Existence of strong symmetric self-orthogonal diagonal Latin squares

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\textbf{A B S T R A C T}

A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. A Latin square is self-orthogonal if it is orthogonal to its transpose. A diagonal Latin square $L$ of order $n$ is strongly symmetric, denoted by $\text{SSSODLS}(n)$, if $L(i,j) + L(n - 1 - i, n - 1 - j) = n - 1$ for all $i, j \in N = \{0, 1, \ldots, n - 1\}$. In this note, we shall prove that an $\text{SSSODLS}(n)$ exists if and only if $n \equiv 0, 1, 3 \pmod{4}$ and $n \neq 3$.

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1. Introduction

Let $N = \{0, 1, \ldots, n - 1\}$. A Latin square of order $n$ is an $n \times n$ array such that every row and every column is a permutation of $N$. A transversal in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals.

Two Latin squares of order $n$ are orthogonal if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A Latin square is self-orthogonal if it is orthogonal to its transpose. Self-orthogonal (diagonal) Latin squares of order $n$ are denoted by $\text{SOLS}(n)$ ($\text{SODLS}(n)$). For the spectra of $\text{SOLS}$ and $\text{SODLS}$, we have the following results.

\textbf{Theorem 1.1} ([3–5,2]). An $\text{SOLS}(n)$ and an $\text{SODLS}(n)$ exist for all positive integers $n$, with the exception of $n \in \{2, 3, 6\}$.

An $\text{SSSODLS}(n)$ exists for each $n \in \{4, 5, 7, 8, 12\}$ and an $\text{SSSSODLS}(n)$ cannot exist for each $n \in \{2, 3, 6, 10\}$ [4].

\textbf{Theorem 1.2}. (1) An $\text{SSSODLS}(n)$ exists for each $n \in \{4, 5, 7, 8, 12\}$.

(2) An $\text{SSSSODLS}(n)$ cannot exist for each $n \in \{2, 3, 6, 10\}$.

Du and Cao proved that an $\text{SSSODLS}(n)$ exists for all positive integers $n \equiv 0, 1, 3 \pmod{4}$ and $n \neq 3, 15$ in 2002 [6]. They also proved that there exists an $\text{SSSODLS}(n)$ for $n \equiv 2 \pmod{4}$ if there exists an $\text{SSSODLS}(r), r \equiv 2 \pmod{4}$. Ref. [6] was written in Chinese. In this note, we shall complete the spectra of $\text{SSSODLS}$ by constructing an $\text{SSSODLS}(15)$ and showing the nonexistence of an $\text{SSSODLS}(n)$ for any $n \equiv 2 \pmod{4}$. We also translate the main part of the proof in [6] and present a full proof of the following theorem.

\textbf{Theorem 1.3}. An $\text{SSSODLS}(n)$ exists if and only if $n \equiv 0, 1, 3 \pmod{4}$ and $n \neq 3$.
2. Proof of Theorem 1.3

We start with the conception of an incomplete self-orthogonal Latin square and some recursive constructions for SSSODLS(n) in [6]. An incomplete SOLS is a self-orthogonal Latin square of order $n$ missing a sub-SOLS of order $k$, denoted by ISOLS(n, k). For the existence of an ISOLS(n, k), see [1,9,7,8,10].

**Theorem 2.1.** There exists an ISOLS(n, k) for all values of $n$ and $k$ satisfying $n \geq 3k + 1$, except for $(n, k) = (6, 1), (8, 2)$ and possibly excepting $n = 3k + 2$ and $k \in \{6, 8, 10\}$.

**Lemma 2.2** ([6]).

1. If $q$ is a prime power, then there exists an SSSODLS(q).
2. If there exists an SSSODLS(m) and an SSSODLS(n), then there exists an SSSODLS(mn).
3. If $n$ is even, SSSODLS(n) and SOLS(m) both exist, then there exists an SSSODLS(mn).
4. If $n$ is even, SSSODLS(n), SSSODLS(k), SOLS(n) and ISOLS(m + k, k) all exist, then there exists an SSSODLS(mn + k).
5. If $n$ and $k$ are even, SSSODLS(n), SSSODLS(k + h), SOLS(m), ISOLS(m + k, k) and ISOLS(m + h, h) all exist, then there exists an SSSODLS(mn + k).
6. Suppose $n$ is odd and there is at most one odd number among $m$, $k$ and $h$. If SSSODLS(n), SSSODLS(m + k + h), SOLS(m), ISOLS(m + k, k) and ISOLS(m + h, h) all exist, then there exists an SSSODLS(mn + k).

**Lemma 2.3.** There exists an SSSODLS(15).

**Proof.** An SSSODLS(15) is constructed in the following table.

\[
\begin{array}{cccccccccccccccc}
0 & 4 & 13 & 5 & 2 & 8 & 7 & 9 & 12 & 3 & 6 & 14 & 11 & 1 & 10 \\
11 & 1 & 5 & 12 & 6 & 3 & 0 & 8 & 4 & 14 & 13 & 10 & 2 & 9 & 7 \\
5 & 10 & 2 & 6 & 11 & 14 & 4 & 0 & 13 & 12 & 9 & 3 & 8 & 7 & 1 \\
12 & 6 & 9 & 3 & 14 & 10 & 13 & 1 & 11 & 8 & 4 & 0 & 7 & 2 & 5 \\
10 & 11 & 14 & 8 & 4 & 13 & 9 & 2 & 0 & 5 & 1 & 7 & 3 & 6 & 12 \\
1 & 9 & 10 & 13 & 0 & 5 & 12 & 3 & 6 & 2 & 7 & 4 & 14 & 11 & 8 \\
14 & 2 & 8 & 9 & 12 & 1 & 6 & 4 & 3 & 7 & 5 & 13 & 10 & 0 & 11 \\
8 & 0 & 1 & 2 & 3 & 4 & 5 & 7 & 9 & 10 & 11 & 12 & 13 & 14 & 6 \\
3 & 14 & 4 & 1 & 9 & 7 & 11 & 10 & 8 & 13 & 2 & 5 & 6 & 12 & 0 \\
6 & 3 & 0 & 10 & 7 & 12 & 8 & 11 & 2 & 9 & 14 & 1 & 4 & 5 & 13 \\
2 & 8 & 11 & 7 & 13 & 9 & 14 & 12 & 5 & 1 & 10 & 6 & 0 & 3 & 4 \\
9 & 12 & 7 & 14 & 10 & 6 & 3 & 13 & 1 & 4 & 0 & 11 & 5 & 8 & 2 \\
13 & 7 & 6 & 11 & 5 & 2 & 1 & 14 & 10 & 0 & 3 & 8 & 12 & 4 & 9 \\
7 & 5 & 12 & 4 & 1 & 0 & 10 & 6 & 14 & 11 & 8 & 2 & 9 & 13 & 3 \\
4 & 13 & 3 & 0 & 8 & 11 & 2 & 5 & 7 & 6 & 12 & 9 & 1 & 10 & 14 \\
\end{array}
\]

**Lemma 2.4.** There does not exist an SSSODLS(n) for any $n \equiv 2 \pmod{4}$.

**Proof.** Suppose $n = 4k + 2$ ($k > 0$) and $L$ is an SSSODLS(n). Let

\[
A = \sum_{i=0}^{2k} \sum_{j=0}^{2k} L(i, j), \quad B = \sum_{i=0}^{2k} \sum_{j=2k+1}^{4k+1} L(i, j), \quad C = \sum_{i=2k+1}^{4k+1} \sum_{j=0}^{2k} L(i, j).
\]

Then we have $A + B = A + C = (n/2) \cdot (\sum_{i=0}^{n-1} i) = n^2(n - 1)/4$ since every row and every column is a permutation of $N = \{0, 1, \ldots, n-1\}$. Further, by the strongly symmetric property, $B + C = (n-1) \cdot (n^2/4) = A + B$. So, $A = C$. We have $2A = n^2(n - 1)/4 = (2k + 1)^2(4k + 1)$. It is a contradiction. \(\Box\)

**Proof of Theorem 1.3.** By Lemma 2.4, there does not exist an SSSODLS(n) for any $n \equiv 2 \pmod{4}$.

1. For $n \equiv 0 \pmod{4}$, $n = 4, 8, 12$ come from **Theorem 1.2**. An SSSODLS(24) can be obtained by using Lemma 2.2(5) with $n = 4, m = 5, k = h = 2$, where the input design ISOLS(7, 2) exists by **Theorem 2.1**. For other values of $n$, start from an SSSODLS(4). Applying Lemma 2.2(3) with an SOLS(n/4) from **Theorem 1.1**, we obtain an SSSODLS(n).

2. For $n \equiv 1 \pmod{4}$, $n = 5, 9, 13, 25$ come from **Lemma 2.2(1)**. An SSSODLS(21) can be obtained by using Lemma 2.2(6) with $n = 5, m = 4, k = 1, h = 0$, where the input designs SOLS(4), SSSODLS(4), SSSODLS(5), and ISOLS(5, 1) exist by **Theorems 1.1, 1.2 and 2.1**. For other values of $n$, start from an SSSODLS(4). Apply Lemma 2.2(4) with $k = 1$ to obtain an SSSODLS(n), where the input designs SOLS(n/4) and ISOLS(n/4 + 1, 1) exist by **Theorems 1.1 and 2.1**.

3. For $n \equiv 3 \pmod{4}$ and $n \geq 7$, let $n = 4k + 7, k \geq 0$. $0 \equiv k \equiv 0, 1, 3, 4, 5, 6, 9, 10, 13$ come from **Lemma 2.2(1)**. $k = 2$ comes from **Lemma 2.3**. $k = 7, 12, 14$ come from **Lemma 2.2(2)** since there exists an SSSODLS(m) for each $m \in \{5, 7, 9, 11\}$. $n = 39$ and $n = 51$ can be obtained from **Lemma 2.2(6)** with $m = 5, n = 7, h = 2, k = 2$ and $m = n = 7, h = 2, k = 0$, respectively. For other values of $k \geq 15$, apply Lemma 2.2(4) to obtain an SSSODLS(n), where the input designs SOLS(k) and ISOLS(k + 7, 7) exist by **Theorems 1.1 and 2.1**. \(\Box\)
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References