^n \to S^n$ is the projection onto the i-th factor for $i = 1, 2$. We obtain the multiplication table

	I	T	P'	P''
I	I	T	P'	P''
T	T	I	P'	P''
P'	P'	P''	P'	P''
P''	P''	P'	P'	P''

and identify the monoid formed by I, T, P' and P'' with the monoid $\mathcal{M}$. Let $\eta_{n+1} \in \pi_{n+1}(S^n)$ be the Hopf element, $i_n \in \pi_n(S^n)$ the identity and $[\eta_{n+1}, i_n] \in \pi_{2n}(S^n)$ the Whitehead product. We know that $\pi_3(S^2) = \mathbb{Z}$ and $\pi_{n+1}(S^n) = \mathbb{Z}_2$, $i \geqslant 3$, are generated by η_{n+1}. Moreover, for small n the Whitehead product satisfies

n	2	3	4	5
$[\eta_{n+1}, i_n]$	0	0	$\neq 0$	$\neq 0$

see [9]. We define the abelian group V_n by

$$V_n = \pi_{2n}\left(S^n\right)/[\eta_{n+1}, i_n].$$

If n is odd, $V_n \oplus V_n$ is an $\mathcal{N}$-bimodule. Namely, for $(x, y) \in V_n \oplus V_n$, the left action of $\alpha = \left[\begin{smallmatrix} a' & a'' \\ b' & b'' \end{smallmatrix}\right] \in \mathcal{N}$ is given by

$$\alpha(x, y) = \left(a'x + a''y, b'x + b''y\right)$$

and the right action is given by

$$(x, y)\alpha = \left(a'b'' + (-1)^n b'a''\right)(x, y).$$

If n is even, $V_n \oplus V_n$ is an $\mathcal{M}$-bimodule by the same formulæ. We define the monoid $\mathcal{M}_n = \mathcal{M} \times (V_n \oplus V_n)$ by the multiplication

$$\big(m, (x, y)\big) \circ \big(m', (x', y')\big) = \big(mm', \big(m(x', y') + (x, y)m'\big)\big),$$

that is, $\mathcal{M}_n$ is a split linear extension of $\mathcal{M}$.

Theorem 1.3. *The set $[S^n \times S^n, S^n \times S^n]^\Delta$ together with composition of maps is a monoid isomorphic to $\mathcal{M}_n$, if n is even. If n is odd, the monoid $\mathcal{N}_n = [S^n \times S^n, S^n \times S^n]^\Delta$ is a linear extension of $\mathcal{N}$ by the bimodule $V_n \oplus V_n$, that is, there is a surjection $\pi : \mathcal{N}_n \to \mathcal{N}$ of monoids and a free action $+$ of $V_n \oplus V_n$ on $\mathcal{N}_n$, such that the linear distributivity law holds, that is,*

$$\big(m + (x, y)\big) \circ \big(m' + (x', y')\big) = m \circ m' + m(x', y') + (x, y)m',$$

and $\pi(m) = \pi(m')$ if and only if there is $(x, y) \in V_n \oplus V_n$ with $m + (x, y) = m'$ for $m, m' \in \mathcal{N}_n$.

For n odd it remains an open question whether the linear extension $\mathcal{N}_n$ splits. Here $\mathcal{N}_n$ splits if and only if the cohomology class $[\mathcal{N}_n] \in H^3(\mathcal{N}, V_n \oplus V_n)$ represented by $\mathcal{N}_n$ is trivial, see [7].

For the proof of Theorem 1.3 we use the fact that the Whitehead product $[i_n, i_n]$ has infinite order if n is even, is trivial for $n = 1, 3, 7$ and otherwise an element of order 2. Moreover, we use the realizability conditions for $u = (u', u'')$ in Corollary 4.4, where $u' + u'' = 1$ implies that either u' or u'' must be even.

Theorem 1.3 was proved for $n = 2$ by different methods in [6]. The special case motived the authors to consider the general case in this paper.

2. The fundamental action

Let D be a space and let $i : D \rightarrowtail X$ be a cofibration. Given a map $u : D \to U$, we consider maps $f : X \to U$ *under* D, that is, maps with $fi = u$. Two maps $f, g : X \to U$ under D are *homotopic relative* D, if there is a homotopy $H : f \simeq g$, such that for each t, $0 \leqslant t \leqslant 1$, the map H_t is also a map under D. Let $[X, U]^D = [X, U]^u$ be the set of homotopy classes relative D of maps under D. For $D = *$ a point, the set $[X, U]^*$ is the usual set of homotopy classes of base point preserving maps. Given a cofibration $E \rightarrowtail D$, the forgetful map

$$\varphi : [X, U]^v \to [X, U]^E$$

takes the homotopy class $[f]$ relative D to the homotopy class $[f]$ relative E. The image of φ is the subset of all elements $[g] \in [X, U]^E$ with $gi \simeq u$ relative E. Let $\Sigma_E D$ be the pushout of $S^1 \times E \twoheadrightarrow E$ and $S^1 \times E \rightarrowtail S^1 \times D$. Then $[\Sigma_E D, U]^v$ is a group acting on $[X, U]^v$ via the *fundamental action* $+$, given by the homotopy extension property of the cofibration $D \rightarrowtail X$. By II (5.17) in [3], $\varphi(f) = \varphi(g)$, for $f, g \in [X, U]^v$, if and only if there is an $\alpha \in [\Sigma_E D, U]^v$ such that $f + \alpha = g$.

In general, the fundamental action is non-trivial. For example, if $E = \emptyset$, the empty set, and $D = *$, the point, then the fundamental action is the action of the fundamental group. If $X = K(G, 1)$ and $U = K(H, 1)$ are Eilenberg–MacLane spaces, then $[X, U]^* = \mathrm{Hom}(G, H)$ and $\pi_1(U) = H$ acts via $(\varphi + \alpha)(g) = -\alpha + \varphi(g) + \alpha$ for $\alpha \in H$ and $\varphi \in \mathrm{Hom}(G, H)$.

3. The pinching action

Choosing a closed ball B^{2n} in the complement, $S^n \times S^n \setminus \Delta(S^n)$, of the diagonal, we obtain the *pinching map*, $\mu : S^n \times S^n \to S^n \times S^n \vee S^{2n}$, by identifying the boundary of the ball to a point. The map μ induces the *pinching action* of the group $\pi_{2n}(U)$ on the set $[S^n \times S^n, U]^u$, where $u : D = S^n \to U$. This action commutes with the fundamental action of Section 2. Since $D = S^n$ is a suspension, there is a homotopy equivalence, $\Sigma_* D \simeq S^{n+1} \vee D$, under D, and the fundamental action on $[S^n \times S^n, U]^u$ is an action of the group $[\Sigma_* D, U]^u = \pi_{n+1}(U)$. The pinching action and the fundamental action define an action of $\pi_{2n}(U) \oplus \pi_{n+1}(U)$ on $[S^n \times S^n, U]^u$. Putting $U = S^n \times S^n$ and denoting the homology functor by H_n, we obtain

Lemma 3.1. *Take $f, g \in [S^n \times S^n, S^n \times S^n]^\Delta$. Then $H_n(f) = H_n(g)$ if and only if there is an $\alpha \in W_n = \pi_{2n}(S^n \times S^n) \oplus \pi_{n+1}(S^n \times S^n)$ with $f + \alpha = g$.*

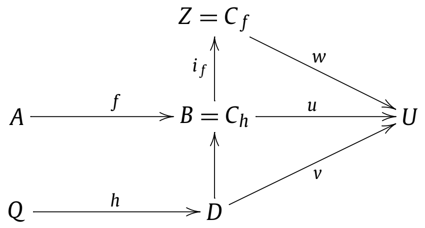
Lemma 3.1 follows from Eq. (1) in Section 4 which is devoted to the computation of the isotropy groups of the action of $\pi_{2n}(U) \oplus \pi_{n+1}(U)$ on $[S^n \times S^n, U]^u$.

4. Computation of the isotropy groups

Let $S = \Sigma T$ be the suspension of a pointed space and assume that the diagonal $\Delta : D = S \subset S \times S$ is a cofibration. There is a homotopy $H : \Delta \simeq \mu$ from the diagonal Δ to the comultiplication $\mu : D = S \to S \vee S \subset S \times S$. Let Z be the mapping cylinder of μ. Then H yields a map $\overline{H} : Z \to S \times S$ under D, which is a homotopy equivalence in **Top***. By Corollary II (2.21) in [3], $\overline{H}$ is also a homotopy equivalence in $\mathbf{Top}^D$ and hence, for a map $v : D = S \to U$,

$$[Z, U]^v = [S \times S, U]^v. \tag{1}$$

We consider the diagram



which corresponds to diagram II (13.1) in [3]. We are interested in the special case where $D = S$, $Q = T \vee T$ and h is the trivial map. Then $B = S' \vee S'' \vee D$ is a 1-point union of three copies of $S = S', S'', D$, and we denote the inclusions of S by e', e'' and e, respectively. Further, we put $A = \Sigma(T \wedge T) \vee S$ and

$$f_1 = f|_{\Sigma(T\wedge T)} = [e', e''], \quad \text{the Whitehead product,}$$
$$f_2 = f|_S = -e + e' + e'', \quad \text{the sum of inclusions.}$$

It is known that the mapping cone C_f coincides with Z, see for example (0.3.3) in [2]. For $S = S^n$, the coaction on C_f yields an action corresponding to the action in Lemma 3.1. The map u in the diagram is an extension of v and thus determined by $u', u'' : S \to U$. In order to apply II (13.10) in [3], we recall the definition of the *difference element* of a map $f : A \to C_h$, where C_h is the mapping cone of a map $h : Q \to D$. The inclusions $i_1 : \Sigma Q \to \Sigma Q \vee C_h$ and $i_2 : C_h \to \Sigma Q \vee C_h$ yield the map $i_2 + i_1 : C_h \to \Sigma Q \vee C_h$ and

$$\nabla f = -f^*(i_2) + f^*(i_2 + i_1) : A \to \Sigma Q \vee C_h.$$

Note that ∇f is trivial on C_h, that is, $(0,1)_*\nabla f = 0$. An extension w of u gives rise to $w^+ : [\Sigma A, U] \to [C_f, U]^v, \alpha \mapsto w + \alpha$, where the action $+$ is given by the pinching map of the mapping cone C_f. Now II (13.10) in [3] yields

Theorem 4.1. *There is an exact sequence*

$$\left[\Sigma^2 Q, U\right] \xrightarrow{\nabla(u,f)} [\Sigma A, U] \xrightarrow{w^+} [C_f, U]^v \xrightarrow{i_f^*} [B, U]^v \xrightarrow{f^*} [A, U],$$

where $\nabla(u, f)(\beta) = (E\nabla f)^*(\beta, u)$ and E is the partial suspension.

Special cases of Theorem 4.1 for $D = *$ correspond to results by Barcus and Barratt [1] and Rutter [8].

For the computation of $\nabla(u, f)$ we must consider $\nabla f : A \to \Sigma Q_1 \vee \Sigma Q_2 \vee D$, where $Q_1 = Q_2 = T \vee T$ and $E\nabla f : \Sigma A \to \Sigma^2 Q_1 \vee \Sigma Q_2 \vee D$. The inclusions e' and e'' yield the corresponding inclusions e'_1, e'_2, e''_1, e''_2 of S in $\Sigma Q_1 \vee \Sigma Q_2 \vee D$ and the inclusions $\Sigma e'_1, \Sigma e''_1$ of ΣS in $\Sigma^2 Q_1 \vee \Sigma Q_2 \vee D$, so that

$$\nabla f_1 = -[e'_2, e''_2] + [e'_2 + e'_1, e''_2 + e''_1],$$
$$\nabla f_2 = -(-e + e'_2 + e''_2) + (-e + e'_2 + e'_1 + e''_2 + e''_1).$$

If T is a co-H-group, the Whitehead product is bilinear and we obtain

$$\begin{aligned}\nabla f_1 &= [e'_1, e''_2] + [e'_2, e''_1] + [e'_1, e''_1] \\ &= [e'_1, e''_2] - (\Sigma\tau)^*[e''_1, e'_2] + [e'_1, e''_1],\end{aligned}$$

where $\tau : T \wedge T \to T \wedge T$ is the interchange map. Then

$$E\nabla f_1 = [\Sigma e'_1, e''_2] - (\Sigma^2\tau)^*[\Sigma e''_1, e'_2],$$

see (3.1.11) [2]. Moreover, $\nabla f_2 = e'_1 + e''_1$, so that

$$E\nabla f_2 = \Sigma e'_1 + \Sigma e''_1.$$

By the definition of $\nabla(u, f)$ in Theorem 4.1, we thus obtain

$$\nabla(u, f_1) : [\Sigma^2(T \vee T), U] \to [\Sigma^2(T \wedge T), U], \qquad \nabla(u, f_1)(\alpha, \beta) = [\alpha, u''] - (\Sigma^2\tau)^*[\beta, u'],$$
$$\nabla(u, f_2) : [\Sigma^2(T \vee T), U] \to [\Sigma^2 T, U], \qquad \nabla(u, f_2)(\alpha, \beta) = \alpha + \beta.$$

We leave it to the reader to compute these functions when T is not a co-H-group.

For the suspension $S = \Sigma T$ with diagonal $\Delta : S \subset S \times S$ and a map $v : S \to U$ as above, consider the commutative diagram

$$\begin{array}{ccc} [Z,U]^v = [S \times S, U]^v & \xrightarrow{\varphi} & [S \times S, U]^* \\ & {\scriptstyle p_\Delta} \searrow \quad \swarrow {\scriptstyle p} & \\ & [S \vee S, U]^*. & \end{array} \tag{2}$$

The group $[\Sigma^2 T \wedge T, U]$ acts on $[S \times S, U]^*$ and the group $[\Sigma^2 T \wedge T, U] \oplus [\Sigma^2 T, U]$ acts on $[Z,U]^v$. For $f, g \in [S \times S, U]^*$ we obtain $p(f) = p(g)$ if and only if there is an element $\alpha \in [\Sigma^2 T \wedge T, U]$ with $f + \alpha = g$. For $f, g \in [Z,U]^v$ we obtain $p_\Delta(f) = p_\Delta(g)$ if and only if there is an element $(\alpha, \beta) \in [\Sigma^2 T \wedge T, U] \oplus [\Sigma^2 T, U]$ with $f + (\alpha, \beta) = g$. An element $u = (u', u'') \in [S \vee S, U]^*$ is in the image of p if and only if $[u', u''] = 0$, and then the isotropy group of the orbit $p^{-1}(u)$ is the image of $\nabla(u, f_1)$, see (3.3.15) in [2]. Moreover, u is in the image of p_Δ if and only if $u' + u'' = v$ and $[u', u''] = 0$, and then the isotropy group of the orbit $p_\Delta^{-1}(u)$ is the image of $\nabla(u, f_1) + \nabla(u, f_2)$.

Theorem 4.2. *Let $S = \Sigma T$ be the suspension of a co-H-group T and let $v : S \to U$ be a map. Choosing a representative in each orbit, we obtain the bijection*

$$[S \times S, U]^v = \bigcup_u \big[\Sigma^2 T \wedge T, U\big]/I_u,$$

where u ranges over the set $\mathcal{I}$ of all $u = (u', u'') \in [S, U] \times [S, U]$ with $[u', u''] = 0$ and $u' + u'' = v$, and

$$I_u = \big\{\big[\alpha, u''\big] + \big(\Sigma^2 \tau\big)^* \big[\alpha, u'\big] \,\big|\, \alpha \in \big[\Sigma^2 T, U\big]\big\}.$$

Proof. Surjectivity of $\nabla(u, f_2)$ yields the commutative diagram

$$\begin{array}{ccccc} \ker \nabla(u, f_2) & \xrightarrow{\nabla(u,f_1)} & [\Sigma^2 T \wedge T, U] & \twoheadrightarrow & [\Sigma^2 T \wedge T, U]/I_u \\ \downarrow & & \downarrow & & \downarrow \cong \\ [\Sigma^2 Q, U] & \xrightarrow{\nabla(u,f)} & [\Sigma A, U] & \twoheadrightarrow & \operatorname{coker} \nabla(u, f) \\ {\scriptstyle \nabla(u,f_2)} \downarrow & & \downarrow & & \downarrow \\ [\Sigma S, U] & = & [\Sigma S, U] & \longrightarrow & 0 \end{array}$$

of short exact columns and exact rows, where $I_u = \nabla(u, f_1)(\ker(\nabla(u, f_2)))$. The formula for I_u follows from the computation of $\nabla(u, f_1)$ and $\nabla(u, f_2)$. □

Theorem 4.2 corresponds to (3.3.15) in [2], where $[S \times S, U]^*$ is computed by the formula

$$[S \times S, U]^* = \bigcup_u \big[\Sigma^2 T \wedge T, U\big]/J_u,$$

where u ranges over the set $\mathcal{J}$ of all $u = (u', u'') \in [S \vee S, U]$ with $[u', u''] = 0$ and

$$J_u = \big\{\big[\alpha, u''\big] - \big(\Sigma^2 \tau\big)^* \big[\beta, u'\big] \,\big|\, \alpha, \beta \in \big[\Sigma^2 T, U\big]\big\}.$$

Moreover, the diagram

$$\begin{array}{ccc} [S \times S, U]^v & = & \bigcup_{u \in \mathcal{I}} [\Sigma^2 T \wedge T, U]/I_u \\ {\scriptstyle \varphi} \downarrow & & \downarrow \\ [S \times S, U]^* & = & \bigcup_{u \in \mathcal{J}} [\Sigma^2 T \wedge T, U]/J_u \end{array}$$

commutes, where $\mathcal{I} \subset \mathcal{J}$ and the arrow on the right-hand side is induced by the identity. Thus we obtain

Corollary 4.3. *The orbits of the fundamental action are given by the quotient groups J_u/I_u acting on $[\Sigma^2 T \wedge T, U]/I_u$. Thus φ is injective if and only if $I_u = J_u$ for all $u \in \mathcal{I}$.*

For the special case of a sphere $S = S^n$, we obtain

Corollary 4.4. *Take a map $v : S^n \to U$. Then*

$$\left[S^n \times S^n, U\right]^v = \bigcup_u \pi_{2n}(U)/I_u,$$

where u ranges over the set $\mathcal{I}$ of all $u = (u', u'') \in \pi_n(U) \times \pi_n(U)$ with $[u', u''] = 0$ and $u' + u'' = v$. Let $w = u'' + (-1)^{n-1}u'$, so that $v = w + (1 + (-1)^n)u'$. Then

$$I_u = \left\{[\alpha, w] \,\middle|\, \alpha \in \pi_{n+1}(U)\right\}$$

and

$$J_u = \left\{[\alpha, w] + \left[\gamma, u'\right] \,\middle|\, \alpha, \gamma \in \pi_{n+1}(U)\right\}.$$

Corollary 4.5. *Let $v = \mathrm{id}_{S^n} : S^n \to S^n$ be the identity. Then the fundamental action on $[S^n \times S^n, S^n]^v$ is trivial.*

Proof. Take $u = (u', u'') \in \mathbb{Z} \oplus \mathbb{Z} = [S^n \vee S^n, S^n]$ with $u'u''[i_n, i_n] = [u', u''] = 0$ and $u' + u'' = v = 1$. Then I_u is the subgroup of all elements of the form $(u'' + (-1)^{n+1}u')[\eta_{n+1}, i_n]$ and J_u is the subgroup generated by $u''[\eta_{n+1}, i_n]$ and $u'[\eta_{n+1}, i_n]$. As $u' + u'' = 1$ implies that either u' or u'' must be odd and $[\eta_{n+1}, i_n] = 0$ for $n = 2$ and is an element of order 2 otherwise, we conclude that $I_u = J_u$. □

5. Computation of I_u and J_u for $S^2 \times S^2$

First recall Whitehead's quadratic functor Γ. A function $\eta : \pi_2 \to \pi_3$ between abelian groups is *quadratic* if $\eta(-a) = \eta(a)$ and $\pi_2 \times \pi_2 \to \pi_3$, $(a, b) \mapsto \eta(a + b) - \eta(a) - \eta(b) = [a, b]_\eta$ is bilinear. There is a *universal quadratic map* $\gamma : \pi_2 \to \Gamma(\pi_2)$, such that there is a unique homomorphism $\hat{\eta} : \Gamma(\pi_2) \to \pi_3$ with $\eta = \gamma\hat{\eta}$. To define the Γ-torsion $\Gamma T(\pi_2)$, take a short free resolution $A_1 \overset{d}{\rightarrowtail} A_0 \twoheadrightarrow \pi_2$ and consider the sequence

$$A_1 \otimes A_1 \xrightarrow{\delta_2} \Gamma(A_1) \oplus A_1 \otimes A_0 \xrightarrow{\delta_1} \Gamma(A_0),$$

where $\delta_1 = (\Gamma(d), [d, 1])$ and $\delta_2 = ([1, 1], -1 \otimes d)$. Then

$$\Gamma T(\pi_2) = \ker(\delta_1)/\mathrm{im}(\delta_2).$$

Moreover, let $M(\eta)$ be the subgroup of $\pi_3 \otimes \mathbb{Z}/2 \oplus \pi_3 \otimes \pi_2$ generated by

$$(\eta x) \otimes x$$

and

$$[x, y]'_\eta \otimes 1 + (\eta x) \otimes y + [y, x]'_\eta \otimes x,$$

where $x, y \in \pi_2$ and $[x, y]'_\eta = \eta(x + y) - \eta(y)$. Putting

$$\Gamma_2^2(\eta) = (\pi_3 \otimes \mathbb{Z}/2 \oplus \pi_3 \otimes \pi_2)/M(\eta),$$

there is a short exact sequence of abelian groups

$$\Gamma_2^2(\eta) \rightarrowtail \Gamma_4(\eta) \twoheadrightarrow \Gamma T(\pi_2), \tag{3}$$

where $\Gamma_4(\eta)$ is the group $\Gamma_4 K(\eta, 2)$ in (12.3.3) which is the natural element contained in $\mathrm{Ext}(\Gamma T(\pi_2), \Gamma_2^2(\eta))$ computed in (11.3.4) and (11.1.21) in [4].

Definition 5.1. Let $\pi_2, \pi_3, \pi_4, H_2, H_3, H_4$ and H_5 be abelian groups with H_5 free abelian and $\pi_2 = H_2$. A (5-*dimensional*) *Γ-sequence* is an exact sequence in the category of abelian groups

$$H_5 \longrightarrow \Gamma_4(\eta) \longrightarrow \pi_4 \longrightarrow H_4 \longrightarrow \Gamma(\pi_2) \xrightarrow{\hat{\eta}} \pi_3 \longrightarrow H_3 \longrightarrow 0. \tag{4}$$

A *morphism φ of Γ-sequences* is given by homomorphisms $H_i(\varphi) : H_i \to H_i'$, $\pi_i(\varphi) : \pi_i \to \pi_i'$ and $\Gamma_4(\varphi) : \Gamma_4(\eta) \to \Gamma_4(\eta')$ compatible with the exact sequences (3) and (4). Here $(\Gamma(\pi_2(\varphi)), \pi_3(\varphi))$ is a morphism $\eta \to \eta'$ inducing a map $\Gamma_2^2(\eta) \to \Gamma_2^2(\eta')$.

The following result is proved in [4].

Theorem 5.2. *There is a representable functor W from the homotopy category of simply connected 5-dimensional CW-complexes to the category of 5-dimensional Γ-sequences. In other words, for every Γ-sequence (4) there is a simply connected 5-dimensional CW-complex U, such that the Γ-sequence $W(U)$ is isomorphic to (4) in the category of Γ-sequences. We call U a* realization *of (4).*

Note that (4) is realizable, but does not determine the homotopy type of the realization U. Complete invariants classifying the homotopy type of U are provided in (12.5.9) in [4]. Theorem 5.2 follows from (3.4.7) and (11.3.4) in [4], see also 7.10 in [5]. The functor W is an enrichment of Whitehead's Certain Exact Sequence [10].

For $n = 2$ we now show that the groups I_u and J_u in Corollary 4.4 depend on the Γ-sequence of U only. Given a Γ-sequence (4) and $y \in \pi_2$, let $[\pi_3, y] \subset \pi_4$ denote the image of

$$\pi_3 \otimes \langle y \rangle \longrightarrow \pi_3 \otimes \pi_2 \longrightarrow \Gamma_2^2(\eta) \rightarrowtail \Gamma_4(\eta) \longrightarrow \pi_4.$$

By (11.3.5) in [4], $[\pi_3, y]$ corresponds to the Whitehead product.

Theorem 5.3. *Given a Γ-sequence (4) and a realization U of (4), take $w, u' \in \pi_2 \cong \pi_2(U)$, such that $[u', u''] = 0$ for $u'' = w + u'$. Then*

$$I_u \cong [\pi_3, w]$$

and

$$J_u \cong [\pi_3, w] + \big[\pi_3, u'\big].$$

In order to find examples with $I_u \neq J_u$, we choose appropriate Γ-sequences.

Corollary 5.4. *Consider a Γ-sequence (4) with realization U and elements $w, u' \in \pi_2$, with $\pi_2 = \pi_2' \oplus \langle w \rangle \oplus \langle u' \rangle$ and $\pi_3 \otimes \langle u' \rangle \neq 0$. Further, assume (4) satisfies $\eta = 0$ and $H_5 = 0$. Then $[u', u''] = 0$ for $u'' = w + u'$,*

$$I_u \cong \pi_3 \otimes \langle w \rangle$$

and

$$J_u \cong \pi_3 \otimes \big(\langle w \rangle \oplus \langle u' \rangle\big),$$

so that $I_u / J_u \cong \pi_3 \otimes \langle u' \rangle \neq 0$. Hence the fundamental action on $[S^2 \times S^2, U]^v$ is non-trivial, where $v : S^2 \to U$ represents $u' + u'' = w + 2u'$. Moreover, the orbits of the fundamental action are of the form $\pi_3 \otimes \langle u' \rangle$, where the abelian group π_3 can be chosen arbitrarily and $\langle u' \rangle$ can be any cyclic group.

Proof. As $H_5 = 0$, the map $\Gamma_4(\eta) \to \pi_4$ is injective. Therefore the map $\Gamma_2^2(\eta) \to \pi_4$ is also injective. Further, $\eta = 0$ implies $M(\eta) = 0$ and hence $\Gamma_2^2(\eta) = \pi_3 \otimes \mathbb{Z}/2 \oplus \pi_3 \otimes \pi_2$. Thus $[\pi_3, w] \cong \pi_3 \otimes \langle w \rangle$ and $[\pi_3, w] + [\pi_3, u'] \cong \pi_3 \otimes (\langle w \rangle \oplus \langle u' \rangle)$. □

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