Discrete Mathematics 311 (2011) 841–843

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Existence of strong symmetric self-orthogonal diagonal Latin squares

H. Cao [a], W. Li [b]

[a] Institute of Mathematics, Nanjing Normal University, Nanjing 210046, China

[b] Department of Mathematics, Yancheng Teachers College, Jiangsu 224002, China

ARTICLE INFO

Article history:
Received 2 October 2010
Received in revised form 18 January 2011
Accepted 2 February 2011
Available online 2 March 2011

Keywords:
Diagonal Latin square
Self-orthogonal
Strongly symmetric

ABSTRACT

A diagonal Latin square is a Latin square whose main diagonal and back diagonal are both transversals. A Latin square is self-orthogonal if it is orthogonal to its transpose. A diagonal Latin square L of order n is *strongly symmetric*, denoted by SSSODLS(n), if $L(i, j) + L(n-1-i, n-1-j) = n-1$ for all $i, j \in N = \{0, 1, \ldots, n-1\}$. In this note, we shall prove that an SSSODLS(n) exists if and only if $n \equiv 0, 1, 3 \pmod 4$ and $n \neq 3$.

1. Introduction

Let $N = \{0, 1, \ldots, n-1\}$. A *Latin square* of order n is an $n \times n$ array such that every row and every column is a permutation of N. A *transversal* in a Latin square is a set of positions, one per row and one per column, among which the symbols occur precisely once each. A *diagonal Latin square* is a Latin square whose main diagonal and back diagonal are both transversals.

Two Latin squares of order n are *orthogonal* if each symbol in the first square meets each symbol in the second square exactly once when they are superposed. A Latin square is *self-orthogonal* if it is orthogonal to its transpose. Self-orthogonal (diagonal) Latin squares of order n are denoted by SOLS(n) (SODLS(n)). For the spectra of SOLS and SODLS, we have the following results.

Theorem 1.1 ([3–5,2]). *An* SOLS(n) *and an* SODLS(n) *exist for all positive integers n, with the exception of* $n \in \{2, 3, 6\}$.

A diagonal Latin square L is *strongly symmetric* if $L(i, j) + L(n-1-i, n-1-j) = n-1$ for all $i, j \in N$. Strongly symmetric self-orthogonal diagonal Latin squares of order n are denoted by SSSODLS(n). The investigation of the existence of an SSSODLS(n) was started by Danhof et al. in 1990. They show that there exists an SSSODLS(n) for each $n \in \{4, 5, 7, 8, 12\}$ and an SSSODLS(n) cannot exist for each $n \in \{2, 3, 6, 10\}$ [4].

Theorem 1.2. (1) *An* SSSODLS(n) *exists for each* $n \in \{4, 5, 7, 8, 12\}$.
(2) *An* SSSODLS(n) *cannot exist for each* $n \in \{2, 3, 6, 10\}$.

Du and Cao proved that an SSSODLS(n) exists for all positive integers $n \equiv 0, 1, 3 \pmod 4$ and $n \neq 3, 15$ in 2002 [6]. They also proved that there exists an SSSODLS(n) for $n \equiv 2 \pmod 4$ and $n \geq 5r+20$ if there exists an SSSODLS(r), $r \equiv 2 \pmod 4$. Ref. [6] was written in Chinese. In this note, we shall complete the spectra of SSSODLS by constructing an SSSODLS(15) and showing the nonexistence of an SSSODLS(n) for any $n \equiv 2 \pmod 4$. We also translate the main part of the proof in [6] and present a full proof of the following theorem.

Theorem 1.3. *An* SSSODLS(n) *exists if and only if* $n \equiv 0, 1, 3 \pmod 4$ *and* $n \neq 3$.

E-mail addresses: caohaitao@njnu.edu.cn (H. Cao), liwen39@hotmail.com (W. Li).

0012-365X/$ – see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2011.02.003

2. Proof of Theorem 1.3

We start with the conception of an incomplete self-orthogonal Latin square and some recursive constructions for SSSODLS(n) in [6]. An *incomplete* SOLS is a self-orthogonal Latin square of order n missing a sub-SOLS of order k, denoted by ISOLS(n, k). For the existence of an ISOLS(n, k), see [1,9,7,8,10].

Theorem 2.1. *There exists an* ISOLS(n, k) *for all values of n and k satisfying $n \geq 3k+1$, except for $(n, k) = (6, 1), (8, 2)$ and possibly excepting $n = 3k+2$ and $k \in \{6, 8, 10\}$.*

Lemma 2.2 ([6])**.**

(1) *If q is a prime power, then there exists an* SSSODLS(q).
(2) *If there exists an* SSSODLS(m) *and an* SSSODLS(n), *then there exists an* SSSODLS(mn).
(3) *If n is even,* SSSODLS(n) *and* SOLS(m) *both exist, then there exists an* SSSODLS(mn).
(4) *If n is even,* SSSODLS(n), SSSODLS(k), SOLS(m) *and* ISOLS($m+k$, k) *all exist, then there exists an* SSSODLS($mn+k$).
(5) *If n and kh are even,* SSSODLS(n), SSSODLS($k+h$), SOLS(m), ISOLS($m+k$, k) *and* ISOLS($m+h$, h) *all exist, then there exists an* SSSODLS($mn+k+h$).
(6) *Suppose n is odd and there is at most one odd number among m, k and h. If* SSSODLS(n), SSSODLS($m+k+h$), SOLS(m), ISOLS($m+k$, k) *and* ISOLS($m+h$, h) *all exist, then there exists an* SSSODLS($mn+k+h$).

Lemma 2.3. *There exists an* SSSODLS(15).

Proof. An SSSODLS(15) is constructed in the following table.

0	4	13	5	2	8	7	9	12	3	6	14	11	1	10
11	1	5	12	6	3	0	8	4	14	13	10	2	9	7
5	10	2	6	11	14	4	0	13	12	9	3	8	7	1
12	6	9	3	14	10	13	1	11	8	4	0	7	2	5
10	11	14	8	4	13	9	2	0	5	1	7	3	6	12
1	9	10	13	0	5	12	3	6	2	7	4	14	11	8
14	2	8	9	12	1	6	4	3	7	5	13	10	0	11
8	0	1	2	3	4	5	7	9	10	11	12	13	14	6
3	14	4	1	9	7	11	10	8	13	2	5	6	12	0
6	3	0	10	7	12	8	11	2	9	14	1	4	5	13
2	8	11	7	13	9	14	12	5	1	10	6	0	3	4
9	12	7	14	10	6	3	13	1	4	0	11	5	8	2
13	7	6	11	5	2	1	14	10	0	3	8	12	4	9
7	5	12	4	1	0	10	6	14	11	8	2	9	13	3
4	13	3	0	8	11	2	5	7	6	12	9	1	10	14

Lemma 2.4. *There does not exist an* SSSODLS(n) *for any $n \equiv 2 \pmod 4$.*

Proof. Suppose $n = 4k+2$ ($k \geq 0$) and L is an SSSODLS(n). Let

$$A = \sum_{i=0}^{2k}\sum_{j=0}^{2k} L(i,j), \qquad B = \sum_{i=0}^{2k}\sum_{j=2k+1}^{4k+1} L(i,j), \qquad C = \sum_{i=2k+1}^{4k-1}\sum_{j=0}^{2k} L(i,j).$$

Then we have $A + B = A + C = (n/2) * (\sum_{i=0}^{n-1} i) = n^2(n-1)/4$ since every row and every column is a permutation of $N = \{0, 1, \ldots, n-1\}$. Further, by the strongly symmetric property, $B + C = (n-1) * (n^2/4) = A + B$. So, $A = C$. We have $2A = n^2(n-1)/4 = (2k+1)^2(4k+1)$. It is a contradiction. □

Proof of Theorem 1.3. By Lemma 2.4, there does not exist an SSSODLS(n) for any $n \equiv 2 \pmod 4$.

(1) For $n \equiv 0 \pmod 4$, $n = 4, 8, 12$ come from Theorem 1.2. An SSSODLS(24) can be obtained by using Lemma 2.2(5) with $n = 4$, $m = 5$, $k = h = 2$, where the input design ISOLS(7, 2) exists by Theorem 2.1. For other values of n, start from an SSSODLS(4). Applying Lemma 2.2(3) with an SOLS($n/4$) from Theorem 1.1, we obtain an SSSODLS(n).
(2) For $n \equiv 1 \pmod 4$, $n = 5, 9, 13, 25$ come from Lemma 2.2(1). An SSSODLS(21) can be obtained by using Lemma 2.2(6) with $n = 5$, $m = 4$, $k = 1$, $h = 0$, where the input designs SOLS(4), SSSODLS(4), SSSODLS(5), and ISOLS(5, 1) exist by Theorems 1.1, 1.2 and 2.1. For other values of n, start from an SSSODLS(4). Apply Lemma 2.2(4) with $k = 1$ to obtain an SSSODLS(n), where the input designs SOLS($\frac{n-1}{4}$) and ISOLS($\frac{n-1}{4}+1$, 1) exist by Theorems 1.1 and 2.1.
(3) For $n \equiv 3 \pmod 4$ and $n \geq 7$, let $n = 4k+7$, $k \geq 0$. $k = 0, 1, 3, 4, 5, 6, 9, 10, 13$ come from Lemma 2.2(1). $k = 2$ comes from Lemma 2.3. $k = 7, 12, 14$ come from Lemma 2.2(2) since there exists an SSSODLS(m) for each $m \in \{5, 7, 9, 11\}$. $n = 39$ and $n = 51$ can be obtained from Lemma 2.2(6) with $m = 5$, $n = 7$, $h = 2$, $k = 2$ and $m = n = 7$, $h = 2$, $k = 0$, respectively. For other values of $k \geq 15$, apply Lemma 2.2(4) to obtain an SSSODLS(n), where the input designs SOLS(k) and ISOLS($k+7$, 7) exist by Theorems 1.1 and 2.1. □

Acknowledgements

The authors would like to thank Professor Zhu Lie of Suzhou University for helpful discussions.
The first author's research was supported by the National Natural Science Foundation of China under Grant No. 10971101.

References

[1] R.J.R. Abel, F.E. Bennett, H. Zhang, L. Zhu, A few more incomplete self-orthogonal Latin squares and related designs, Australas. J. Combin. 21 (2000) 85–94.
[2] F.E. Bennett, B. Du, H. Zhang, Existence of self-orthogonal diagonal Latin squares with a missing subsquare, Discrete Math. 261 (2003) 69–86.
[3] R.K. Brayton, D. Coppersmith, A.J. Hoffman, Self-orthogonal Latin squares of all orders $n \neq 2$, 3 or 6, Bull. Amer. Math. Soc. 80 (1974) 116–118.
[4] K.J. Danhof, N.C.K. Phillips, W.D. Wallis, On self-orthogonal diagonal Latin squares, J. Combin. Math. Combin. Comput. 8 (1990) 3–8.
[5] B. Du, On the existence of self-orthogonal diagonal Latin squares, Australas. J. Combin. 10 (1994) 45–50.
[6] B. Du, H. Cao, Existence of strongly symmetric self-orthogonal diagonal Latin squares, Acta Math. Appl. Sin. 25 (2002) 187–189.
[7] K. Heinrich, L. Wu, L. Zhu, Incomplete self-orthogonal Latin squares ISOLS$(6m + 6, 2m)$ exist for all m, Discrete Math. 87 (1991) 281–290.
[8] K. Heinrich, L. Zhu, Incomplete self-orthogonal Latin squares, J. Aust. Math. Soc. Ser. A 42 (1987) 365–384.
[9] Y. Xu, H. Zhang, L. Zhu, Existence of frame SOLS of type $a^n b^1$, Discrete Math. 250 (2002) 211–230.
[10] L. Zhu, Existence of self-orthogonal Latin squares ISOLS$(6m + 2, 2m)$, Ars Combin. 39 (1995) 65–74.