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A criterion for the integrality of the Taylor coefficients of mirror maps in several variables

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Abstract

We give a necessary and sufficient condition for the integrality of the Taylor coefficients at the origin of formal power series $q_i(\mathbf{z}) = z_i \exp(G_i(\mathbf{z})/F(\mathbf{z}))$, with $\mathbf{z} = (z_1, \ldots, z_d)$ and where $F(\mathbf{z})$ and $G_i(\mathbf{z}) + \log(z_i)F(\mathbf{z})$, $i = 1, \ldots, d$ are particular solutions of certain A-systems of differential equations. This criterion is based on the analytical properties of Landau's function (which is classically associated with sequences of factorial ratios) and it generalizes the criterion in the case of one variable presented in [E. Delaygue, Critère pour l'intégralité des coefficients de Taylor des applications miroir, J. Reine Angew. Math. 662 (2012) 205–252]. One of the techniques used to prove this criterion is a generalization of a version of a theorem of Dwork on formal congruences between formal series, proved by Krattenthaler and Rivoal in [C. Krattenthaler, T. Rivoal, Multivariate *p-adic* formal congruences and integrality of Taylor coefficients of mirror maps, in: L. Di Vizio, T. Rivoal (Eds.), Théories Galoisiennes et Arithmétiques des Équations Différentielles, in: Séminaire et Congrés, vol. 27, Soc. Math. France, Paris, 2011, pp. 279–307].

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1. Introduction

The mirror maps considered in this article are formal series of d variables $z_i(\mathbf{x})$, $i = 1, \ldots, d$, with $\mathbf{x} = (x_1, \ldots, x_d)$. The map $\mathbf{x} \mapsto (z_1(\mathbf{x}), \ldots, z_d(\mathbf{x}))$ is the compositional inverse of the

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map $\mathbf{x} \mapsto (q_1(\mathbf{x}), \ldots, q_d(\mathbf{x}))$, with $q_i(\mathbf{x}) = x_i \exp(G_i(\mathbf{x})/F(\mathbf{x}))$ and where $F(\mathbf{x})$ and $G_i(\mathbf{x}) + \log(x_i)F(\mathbf{x})$ are particular solutions of a certain A-system of linear differential equations. These objects are geometric in nature because the series $F(\mathbf{x})$ are A-hypergeometric functions ([1]) which can be viewed as the period of certain multi-parameter families of algebraic varieties in a product of weighted projective spaces (see [6] for details).

A classical example of multivariate mirror maps, studied in [2,13,9] is related to the series

$$F(z_1, z_2) = \sum_{m,n \geq 0} \frac{(3m+3n)!}{m!^3 n!^3} z_1^m z_2^n, \tag{1.1}$$

which is a solution of the system of differential equations

$$\begin{cases} D_1^3 y - z_1 (3D_1 + 3D_2 + 1)(3D_1 + 3D_2 + 2)(3D_1 + 3D_2 + 3)\, y = 0, \\ D_2^3 y - z_2 (3D_1 + 3D_2 + 1)(3D_1 + 3D_2 + 2)(3D_1 + 3D_2 + 3)\, y = 0, \end{cases}$$

where $D_1 = z_1 \frac{d}{dz_1}$ and $D_2 = z_2 \frac{d}{dz_2}$. We find two other solutions of this system $G_1(z_1, z_2) + \log(z_1)F(z_1, z_2)$ and $G_2(z_1, z_2) + \log(z_2)F(z_1, z_2)$ where

$$G_1(z_1, z_2) = \sum_{m,n \geq 0} \frac{(3m+3n)!}{m!^3 n!^3} (3H_{3m+3n} - 3H_m) z_1^m z_2^n$$

and $G_2(z_1, z_2) = G_1(z_2, z_1)$. This set of solutions enables us to define two *canonical coordinates* $q_1(z_1, z_2) = z_1 \exp(G_1(z_1, z_2)/F(z_1, z_2))$ and $q_2(z_1, z_2) = z_2 \exp(G(z_1, z_2)/F(z_1, z_2))$.

The associated *mirror maps* are defined by the formal series $z_1(q_1, q_2)$ and $z_2(q_1, q_2)$ such that the map $(q_1, q_2) \mapsto (z_1(q_1, q_2), z_2(q_1, q_2))$ is the compositional inverse of the map $(z_1, z_2) \mapsto (q_1(z_1, z_2), q_2(z_1, z_2))$.

According to Corollary 1 of [9], the series $q_1(z_1, z_2), q_2(z_1, z_2), z_1(q_1, q_2)$ and $z_2(q_1, q_2)$ have integral Taylor coefficients.

Mirror maps are of interest in Mathematical Physics and Algebraic Geometry. In particular, within Mirror Symmetry Theory, it has been observed that the Taylor coefficients of mirror maps are integers. This surprising observation has led to the study of these objects within Number Theory, which has led to its proof in many cases (see further down in the introduction). The aim of this article is to establish a necessary and sufficient condition for the integrality of all the Taylor coefficients of mirror maps defined by ratios of factorials of linear forms.

1.1. Definition of mirror maps

In order to define the mirror maps considered in this article, we introduce some standard multi-index notation, which we use throughout the article. Namely, given a positive integer $d, k \in \{1, \ldots, d\}$ and vectors $\mathbf{m} := (m_1, \ldots, m_d)$ and $\mathbf{n} := (n_1, \ldots, n_d)$ in $\mathbb{R}^d$, we write $\mathbf{m} \cdot \mathbf{n}$ for the scalar product $m_1 n_1 + \cdots + m_d n_d$ and $\mathbf{m}^{(k)}$ for m_k. We write $\mathbf{m} \geq \mathbf{n}$ if and only if $m_i \geq n_i$ for all $i \in \{1, \ldots, d\}$. In addition, if $\mathbf{z} := (z_1, \ldots, z_d)$ is a vector of variables and if $\mathbf{n} := (n_1, \ldots, n_d) \in \mathbb{Z}^d$, then we write $\mathbf{z}^\mathbf{n}$ for the product $z_1^{n_1} \cdots z_d^{n_d}$. Finally, we write $\mathbf{0}$ for the vector $(0, \ldots, 0) \in \mathbb{Z}^d$.

Given two sequences of vectors in $\mathbb{N}^d$, $e := (\mathbf{e}_1, \ldots, \mathbf{e}_{q_1})$ and $f := (\mathbf{f}_1, \ldots, \mathbf{f}_{q_2})$, we write $|e| := \sum_{i=1}^{q_1} \mathbf{e}_i$ and $|f| := \sum_{i=1}^{q_2} \mathbf{f}_i \in \mathbb{N}^d$ so that, for all $k \in \{1, \ldots, d\}$, we have $|e|^{(k)} = \sum_{i=1}^{q_1} \mathbf{e}_i^{(k)}$

[1] The A-hypergeometric series are also called GKZ hypergeometric series. See [13] for an introduction to these series, which generalize the classical hypergeometric series to the multivariate case.

and $|f|^{(k)}=\sum_{i=1}^{q_2}\mathbf{f}_i^{(k)}$. For all $\mathbf{n}\in\mathbb{N}^d$, we write

$$\mathcal{Q}_{e,f}(\mathbf{n}):=\frac{(\mathbf{e}_1\cdot\mathbf{n})!\cdots(\mathbf{e}_{q_1}\cdot\mathbf{n})!}{(\mathbf{f}_1\cdot\mathbf{n})!\cdots(\mathbf{f}_{q_2}\cdot\mathbf{n})!}.$$

We define the formal series

$$F_{e,f}(\mathbf{z}):=\sum_{\mathbf{n}\geq\mathbf{0}}\frac{(\mathbf{e}_1\cdot\mathbf{n})!\cdots(\mathbf{e}_{q_1}\cdot\mathbf{n})!}{(\mathbf{f}_1\cdot\mathbf{n})!\cdots(\mathbf{f}_{q_2}\cdot\mathbf{n})!}\mathbf{z}^{\mathbf{n}}$$

and

$$G_{e,f,k}(\mathbf{z}):=\sum_{\mathbf{n}\geq\mathbf{0}}\frac{(\mathbf{e}_1\cdot\mathbf{n})!\cdots(\mathbf{e}_{q_1}\cdot\mathbf{n})!}{(\mathbf{f}_1\cdot\mathbf{n})!\cdots(\mathbf{f}_{q_2}\cdot\mathbf{n})!}\left(\sum_{i=1}^{q_1}\mathbf{e}_i^{(k)}H_{\mathbf{e}_i\cdot\mathbf{n}}-\sum_{j=1}^{q_2}\mathbf{f}_j^{(k)}H_{\mathbf{f}_j\cdot\mathbf{n}}\right)\mathbf{z}^{\mathbf{n}},\tag{1.2}$$

where $k\in\{1,\ldots,d\}$ and, for all $m\in\mathbb{N}$, $H_m:=\sum_{i=1}^{m}\frac{1}{i}$ is the m-th harmonic number. The series $F_{e,f}(\mathbf{z})$ is an A-hypergeometric series and is therefore a solution of an A-system of linear differential equations. In some cases, we find d additional solutions of this system together with at most logarithmic singularities at the origin, the series $G_{e,f,k}(\mathbf{z})+\log(z_k)F(\mathbf{z})$ for $k\in\{1,\ldots,d\}$.

In the context of mirror symmetry, when $|e|=|f|$, the d functions

$$q_{e,f,k}(\mathbf{z}):=z_k\exp(G_{e,f,k}(\mathbf{z})/F_{e,f}(\mathbf{z})),\quad k\in\{1,\ldots,d\},$$

are *canonical coordinates*. The compositional inverse of the map $\mathbf{z}\mapsto(q_{e,f,1}(\mathbf{z}),\ldots,q_{e,f,d}(\mathbf{z}))$ defines the vector $(z_{e,f,1}(\mathbf{q}),\ldots,z_{e,f,d}(\mathbf{q}))$ of *mirror maps*.

The aim of this article is to establish a necessary and sufficient condition for the integrality of the coefficients of the d mirror maps $z_{e,f,k}(\mathbf{q})$, that is, to determine under which conditions, for all $k\in\{1,\ldots,d\}$, we have $z_{e,f,k}(\mathbf{q})\in\mathbb{Z}[[\mathbf{q}]]$. In the context of Number Theory of this article, the mirror map $z_{e,f,k}(\mathbf{q})$ and the corresponding canonical coordinate $q_{e,f,k}(\mathbf{z})$ play strictly the same role because, for all $k\in\{1,\ldots,d\}$, we have $q_{e,f,k}(\mathbf{z})\in z_k\mathbb{Z}[[\mathbf{z}]]$ if and only if, for all $k\in\{1,\ldots,d\}$, we have $z_{e,f,k}(\mathbf{q})\in q_k\mathbb{Z}[[\mathbf{q}]]$ (see [9, Partie 1.2]). Therefore, we shall formulate the criterion only for canonical coordinates but it also holds for the corresponding mirror maps.

1.2. Statement of the criterion

Before stating the criterion for the integrality of the Taylor coefficients of $q_{e,f,k}(\mathbf{z})$, we recall the definition of *Landau's function* associated with a ratio of factorials of linear forms. Given two sequences of vectors in $\mathbb{N}^d$ $e:=(\mathbf{e}_1,\ldots,\mathbf{e}_{q_1})$ and $f:=(\mathbf{f}_1,\ldots,\mathbf{f}_{q_2})$, we write $\Delta_{e,f}$ the function of Landau associated with $\mathcal{Q}_{e,f}$, which is defined, for all $\mathbf{x}\in\mathbb{R}^d$, by

$$\Delta_{e,f}(\mathbf{x}):=\sum_{i=1}^{q_1}\lfloor\mathbf{e}_i\cdot\mathbf{x}\rfloor-\sum_{j=1}^{q_2}\lfloor\mathbf{f}_j\cdot\mathbf{x}\rfloor,$$

where $\lfloor\cdot\rfloor$ denotes the floor function. We also write $\{\cdot\}$ for the fractional part function. We still write $\lfloor\cdot\rfloor$, respectively $\{\cdot\}$, for the function defined, for all $\mathbf{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$, by $\lfloor\mathbf{x}\rfloor:=(\lfloor x_1\rfloor,\ldots,\lfloor x_d\rfloor)$, respectively by $\{\mathbf{x}\}:=(\{x_1\},\ldots,\{x_d\})$. For all $\mathbf{c}\in\mathbb{N}^d$, we have $\lfloor\mathbf{c}\cdot\mathbf{x}\rfloor=\lfloor\mathbf{c}\cdot\{\mathbf{x}\}\rfloor+\mathbf{c}\cdot\lfloor\mathbf{x}\rfloor$ and therefore $\Delta_{e,f}(\mathbf{x})=\Delta_{e,f}(\{\mathbf{x}\})+(|e|-|f|)\cdot\lfloor\mathbf{x}\rfloor$. So, we have $|e|=|f|$ if and only if $\Delta_{e,f}$ is 1-periodic in each of its variables. We write $\mathcal{D}_{e,f}$ for the semi-algebraic set of all $\mathbf{x}\in[0,1[^d$ such that there exists $\mathbf{d}\in\{\mathbf{e}_1,\ldots,\mathbf{e}_{q_1},\mathbf{f}_1,\ldots,\mathbf{f}_{q_2}\}$ verifying $\mathbf{d}\cdot\mathbf{x}\geq 1$. The set $[0,1[^d\setminus\mathcal{D}_{e,f}$ is nonempty and the function $\Delta_{e,f}$ vanishes on $[0,1[^d\setminus\mathcal{D}_{e,f}$. The following proposition shows that Landau's function provides a characterization of the sequences e and f such that, for all $\mathbf{n} \in \mathbb{N}^d$, $\mathcal{Q}_{e,f}(\mathbf{n})$ is an integer.

Landau's criterion. *Let e and f be two sequences of vectors in $\mathbb{N}^d$. We have the following dichotomy.*

(i) *If, for all $\mathbf{x} \in [0,1]^d$, we have $\Delta_{e,f}(\mathbf{x}) \geq 0$, then, for all $\mathbf{n} \in \mathbb{N}^d$, we have $\mathcal{Q}_{e,f}(\mathbf{n}) \in \mathbb{N}$.*
(ii) *If there exists $\mathbf{x} \in [0,1]^d$ such that $\Delta_{e,f}(\mathbf{x}) \leq -1$, then there are only finitely many prime numbers p such that all terms of the family $\mathcal{Q}_{e,f}$ are in $\mathbb{Z}_p$.*

Remark. Assertion (i) is a result of Landau in [10]: he has proved that it is in fact a necessary and sufficient condition. We prove Landau's criterion assertion (ii) in Section 2.

In the literature, one can find several results proving the integrality of the Taylor coefficients of univariate mirror maps (*i.e.* $d = 1$) when $|e| = |f|$. One can find them, in an increasing order of generality, in [12,14,8,4]. We refer the reader to the introduction of [4] for a detailed statement of all these results. In the univariate case, the most general result builds up a criterion for the integrality of the Taylor coefficients of mirror maps defined by sequences of ratios of factorials [4, Theorem 1]. In the multivariate case, Krattenthaler and Rivoal proved in [9] the integrality of the Taylor coefficients of mirror maps belonging to large infinite families. In order to state this result, for all $k \in \{1,\dots,d\}$, we write $\mathbf{1}_k$ for the vector in $\mathbb{N}^d$, all coordinates of which are equal to zero except the k-th which is equal to 1.

Theorem (*Corollary 1 of [9]*)**.** *Let e and f be two sequences of vectors in $\mathbb{N}^d$ verifying $|e| = |f|$ such that f is only composed of vectors of the form $\mathbf{1}_k$ with $k \in \{1,\dots,d\}$. Then, for all $k \in \{1,\dots,d\}$, we have $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}[[\mathbf{z}]]$.*

The purpose of this article is to prove the following theorems, which provide a characterization of the multivariate mirror maps associated with integral ratios of factorials of linear forms for which all the Taylor coefficients are integers. We prove in Section 1.3 that they contain the results of other authors who worked on this subject previously. First, we consider the case $|e| = |f|$ and then we state the results when there exists $k \in \{1,\dots,d\}$ such that $|e|^{(k)} > |f|^{(k)}$. When there exists $k \in \{1,\dots,d\}$ such that $|e|^{(k)} < |f|^{(k)}$, the family $\mathcal{Q}_{e,f}$ has a term that is not an integer and the question of the integrality of the Taylor coefficients of $q_{e,f,k}(\mathbf{z})$ is still open.

Theorem 1. *Let e and f be two disjoint sequences of nonzero vectors in $\mathbb{N}^d$ such that $\mathcal{Q}_{e,f}$ is a family of integers (equivalent to $\Delta_{e,f} \geq 0$ on $[0,1]^d$) and which satisfy $|e| = |f|$. Then we have the following dichotomy.*

(i) *If, for all $\mathbf{x} \in \mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x}) \geq 1$, then, for all $k \in \{1,\dots,d\}$, we have $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}[[\mathbf{z}]]$.*
(ii) *If there exists $\mathbf{x} \in \mathcal{D}_{e,f}$ such that $\Delta_{e,f}(\mathbf{x}) = 0$, then there exists $k \in \{1,\dots,d\}$ such that there are only finitely many prime numbers p such that $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}_p[[\mathbf{z}]]$.*

Remarks. • Note the similarity between Landau's criterion and Theorem 1.

• We assume that the terms of the sequences e and f are nonzero and that these sequences are disjoint in order to rule out the possibility that $\Delta_{e,f}$ vanishes identically, which corresponds to the formal series $F_{e,f}(\mathbf{z}) = (1-z_1)^{-1}\cdots(1-z_d)^{-1}$, $G_{e,f,k}(\mathbf{z}) = 0$ and $q_{e,f,k}(\mathbf{z}) = z_k$.

- Assertion (ii) of Theorem 1 is optimal since, if $\Delta_{e,f}$ vanishes on $\mathcal{D}_{e,f}$ and if $d \geq 2$, then there may exist $k \in \{1,\dots,d\}$ such that $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}[[\mathbf{z}]]$. Indeed, if one chooses $d = 2$, $e = ((3,0))$ and $f = ((2,0),(1,0))$. Then we have $\mathcal{D}_{e,f} = \{(x_1,x_2) \in [0,1[^2 : x_1 \geq 1/3\}$, $\Delta_{e,f}((1/2,0)) = 0$ and $q_{e,f,2}(\mathbf{z}) = z_2$.
- Theorem 1 generalizes the criterion for univariate mirror maps and Corollary 1 of [9] (see Section 1.3).

We will now state a criterion for the integrality of the Taylor coefficients of *mirror-type* maps $q_{\mathbf{L},e,f}$ defined, for all $\mathbf{L} \in \mathbb{N}^d$, by $q_{\mathbf{L},e,f}(\mathbf{z}) := \exp(G_{\mathbf{L},e,f}(\mathbf{z})/F_{e,f}(\mathbf{z}))$, where $G_{\mathbf{L},e,f}$ is the formal power series

$$G_{\mathbf{L},e,f}(\mathbf{z}) := \sum_{\mathbf{n}\geq\mathbf{0}} \frac{(\mathbf{e}_1\cdot\mathbf{n})!\cdots(\mathbf{e}_{q_1}\cdot\mathbf{n})!}{(\mathbf{f}_1\cdot\mathbf{n})!\cdots(\mathbf{f}_{q_2}\cdot\mathbf{n})!} H_{\mathbf{L}\cdot\mathbf{n}}\,\mathbf{z}^{\mathbf{n}}. \tag{1.3}$$

We write $\mathcal{E}_{e,f}$ for the set of all $\mathbf{L} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$ such that there is a $\mathbf{d} \in \{\mathbf{e}_1,\dots,\mathbf{e}_{q_1},\mathbf{f}_1,\dots,\mathbf{f}_{q_2}\}$ satisfying $\mathbf{L} \leq \mathbf{d}$. We have $q_{\mathbf{L},e,f}(\mathbf{z}) \in 1 + \sum_{j=1}^d z_j\mathbb{Q}[[\mathbf{z}]]$ and

$$z_k^{-1} q_{e,f,k}(\mathbf{z}) = \left(\prod_{i=1}^{q_1}\left(q_{\mathbf{e}_i,e,f}(\mathbf{z})\right)^{\mathbf{e}_i^{(k)}}\right) \Big/ \left(\prod_{j=1}^{q_2}\left(q_{\mathbf{f}_j,e,f}(\mathbf{z})\right)^{\mathbf{f}_j^{(k)}}\right), \tag{1.4}$$

so that if, for all $\mathbf{L} \in \mathcal{E}_{e,f}$, we have $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}[[\mathbf{z}]]$, then, for all $k \in \{1,\dots,d\}$, we have $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}[[\mathbf{z}]]$. Thus, assertion (i) of Theorem 2 implies assertion (i) of Theorem 1. Assertion (ii) of Theorem 2 adds details to assertion (ii) of Theorem 1. To be more precise, it proves that there exists $k \in \{1,\dots,d\}$ such that $q_{e,f,k}(\mathbf{z}) \notin z_k\mathbb{Z}[[\mathbf{z}]]$ and that all the mirror-type maps indeed involved in (1.4) have at least one Taylor coefficient which is not an integer. Thus Theorem 1 can be seen as a corollary of Theorem 2.

Theorem 2. *Let e and f be two disjoint sequences of nonzero vectors in $\mathbb{N}^d$ such that $\mathcal{Q}_{e,f}$ is a family of integers (which is equivalent to $\Delta_{e,f} \geq 0$ on $[0,1]^d$) and which satisfy $|e| = |f|$. Then we have the following dichotomy.*

(i) *If, for all $\mathbf{x} \in \mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x}) \geq 1$, then, for all $\mathbf{L} \in \mathcal{E}_{e,f}$, we have $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}[[\mathbf{z}]]$.*
(ii) *If there exists $\mathbf{x} \in \mathcal{D}_{e,f}$ such that $\Delta_{e,f}(\mathbf{x}) = 0$, then there exists $k \in \{1,\dots,d\}$ such that, if $\mathbf{L} \in \mathcal{E}_{e,f}$ verifies $\mathbf{L}^{(k)} \geq 1$, then there are only finitely many prime numbers p such that $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$. Furthermore, there are only finitely many prime numbers p such that $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}_p[[\mathbf{z}]]$.*

In Section 9 of the preprint version [3] of this paper, we show that Theorem 2 implies the integrality of the Taylor coefficients of new univariate mirror maps listed in [1]. Theorem 2 generalizes Theorem 2 of [4] and Theorem 2 of [9] (see Section 1.3). If there exists $k \in \{1,\dots,d\}$ such that $|e|^{(k)} > |f|^{(k)}$, we have the following theorem which generalizes Theorem 3 of [4].

Theorem 3. *Let e and f be two disjoint sequences of nonzero vectors in $\mathbb{N}^d$ and such that $\mathcal{Q}_{e,f}$ is a family of integers (which is equivalent to $\Delta_{e,f} \geq 0$ on $[0,1]^d$) and such that there exists $k \in \{1,\dots,d\}$ verifying $|e|^{(k)} > |f|^{(k)}$. Then,*

(a) *there are only finitely many prime numbers p such that $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}_p[[\mathbf{z}]]$;*
(b) *for all $\mathbf{L} \in \mathcal{E}_{e,f}$ verifying $\mathbf{L}^{(k)} \geq 1$, there are only finitely many prime numbers p such that $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$.*

1.3. Comparison of Theorems 1–3 with previous results

First, we prove that Theorems 1 and 2 generalize Corollary 1 and Theorem 2 of [9]. We only have to prove that, if e and f are two disjoint sequences of nonzero vectors in $\mathbb{N}^d$, verifying $|e| = |f|$ and such that f is only constituted by vectors $\mathbf{1}_k$ with $k \in \{1, \dots, d\}$, then, for all $\mathbf{x} \in \mathcal{D}_{e,f}$, we have $\Delta_{e,f}(\mathbf{x}) \geq 1$. Indeed, if $\mathbf{x} \in \mathcal{D}_{e,f}$, then $\mathbf{x} \in [0, 1[^d$ and, for all $k \in \{1, \dots, d\}$, we have $\mathbf{1}_k \cdot \mathbf{x} = 0$. Thus, there exists an element $\mathbf{d}$ in e such that $\mathbf{d} \cdot \mathbf{x} \geq 1$ and we have $\Delta_{e,f}(\mathbf{x}) = \sum_{i=1}^{q_1} \lfloor \mathbf{e}_i \cdot \mathbf{x} \rfloor - \sum_{j=1}^{q_2} \lfloor \mathbf{f}_j \cdot \mathbf{x} \rfloor = \sum_{i=1}^{q_1} \lfloor \mathbf{e}_i \cdot \mathbf{x} \rfloor \geq 1$.

Let us now prove that Theorems 2 and 3 generalize Theorems 2 and 3 of [4]. It is sufficient to note that if $d = 1$, then e and f are two sequences of positive integers and, writing $M_{e,f}$ for the greatest element in the sequences e and f, we obtain $\mathcal{E}_{e,f} = \{1, \dots, M_{e,f}\}$ and $\mathcal{D}_{e,f} = [1/M_{e,f}, 1[$.

1.4. Structure of proofs

First, we prove assertion (ii) of Landau's criterion in Section 2.

Section 3 is dedicated to the statement and the proof of Theorem 4, which generalizes criteria of formal congruences proved by Dwork and by Krattenthaler and Rivoal. These criteria were crucial for the previous results about the integrality of the Taylor coefficients of mirror maps. Theorem 4 is central to the proofs of Theorems 1 and 2.

In Section 4, we reduce the proofs of Theorems 1–3 to the proofs of p-adic relations.

Section 5 is dedicated to the statement and the proof of a technical lemma which we will use to prove both assertions of Theorems 1 and 2.

We prove assertion (i) of Theorems 1 and 2 in Section 6, this is by far the longest and the most technical part of this article. Particularly, we have to prove a certain number of delicate p-adic estimations in order to be able to apply Theorem 4.

In Sections 7 and 8, we prove assertion (ii) of Theorems 1 and 2 and Theorem 3, which ensue rather fast from reformulations of these theorems established in Section 4.

2. Proof of assertion (ii) of Landau's criterion

First, let us introduce some additional notations which we will use throughout this article. Given $d \in \mathbb{N}, d \geq 1, \lambda \in \mathbb{R}, k \in \{1, \dots, d\}$ and vectors $\mathbf{m} := (m_1, \dots, m_d)$ and $\mathbf{n} := (n_1, \dots, n_d)$ in $\mathbb{R}^d$, we write $\mathbf{m} + \mathbf{n}$ for $(m_1 + n_1, \dots, m_d + n_d)$, $\lambda\mathbf{m}$ or $\mathbf{m}\lambda$ for $(\lambda m_1, \dots, \lambda m_d)$, and $\mathbf{m}/\lambda$ for $(m_1/\lambda, \dots, m_d/\lambda)$ when λ is nonzero.

To prove assertion (ii) of Landau's criterion, we will use the fact that, for all primes p and all $\mathbf{n} \in \mathbb{N}^d$, we have $v_p(\mathcal{Q}_{e,f}(\mathbf{n})) = \sum_{\ell=1}^{\infty} \Delta(\mathbf{n}/p^\ell)$. Indeed, we recall that, for all $m \in \mathbb{N}$, we have the formula $v_p(m!) = \sum_{\ell=1}^{\infty} \lfloor m/p^\ell \rfloor$. Thereby, we get

$$v_p(\mathcal{Q}_{e,f}(\mathbf{n})) = \sum_{\ell=1}^{\infty} \left(\sum_{i=1}^{q_1} \lfloor \mathbf{e}_i \cdot \mathbf{n}/p^\ell \rfloor - \sum_{j=1}^{q_2} \lfloor \mathbf{f}_j \cdot \mathbf{n}/p^\ell \rfloor \right) = \sum_{\ell=1}^{\infty} \Delta\left(\frac{\mathbf{n}}{p^\ell}\right).$$

We will need the following lemma, which we will also use for the proofs of assertion (ii) of Theorems 1 and 2. In the rest of the article, we write $\mathbf{1}$ for the vector $(1, \dots, 1) \in \mathbb{N}^d$.

Lemma 1. *Let $u := (\mathbf{u}_1, \dots, \mathbf{u}_n)$ be a sequence of vectors in $\mathbb{N}^d$ and $\mathbf{x}_0 \in \mathbb{R}^d$. Then, there exists $\mu > 0$ such that, for all $\mathbf{x} \in \mathbb{R}^d$ satisfying $\mathbf{0} \leq \mathbf{x} \leq \mu\mathbf{1}$ and all $i \in \{1, \dots, n\}$, we have $\lfloor \mathbf{u}_i \cdot (\mathbf{x}_0 + \mathbf{x}) \rfloor = \lfloor \mathbf{u}_i \cdot \mathbf{x}_0 \rfloor$.*

Proof. For all $y \in \mathbb{R}$ there exists $\nu_y > 0$ such that $\lfloor y + \nu_y \rfloor = \lfloor y \rfloor$. Thus, writing $\nu := \min\{\nu_{\mathbf{u}_i \cdot \mathbf{x}_0} : 1 \le i \le n\} > 0$, we obtain that, for all $i \in \{1, \dots, n\}$, we have $\lfloor \mathbf{u}_i.\mathbf{x}_0 + \nu \rfloor = \lfloor \mathbf{u}_i.\mathbf{x}_0 \rfloor$. Therefore, writing $\mu := \min\{\nu/|\mathbf{u}_i| : 1 \le i \le n,\ \mathbf{u}_i \ne \mathbf{0}\} > 0$, we get that, for all $\mathbf{0} \le \mathbf{x} \le \mu\mathbf{1}$ and all $i \in \{1, \dots, n\}$, we have $\mathbf{u}_i \cdot \mathbf{x} \le \mu|\mathbf{u}_i| \le \nu$ so $\lfloor \mathbf{u}_i \cdot (\mathbf{x}_0 + \mathbf{x}) \rfloor = \lfloor \mathbf{u}_i \cdot \mathbf{x}_0 \rfloor$. This completes the proof of the lemma. □

Proof of assertion (ii) of Landau's criterion. Given $\mathbf{x}_0 \in [0, 1]^d$ satisfying $\Delta_{e,f}(\mathbf{x}_0) \le -1$ and applying Lemma 1 with, instead of u, the sequence constituted by the elements of e and f, we obtain that there exists $\mu > 0$ such that, for all $\mathbf{x} \in \mathbb{R}^d$ verifying $\mathbf{0} \le \mathbf{x} \le \mu\mathbf{1}$, we have $\Delta_{e,f}(\mathbf{x}_0 + \mathbf{x}) = \Delta_{e,f}(\mathbf{x}_0) \le -1$. We write $\mathcal{U} := \{\mathbf{x}_0 + \mathbf{x} : \mathbf{0} \le \mathbf{x} \le \mu\mathbf{1}\}$ during the proof.

There exists a constant $\mathcal{N}_1$ such that, for all primes $p \ge \mathcal{N}_1$, there is $\mathbf{n}_p \in \mathbb{N}^d$ such that $\mathbf{n}_p/p \in \mathcal{U}$. There exists a constant $\mathcal{N}_2$ such that, for all primes $p \ge \mathcal{N}_2$ and all $\mathbf{d} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$, we have $|\mathbf{d}|(\mu + 1)/p < 1$.

Thus, for all prime numbers $p \ge \mathcal{N} := \max(\mathcal{N}_1, \mathcal{N}_2)$ and all integers $\ell \ge 2$, we have $\Delta_{e,f}(\mathbf{n}_p/p) \le -1$ and, since $\mathbf{n}_p/p \in \mathcal{U}$, we have $\mathbf{n}_p/p \le (1 + \mu)\mathbf{1}$ and $\mathbf{n}_p/p^\ell \le \mathbf{n}_p/p^2 \le (\mu + 1)/p\mathbf{1}$. As a result, for all $\mathbf{d} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$, we obtain $\mathbf{d} \cdot \mathbf{n}_p/p^\ell \le |\mathbf{d}|(\mu + 1)/p < 1$, which leads to $\mathbf{n}_p/p^\ell \in [0, 1[^d \setminus \mathcal{D}_{e,f}$ and so $\Delta_{e,f}(\mathbf{n}_p/p^\ell) = 0$.

Thus, for all primes $p \ge \mathcal{N}$, we have $v_p(\mathcal{Q}_{e,f}(\mathbf{n}_p)) = \sum_{\ell=1}^{\infty} \Delta_{e,f}(\mathbf{n}_p/p^\ell) \le -1$, which finishes the proof of Landau's criterion. □

3. Formal congruences

The proof of assertion (i) of Theorem 2 is essentially based on the generalization (Theorem 4) of a theorem of Krattenthaler and Rivoal [9, Theorem 1, p. 3] which is a multivariate adaptation of a theorem of Dwork [5, Theorem 1, p. 296].

Before stating Theorem 4, we introduce some notations. Let p be a prime number and $d \in \mathbb{N}, d \ge 1$. We write Ω for the completion of the algebraic closure of $\mathbb{Q}_p$ and $\mathcal{O}$ for the ring of integers of Ω.

If $\mathcal{N}$ is a subset of $\bigcup_{t \ge 1} \left(\{0, \dots, p^t - 1\}^d \times \{t\}\right)$, then, for all $s \in \mathbb{N}$, we write $\Psi_s(\mathcal{N})$ for the set of all $\mathbf{u} \in \{0, \dots, p^s - 1\}^d$ such that, for all $(\mathbf{n}, t) \in \mathcal{N}$, with $t \le s$, and all $\mathbf{j} \in \{0, \dots, p^{s-t} - 1\}^d$, we have $\mathbf{u} \ne \mathbf{j} + p^{s-t}\mathbf{n}$.

Given $\mathbf{u} \in \{0, \dots, p^s - 1\}^d$, $\mathbf{u} := \sum_{k=0}^{s-1} \mathbf{u}_\mathbf{k} p^k$ with $\mathbf{u}_k \in \{0, \dots, p - 1\}^d$, we write $\mathcal{M}_s(\mathbf{u})$ for the word $\mathbf{u}_0 \cdots \mathbf{u}_{s-1}$ of length s over the alphabet $\{0, \dots, p - 1\}^d$. According to this definition, the following properties hold: $\mathbf{u} \in \Psi_s(\mathcal{N})$ if and only if none of the words $\mathcal{M}_t(\mathbf{n})$, $(\mathbf{n}, t) \in \mathcal{N}$, is a suffix of $\mathcal{M}_s(\mathbf{u})$.

For example, let us take $\mathcal{N} := \{(\mathbf{0}, t) : t \ge 1\}$. In this case, $\Psi_s(\mathcal{N})$ is the set of all $\mathbf{u} = \sum_{k=0}^{s-1} \mathbf{u}_k p^k$ such that $\mathbf{u}_{s-1} \ne \mathbf{0}$. We observe that $\Psi_s(\mathcal{N}) = \Psi_s(\mathcal{N}')$ with $\mathcal{N}' = \{(\mathbf{0}, 1)\}$.

Theorem 4. *Let us fix a prime number p. Let $(\mathbf{A}_r)_{r \ge 0}$ be a sequence of maps from $\mathbb{N}^d$ to $\Omega \setminus \{0\}$ and $(\mathbf{g}_r)_{r \ge 0}$ be a sequence of maps from $\mathbb{N}^d$ to $\mathcal{O} \setminus \{0\}$. We assume that there exists $\mathcal{N} \subset \bigcup_{t \ge 1} \left(\{0, \dots, p^t - 1\}^d \times \{t\}\right)$ such that, for all $r \ge 0$, we have*

(i) $|\mathbf{A}_r(\mathbf{0})|_p = 1$;
(ii) *for all* $\mathbf{m} \in \mathbb{N}^d$, *we have* $\mathbf{A}_r(\mathbf{m}) \in \mathbf{g}_r(\mathbf{m})\mathcal{O}$;
(iii) *for all* $s \in \mathbb{N}$ *and* $\mathbf{m} \in \mathbb{N}^d$, *we have:*
  (a) *for all* $\mathbf{u} \in \Psi_s(\mathcal{N})$ *and* $\mathbf{v} \in \{0, \dots, p - 1\}^d$, *we have*

$$\frac{\mathbf{A}_r(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})}{\mathbf{A}_r(\mathbf{v} + \mathbf{u}p)} - \frac{\mathbf{A}_{r+1}(\mathbf{u} + \mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{u})} \in p^{s+1}\frac{\mathbf{g}_{r+s+1}(\mathbf{m})}{\mathbf{A}_r(\mathbf{v} + \mathbf{u}p)}\mathcal{O};$$

(a_1) *furthermore, if* $\mathbf{v}+p\mathbf{u}\in\Psi_{s+1}(\mathcal{N})$*, then we have*

$$\frac{\mathbf{A}_r(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}{\mathbf{A}_r(\mathbf{v}+\mathbf{u}p)}-\frac{\mathbf{A}_{r+1}(\mathbf{u}+\mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{u})}\in p^{s+1}\frac{\mathbf{g}_{r+s+1}(\mathbf{m})}{\mathbf{g}_r(\mathbf{v}+\mathbf{u}p)}\mathcal{O};$$

(a_2) *on the other hand, if* $\mathbf{v}+p\mathbf{u}\notin\Psi_{s+1}(\mathcal{N})$*, then we have*

$$\frac{\mathbf{A}_{r+1}(\mathbf{u}+p^s\mathbf{m})}{\mathbf{A}_{r+1}(\mathbf{u})}\in p^{s+1}\frac{\mathbf{g}_{s+r+1}(\mathbf{m})}{\mathbf{g}_r(\mathbf{v}+p\mathbf{u})}\mathcal{O};$$

(b) *for all* $(\mathbf{n},t)\in\mathcal{N}$*, we have* $\mathbf{g}_r\left(\mathbf{n}+p^t\mathbf{m}\right)\in p^t\mathbf{g}_{r+t}(\mathbf{m})\mathcal{O}$.

Then, for all $\mathbf{a}\in\{0,\dots,p-1\}^d$, $\mathbf{m}\in\mathbb{N}^d$, $s,r\in\mathbb{N}$ *and* $\mathbf{K}\in\mathbb{Z}^d$*, we have*

$$\mathbf{S}_r(\mathbf{a},\mathbf{K},s,p,\mathbf{m}):=\sum_{\mathbf{m}p^s\le\mathbf{j}\le(\mathbf{m}+\mathbf{1})p^s-\mathbf{1}}\big(\mathbf{A}_r(\mathbf{a}+p(\mathbf{K}-\mathbf{j}))\mathbf{A}_{r+1}(\mathbf{j})-\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{j})\mathbf{A}_r(\mathbf{a}+\mathbf{j}p)\big)\in p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O},\tag{3.1}$$

where we extend $\mathbf{A}_r$ *to* $\mathbb{Z}^d$ *by* $\mathbf{A}_r(\mathbf{n})=0$ *if there is an* $i\in\{1,\dots,d\}$ *such that* $n_i<0$.

This theorem generalizes Theorem 1 of [9]. Indeed, let $A:\mathbb{N}^d\mapsto\mathbb{Z}_p\setminus\{0\}$ and $g:\mathbb{N}^d\mapsto\mathbb{Z}_p\setminus\{0\}$ be two maps verifying conditions (i), (ii) and (iii) of Theorem 1 of [9]. Let $(\mathbf{A}_r)_{r\ge0}$ be the constant sequence of value A and $(\mathbf{g}_r)_{r\ge0}$ be the constant sequence of value g. These two sequences verify conditions (i) and (ii) of Theorem 4. Let us choose $\mathcal{N}:=\emptyset$ so that, for all $s\in\mathbb{N}$, we have $\Psi_s(\mathcal{N})=\{0,\dots,p^s-1\}^d$. In particular, conditions (a_2) and (b) of Theorem 4 are empty. Thus we only have to prove that $(\mathbf{A}_r)_{r\ge0}$ and $(\mathbf{g}_r)_{r\ge0}$ verify assertions (a) and (a_1) of Theorem 4. The equality $\Psi_{s+1}(\mathcal{N})=\{0,\dots,p^{s+1}-1\}^d$, associated with assertion (ii), proves that condition (a_1) implies assertion (a). But assertion (a_1) corresponds to (iii) of Theorem 1 of [9]. Thus the conditions of Theorem 4 are valid and we have the conclusion of Theorem 1 of [9].

The aim of the end of this section is to prove Theorem 4.

3.1. Proof of Theorem 4

The structure of the proof is based on the structure of the proofs of the theorems of Dwork and Krattenthaler and Rivoal, but it rather significantly differs in details.

For all $s\in\mathbb{N}$, $s\ge1$, we write α_s for the following assertion: for all $\mathbf{a}\in\{0,\dots,p-1\}^d$, $u\in\{0,\dots,s-1\}$, $\mathbf{m}\in\mathbb{N}^d$, $r\ge0$ and $\mathbf{K}\in\mathbb{Z}^d$, we have the congruence $\mathbf{S}_r(\mathbf{a},\mathbf{K},u,p,\mathbf{m})\in p^{u+1}\mathbf{g}_{u+r+1}(\mathbf{m})\mathcal{O}$.

For all $s\in\mathbb{N}$, $s\ge1$ and $t\in\{0,\dots,s\}$, we write $\beta_{t,s}$ for the following assertion: for all $\mathbf{a}\in\{0,\dots,p-1\}^d$, $\mathbf{m}\in\mathbb{N}^d$, $r\ge0$ and $\mathbf{K}\in\mathbb{Z}^d$, we have the congruence

$$\mathbf{S}_r(\mathbf{a},\mathbf{K}+\mathbf{m}p^s,s,p,\mathbf{m})\equiv\sum_{\mathbf{j}\in\Psi_{s-t}(\mathcal{N})}\frac{\mathbf{A}_{t+r+1}(\mathbf{j}+\mathbf{m}p^{s-t})}{\mathbf{A}_{t+r+1}(\mathbf{j})}\mathbf{S}_r(\mathbf{a},\mathbf{K},t,p,\mathbf{j})\bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.$$

For all $\mathbf{a}\in\{0,\dots,p-1\}^d$, $\mathbf{K}\in\mathbb{Z}^d$, $r\in\mathbb{N}$ and $\mathbf{j}\in\mathbb{N}^d$, we set

$$\mathbf{U}_r(\mathbf{a},\mathbf{K},p,\mathbf{j}):=\mathbf{A}_r(\mathbf{a}+p(\mathbf{K}-\mathbf{j}))\mathbf{A}_{r+1}(\mathbf{j})-\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{j})\mathbf{A}_r(\mathbf{a}+\mathbf{j}p).$$

Then we have $\mathbf{S}_r(\mathbf{a},\mathbf{K},s,p,\mathbf{m})=\sum_{\mathbf{0}\le\mathbf{j}\le(p^s-1)\mathbf{1}}\mathbf{U}_r(\mathbf{a},\mathbf{K},p,\mathbf{j}+\mathbf{m}p^s)$.

We state now four lemmas enabling us to prove (3.1).

Lemma 2. *Assertion α_1 is true.*

Lemma 3. *For all $s, r \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^d$, $a \in \{0, \dots, p-1\}^d$, $\mathbf{j} \in \Psi_s(\mathcal{N})$ and $\mathbf{K} \in \mathbb{Z}^d$, we have*

$$\mathbf{U}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, p, \mathbf{j} + \mathbf{m}p^s) \equiv \frac{\mathbf{A}_{r+1}(\mathbf{j} + \mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{j})} \times \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.$$

Lemma 4. *For all $s \in \mathbb{N}$, $s \geq 1$, if α_s is true, then, for all $\mathbf{a} \in \{0, \dots, p-1\}^d$, $\mathbf{K} \in \mathbb{Z}^d$, $r \geq 0$ and $\mathbf{m} \in \mathbb{N}^d$, we have*

$$\mathbf{S}_r(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m}) \equiv \sum_{\mathbf{j} \in \Psi_s(\mathcal{N})} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j} + \mathbf{m}p^s) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.$$

Lemma 5. *For all $s \in \mathbb{N}$, $s \geq 1$, and all $t \in \{0, \dots, s-1\}$, assertions α_s and $\beta_{t,s}$ imply assertion $\beta_{t+1,s}$.*

Before proving these lemmas, we check that their validity implies (3.1). We prove by induction on s that α_s is true for all $s \geq 1$, which leads to the conclusion of Theorem 4. According to Lemma 2, α_1 is true. Let us assume that α_s is true for a fixed $s \geq 1$. We note that $\beta_{0,s}$ is the assertion

$$\beta_{0,s} : \mathbf{S}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, s, p, \mathbf{m}) \equiv \sum_{\mathbf{j} \in \Psi_s(\mathcal{N})} \frac{\mathbf{A}_{r+1}(\mathbf{j} + \mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{j})} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, 0, p, \mathbf{j}) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.$$

As $\mathbf{S}_r(\mathbf{a}, \mathbf{K}, 0, p, \mathbf{j}) = \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j})$, we have

$$\sum_{\mathbf{j} \in \Psi_s(\mathcal{N})} \frac{\mathbf{A}_{r+1}(\mathbf{j} + \mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{j})} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, 0, p, \mathbf{j}) = \sum_{\mathbf{j} \in \Psi_s(\mathcal{N})} \frac{\mathbf{A}_{r+1}(\mathbf{j} + \mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{j})} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j})$$

and, according to Lemma 3, we get

$$\begin{aligned} \sum_{\mathbf{j} \in \Psi_s(\mathcal{N})} & \frac{\mathbf{A}_{r+1}(\mathbf{j} + \mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{j})} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}) \\ & \equiv \sum_{\mathbf{j} \in \Psi_s(\mathcal{N})} \mathbf{U}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, p, \mathbf{j} + \mathbf{m}p^s) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O} \\ & \equiv \mathbf{S}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, s, p, \mathbf{m}) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}, \qquad (3.2) \end{aligned}$$

where (3.2) is obtained *via* Lemma 4.

Hence, assertion $\beta_{0,s}$ is true. Then we get, according to Lemma 5, the validity of $\beta_{1,s}$. By iteration of Lemma 5, we finally obtain $\beta_{s,s}$ which is

$$\begin{aligned} & \mathbf{S}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, s, p, \mathbf{m}) \\ & \equiv \sum_{\mathbf{j} \in \Psi_0(\mathcal{N})} \frac{\mathbf{A}_{s+r+1}(\mathbf{j} + \mathbf{m})}{\mathbf{A}_{s+r+1}(\mathbf{j})} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, s, p, \mathbf{j}) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O} \\ & \equiv \frac{\mathbf{A}_{s+r+1}(\mathbf{m})}{\mathbf{A}_{s+r+1}(\mathbf{0})} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, s, p, \mathbf{0}) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}, \qquad (3.3) \end{aligned}$$

where we used the fact that $\Psi_0(\mathcal{N}) = \{\mathbf{0}\}$ for (3.3).

We will now prove that, for all $\mathbf{a} \in \{0, \ldots, p-1\}^d, r \in \mathbb{N}$ and $\mathbf{K} \in \mathbb{Z}^d$, we have $\mathbf{S}_r(\mathbf{a}, \mathbf{K}, s, p, \mathbf{0}) \in p^{s+1}\mathcal{O}$. For all $\mathbf{N} \in \mathbb{Z}^d$, we write $P_{\mathbf{N}}$ for the assertion: "for all $\mathbf{a} \in \{0, \ldots, p-1\}^d$ and $r \in \mathbb{N}$, we have $S_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{0}) \in p^{s+1}\mathcal{O}$". If there exists $i \in \{1, \ldots, d\}$ such that $N_i < 0$, then, for all $\mathbf{j} \in \{0, \ldots, p^s - 1\}^d$, we have $\mathbf{A}_r(\mathbf{a} + p(\mathbf{N} - \mathbf{j})) = 0$ and $\mathbf{A}_{r+1}(\mathbf{N} - \mathbf{j}) = 0$ so that $S_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{0}) = 0 \in p^{s+1}\mathcal{O}$. First, we prove by contradiction that, for all $\mathbf{N} \in \mathbb{Z}^d$, $P_{\mathbf{N}}$ is true. Let us assume that there is a minimal element $\mathbf{N} \in \mathbb{N}^d$ such that $P_{\mathbf{N}}$ is false. Given $\mathbf{m} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$ and $\mathbf{N}' := \mathbf{N} - \mathbf{m}p^s$ and applying (3.3) with $\mathbf{N}'$ instead of $\mathbf{K}$, we get

$$\mathbf{S}_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{m}) \equiv \frac{\mathbf{A}_{s+r+1}(\mathbf{m})}{\mathbf{A}_{s+r+1}(\mathbf{0})}\mathbf{S}_r(\mathbf{a}, \mathbf{N}', s, p, \mathbf{0}) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.$$

As $\mathbf{m} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$, we have $\mathbf{N}' < \mathbf{N}$, which, according to the definition of $\mathbf{N}$, leads to $\mathbf{S}_r(\mathbf{a}, \mathbf{N}', s, p, \mathbf{0}) \in p^{s+1}\mathcal{O}$. According to conditions (i) and (ii), we have $|\mathbf{A}_{s+r+1}(\mathbf{0})|_p = 1$ and $\mathbf{A}_{s+r+1}(\mathbf{m}) \in \mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}$, so we get $\mathbf{S}_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{m}) \in p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O} \subset p^{s+1}\mathcal{O}$. Thereby, for all $\mathbf{m} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$, we have $\mathbf{S}_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{m}) \in p^{s+1}\mathcal{O}$. Given $\mathbf{T} \in \mathbb{N}^d$ such that, for all $i \in \{1, \ldots, d\}$ we have $(T_i + 1)p^s > N_i$, we get

$$\begin{aligned}
&\sum_{\mathbf{0} \le \mathbf{m} \le \mathbf{T}} \mathbf{S}_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{m}) \\
&\quad = \sum_{\mathbf{0} \le \mathbf{m} \le \mathbf{T}} \sum_{\mathbf{m}p^s \le \mathbf{j} \le (\mathbf{m}+\mathbf{1})p^s - \mathbf{1}} (\mathbf{A}_r(\mathbf{a} + p(\mathbf{N} - \mathbf{j}))\mathbf{A}_{r+1}(\mathbf{j}) - \mathbf{A}_{r+1}(\mathbf{N} - \mathbf{j})\mathbf{A}_r(\mathbf{a} + \mathbf{j}p)) \\
&\quad = \sum_{\mathbf{0} \le \mathbf{j} \le \mathbf{N}} (\mathbf{A}_r(\mathbf{a} + p(\mathbf{N} - \mathbf{j}))\mathbf{A}_{r+1}(\mathbf{j}) - \mathbf{A}_{r+1}(\mathbf{N} - \mathbf{j})\mathbf{A}_r(\mathbf{a} + \mathbf{j}p)) \qquad (3.4) \\
&\quad = 0, \qquad (3.5)
\end{aligned}$$

where we used the fact that $\mathbf{A}_r(\mathbf{n}) = 0$ when there is an $i \in \{1, \ldots, d\}$ such that $n_i < 0$ for (3.4), and (3.5) occurs because the term of sum (3.4) is changed into its opposite when changing the index $\mathbf{j}$ in $\mathbf{N} - \mathbf{j}$. So we obtain $\mathbf{S}_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{0}) = -\sum_{\mathbf{0} < \mathbf{m} \le \mathbf{T}} \mathbf{S}_r(\mathbf{a}, \mathbf{N}, s, p, \mathbf{m}) \in p^{s+1}\mathcal{O}$, which is contradictory to the status of $\mathbf{N}$. Thus, for all $\mathbf{N} \in \mathbb{Z}^d$, $P_{\mathbf{N}}$ is true.

Conditions (i) and (ii) lead to $|\mathbf{A}_{s+r+1}(\mathbf{0})|_p = 1$ and $\mathbf{A}_{s+r+1}(\mathbf{m}) \in \mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}$. Then we obtain, according to (3.3), that $\mathbf{S}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, s, p, \mathbf{m}) \in p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}$. This latest congruence is valid for all $\mathbf{a} \in \{0, \ldots, p-1\}^d$, $\mathbf{K} \in \mathbb{Z}^d$, $\mathbf{m} \in \mathbb{N}^d$ and $r \ge 0$, which proves that the assertion α_{s+1} is true and completes the induction on s. We now have to prove Lemmas 2–5.

3.1.1. *Proof of Lemma 2*

Given $\mathbf{a} \in \{0, \ldots, p-1\}^d$, $\mathbf{K} \in \mathbb{Z}^d$, $\mathbf{m} \in \mathbb{N}^d$ and $r \ge 0$, we have

$$\mathbf{S}_r(\mathbf{a}, \mathbf{K}, 0, p, \mathbf{m}) = \mathbf{A}_r(\mathbf{a} + p(\mathbf{K} - \mathbf{m}))\mathbf{A}_{r+1}(\mathbf{m}) - \mathbf{A}_{r+1}(\mathbf{K} - \mathbf{m})\mathbf{A}_r(\mathbf{a} + p\mathbf{m}). \qquad (3.6)$$

If $\mathbf{K} - \mathbf{m} \notin \mathbb{N}^d$, then we have $\mathbf{A}_r(\mathbf{a} + p(\mathbf{K} - \mathbf{m})) = 0$ and $\mathbf{A}_{r+1}(\mathbf{K} - \mathbf{m}) = 0$ so that $\mathbf{S}_r(\mathbf{a}, \mathbf{K}, 0, p, \mathbf{m}) = 0 \in p\mathbf{g}_{r+1}(\mathbf{m})\mathcal{O}$, as expected. Thus we can assume that $\mathbf{K} - \mathbf{m} \in \mathbb{N}^d$. We rewrite (3.6) as follows.

$$\begin{aligned}
\mathbf{S}_r(\mathbf{a}, \mathbf{K}, 0, p, \mathbf{m}) = \mathbf{A}_r(\mathbf{a})\Bigg(&\mathbf{A}_{r+1}(\mathbf{m})\left(\frac{\mathbf{A}_r(\mathbf{a} + p(\mathbf{K} - \mathbf{m}))}{\mathbf{A}_r(\mathbf{a})} - \frac{\mathbf{A}_{r+1}(\mathbf{K} - \mathbf{m})}{\mathbf{A}_{r+1}(\mathbf{0})}\right) \\
&- \mathbf{A}_{r+1}(\mathbf{K} - \mathbf{m})\left(\frac{\mathbf{A}_r(\mathbf{a} + \mathbf{m}p)}{\mathbf{A}_r(\mathbf{a})} - \frac{\mathbf{A}_{r+1}(\mathbf{m})}{\mathbf{A}_{r+1}(\mathbf{0})}\right)\Bigg). \qquad (3.7)
\end{aligned}$$

As $\Psi_0(\mathcal{N}) = \{\mathbf{0}\}$, we can use (a), with $\mathbf{0}$ instead of $\mathbf{u}$ and $\mathbf{a}$ instead of $\mathbf{v}$, to obtain

$$\frac{\mathbf{A}_r(\mathbf{a}+p(\mathbf{K}-\mathbf{m}))}{\mathbf{A}_r(\mathbf{a})} - \frac{\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{m})}{\mathbf{A}_{r+1}(\mathbf{0})} \in p\frac{\mathbf{g}_{r+1}(\mathbf{K}-\mathbf{m})}{\mathbf{A}_r(\mathbf{a})}\mathcal{O}$$

and

$$\frac{\mathbf{A}_r(\mathbf{a}+\mathbf{m}p)}{\mathbf{A}_r(\mathbf{a})} - \frac{\mathbf{A}_{r+1}(\mathbf{m})}{\mathbf{A}_{r+1}(\mathbf{0})} \in p\frac{\mathbf{g}_{r+1}(\mathbf{m})}{\mathbf{A}_r(\mathbf{a})}\mathcal{O}.$$

This leads to

$$\mathbf{A}_r(\mathbf{a})\mathbf{A}_{r+1}(\mathbf{m})\left(\frac{\mathbf{A}_r(\mathbf{a}+p(\mathbf{K}-\mathbf{m}))}{\mathbf{A}_r(\mathbf{a})} - \frac{\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{m})}{\mathbf{A}_{r+1}(\mathbf{0})}\right)$$
$$\in p\mathbf{g}_{r+1}(\mathbf{K}-\mathbf{m})\mathbf{A}_{r+1}(\mathbf{m})\mathcal{O} \subset p\mathbf{g}_{r+1}(\mathbf{m})\mathcal{O} \tag{3.8}$$

and

$$\mathbf{A}_r(\mathbf{a})\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{m})\left(\frac{\mathbf{A}_r(\mathbf{a}+\mathbf{m}p)}{\mathbf{A}_r(\mathbf{a})} - \frac{\mathbf{A}_{r+1}(\mathbf{m})}{\mathbf{A}_{r+1}(\mathbf{0})}\right) \in p\mathbf{g}_{r+1}(\mathbf{m})\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{m})\mathcal{O}$$
$$\subset p\mathbf{g}_{r+1}(\mathbf{m})\mathcal{O}, \tag{3.9}$$

where we used condition (ii) for (3.8) and (3.9), which leads to $\mathbf{A}_{r+1}(\mathbf{m}) \in \mathbf{g}_{r+1}(\mathbf{m})\mathcal{O}$ and $\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{m}) \in \mathbf{g}_{r+1}(\mathbf{K}-\mathbf{m})\mathcal{O} \subset \mathcal{O}$. Applying (3.8) and (3.9) to (3.7), we obtain $\mathbf{S}_r(\mathbf{a}, \mathbf{K}, 0, p, \mathbf{m}) \in p\mathbf{g}_{r+1}(\mathbf{m})$, which finishes the proof of the lemma.

3.1.2. Proof of Lemma 3

We have

$$\mathbf{U}_r(\mathbf{a}, \mathbf{K}+\mathbf{m}p^s, p, \mathbf{j}+\mathbf{m}p^s) - \frac{\mathbf{A}_{r+1}(\mathbf{j}+\mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{j})}\mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j})$$
$$= -\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{j})\mathbf{A}_r(\mathbf{a}+\mathbf{j}p)\left(\frac{\mathbf{A}_r(\mathbf{a}+\mathbf{j}p+\mathbf{m}p^{s+1})}{\mathbf{A}_r(\mathbf{a}+\mathbf{j}p)} - \frac{\mathbf{A}_{r+1}(\mathbf{j}+\mathbf{m}p^s)}{\mathbf{A}_{r+1}(\mathbf{j})}\right). \tag{3.10}$$

As $\mathbf{j} \in \Psi_s(\mathcal{N})$, hypothesis (a) implies that the right-hand side of equality (3.10) lies in

$$\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{j})\mathbf{A}_r(\mathbf{a}+\mathbf{j}p)p^{s+1}\frac{\mathbf{g}_{s+r+1}(\mathbf{m})}{\mathbf{A}_r(\mathbf{a}+\mathbf{j}p)}\mathcal{O}.$$

Furthermore, according to condition (ii), we have $\mathbf{A}_{r+1}(\mathbf{K}-\mathbf{j}) \in \mathbf{g}_{r+1}(\mathbf{K}-\mathbf{j})\mathcal{O} \subset \mathcal{O}$. These estimates prove that the left-hand side of (3.10) lies in $p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}$, which completes the proof of the lemma.

3.1.3. Proof of Lemma 4

Let us fix $r, s \in \mathbb{N}$, $s \geq 1$, such that α_s is true.

For all $u \in \{0, \dots, s\}$, we write $\mathcal{A}_u$ for the assertion: for all $\mathbf{n} \in \{0, \dots, p^{s-u}-1\}^d$, we have $\sum_{\mathbf{0}\leq\mathbf{j}\leq(p^u-1)\mathbf{1}} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+\mathbf{n}p^u+\mathbf{m}p^s) = \mathbf{S}_r(\mathbf{a}, \mathbf{K}, u, p, \mathbf{n}+\mathbf{m}p^{s-u})$.

We will prove by induction on u that, for all $u \in \{0, \dots, s\}$, the assertion $\mathcal{A}_u$ is true.

If $u = 0$, then there is nothing to prove so $\mathcal{A}_0$ is true. Let $u \in \{0, \dots, s-1\}$ such that $\mathcal{A}_u$ is true. Let us prove that $\mathcal{A}_{u+1}$ is true. For all $\mathbf{n} \in \{0, \dots, p^{s-u-1}-1\}^d$, we have

$$\mathbf{S}_r(\mathbf{a}, \mathbf{K}, u+1, p, \mathbf{n}+\mathbf{m}p^{s-u-1}) = \sum_{\mathbf{0}\leq\mathbf{v}\leq(p-1)\mathbf{1}} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, u, p, \mathbf{v}+\mathbf{n}p+\mathbf{m}p^{s-u})$$

$$= \sum_{\mathbf{0}\leq\mathbf{v}\leq(p-1)\mathbf{1}} \sum_{\mathbf{0}\leq\mathbf{j}\leq(p^u-1)\mathbf{1}} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+\mathbf{v}p^u+\mathbf{n}p^{u+1}+\mathbf{m}p^s) \tag{3.11}$$

$$= \sum_{\mathbf{0}\leq\mathbf{j}\leq(p^{u+1}-1)\mathbf{1}} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+\mathbf{n}p^{u+1}+\mathbf{m}p^s), \tag{3.12}$$

where we used assertion $\mathcal{A}_u$ for (3.11). Equality (3.12) proves that $\mathcal{A}_{u+1}$ is true, which finishes the induction on u.

If $\Psi_s(\mathcal{N}) = \{0, \dots, p^s-1\}^d$, then Lemma 4 is trivial. In the sequel of this proof, we assume that $\Psi_s(\mathcal{N}) \neq \{0, \dots, p^s-1\}^d$. We have $\mathbf{u} \in \{0, \dots, p^s-1\}^d \setminus \Psi_s(\mathcal{N})$ if and only if there exists $(\mathbf{n}, t) \in \mathcal{N}$, $t \leq s$, and $\mathbf{j} \in \{0, \dots, p^{s-t}-1\}^d$ such that $\mathbf{u} = \mathbf{j} + p^{s-t}\mathbf{n}$. We write $\mathcal{N}_s$ the set of all $(\mathbf{n}, t) \in \mathcal{N}$ with $t \leq s$. So we have

$$\{0, \dots, p^s-1\}^d \setminus \Psi_s(\mathcal{N}) = \bigcup_{(\mathbf{n},t)\in\mathcal{N}_s} \{\mathbf{j}+p^{s-t}\mathbf{n} : \mathbf{j} \in \{0, \dots, p^{s-t}-1\}^d\}.$$

In particular, the set $\mathcal{N}_s$ is nonempty.

We will prove that there exist $k \in \mathbb{N}$, $k \geq 1$, and $(\mathbf{n}_1, t_1), \dots, (\mathbf{n}_k, t_k) \in \mathcal{N}_s$ such that the sets $J(\mathbf{n}_i, t_i) := \{\mathbf{j}+p^{s-t_i}\mathbf{n}_i : \mathbf{j} \in \{0, \dots, p^{s-t_i}-1\}^d\}$ induce a partition of $\{0, \dots, p^s-1\}^d \setminus \Psi_s(\mathcal{N})$. We observe that $\mathcal{N}_s \subset \bigcup_{t=1}^s \big(\{0, \dots, p^t-1\} \times \{t\}\big)$ and thus $\mathcal{N}_s$ is finite. Therefore, we only have to prove that if $(\mathbf{n}, t), (\mathbf{n}', t') \in \mathcal{N}_s$, $\mathbf{j} \in \{0, \dots, p^{s-t}-1\}^d$ and $\mathbf{j}' \in \{0, \dots, p^{s-t'}-1\}^d$ verify $\mathbf{j}+p^{s-t}\mathbf{n} = \mathbf{j}'+p^{s-t'}\mathbf{n}'$, then we have $J(\mathbf{n}, t) \subset J(\mathbf{n}', t')$ or $J(\mathbf{n}', t') \subset J(\mathbf{n}, t)$. Let us assume, for example, that $t \leq t'$. Then there exists $\mathbf{j}_0 \in \{0, \dots, p^{t'-t}-1\}^d$ such that $\mathbf{j} = \mathbf{j}'+p^{s-t'}\mathbf{j}_0$, so that $p^{s-t'}\mathbf{n}' = p^{s-t}\mathbf{n}+p^{s-t'}\mathbf{j}_0$ and thus $J(\mathbf{n}', t') \subset J(\mathbf{n}, t)$. Also, if $t \geq t'$, then we have $J(\mathbf{n}, t) \subset J(\mathbf{n}', t')$. Thus, we get

$$\mathbf{S}_r(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m}) = \sum_{\mathbf{j}\in\Psi_s(\mathcal{N})} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+\mathbf{m}p^s) + \sum_{\mathbf{j}\in\{0,\dots,p^s-1\}^d\setminus\Psi_s(\mathcal{N})} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+\mathbf{m}p^s), \tag{3.13}$$

with

$$\sum_{\mathbf{j}\in\{0,\dots,p^s-1\}^d\setminus\Psi_s(\mathcal{N})} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+\mathbf{m}p^s) = \sum_{i=1}^{k} \sum_{\mathbf{j}\in\{0,\dots,p^{s-t_i}-1\}^d} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+p^{s-t_i}\mathbf{n}_i+\mathbf{m}p^s). \tag{3.14}$$

We now prove that, for all $i \in \{1, \dots, k\}$, we have

$$\sum_{\mathbf{j}\in\{0,\dots,p^{s-t_i}-1\}^d} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+p^{s-t_i}\mathbf{n}_i+\mathbf{m}p^s) \in p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}. \tag{3.15}$$

Given $i \in \{1, \dots, k\}$, assertion $\mathcal{A}_{s-t_i}$ leads to

$$\sum_{\mathbf{0}\leq\mathbf{j}\leq(p^{s-t_i}-1)\mathbf{1}} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j}+p^{s-t_i}\mathbf{n}_i+\mathbf{m}p^s) = \mathbf{S}_r(\mathbf{a}, \mathbf{K}, s-t_i, p, \mathbf{n}_i+\mathbf{m}p^{t_i}).$$

As $t_i \geq 1$, we get, *via* α_s, that $\mathbf{S}_r(\mathbf{a}, \mathbf{K}, s - t_i, p, \mathbf{n}_i + \mathbf{m}p^{t_i}) \in p^{s-t_i+1}\mathbf{g}_{s-t_i+r+1}(\mathbf{n}_i + \mathbf{m}p^{t_i})\mathcal{O}$. Applying assertion (b) with t_i instead of t and $r + s - t_i + 1$ instead of r, we obtain

$$p^{s-t_i+1}\mathbf{g}_{s-t_i+r+1}(\mathbf{n}_i + \mathbf{m}p^{t_i}) \in p^{s-t_i+1}p^{t_i}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O} = p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.$$

Thus, for all $i \in \{1, \ldots, k\}$, we have (3.15).

Congruence (3.15), associated with (3.14) and (3.13), proves that

$$\mathbf{S}_r(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m}) \equiv \sum_{\mathbf{j} \in \Psi_s(\mathcal{N})} \mathbf{U}_r(\mathbf{a}, \mathbf{K}, p, \mathbf{j} + \mathbf{m}p^s) \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O},$$

which completes the proof of Lemma 4.

3.1.4. *Proof of Lemma 5*

During this proof, $\mathbf{i}$ indicates an element of $\{0, \ldots, p-1\}^d$ and $\mathbf{u}$ indicates an element of $\{0, \ldots, p^{s-t-1} - 1\}^d$. For $t < s$, we write $\beta_{t,s}$ as follows

$$\begin{aligned}&\mathbf{S}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, s, p, \mathbf{m})\\ &\quad\equiv \sum_{\mathbf{i}+\mathbf{u}p \in \Psi_{s-t}(\mathcal{N})} \frac{\mathbf{A}_{t+r+1}(\mathbf{i} + \mathbf{u}p + \mathbf{m}p^{s-t})}{\mathbf{A}_{t+r+1}(\mathbf{i} + \mathbf{u}p)} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, t, p, \mathbf{i} + \mathbf{u}p)\\ &\qquad \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}. \end{aligned} \tag{3.16}$$

We want to prove $\beta_{t+1,s}$, which is

$$\begin{aligned}&\mathbf{S}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, s, p, \mathbf{m})\\ &\quad\equiv \sum_{\mathbf{u} \in \Psi_{s-t-1}(\mathcal{N})} \frac{\mathbf{A}_{t+r+2}(\mathbf{u} + \mathbf{m}p^{s-t-1})}{\mathbf{A}_{t+r+2}(\mathbf{u})} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, t+1, p, \mathbf{u})\\ &\qquad \bmod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.\end{aligned}$$

We note that $\mathbf{S}_r(\mathbf{a}, \mathbf{K}, t+1, p, \mathbf{u}) = \sum_{\mathbf{0} \leq \mathbf{i} \leq (p-1)\mathbf{1}} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, t, p, \mathbf{i} + \mathbf{u}p)$. Thus, writing

$$\begin{aligned}X &:= \mathbf{S}_r(\mathbf{a}, \mathbf{K} + \mathbf{m}p^s, s, p, \mathbf{m})\\ &\quad - \sum_{\mathbf{0} \leq \mathbf{i} \leq (p-1)\mathbf{1}} \sum_{\mathbf{u} \in \Psi_{s-t-1}(\mathcal{N})} \frac{\mathbf{A}_{t+r+2}(\mathbf{u} + \mathbf{m}p^{s-t-1})}{\mathbf{A}_{t+r+2}(\mathbf{u})} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, t, p, \mathbf{i} + \mathbf{u}p),\end{aligned}$$

we only have to prove that $X \in p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}$. We have

$$\mathbf{i} + \mathbf{u}p \in \Psi_{s-t}(\mathcal{N}) \Rightarrow \mathbf{u} \in \Psi_{s-t-1}(\mathcal{N}). \tag{3.17}$$

Indeed, if $\mathbf{u} \notin \Psi_{s-t-1}(\mathcal{N})$, then there exist $(\mathbf{n}, k) \in \mathcal{N}$, $k \leq s-t-1$, and $\mathbf{j} \in \{0, \ldots, p^{s-t-1-k} - 1\}^d$ such that $\mathbf{u} = \mathbf{j} + p^{s-t-1-k}\mathbf{n}$. Thus we have $\mathbf{i} + \mathbf{u}p = \mathbf{i} + \mathbf{j}p + p^{s-t-k}\mathbf{n}$, which leads to $\mathbf{i} + \mathbf{u}p \notin \Psi_{s-t}(\mathcal{N})$. Hence, according to $\beta_{t,s}$ written as (3.16), we obtain

$$\begin{aligned}X &\equiv \sum_{\mathbf{i}+\mathbf{u}p \in \Psi_{s-t}(\mathcal{N})} \mathbf{S}_r(\mathbf{a}, \mathbf{K}, t, p, \mathbf{i} + \mathbf{u}p)\\ &\quad \times \left(\frac{\mathbf{A}_{t+r+1}(\mathbf{i} + \mathbf{u}p + \mathbf{m}p^{s-t})}{\mathbf{A}_{t+r+1}(\mathbf{i} + \mathbf{u}p)} - \frac{\mathbf{A}_{t+r+2}(\mathbf{u} + \mathbf{m}p^{s-t-1})}{\mathbf{A}_{t+r+2}(\mathbf{u})}\right)\end{aligned}$$

$$+\sum_{\substack{\mathbf{u}\in\Psi_{s-t-1}(\mathcal{N})\\ \mathbf{i}+\mathbf{u}p\notin\Psi_{s-t}(\mathcal{N})}}\frac{\mathbf{A}_{t+r+2}(\mathbf{u}+\mathbf{m}p^{s-t-1})}{\mathbf{A}_{t+r+2}(\mathbf{u})}\mathbf{S}_r(\mathbf{a},\mathbf{K},t,p,\mathbf{i}+\mathbf{u}p)$$

$$\mod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}.$$

Furthermore, applying (a_1) with $s-t-1$ instead of s and $t+r+1$ instead of r, we get

$$\frac{\mathbf{A}_{t+r+1}(\mathbf{i}+\mathbf{u}p+\mathbf{m}p^{s-t})}{\mathbf{A}_{t+r+1}(\mathbf{i}+\mathbf{u}p)}-\frac{\mathbf{A}_{t+r+2}(\mathbf{u}+\mathbf{m}p^{s-t-1})}{\mathbf{A}_{t+r+2}(\mathbf{u})}\in p^{s-t}\frac{\mathbf{g}_{s+r+1}(\mathbf{m})}{\mathbf{g}_{t+r+1}(\mathbf{i}+\mathbf{u}p)}\mathcal{O}.$$

In addition, since $t<s$ and α_s is true, we have

$$\mathbf{S}_r(\mathbf{a},\mathbf{K},t,p,\mathbf{i}+\mathbf{u}p)\in p^{t+1}\mathbf{g}_{t+r+1}(\mathbf{i}+\mathbf{u}p)\mathcal{O} \tag{3.18}$$

and we obtain

$$X\equiv\sum_{\substack{\mathbf{u}\in\Psi_{s-t-1}(\mathcal{N})\\ \mathbf{i}+\mathbf{u}p\notin\Psi_{s-t}(\mathcal{N})}}\frac{\mathbf{A}_{t+r+2}(\mathbf{u}+\mathbf{m}p^{s-t-1})}{\mathbf{A}_{t+r+2}(\mathbf{u})}\mathbf{S}_r(\mathbf{a},\mathbf{K},t,p,\mathbf{i}+\mathbf{u}p)$$

$$\mod p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}. \tag{3.19}$$

Finally, when $\mathbf{i}+\mathbf{u}p\notin\Psi_{s-t}(\mathcal{N})$, we can apply condition (a_2) with $s-t-1$ instead of s, $\mathbf{i}$ instead of $\mathbf{v}$ and $r+t+1$ instead of r, which leads to

$$\frac{\mathbf{A}_{t+r+2}(\mathbf{u}+\mathbf{m}p^{s-t-1})}{\mathbf{A}_{t+r+2}(\mathbf{u})}\in p^{s-t}\frac{\mathbf{g}_{s+r+1}(\mathbf{m})}{\mathbf{g}_{t+r+1}(\mathbf{i}+\mathbf{u}p)}\mathcal{O}. \tag{3.20}$$

Applying (3.18) and (3.20) to (3.19), we obtain $X\in p^{s+1}\mathbf{g}_{s+r+1}(\mathbf{m})\mathcal{O}$. This finishes the proof of Lemma 5 and thus the one of Theorem 4.

4. A p-adic reformulation of Theorems 1–3

Let e and f be two disjoint sequences of nonzero vectors in $\mathbb{N}^d$ such that $\mathcal{Q}_{e,f}$ is a family of integers. We fix $\mathbf{L}\in\mathcal{E}_{e,f}$ throughout this section. We recall that, for all $k\in\{1,\dots,d\}$, we have $q_{e,f,k}(\mathbf{z})\in z_k\mathbb{Z}[[\mathbf{z}]]$, respectively $q_{\mathbf{L},e,f}(\mathbf{z})\in\mathbb{Z}[[\mathbf{z}]]$, if and only if, for all prime numbers p, we have $q_{e,f,k}(\mathbf{z})\in z_k\mathbb{Z}_p[[\mathbf{z}]]$, respectively $q_{\mathbf{L},e,f}(\mathbf{z})\in\mathbb{Z}_p[[\mathbf{z}]]$.

We will define, for all prime numbers p, two elements $\Phi_{p,k}(\mathbf{a}+p\mathbf{K})$ and $\Phi_{\mathbf{L},p}(\mathbf{a}+p\mathbf{K})$ of $\mathbb{Q}_p$, where $\mathbf{a}\in\{0,\dots,p-1\}^d$ and $\mathbf{K}\in\mathbb{N}^d$, and we will prove that $q_{e,f,k}(\mathbf{z})\in z_k\mathbb{Z}_p[[\mathbf{z}]]$, respectively $q_{\mathbf{L},e,f}(\mathbf{z})\in\mathbb{Z}_p[[\mathbf{z}]]$, if and only if, for all $\mathbf{a}\in\{0,\dots,p-1\}^d$ and all $\mathbf{K}\in\mathbb{N}^d$, we have $\Phi_{p,k}(\mathbf{a}+p\mathbf{K})\in p\mathbb{Z}_p$, respectively $\Phi_{\mathbf{L},p}(\mathbf{a}+p\mathbf{K})\in p\mathbb{Z}_p$.

To simplify notations, we will write $\mathcal{E}:=\mathcal{E}_{e,f}$, $\mathcal{D}:=\mathcal{D}_{e,f}$, $\Delta:=\Delta_{e,f}$, $\mathcal{Q}:=\mathcal{Q}_{e,f}$, $F:=F_{e,f}$, $G_k:=G_{e,f,k}$, $G_{\mathbf{L}}:=G_{\mathbf{L},e,f}$, $q_k:=q_{e,f,k}$ and $q_{\mathbf{L}}:=q_{\mathbf{L},e,f}$, throughout the rest of the article. We fix a prime number p in this section.

Before proving Theorems 1–3, we will reformulate them. The following result is due to Krattenthaler and Rivoal's Lemma [9, Lemma 2, p. 7]; it is the analogue in several variables of a lemma of Dieudonné and Dwork [7, Chapter IV, Section 2, Lemma 3]; [11, Chapter 14, Section 2].

Lemma 6. *Given two formal power series $F(\mathbf{z})\in 1+\sum_{i=1}^d z_i\mathbb{Z}[[\mathbf{z}]]$ and $G(\mathbf{z})\in\sum_{i=1}^d z_i\mathbb{Q}[[\mathbf{z}]]$, we define $q(\mathbf{z}):=\exp(G(\mathbf{z})/F(\mathbf{z}))$. Then we have $q(\mathbf{z})\in 1+\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$ if and only if $F(\mathbf{z})G(\mathbf{z}^p)-pF(\mathbf{z}^p)G(\mathbf{z})\in p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$.*

Lemma 6 will enable us to “eliminate” the exponential in $q_k(\mathbf{z}) = z_k \exp(G_k(\mathbf{z})/F(\mathbf{z}))$ and $q_{\mathbf{L}}(\mathbf{z}) = \exp(G_{\mathbf{L}}(\mathbf{z})/F(\mathbf{z}))$. Since $\Delta \geq 0$ on $[0,1]^d$, we obtain, according to Landau's criterion, $\mathcal{Q}$ as a family of integers and thus $F(\mathbf{z}) \in 1+\sum_{i=1}^d z_i\mathbb{Z}[[\mathbf{z}]]$. Furthermore, according to identities (1.2) and (1.3) defining the power series G_k and $G_{\mathbf{L}}$, we have $G_k(\mathbf{0}) = G_{\mathbf{L}}(\mathbf{0}) = 0$ and so $G_k(\mathbf{z})$ and $G_{\mathbf{L}}(\mathbf{z})$ lie in $\sum_{i=1}^d z_i\mathbb{Q}[[\mathbf{z}]]$. Thereby, according to Lemma 6, we have $q_k(\mathbf{z}) \in z_k\mathbb{Z}_p[[\mathbf{z}]]$, respectively $q_{\mathbf{L}}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$, if and only if we have $F(\mathbf{z})G_k(\mathbf{z}^p) - pF(\mathbf{z}^p)G_k(\mathbf{z}) \in p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$, respectively $F(\mathbf{z})G_{\mathbf{L}}(\mathbf{z}^p) - pF(\mathbf{z}^p)G_{\mathbf{L}}(\mathbf{z}) \in p\sum_{i=1}^d z_i\mathbb{Z}_p[[\mathbf{z}]]$.

According to identity (1.2) which defines G_k, the coefficient of $\mathbf{z}^{\mathbf{a}+p\mathbf{K}}$ in $F(\mathbf{z})G_k(\mathbf{z}^p) - pF(\mathbf{z}^p)G_k(\mathbf{z})$ is

$$\Phi_{p,k}(\mathbf{a}+p\mathbf{K}) := \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}} \mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}+p\mathbf{j}) \times \left(\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)}(H_{(\mathbf{K}-\mathbf{j})\cdot\mathbf{e}_i} - pH_{(\mathbf{a}+p\mathbf{j})\cdot\mathbf{e}_i}) - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)}(H_{(\mathbf{K}-\mathbf{j})\cdot\mathbf{f}_i} - pH_{(\mathbf{a}+p\mathbf{j})\cdot\mathbf{f}_i})\right)$$

and, according to identity (1.3) defining $G_{\mathbf{L}}$, the coefficient of $\mathbf{z}^{\mathbf{a}+p\mathbf{K}}$ in $F(\mathbf{z})G_{\mathbf{L}}(\mathbf{z}^p) - pF(\mathbf{z}^p)G_{\mathbf{L}}(\mathbf{z})$ is

$$\Phi_{\mathbf{L},p}(\mathbf{a}+\mathbf{K}p) := \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}} \mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}+\mathbf{j}p)(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{j})} - pH_{\mathbf{L}\cdot(\mathbf{a}+\mathbf{j}p)}).$$

Thus we have $q_k(\mathbf{z}) \in z_k\mathbb{Z}_p[[\mathbf{z}]]$, respectively $q_{\mathbf{L}}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$, if and only if, for all $\mathbf{a} \in \{0,\ldots,p-1\}^d$ and $\mathbf{K} \in \mathbb{N}^d$, we have $\Phi_{p,k}(\mathbf{a}+p\mathbf{K}) \in p\mathbb{Z}_p$, respectively $\Phi_{\mathbf{L},p}(\mathbf{a}+p\mathbf{K}) \in p\mathbb{Z}_p$.

5. A technical lemma

The aim of this section is to prove the following lemma which we will use for the proofs of assertions (i) and (ii) of Theorem 2.

Lemma 7. *Let e and f be two sequences of vectors in $\mathbb{N}^d$ such that $|e| = |f|$. Then, for all $s \in \mathbb{N}$, $\mathbf{c} \in \{0,\ldots,p^s-1\}^d$ and $\mathbf{m} \in \mathbb{N}^d$, we have*

$$\frac{\mathcal{Q}_{e,f}(\mathbf{c})}{\mathcal{Q}_{e,f}(\mathbf{c}p)}\frac{\mathcal{Q}_{e,f}(\mathbf{c}p+\mathbf{m}p^{s+1})}{\mathcal{Q}_{e,f}(\mathbf{c}+\mathbf{m}p^s)} \in 1+p^{s+1}\mathbb{Z}_p.$$

To prove Lemma 7, we will use certain properties of the p-adic gamma function which is defined by $\Gamma_p(n) := (-1)^n\gamma_p(n)$, where $\gamma_p(n) := \prod_{\substack{k=1\\(k,p)=1}}^{n-1} k$. The function Γ_p can be extended to the whole set $\mathbb{Z}_p$ but we shall not need this here.

Lemma 8. (i) *For all $n \in \mathbb{N}$, we have $\frac{(np)!}{n!} = p^n\gamma_p(1+np)$.*
(ii) *For all $k, n, s \in \mathbb{N}$, we have $\Gamma_p(k+np^s) \equiv \Gamma_p(k) \mod p^s$.*

Assertion (i) of Lemma 8 is obtained by observing that $\gamma_p(1+np) = \frac{(np)!}{n!p^n}$. Assertion (ii) of Lemma 8 is Lemma 1.1 of [11]. We are now able to prove Lemma 7.

Proof of Lemma 7. We have

$$\frac{\mathcal{Q}_{e,f}(\mathbf{c}p+\mathbf{m}p^{s+1})}{\mathcal{Q}_{e,f}(\mathbf{c}+\mathbf{m}p^s)} = \prod_{i=1}^{q_1}\frac{(\mathbf{e}_i\cdot(\mathbf{c}p+\mathbf{m}p^{s+1}))!}{(\mathbf{e}_i\cdot(\mathbf{c}+\mathbf{m}p^s))!}\prod_{i=1}^{q_2}\frac{(\mathbf{f}_i\cdot(\mathbf{c}+\mathbf{m}p^s))!}{(\mathbf{f}_i\cdot(\mathbf{c}p+\mathbf{m}p^{s+1}))!}$$

$$= \frac{\prod_{i=1}^{q_1} p^{\mathbf{e}_i\cdot(\mathbf{c}+\mathbf{m}p^s)}\gamma_p(1+p\mathbf{e}_i\cdot(\mathbf{c}+\mathbf{m}p^s))}{\prod_{i=1}^{q_2} p^{\mathbf{f}_i\cdot(\mathbf{c}+\mathbf{m}p^s)}\gamma_p(1+p\mathbf{f}_i\cdot(\mathbf{c}+\mathbf{m}p^s))}$$

$$= p^{(|e|-|f|)\cdot\mathbf{m}p^s}\frac{\prod_{i=1}^{q_1}(p^{\mathbf{e}_i\cdot\mathbf{c}}(-1)^{1+p\mathbf{e}_i\cdot(\mathbf{c}+\mathbf{m}p^s)}\Gamma_p(1+p\mathbf{e}_i\cdot(\mathbf{c}+\mathbf{m}p^s)))}{\prod_{i=1}^{q_2}\left(p^{\mathbf{f}_i\cdot\mathbf{c}}(-1)^{1+p\mathbf{f}_i\cdot(\mathbf{c}+\mathbf{m}p^s)}\Gamma_p(1+p\mathbf{f}_i\cdot(\mathbf{c}+\mathbf{m}p^s))\right)}$$

$$= (-1)^{(|e|-|f|)\cdot\mathbf{m}p^{s+1}}\frac{\prod_{i=1}^{q_1}\left(p^{\mathbf{e}_i\cdot\mathbf{c}}(-1)^{1+\mathbf{e}_i\cdot\mathbf{c}p}\,\Gamma_p(1+p\mathbf{e}_i\cdot(\mathbf{c}+\mathbf{m}p^s))\right)}{\prod_{i=1}^{q_2}\left(p^{\mathbf{f}_i\cdot\mathbf{c}}(-1)^{1+\mathbf{f}_i\cdot\mathbf{c}p}\,\Gamma_p(1+p\mathbf{f}_i\cdot(\mathbf{c}+\mathbf{m}p^s))\right)} \tag{5.1}$$

$$= \frac{\prod_{i=1}^{q_1} p^{\mathbf{e}_i\cdot\mathbf{c}}(-1)^{1+\mathbf{e}_i\cdot\mathbf{c}p}}{\prod_{i=1}^{q_2} p^{\mathbf{f}_i\cdot\mathbf{c}}(-1)^{1+\mathbf{f}_i\cdot\mathbf{c}p}}\cdot\frac{\prod_{i=1}^{q_1}\Gamma_p(1+p\mathbf{e}_i\cdot(\mathbf{c}+\mathbf{m}p^s))}{\prod_{i=1}^{q_2}\Gamma_p(1+p\mathbf{f}_i\cdot(\mathbf{c}+\mathbf{m}p^s))}, \tag{5.2}$$

where we used the identity $|e|-|f|=\mathbf{0}$ for (5.1) and (5.2). According to assertion (ii) of Lemma 8, for all $\mathbf{n}\in\mathbb{N}^d$, we have $\Gamma_p(1+\mathbf{n}\cdot\mathbf{c}p+\mathbf{n}\cdot\mathbf{m}p^{s+1})\equiv\Gamma_p(1+\mathbf{n}\cdot\mathbf{c}p) \mod p^{s+1}$. So we get

$$\frac{\prod_{i=1}^{q_1}\Gamma_p(1+\mathbf{e}_i\cdot\mathbf{c}p+\mathbf{e}_i\cdot\mathbf{m}p^{s+1})}{\prod_{i=1}^{q_2}\Gamma_p(1+\mathbf{f}_i\cdot\mathbf{c}p+\mathbf{f}_i\cdot\mathbf{m}p^{s+1})} = \frac{\prod_{i=1}^{q_1}\left(\Gamma_p(1+\mathbf{e}_i\cdot\mathbf{c}p)+O(p^{s+1})\right)}{\prod_{i=1}^{q_2}\left(\Gamma_p(1+\mathbf{f}_i\cdot\mathbf{c}p)+O(p^{s+1})\right)},$$

where we write $x=O(p^k)$ when $x\in p^k\mathbb{Z}_p$. Furthermore, according to the definition of Γ_p, for all $\mathbf{n}\in\mathbb{N}^d$, we have $\Gamma_p(1+\mathbf{n}\cdot\mathbf{c}p)\in\mathbb{Z}_p^\times$. Then we obtain

$$\frac{\prod_{i=1}^{q_1}\left(\Gamma_p(1+\mathbf{e}_i\cdot\mathbf{c}p)+O(p^{s+1})\right)}{\prod_{i=1}^{q_2}\left(\Gamma_p(1+\mathbf{f}_i\cdot\mathbf{c}p)+O(p^{s+1})\right)} = \frac{\prod_{i=1}^{q_1}\Gamma_p(1+\mathbf{e}_i\cdot\mathbf{c}p)}{\prod_{i=1}^{q_2}\Gamma_p(1+\mathbf{f}_i\cdot\mathbf{c}p)}(1+O(p^{s+1}))$$

and thus,

$$\frac{\mathcal{Q}_{e,f}(\mathbf{c}p+\mathbf{m}p^{s+1})}{\mathcal{Q}_{e,f}(\mathbf{c}+\mathbf{m}p^s)} = \frac{\prod_{i=1}^{q_1} p^{\mathbf{e}_i\cdot\mathbf{c}}(-1)^{1+\mathbf{e}_i\cdot\mathbf{c}p}}{\prod_{i=1}^{q_2} p^{\mathbf{f}_i\cdot\mathbf{c}}(-1)^{1+\mathbf{f}_i\cdot\mathbf{c}p}}\cdot\frac{\prod_{i=1}^{q_1}\Gamma_p(1+\mathbf{e}_i\cdot\mathbf{c}p)}{\prod_{i=1}^{q_2}\Gamma_p(1+\mathbf{f}_i\cdot\mathbf{c}p)}(1+O(p^{s+1}))$$

$$= \frac{\prod_{i=1}^{q_1} p^{\mathbf{e}_i\cdot\mathbf{c}}\gamma_p(1+\mathbf{e}_i\cdot\mathbf{c}p)}{\prod_{i=1}^{q_2} p^{\mathbf{f}_i\cdot\mathbf{c}}\gamma_p(1+\mathbf{f}_i\cdot\mathbf{c}p)}(1+O(p^{s+1}))$$

$$= \frac{\mathcal{Q}_{e,f}(\mathbf{c}p)}{\mathcal{Q}_{e,f}(\mathbf{c})}(1+O(p^{s+1})).$$

This completes the proof of the lemma. □

6. Proofs of assertion (i) of Theorems 1 and 2

We assume the hypothesis of Theorems 1 and 2. Furthermore, we assume that, for all $\mathbf{x} \in \mathcal{D}$, we have $\Delta(\mathbf{x}) \geq 1$. As we said in Section 1.2, assertion (i) of Theorem 2 implies assertion (i) of Theorem 1. So the aim of this section is to prove that, for all $\mathbf{L} \in \mathcal{E}$, we have $q_{\mathbf{L}}(\mathbf{z}) \in \mathbb{Z}[[\mathbf{z}]]$. Following Section 4, we only have to prove that, for all $\mathbf{L} \in \mathcal{E}$, all prime numbers p, all $\mathbf{a} \in \{0,\dots,p-1\}^d$ and $\mathbf{K} \in \mathbb{N}^d$, we have $\Phi_{\mathbf{L},p}(\mathbf{a}+p\mathbf{K}) \in p\mathbb{Z}_p$. We fix a $\mathbf{L} \in \mathcal{E}$ in this section.

6.1. New reformulation of the problem

For all prime numbers p, all $s \in \mathbb{N}$, $\mathbf{a} \in \{0,\dots,p-1\}^d$ and $\mathbf{K},\mathbf{m} \in \mathbb{N}^d$, we define

$$S(\mathbf{a},\mathbf{K},s,p,\mathbf{m}) := \sum_{\mathbf{m}p^s \leq \mathbf{j} \leq (\mathbf{m}+\mathbf{1})p^s-\mathbf{1}} (\mathcal{Q}(\mathbf{a}+\mathbf{j}p)\mathcal{Q}(\mathbf{K}-\mathbf{j}) - \mathcal{Q}(\mathbf{j})\mathcal{Q}(\mathbf{a}+(\mathbf{K}-\mathbf{j})p)),$$

where we extend $\mathcal{Q}$ to $\mathbb{Z}^d$ by $\mathcal{Q}(\mathbf{n}) = 0$ if there is an $i \in \{1,\dots,d\}$ such that $n_i < 0$.

The aim of this section is to produce, for all prime numbers p, a function g_p from $\mathbb{N}^d$ to $\mathbb{Z}_p$ such that: if, for all primes p, all $s \in \mathbb{N}$, $\mathbf{a} \in \{0,\dots,p-1\}^d$ and $\mathbf{K},\mathbf{m} \in \mathbb{N}^d$, we have $S(\mathbf{a},\mathbf{K},s,p,\mathbf{m}) \in p^{s+1}g_p(\mathbf{m})\mathbb{Z}_p$, then we have $\Phi_{\mathbf{L},p}(\mathbf{a}+\mathbf{K}p) \in p\mathbb{Z}_p$. Thus the proof of assertion (i) of Theorem 2 will amount to finding a suitable lower bound of the p-adic valuation of $S(\mathbf{a},\mathbf{K},s,p,\mathbf{m})$ for all primes p. This reduction method is an adaptation of the approach to the problem made by Dwork in [5].

6.1.1. A reformulation of $\Phi_{\mathbf{L},p}(\mathbf{a}+\mathbf{K}p)$ modulo $p\mathbb{Z}_p$

This step is the analogue of a reformulation made by Krattenthaler and Rivoal in Section 2 of [8]. We fix a prime number p. We will prove that

$$\Phi_{\mathbf{L},p}(\mathbf{a}+\mathbf{K}p) \equiv - \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}} H_{\mathbf{L}\cdot\mathbf{j}}\left(\mathcal{Q}(\mathbf{a}+\mathbf{j}p)\mathcal{Q}(\mathbf{K}-\mathbf{j}) - \mathcal{Q}(\mathbf{j})\mathcal{Q}(\mathbf{a}+(\mathbf{K}-\mathbf{j})p)\right) \mod p\mathbb{Z}_p. \tag{6.1}$$

For all $\mathbf{a} \in \{0,\dots,p-1\}^d$ and $\mathbf{j} \in \mathbb{N}^d$, we have

$$\begin{aligned} pH_{\mathbf{L}\cdot(\mathbf{a}+\mathbf{j}p)} &= p\left(\sum_{i=1}^{\mathbf{L}\cdot\mathbf{j}p}\frac{1}{i} + \sum_{i=1}^{\mathbf{L}\cdot\mathbf{a}}\frac{1}{\mathbf{L}\cdot\mathbf{j}p+i}\right) \\ &\equiv p\left(\sum_{i=1}^{\mathbf{L}\cdot\mathbf{j}}\frac{1}{ip} + \sum_{i=1}^{\lfloor\mathbf{L}\cdot\mathbf{a}/p\rfloor}\frac{1}{\mathbf{L}\cdot\mathbf{j}p+ip}\right) \mod p\mathbb{Z}_p \\ &\equiv H_{\mathbf{L}\cdot\mathbf{j}} + \sum_{i=1}^{\lfloor\mathbf{L}\cdot\mathbf{a}/p\rfloor}\frac{1}{\mathbf{L}\cdot\mathbf{j}+i} \mod p\mathbb{Z}_p. \end{aligned} \tag{6.2}$$

We need a result that we shall prove below by means of Lemma 10 stated in Section 6.1.2.

For all $\mathbf{L} \in \mathcal{E}$, $\mathbf{a} \in \{0, \ldots, p-1\}^d$ and $\mathbf{j} \in \mathbb{N}^d$, we have

$$\mathcal{Q}(\mathbf{a}+\mathbf{j}p) \sum_{i=1}^{\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor} \frac{1}{\mathbf{L}\cdot\mathbf{j}+i} \in p\mathbb{Z}_p. \tag{6.3}$$

Applying (6.3) to (6.2) and with the fact that $\mathcal{Q}(\mathbf{a}+\mathbf{j}p) \in \mathbb{Z}_p$ and $\mathcal{Q}(\mathbf{K}-\mathbf{j}) \in \mathbb{Z}_p$, we obtain $\mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}+\mathbf{j}p)pH_{\mathbf{L}\cdot(\mathbf{a}+\mathbf{j}p)} \equiv \mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}+\mathbf{j}p)H_{\mathbf{L}\cdot\mathbf{j}} \bmod p\mathbb{Z}_p$. This leads to

$$\begin{aligned}
\Phi_{\mathbf{L},p}(\mathbf{a}+\mathbf{K}p) &= \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}} \mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}+\mathbf{j}p)(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{j})} - pH_{\mathbf{L}\cdot(\mathbf{a}+\mathbf{j}p)}) \\
&\equiv \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}} \mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}+\mathbf{j}p)(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{j})} - H_{\mathbf{L}\cdot\mathbf{j}}) \bmod p\mathbb{Z}_p \\
&\equiv -\sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}} H_{\mathbf{L}\cdot\mathbf{j}} \\
&\quad \times (\mathcal{Q}(\mathbf{a}+\mathbf{j}p)\mathcal{Q}(\mathbf{K}-\mathbf{j}) - \mathcal{Q}(\mathbf{j})\mathcal{Q}(\mathbf{a}+(\mathbf{K}-\mathbf{j})p)) \bmod p\mathbb{Z}_p,
\end{aligned}$$

which is the expected equation (6.1).

We now use a Krattenthaler and Rivoal's combinatorial lemma (see [9, Lemma 5, p. 14]) which enables us to write

$$\begin{aligned}
&\sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}} H_{\mathbf{L}\cdot\mathbf{j}} \left(\mathcal{Q}(\mathbf{a}+\mathbf{j}p)\mathcal{Q}(\mathbf{K}-\mathbf{j}) - \mathcal{Q}(\mathbf{j})\mathcal{Q}(\mathbf{a}+(\mathbf{K}-\mathbf{j})p)\right) \\
&\quad = \sum_{s=0}^{r-1} \sum_{\mathbf{0}\leq\mathbf{m}\leq(p^{r-s}-1)\mathbf{1}} W_{\mathbf{L}}(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m}),
\end{aligned}$$

where r is such that $p^{r-1} > \max(K_1, \ldots, K_d)$ and

$$W_{\mathbf{L}}(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m}) := S(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m})(H_{\mathbf{L}\cdot\mathbf{m}p^s} - H_{\mathbf{L}\cdot\lfloor\mathbf{m}/p\rfloor p^{s+1}}).$$

If we prove that, for all $s \in \mathbb{N}$ and $\mathbf{m} \in \mathbb{N}^d$, we have $W_{\mathbf{L}}(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m}) \in p\mathbb{Z}_p$, then we will have $\Phi_{\mathbf{L},p}(\mathbf{a}+\mathbf{K}p) \in p\mathbb{Z}_p$, as expected.

For all $\mathbf{m} \in \mathbb{N}^d$, we set $\mu_p(\mathbf{m}) := \sum_{\ell=1}^{\infty} \mathbf{1}_{\mathcal{D}}(\{\mathbf{m}/p^\ell\})$ and $g_p(\mathbf{m}) := p^{\mu_p(\mathbf{m})}$, where $\mathbf{1}_{\mathcal{D}}$ is the characteristic function of $\mathcal{D}$. We now use the following lemma which we will prove in Section 6.1.2.

Lemma 9. *For all prime numbers p, all $\mathbf{L} \in \mathcal{E}$, $\mathbf{m} \in \mathbb{N}^d$ and $s \in \mathbb{N}$, we have*

$$p^{s+1} g_p(\mathbf{m}) \left(H_{\mathbf{L}\cdot\mathbf{m}p^s} - H_{\mathbf{L}\cdot\lfloor\mathbf{m}/p\rfloor p^{s+1}}\right) \in p\mathbb{Z}_p.$$

According to Lemma 9, if we prove that, for all $\mathbf{a} \in \{0, \ldots, p-1\}^d$, $\mathbf{K}, \mathbf{m} \in \mathbb{N}^d$ and $s \in \mathbb{N}$, we have $S(\mathbf{a}, \mathbf{K}, s, p, \mathbf{m}) \in p^{s+1} g_p(\mathbf{m})\mathbb{Z}_p$, then we will have $q_{\mathbf{L}}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$, which is the announced reformulation.

6.1.2. Proofs of (6.3) and Lemma 9

We state a result which enables us to prove (6.3) and Lemma 9.

Lemma 10. *Given $s \in \mathbb{N}, s \geq 1$, $\mathbf{a} \in \{0, \ldots, p^s-1\}^d$, $\mathbf{m} \in \mathbb{N}^d$ and $\mathbf{L} \in \mathcal{E}$. If we have $\lfloor \mathbf{L}\cdot\mathbf{a}/p^s \rfloor \geq 1$, then, for all $u \in \{1, \ldots, \lfloor \mathbf{L}\cdot\mathbf{a}/p^s \rfloor\}$ and $\ell \in \{s, \ldots, s+v_p(\mathbf{L}\cdot\mathbf{m}+u)\}$, we have $\left\{(\mathbf{a}+\mathbf{m}p^s)/p^\ell\right\} \in \mathcal{D}$.*

Proof. We recall that $\mathcal{D}$ is the set of all $\mathbf{x} \in [0,1[^d$ such that there exists an element $\mathbf{d}$ of e or f satisfying $\mathbf{d} \cdot \mathbf{x} \geq 1$. We have $\left\{(\mathbf{a}+\mathbf{m}p^s)/p^\ell\right\} \in [0,1[^d$, so we only have to prove that $\mathbf{L} \cdot \left\{(\mathbf{a}+\mathbf{m}p^s)/p^\ell\right\} \geq 1$. Indeed, since $\mathbf{L} \in \mathcal{E}$, there exists $\mathbf{d} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$ such that $\mathbf{d} \geq \mathbf{L}$, which leads to $\mathbf{L} \cdot \left\{(\mathbf{a}+p^s\mathbf{m})/p^\ell\right\} \geq 1$, thus $\mathbf{d} \cdot \left\{(\mathbf{a}+p^s\mathbf{m})/p^\ell\right\} \geq 1$ and $\left\{(\mathbf{a}+p^s\mathbf{m})/p^\ell\right\} \in \mathcal{D}$. We write $\mathbf{m} = \sum_{j=0}^{\infty} \mathbf{m}_j p^j$ with $\mathbf{m}_j \in \{0, \dots, p-1\}^d$. We have

$$\left\{\frac{\mathbf{a}+\mathbf{m}p^s}{p^\ell}\right\} = \frac{\mathbf{a}+p^s \sum_{j=0}^{\ell-s-1} \mathbf{m}_j p^j}{p^\ell}.$$

We have that $p^{\ell-s}$ divides $(u+\mathbf{L}\cdot\mathbf{m})$ and so $p^{\ell-s}$ divides $u+\mathbf{L}\cdot\mathbf{m}-\mathbf{L}\cdot\left(\sum_{j=\ell-s}^{\infty} \mathbf{m}_j p^j\right) = u+\mathbf{L}\cdot\left(\sum_{j=0}^{\ell-s-1} \mathbf{m}_j p^j\right)$. We obtain $p^{\ell-s} \leq u+\mathbf{L}\cdot\left(\sum_{j=0}^{\ell-s-1} \mathbf{m}_j p^j\right) \leq \frac{1}{p^s}\mathbf{L}\cdot\mathbf{a}+\mathbf{L}\cdot\left(\sum_{j=0}^{\ell-s-1} \mathbf{m}_j p^j\right)$ and we get $1 \leq \mathbf{L}\cdot\mathbf{a}p^{-\ell} + p^{s-\ell}\mathbf{L}\cdot\left(\sum_{j=0}^{\ell-s-1} \mathbf{m}_j p^j\right) = \mathbf{L}\cdot\left\{(\mathbf{a}+\mathbf{m}p^s)/p^\ell\right\}$. □

We will now apply Lemma 10 to prove (6.3).

Proof of (6.3). Given $\mathbf{L} \in \mathcal{E}$, $\mathbf{a} \in \{0, \dots, p-1\}^d$ and $j \in \mathbb{N}^d$, we have to prove that $\mathcal{Q}(\mathbf{a}+\mathbf{j}p) \times \sum_{i=1}^{\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor} \frac{1}{\mathbf{L}\cdot\mathbf{j}+i} \in p\mathbb{Z}_p$. If $\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor = 0$, this is evident. Thus let us assume that $\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor \geq 1$. Applying Lemma 10 with $s=1$ and $\mathbf{m}=\mathbf{j}$, we obtain that, for all $i \in \{1, \dots, \lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor\}$ and $\ell \in \{1, \dots, 1+v_p(i+\mathbf{L}\cdot\mathbf{j})\}$, we have $\{(\mathbf{a}+\mathbf{j}p)/p^\ell\} \in \mathcal{D}$ and so $\Delta((\mathbf{a}+\mathbf{j}p)/p^\ell) \geq 1$. Since $\Delta \geq 0$ on $\mathbb{R}^d$, we get

$$\begin{aligned} v_p(\mathcal{Q}(\mathbf{a}+\mathbf{j}p)) &= \sum_{\ell=1}^{\infty} \Delta\left(\left\{\frac{\mathbf{a}+\mathbf{j}p}{p^\ell}\right\}\right) \geq \sum_{\ell=1}^{1+v_p(\mathbf{L}\cdot\mathbf{j}+i)} \Delta\left(\left\{\frac{\mathbf{a}+\mathbf{j}p}{p^\ell}\right\}\right) \\ &\geq 1+v_p(\mathbf{L}\cdot\mathbf{j}+i), \end{aligned}$$

which finishes the proof of (6.3). □

Proof of Lemma 9. Given $\mathbf{L} \in \mathcal{E}$, $\mathbf{m} \in \mathbb{N}^d$ and $s \in \mathbb{N}$, we have to prove that

$$p^{s+1} g_p(\mathbf{m})(H_{\mathbf{L}\cdot\mathbf{m}p^s} - H_{\mathbf{L}\cdot\lfloor \mathbf{m}/p \rfloor p^{s+1}}) \in p\mathbb{Z}_p.$$

We write $\mathbf{m} = \mathbf{b}+\mathbf{q}p$ where $\mathbf{b} \in \{0, \dots, p-1\}^d$ and $\mathbf{q} \in \mathbb{N}^d$. Then we have $\mathbf{L}\cdot\mathbf{m}p^s = \mathbf{L}\cdot\mathbf{b}p^s + \mathbf{L}\cdot\mathbf{q}p^{s+1}$ and $\mathbf{L}\cdot\lfloor \mathbf{m}/p \rfloor p^{s+1} = \mathbf{L}\cdot\mathbf{q}p^{s+1}$. Therefore, we get

$$\begin{aligned} H_{\mathbf{L}\cdot\mathbf{m}p^s} - H_{\mathbf{L}\cdot\lfloor \mathbf{m}/p \rfloor p^{s+1}} &= \sum_{j=1}^{\mathbf{L}\cdot\mathbf{b}p^s} \frac{1}{\mathbf{L}\cdot\mathbf{q}p^{s+1}+j} \\ &\equiv \sum_{i=1}^{\lfloor \mathbf{L}\cdot\mathbf{b}/p \rfloor} \frac{1}{\mathbf{L}\cdot\mathbf{q}p^{s+1}+ip^{s+1}} \mod \frac{1}{p^s}\mathbb{Z}_p \end{aligned}$$

and so $p^{s+1} g_p(\mathbf{m})(H_{\mathbf{L}\cdot\mathbf{m}p^s} - H_{\mathbf{L}\cdot\lfloor \mathbf{m}/p \rfloor p^{s+1}}) \equiv g_p(\mathbf{b}+\mathbf{q}p) \sum_{i=1}^{\lfloor \mathbf{L}\cdot\mathbf{b}/p \rfloor} \frac{1}{\mathbf{L}\cdot\mathbf{q}+i} \mod p\mathbb{Z}_p$. We now have to prove that $g_p(\mathbf{b}+\mathbf{q}p) \sum_{i=1}^{\lfloor \mathbf{L}\cdot\mathbf{b}/p \rfloor} \frac{1}{\mathbf{L}\cdot\mathbf{q}+i} \in p\mathbb{Z}_p$. If $\lfloor \mathbf{L}\cdot\mathbf{b}/p \rfloor = 0$, this is evident. Let us assume that $\lfloor \mathbf{L}\cdot\mathbf{b}/p \rfloor \geq 1$. Applying Lemma 10 with $s=1$ and $\mathbf{q}$ instead of $\mathbf{m}$, we obtain that, for all $i \in \{1, \dots, \lfloor \mathbf{L}\cdot\mathbf{b}/p \rfloor\}$ and all $\ell \in \{1, \dots, 1+v_p(i+\mathbf{L}\cdot\mathbf{q})\}$, we have $\{(\mathbf{b}+\mathbf{q}p)/p^\ell\} \in \mathcal{D}$ and thus

$$v_p(g_p(\mathbf{b}+\mathbf{q}p)) = \mu_p(\mathbf{b}+\mathbf{q}p) = \sum_{\ell=1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{b}+\mathbf{q}p}{p^\ell}\right\}\right)$$

$$\geq \sum_{\ell=1}^{1+v_p(\mathbf{L}\cdot\mathbf{q}+i)} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{b}+\mathbf{q}p}{p^\ell}\right\}\right) \geq 1+v_p(\mathbf{L}\cdot\mathbf{q}+i),$$

which completes the proof of Lemma 9. □

6.2. Application of Theorem 4

We use Theorem 4 to finish the proofs of assertion (i) of Theorems 1 and 2. In the following sections, we will prove that, setting $\mathbf{A}_r = \mathcal{Q}$ and $\mathbf{g}_r = g_p$ for all $r \geq 0$, then there exists $\mathcal{N} \subset \bigcup_{t\geq 1}\big(\{0,\dots,p^t-1\}^d \times \{t\}\big)$ such that the sequences $(\mathbf{A}_r)_{r\geq 0}$ and $(\mathbf{g}_r)_{r\geq 0}$ satisfy assertions (i), (ii) and (iii) of Theorem 4. Thus, we will obtain $S(\mathbf{a},\mathbf{K},s,p,\mathbf{m}) \in p^{s+1}g_p(\mathbf{m})\mathbb{Z}_p$, as expected.

In the following sections, we check the assumptions for the application of Theorem 4.

6.3. Verification of assertions (i) and (ii) of Theorem 4

We fix a prime number p and we write $g := g_p$ and $\mu := \mu_p$. For all $r \geq 0$, we set $\mathbf{A}_r = \mathcal{Q}$ and $\mathbf{g}_r = g$. In this section, we will prove that the sequences $(\mathbf{A}_r)_{r\geq 0}$ and $(\mathbf{g}_r)_{r\geq 0}$ verify assertions (i) and (ii) of Theorem 4.

For all $r \geq 0$, we have $|\mathbf{A}_r(\mathbf{0})|_p = |\mathcal{Q}(\mathbf{0})|_p = 1$. Furthermore, for all $\mathbf{m} \in \mathbb{N}^d$, we have $v_p(g(\mathbf{m})) = \mu(\mathbf{m}) \geq 0$, so we get $g(\mathbf{m}) \in \mathbb{Z}_p \setminus \{0\}$. We now have to prove that $A(\mathbf{m}) \in g(\mathbf{m})\mathbb{Z}_p$, which amounts to proving that $\mu_p(\mathbf{m}) \leq v_p(\mathcal{Q}(\mathbf{m}))$. This is true because, for all $\ell \in \mathbb{N}$, $\ell \geq 1$, we have $\Delta(\mathbf{m}/p^\ell) = \Delta(\{\mathbf{m}/p^\ell\}) \geq \mathbf{1}_{\mathcal{D}}(\{\mathbf{m}/p^\ell\})$, because $\Delta(\mathbf{x}) \geq 1$ for $\mathbf{x} \in \mathcal{D}$.

6.4. Verification of assertion (iii) of Theorem 4

We fix a prime number p and we set

$$\mathcal{N} := \bigcup_{t\geq 1}\left(\left\{\mathbf{n} \in \{0,\dots,p^t-1\}^d : \forall \ell \in \{1,\dots,t\}, \left\{\frac{\mathbf{n}}{p^\ell}\right\} \in \mathcal{D}\right\} \times \{t\}\right).$$

6.4.1. Verification of assertion (b)

Let $(\mathbf{n},t) \in \mathcal{N}$ and $\mathbf{m} \in \mathbb{N}^d$. We have to prove that $g(\mathbf{n}+p^t\mathbf{m}) \in p^t g(\mathbf{m})\mathbb{Z}_p$. We have

$$\begin{aligned} v_p(g(\mathbf{n}+p^t\mathbf{m})) &= \sum_{\ell=1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{n}+p^t\mathbf{m}}{p^\ell}\right\}\right) = \sum_{\ell=1}^{t} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{n}}{p^\ell}\right\}\right) \\ &\quad + \sum_{\ell=t+1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{n}+p^t\mathbf{m}}{p^\ell}\right\}\right) \\ &= t + \sum_{\ell=t+1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{n}+p^t\mathbf{m}}{p^\ell}\right\}\right). \end{aligned} \tag{6.4}$$

Let us write $\mathbf{m} = \sum_{k=0}^{\infty} \mathbf{m}_k p^k$, where the $\mathbf{m}_k \in \{0,\dots,p-1\}^d$ are zero except for a finite number of k. For all $\ell \geq t+1$, we have

$$\left\{\frac{\mathbf{n}+p^t\mathbf{m}}{p^\ell}\right\} = \frac{\mathbf{n}+p^t\left(\sum_{k=0}^{\ell-t-1}\mathbf{m}_k p^k\right)}{p^\ell} \geq \frac{p^t\left(\sum_{k=0}^{\ell-t-1}\mathbf{m}_k p^k\right)}{p^\ell} = \left\{\frac{\mathbf{m}}{p^{\ell-t}}\right\}.$$

Thus, for all $\ell \geq t+1$, if $\left\{\frac{\mathbf{m}}{p^{\ell-t}}\right\} \in \mathcal{D}$, then there exists $\mathbf{L} \in \mathcal{E}$ such that

$$1 \leq \mathbf{L} \cdot \left\{\frac{\mathbf{m}}{p^{\ell-t}}\right\} \leq \mathbf{L} \cdot \left\{\frac{\mathbf{n}+p^t\mathbf{m}}{p^\ell}\right\},$$

which gives us $\left\{\frac{\mathbf{n}+p^t\mathbf{m}}{p^\ell}\right\} \in \mathcal{D}$. We get

$$\sum_{\ell=t+1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{n}+p^t\mathbf{m}}{p^\ell}\right\}\right) \geq \sum_{\ell=t+1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{m}}{p^{\ell-t}}\right\}\right) = \sum_{\ell=1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{m}}{p^{\ell}}\right\}\right) = v_p(g(\mathbf{m})),$$

which, associated with (6.4), leads to $v_p(g(\mathbf{n}+p^t\mathbf{m})) \geq t + v_p(g(\mathbf{m}))$, *i.e.* $g(\mathbf{n}+p^t\mathbf{m}) \in p^t g(\mathbf{m})\mathbb{Z}_p$, as expected.

6.4.2. Verification of assertion (a_2)

Given $s \in \mathbb{N}$, $\mathbf{u} \in \Psi_s(\mathcal{N})$ and $\mathbf{v} \in \{0, \ldots, p-1\}^d$ such that $\mathbf{v}+p\mathbf{u} \notin \Psi_{s+1}(\mathcal{N})$, we have to prove that

$$\frac{\mathcal{Q}(\mathbf{u}+p^s\mathbf{m})}{\mathcal{Q}(\mathbf{u})} \in p^{s+1}\frac{g(\mathbf{m})}{g(\mathbf{v}+p\mathbf{u})}\mathbb{Z}_p. \tag{6.5}$$

First, we give another expression for

$$\Psi_s(\mathcal{N}) = \{\mathbf{u} \in \{0, \ldots, p^s-1\}^d : \forall (\mathbf{n}, t) \in \mathcal{N}, t \leq s, \forall \mathbf{j} \in \{0, \ldots, p^{s-t}-1\}^d, \\ \mathbf{u} \neq \mathbf{j} + p^{s-t}\mathbf{n}\}.$$

For that purpose, we need the following lemma.

Lemma 11. *Given* $s \in \mathbb{N}, s \geq 1$, *and* $\mathbf{u} \in \{0, \ldots, p^s-1\}^d$, *we write* $\mathbf{u} = \sum_{k=0}^{s-1}\mathbf{u}_k p^k$, *with* $\mathbf{u}_k \in \{0, \ldots, p-1\}^d$. *Then, the following assertions are equivalent.*

(1) *We have* $\{\mathbf{u}/p^s\} \in \mathcal{D}$.
(2) *There exist* $(\mathbf{n}, t) \in \mathcal{N}$, $t \leq s$ *and* $\mathbf{j} \in \{0, \ldots, p^{s-t}-1\}^d$ *such that* $\mathbf{u} = \mathbf{j} + p^{s-t}\mathbf{n}$.

Proof of Lemma 11. (1) ⇒ (2): For all $s \geq 1$, $\mathbf{u} \in \{0, \ldots, p^s-1\}^d$ such that $\{\mathbf{u}/p^s\} \in \mathcal{D}$ and all $i \in \{0, \ldots, s-1\}$, we write $\mathcal{A}_{s,i}(\mathbf{u})$ for the assertion: for all $\ell \in \{1, \ldots, s-i\}$, we have $\left\{\left(\sum_{k=i}^{s-1}\mathbf{u}_k p^{k-i}\right)/p^\ell\right\} \in \mathcal{D}$.

For all $s \geq 1$, we write $\mathcal{B}_s$ for the assertion: for all $\mathbf{u} \in \{0, \ldots, p^s-1\}^d$ such that $\{\mathbf{u}/p^s\} \in \mathcal{D}$, there exists $i \in \{0, \ldots, s-1\}$, such that $\mathcal{A}_{s,i}(\mathbf{u})$ is true.

First, by induction on s we will prove that, for all $s \geq 1$, $\mathcal{B}_s$ is true.

If $s = 1$, then, for all $\mathbf{u} \in \{0, \ldots, p-1\}^d$ such that $\{\mathbf{u}/p\} \in \mathcal{D}$, assertion $\mathcal{A}_{1,0}(\mathbf{u})$ corresponds to the assertion that $\{\mathbf{u}/p\} \in \mathcal{D}$ and thus is true. Hence, $\mathcal{B}_1$ is true.

Given $s \geq 2$ such that $\mathcal{B}_1, \ldots, \mathcal{B}_{s-1}$ are true, and $\mathbf{u} \in \{0, \ldots, p^s-1\}^d$ verifying $\{\mathbf{u}/p^s\} \in \mathcal{D}$ such that $\mathcal{A}_{s,1}(\mathbf{u}), \ldots, \mathcal{A}_{s,s-1}(\mathbf{u})$ are false, we will prove that assertion $\mathcal{A}_{s,0}(\mathbf{u})$ is true. This will imply the validity of $\mathcal{B}_s$ and will finish the induction on s.

We give a proof by contradiction, assuming that there exists $\ell \in \{1, \ldots, s\}$ such that

$$\mathbf{a}_\ell := \frac{\sum\limits_{k=0}^{\ell-1}\mathbf{u}_k p^k}{p^\ell} = \left\{\frac{\sum\limits_{k=0}^{s-1}\mathbf{u}_k p^k}{p^\ell}\right\} \notin \mathcal{D}.$$

We actually have $\ell \in \{1, \dots, s-1\}$ because $\{\mathbf{u}/p^s\} \in \mathcal{D}$. For all $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$, we have $\mathbf{L} \cdot \mathbf{a}_\ell < 1$. We write

$$\left\{\frac{\mathbf{u}}{p^s}\right\} = \frac{\mathbf{u}}{p^s} = \frac{p^\ell \mathbf{a}_\ell + p^\ell \sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}}{p^s} = \frac{\mathbf{a}_\ell}{p^{s-\ell}} + \frac{\sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}}{p^{s-\ell}}.$$

Since $\{\mathbf{u}/p^s\} \in \mathcal{D}$, there exists $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$ such that

$$1 \leq \mathbf{L} \cdot \left\{\frac{\mathbf{u}}{p^s}\right\} = \frac{\mathbf{L} \cdot \mathbf{a}_\ell}{p^{s-\ell}} + \mathbf{L} \cdot \frac{\sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}}{p^{s-\ell}} < \frac{1}{p^{s-\ell}} + \mathbf{L} \cdot \frac{\sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}}{p^{s-\ell}},$$

which leads to $\mathbf{L} \cdot \left(\sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}\right) > p^{s-\ell} - 1$. Since $\mathbf{L} \cdot \left(\sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}\right)$ is an integer, we get $\mathbf{L} \cdot \left(\sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}\right) \geq p^{s-\ell}$, *i.e.* $\left\{\left(\sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell}\right) / p^{s-\ell}\right\} \in \mathcal{D}$. We write $\mathbf{v} := \sum_{k=\ell}^{s-1} \mathbf{u}_k p^{k-\ell} \in \{0, \dots, p^{s-\ell} - 1\}^d$. Thus we have $\{\mathbf{v}/p^{s-\ell}\} \in \mathcal{D}$ and, applying $\mathcal{B}_{s-\ell}$, we obtain that there exists $i \in \{0, \dots, s-\ell-1\}$ such that $\mathcal{A}_{s-\ell,i}(\mathbf{v})$ is true, *i.e.*, for all $r \in \{1, \dots, s-\ell-i\}$, we have $\left\{\left(\sum_{k=i}^{s-\ell-1} \mathbf{v}_k p^{k-i}\right) / p^r\right\} \in \mathcal{D}$. Furthermore, for all k, we have $\mathbf{v}_k = \mathbf{u}_{\ell+k}$ and therefore $\sum_{k=i}^{s-\ell-1} \mathbf{v}_k p^{k-i} = \sum_{k=i+\ell}^{s-1} \mathbf{u}_k p^{k-i-\ell}$. Thereby, assertion $\mathcal{A}_{s-\ell,i}(\mathbf{v})$ becomes: for all $r \in \{1, \dots, s-\ell-i\}$, we have $\left\{\left(\sum_{k=i+\ell}^{s-1} \mathbf{u}_k p^{k-i-\ell}\right) / p^r\right\} \in \mathcal{D}$, which corresponds to the assertion $\mathcal{A}_{s,i+\ell}(\mathbf{u})$. Since we assumed that $\mathcal{A}_{s,1}(\mathbf{u}), \dots, \mathcal{A}_{s,s-1}(\mathbf{u})$ are false, we get a contradiction. Hence $\mathcal{A}_{s,0}(\mathbf{u})$ is true and $\mathcal{B}_s$ is also true, which finishes the induction on s.

As $\{\mathbf{u}/p^s\} \in \mathcal{D}$, assertion $\mathcal{B}_s$ tells us that an $i \in \{0, \dots, s-1\}$ exists such that $\mathcal{A}_{s,i}(\mathbf{u})$ is true, *i.e.* for all $\ell \in \{1, \dots, s-i\}$, we have $\left\{\left(\sum_{k=i}^{s-1} p^{k-i} \mathbf{u}_k\right) / p^\ell\right\} \in \mathcal{D}$. Thus we have $\left(\sum_{k=i}^{s-1} p^{k-i} \mathbf{u}_k, s-i\right) \in \mathcal{N}$ and $\mathbf{u} = \sum_{k=0}^{i-1} p^k \mathbf{u}_k + p^i \sum_{k=i}^{s-1} p^{k-i} \mathbf{u}_k$. Therefore, the assertion (2) is valid with $s-i$ instead of t, $\sum_{k=i}^{s-1} p^{k-i} \mathbf{u}_k$ instead of $\mathbf{n}$ and $\sum_{k=0}^{i-1} p^k \mathbf{u}_k$ instead of $\mathbf{j}$.

(2) $\Rightarrow$ (1): We have $\{\mathbf{u}/p^s\} = \mathbf{u}/p^s = (\mathbf{j} + p^{s-t}\mathbf{n})/p^s \geq \mathbf{n}/p^t = \{\mathbf{n}/p^t\} \in \mathcal{D}$ and so $\{\mathbf{u}/p^s\} \in \mathcal{D}$, as expected. □

According to Lemma 11, we obtain

$$\Psi_s(\mathcal{N}) = \{\mathbf{u} \in \{0, \dots, p^s - 1\}^d : \{\mathbf{u}/p^s\} \notin \mathcal{D}\}. \tag{6.6}$$

Thus, for all $\mathbf{u} \in \Psi_s(\mathcal{N})$ and $\ell \geq s$, we have $\{\mathbf{u}/p^\ell\} = \mathbf{u}/p^\ell \leq \mathbf{u}/p^s = \{\mathbf{u}/p^s\}$, which implies that, for all $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$ and $\ell \geq s$, we have $\mathbf{L} \cdot \{\mathbf{u}/p^\ell\} \leq \mathbf{L} \cdot \{\mathbf{u}/p^s\} < 1$ and so $\{\mathbf{u}/p^\ell\} \notin \mathcal{D}$. As a result, for all $\ell \geq s$, we have $\Delta(\{\mathbf{u}/p^\ell\}) = 0$ and thus

$$v_p\left(\mathcal{Q}(\mathbf{u})\right) = \sum_{\ell=1}^{\infty} \Delta\left(\left\{\frac{\mathbf{u}}{p^\ell}\right\}\right) = \sum_{\ell=1}^{s} \Delta\left(\left\{\frac{\mathbf{u}}{p^\ell}\right\}\right).$$

Furthermore, we have

$$v_p\left(\mathcal{Q}(\mathbf{u} + p^s\mathbf{m})\right) = \sum_{\ell=1}^{\infty} \Delta\left(\left\{\frac{\mathbf{u} + p^s\mathbf{m}}{p^\ell}\right\}\right) = \sum_{\ell=1}^{s} \Delta\left(\left\{\frac{\mathbf{u}}{p^\ell}\right\}\right) + \sum_{\ell=s+1}^{\infty} \Delta\left(\left\{\frac{\mathbf{u} + p^s\mathbf{m}}{p^\ell}\right\}\right),$$

which leads to

$$v_p\left(\frac{\mathcal{Q}(\mathbf{u}+p^s\mathbf{m})}{\mathcal{Q}(\mathbf{u})}\right)=\sum_{\ell=s+1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}+p^s\mathbf{m}}{p^\ell}\right\}\right). \tag{6.7}$$

We write $\mathbf{m}=\sum_{k=0}^{\infty}p^k\mathbf{m}_k$, with $\mathbf{m}_k\in\{0,\ldots,p-1\}^d$. For all $\ell\geq s+1$, we have

$$\left\{\frac{\mathbf{u}+p^s\mathbf{m}}{p^\ell}\right\}=\frac{\mathbf{u}+p^s\sum_{k=0}^{\ell-1-s}p^k\mathbf{m}_k}{p^\ell}\geq\frac{\sum_{k=0}^{\ell-1-s}p^k\mathbf{m}_k}{p^{\ell-s}}=\left\{\frac{\mathbf{m}}{p^{\ell-s}}\right\}$$

and thus

$$\sum_{\ell=s+1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}+p^s\mathbf{m}}{p^\ell}\right\}\right)\geq\sum_{\ell=s+1}^{\infty}\mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{u}+p^s\mathbf{m}}{p^\ell}\right\}\right) \tag{6.8}$$

$$\geq\sum_{\ell=s+1}^{\infty}\mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{m}}{p^{\ell-s}}\right\}\right)$$

$$=\sum_{\ell=1}^{\infty}\mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{m}}{p^\ell}\right\}\right)=v_p(g(\mathbf{m})), \tag{6.9}$$

where inequality (6.8) is true because, for all $\mathbf{x}\in\mathcal{D}$, we have $\Delta(\mathbf{x})\geq 1$. Applying (6.9) to (6.7), we get $v_p\left(\mathcal{Q}(\mathbf{u}+p^s\mathbf{m})/\mathcal{Q}(\mathbf{u})\right)\geq v_p(g(\mathbf{m}))$.

Thus, to verify assertion (a_2), we only have to prove that, for all $\mathbf{u}\in\Psi_s(\mathcal{N})$ and $\mathbf{v}\in\{0,\ldots,p-1\}^d$ such that $\mathbf{v}+p\mathbf{u}\notin\Psi_{s+1}(\mathcal{N})$, we have $g(\mathbf{v}+p\mathbf{u})\in p^{s+1}\mathbb{Z}_p$.

We write $\mathbf{u}=\sum_{k=0}^{s-1}p^k\mathbf{u}_k$, with $\mathbf{u}_k\in\{0,\ldots,p-1\}^d$. We have $\{(\mathbf{v}+p\mathbf{u})/p\}=\mathbf{v}/p$ and, for all $\ell\geq 2$, we have $\{(\mathbf{v}+p\mathbf{u})/p^\ell\}=\left(\mathbf{v}+p\sum_{k=0}^{\ell-2}p^k\mathbf{u}_k\right)/p^\ell$. We get

$$v_p(g(\mathbf{v}+p\mathbf{u}))=\sum_{\ell=1}^{\infty}\mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{v}+p\mathbf{u}}{p^\ell}\right\}\right)\geq\mathbf{1}_{\mathcal{D}}\left(\frac{\mathbf{v}}{p}\right)+\sum_{\ell=2}^{s+1}\mathbf{1}_{\mathcal{D}}\left(\frac{\mathbf{v}+p\sum_{k=0}^{\ell-2}p^k\mathbf{u}_k}{p^\ell}\right).$$

Thus, if we prove that $\mathbf{v}/p\in\mathcal{D}$ and that $\left(\mathbf{v}+p\sum_{k=0}^{\ell-2}p^k\mathbf{u}_k\right)/p^\ell\in\mathcal{D}$ for all $\ell\in\{2,\ldots,s+1\}$, then we would have $v_p(g(\mathbf{v}+p\mathbf{u}))\geq s+1$.

- Let us prove that $\mathbf{v}/p\in\mathcal{D}$.

As $\mathbf{v}+p\mathbf{u}\notin\Psi_{s+1}(\mathcal{N})$, we obtain, according to (6.6), that $\{(\mathbf{v}+p\mathbf{u})/p^{s+1}\}\in\mathcal{D}$. Thus there exists $\mathbf{L}\in\{\mathbf{e}_1,\ldots,\mathbf{e}_{q_1},\mathbf{f}_1,\ldots,\mathbf{f}_{q_2}\}$ such that $\mathbf{L}\cdot\{(\mathbf{v}+p\mathbf{u})/p^{s+1}\}\geq 1$. We get

$$1\leq\mathbf{L}\cdot\frac{\mathbf{v}+p\sum_{k=0}^{s-1}p^k\mathbf{u}_k}{p^{s+1}}=\mathbf{L}\cdot\frac{\mathbf{v}}{p^{s+1}}+\mathbf{L}\cdot\frac{\sum_{k=0}^{s-1}p^k\mathbf{u}_k}{p^s}=\mathbf{L}\cdot\frac{\mathbf{v}}{p^{s+1}}+\mathbf{L}\cdot\left\{\frac{\mathbf{u}}{p^s}\right\}. \tag{6.10}$$

As $\mathbf{u}\in\Psi_s(\mathcal{N})$, we have $\{\mathbf{u}/p^s\}\notin\mathcal{D}$ and so $\mathbf{L}\cdot\{\mathbf{u}/p^s\}<1$. We have $\mathbf{L}\cdot\{\mathbf{u}/p^s\}\in\frac{1}{p^s}\mathbb{N}$ thus $\mathbf{L}\cdot\{\mathbf{u}/p^s\}\leq(p^s-1)/p^s$ and we get, *via* inequality (6.10), that $\mathbf{L}\cdot\mathbf{v}/p^{s+1}\geq 1/p^s$, *i.e.* $\mathbf{L}\cdot\mathbf{v}/p\geq 1$. Thereby, we have $\mathbf{v}/p\in\mathcal{D}$.

- Let us prove that, for all $\ell \in \{2, \ldots, s+1\}$, we have $\left(\mathbf{v} + p\sum_{k=0}^{\ell-2} p^k \mathbf{u}_k\right)/p^\ell \in \mathcal{D}$.

  We assume that $s \geq 1$. Given $\ell \in \{2, \ldots, s+1\}$, we have

$$1 \leq \mathbf{L} \cdot \frac{\mathbf{v} + p\sum_{k=0}^{s-1} p^k \mathbf{u}_k}{p^{s+1}} = \mathbf{L} \cdot \frac{\mathbf{v} + p\sum_{k=0}^{\ell-2} p^k \mathbf{u}_k}{p^{s+1}} + \mathbf{L} \cdot \frac{p\sum_{k=\ell-1}^{s-1} p^k \mathbf{u}_k}{p^{s+1}}. \tag{6.11}$$

We have $\mathbf{u} \in \Psi_s(\mathcal{N})$ and $\mathbf{u} = \mathbf{u}_0 + p\sum_{k=1}^{s-1} p^{k-1}\mathbf{u}_k$. Thus, applying (3.17) with $t = 0$, we obtain $\sum_{k=1}^{s-1} p^{k-1}\mathbf{u}_k \in \Psi_{s-1}(\mathcal{N})$. Iterating (3.17), we finally get that $\sum_{k=\ell-1}^{s-1} p^{k-\ell+1}\mathbf{u}_k \in \Psi_{s-\ell+1}(\mathcal{N})$. Following Lemma 11, we get

$$\frac{p\sum_{k=\ell-1}^{s-1} p^k \mathbf{u}_k}{p^{s+1}} = \frac{\sum_{k=\ell-1}^{s-1} p^{k-\ell+1}\mathbf{u}_k}{p^{s-\ell+1}} = \left\{\frac{\sum_{k=\ell-1}^{s-1} p^{k-\ell+1}\mathbf{u}_k}{p^{s-\ell+1}}\right\} \notin \mathcal{D}.$$

In particular, we obtain $1 > \mathbf{L} \cdot \left(\sum_{k=\ell-1}^{s-1} p^{k-\ell+1}\mathbf{u}_k\right)/p^{s-\ell+1} \in \frac{1}{p^{s-\ell+1}}\mathbb{N}$. Thus we have $\mathbf{L} \cdot \left(\sum_{k=\ell-1}^{s-1} p^{k-\ell+1}\mathbf{u}_k\right)/p^{s-\ell+1} \leq (p^{s-\ell+1} - 1)/p^{s-\ell+1}$. Using this last inequality in (6.11), we get $\mathbf{L} \cdot \left(\mathbf{v} + p\sum_{k=0}^{\ell-2} p^k \mathbf{u}_k\right)/p^{s+1} \geq p^{\ell-s-1}$. Therefore, for all $\ell \in \{2, \ldots, s+1\}$, we have

$$\mathbf{L} \cdot \left\{\frac{\mathbf{v} + p\mathbf{u}}{p^\ell}\right\} = \mathbf{L} \cdot \frac{\mathbf{v} + p\sum_{k=0}^{\ell-2} p^k \mathbf{u}_k}{p^\ell} \geq 1 \tag{6.12}$$

and, for all $\ell \in \{2, \ldots, s+1\}$, we obtain $\left\{(\mathbf{v} + p\mathbf{u})/p^\ell\right\} \in \mathcal{D}$. This completes the verification of assertion (a_2).

6.4.3. Verification of assertions (a) and (a_1)

For all $s \in \mathbb{N}$, $\mathbf{v} \in \{0, \ldots, p-1\}^d$ and $\mathbf{u} \in \Psi_s(\mathcal{N})$, we set $\theta_s(\mathbf{v} + \mathbf{u}p) := \mathcal{Q}(\mathbf{v} + \mathbf{u}p)$ if $\mathbf{v} + \mathbf{u}p \notin \Psi_{s+1}(\mathcal{N})$, and $\theta_s(\mathbf{v} + \mathbf{u}p) := g(\mathbf{v} + \mathbf{u}p)$ if $\mathbf{v} + \mathbf{u}p \in \Psi_{s+1}(\mathcal{N})$.

The aim of this section is to prove the following assertion: for all $s \in \mathbb{N}$, $\mathbf{v} \in \{0, \ldots, p-1\}^d$, $\mathbf{u} \in \Psi_s(\mathcal{N})$ and $\mathbf{m} \in \mathbb{N}^d$, we have

$$\frac{\mathcal{Q}(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})}{\mathcal{Q}(\mathbf{v} + \mathbf{u}p)} - \frac{\mathcal{Q}(\mathbf{u} + \mathbf{m}p^s)}{\mathcal{Q}(\mathbf{u})} \in p^{s+1}\frac{g(\mathbf{m})}{\theta_s(\mathbf{v} + \mathbf{u}p)}\mathbb{Z}_p, \tag{6.13}$$

which will prove assertions (a) and (a_1) of Theorem 4. Indeed, for all $\mathbf{v} \in \{0, \ldots, p-1\}^d$ and $\mathbf{u} \in \Psi_s(\mathcal{N})$, we have $\mathcal{Q}(\mathbf{v} + \mathbf{u}p) \in g(\mathbf{v} + \mathbf{u}p)\mathbb{Z}_p$ so that (6.13) implies (a). Furthermore, according to the definition of θ_s, when $\mathbf{v} + \mathbf{u}p \in \Psi_{s+1}(\mathcal{N})$, congruence (6.13) implies (a_1).

Congruence (6.13) is valid if and only if, for all $\mathbf{v} \in \{0, \ldots, p-1\}^d$, $\mathbf{u} \in \Psi_s(\mathcal{N})$ and $\mathbf{m} \in \mathbb{N}^d$, we have

$$\left(1 - \frac{\mathcal{Q}(\mathbf{v} + \mathbf{u}p)}{\mathcal{Q}(\mathbf{u})}\frac{\mathcal{Q}(\mathbf{u} + \mathbf{m}p^s)}{\mathcal{Q}(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})}\right)\frac{\mathcal{Q}(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})}{\mathcal{Q}(\mathbf{v} + \mathbf{u}p)} \in p^{s+1}\frac{g(\mathbf{m})}{\theta_s(\mathbf{v} + \mathbf{u}p)}\mathbb{Z}_p.$$

In the sequel of the proof, we set

$$X_s(\mathbf{v},\mathbf{u},\mathbf{m}) := \frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\mathcal{Q}(\mathbf{u})}\frac{\mathcal{Q}(\mathbf{u}+\mathbf{m}p^s)}{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}.$$

Thus, to prove (6.13), we only have to prove that

$$(X_s(\mathbf{v},\mathbf{u},\mathbf{m})-1)\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}{g(\mathbf{m})} \in p^{s+1}\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\theta_s(\mathbf{v}+\mathbf{u}p)}\mathbb{Z}_p. \tag{6.14}$$

In order to estimate the valuation of $X_s(\mathbf{v},\mathbf{u},\mathbf{m})-1$, let us set, for all $\mathbf{v}\in\{0,\dots,p-1\}^d$, $\mathbf{u}\in\{0,\dots,p^s-1\}^d$, $s\in\mathbb{N}$ and $\mathbf{m}\in\mathbb{N}^d$,

$$Y_s(\mathbf{v},\mathbf{u},\mathbf{m}) := \frac{\prod_{i=1}^{q_2}\prod_{j=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{m}p^s}{\mathbf{f}_i\cdot\mathbf{u}+j}\right)}{\prod_{i=1}^{q_1}\prod_{j=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{m}p^s}{\mathbf{e}_i\cdot\mathbf{u}+j}\right)}.$$

Given $s\in\mathbb{N}$, $\mathbf{m}\in\mathbb{N}^d$ and $\mathbf{a}\in\{0,\dots,p^s-1\}^d$, we write $\eta_s(\mathbf{a},\mathbf{m}) := \sum_{\ell=s+1}^{\infty}\Delta\left(\left\{\frac{\mathbf{a}+\mathbf{m}p^s}{p^\ell}\right\}\right)$. We state four lemmas, which we prove in Section 6.4.4.

Lemma 12. *For all $s\in\mathbb{N}$, $\mathbf{v}\in\{0,\dots,p-1\}^d$, $\mathbf{u}\in\Psi_s(\mathcal{N})$ and $\mathbf{m}\in\mathbb{N}^d$, we have $X_s(\mathbf{v},\mathbf{u},\mathbf{m}) \in Y_s(\mathbf{v},\mathbf{u},\mathbf{m})\left(1+p^{s+1}\mathbb{Z}_p\right)$ and $v_p(Y_s(\mathbf{v},\mathbf{u},\mathbf{m}))\geq \eta_s(\mathbf{u},\mathbf{m})-\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{m})$.*

Lemma 13. *Given $s\in\mathbb{N}$, $\mathbf{v}\in\{0,\dots,p-1\}^d$ and $\mathbf{u}\in\{0,\dots,p^s-1\}^d$, if there exists $j\in\{1,\dots,s+1\}$ such that $\{(\mathbf{v}+\mathbf{u}p)/p^j\}\notin\mathcal{D}$, then we have $Y_s(\mathbf{v},\mathbf{u},\mathbf{m})\in 1+p^{s-j+2}\mathbb{Z}_p$.*

Lemma 14. *For all $s\in\mathbb{N}$, $\mathbf{a}\in\{0,\dots,p^{s+1}-1\}^d$ and $\mathbf{m}\in\mathbb{N}^d$, we have*

$$\eta_{s+1}(\mathbf{a},\mathbf{m})\geq\mu(\mathbf{m}) \tag{6.15}$$

and

$$v_p\left(\frac{\mathcal{Q}(\mathbf{a}+\mathbf{m}p^{s+1})}{g(\mathbf{m})}\right)\geq\sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{a}}{p^\ell}\right\}\right). \tag{6.16}$$

Lemma 15. *Given $s\in\mathbb{N}$ and $\mathbf{a}\in\Psi_s(\mathcal{N})$, we have $v_p(\mathcal{Q}(\mathbf{a}))=\sum_{\ell=1}^{s}\Delta\left(\left\{\frac{\mathbf{a}}{p^\ell}\right\}\right)$.*

In order to prove (6.14), we will now distinguish two cases.

- *Case* 1. Let us assume that there exists $j\in\{1,\dots,s+1\}$ such that $\left\{(\mathbf{v}+\mathbf{u}p)/p^j\right\}\notin\mathcal{D}$.

Let j_0 be the smallest $j\in\{1,\dots,s+1\}$ such that $\left\{(\mathbf{v}+\mathbf{u}p)/p^j\right\}\notin\mathcal{D}$. According to Lemma 13 applied with j_0 in place of j, we get $Y_s(\mathbf{v},\mathbf{u},\mathbf{m})\in 1+p^{s-j_0+2}\mathbb{Z}_p$ and thus, according to Lemma 12, $v_p(X_s(\mathbf{v},\mathbf{u},\mathbf{m})-1)\geq s-j_0+2$. According to (6.16), we get

$$\begin{aligned} v_p&\left((X_s(\mathbf{v},\mathbf{u},\mathbf{m})-1)\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}{g(\mathbf{m})}\right)\\ &\geq v_p(X_s(\mathbf{v},\mathbf{u},\mathbf{m})-1)+\sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right)\\ &\geq s-j_0+2+\sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right). \end{aligned} \tag{6.17}$$

For all $\ell \in \{1, \dots, j_0 - 1\}$, we have $\{(\mathbf{v} + \mathbf{u}p)/p^\ell\} \in \mathcal{D}$ and so $\Delta(\{(\mathbf{v} + \mathbf{u}p)/p^\ell\}) \geq 1$. We get $\sum_{\ell=1}^{s+1} \Delta(\{(\mathbf{v} + \mathbf{u}p)/p^\ell\}) \geq j_0 - 1$ which, in combination with (6.17), leads to

$$v_p\left((X_s(\mathbf{v}, \mathbf{u}, \mathbf{m}) - 1)\frac{\mathcal{Q}(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})}{g(\mathbf{m})}\right) \geq s + 1. \tag{6.18}$$

If $\mathbf{v} + \mathbf{u}p \notin \Psi_{s+1}(\mathcal{N})$, then we have $\theta_s(\mathbf{v} + \mathbf{u}p) = \mathcal{Q}(\mathbf{v} + \mathbf{u}p)$ and $p^{s+1}\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\theta_s(\mathbf{v}+\mathbf{u}p)} = p^{s+1}$. Hence, when $\mathbf{v} + \mathbf{u}p \notin \Psi_{s+1}(\mathcal{N})$, inequality (6.18) implies (6.14).

We assume, throughout the end of the proof of Case 1, that $\mathbf{v} + \mathbf{u}p \in \Psi_{s+1}(\mathcal{N})$, thus $\theta_s(\mathbf{v} + \mathbf{u}p) = g(\mathbf{v} + \mathbf{u}p)$. Let us prove that we have $v_p(g(\mathbf{v} + \mathbf{u}p)) \geq j_0 - 1$. Indeed, for all $\ell \in \{1, \dots, j_0 - 1\}$, we have $\{(\mathbf{v} + \mathbf{u}p)/p^\ell\} \in \mathcal{D}$ and thus $v_p(g(\mathbf{v} + \mathbf{u}p)) = \sum_{\ell=1}^{\infty} \mathbf{1}_{\mathcal{D}}\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right) \geq j_0 - 1$. Following (6.17), we get

$$\begin{aligned}
&v_p\left((X_s(\mathbf{v}, \mathbf{u}, \mathbf{m}) - 1)\frac{\mathcal{Q}(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})}{g(\mathbf{m})}\right)\\
&\quad\geq s - j_0 + 2 + v_p(g(\mathbf{v} + \mathbf{u}p)) + \left(\sum_{\ell=1}^{s+1} \Delta\left(\left\{\frac{\mathbf{v} + \mathbf{u}p}{p^\ell}\right\}\right) - v_p(g(\mathbf{v} + \mathbf{u}p))\right)\\
&\quad\geq (s - j_0 + 2) + j_0 - 1 + v_p\left(\frac{\mathcal{Q}(\mathbf{v} + \mathbf{u}p)}{g(\mathbf{v} + \mathbf{u}p)}\right) \geq s + 1 + v_p\left(\frac{\mathcal{Q}(\mathbf{v} + \mathbf{u}p)}{g(\mathbf{v} + \mathbf{u}p)}\right),
\end{aligned} \tag{6.19}$$

where the first inequality in (6.19) is valid because, applying Lemma 15 with $s + 1$ instead of s and $\mathbf{v} + \mathbf{u}p$ instead of $\mathbf{a}$, we get $v_p(\mathcal{Q}(\mathbf{v} + \mathbf{u}p)) = \sum_{\ell=1}^{s+1} \Delta(\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\})$. Thus we have (6.14) in this case.

- *Case* 2. Let us assume that, for all $j \in \{1, \dots, s + 1\}$, we have $\{(\mathbf{v} + \mathbf{u}p)/p^j\} \in \mathcal{D}$.

In particular, we have $\mathbf{v} + \mathbf{u}p \notin \Psi_{s+1}(\mathcal{N})$ and thus $\theta_s(\mathbf{v} + \mathbf{u}p) = \mathcal{Q}(\mathbf{v} + \mathbf{u}p)$. Furthermore, we obtain $\sum_{\ell=1}^{s+1} \Delta(\{(\mathbf{v} + \mathbf{u}p)/p^\ell\}) \geq s + 1$.

If $v_p(Y_s(\mathbf{v}, \mathbf{u}, \mathbf{m})) \geq 0$, then, according to Lemma 12, $v_p(X_s(\mathbf{v}, \mathbf{u}, \mathbf{m}) - 1) \geq 0$ and, according to (6.16), we have $v_p\left(\mathcal{Q}(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})/g(\mathbf{m})\right) \geq \sum_{\ell=1}^{s+1} \Delta\left(\{(\mathbf{v} + \mathbf{u}p)/p^\ell\}\right) \geq s + 1$, thus we have (6.14).

Let us now assume that $v_p(Y_s(\mathbf{v}, \mathbf{u}, \mathbf{m})) < 0$. In this case, according to Lemma 12, we have $v_p(X_s(\mathbf{v}, \mathbf{u}, \mathbf{m}) - 1) = v_p(Y_s(\mathbf{v}, \mathbf{u}, \mathbf{m})) \geq \eta_s(\mathbf{u}, \mathbf{m}) - \eta_{s+1}(\mathbf{v} + \mathbf{u}p, \mathbf{m})$. Furthermore,

$$\begin{aligned}
v_p(\mathcal{Q}(\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1})) &= \sum_{\ell=1}^{\infty} \Delta\left(\left\{\frac{\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\\
&= \sum_{\ell=1}^{s+1} \Delta\left(\left\{\frac{\mathbf{v} + \mathbf{u}p}{p^\ell}\right\}\right)\\
&\quad + \sum_{\ell=s+2}^{\infty} \Delta\left(\left\{\frac{\mathbf{v} + \mathbf{u}p + \mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\\
&= \sum_{\ell=1}^{s+1} \Delta\left(\left\{\frac{\mathbf{v} + \mathbf{u}p}{p^\ell}\right\}\right) + \eta_{s+1}(\mathbf{v} + \mathbf{u}p, \mathbf{m}).
\end{aligned}$$

Thereby, we get

$$\begin{aligned} &v_p\left((X_s(\mathbf{v},\mathbf{u},\mathbf{m})-1)\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}{g(\mathbf{m})}\right)\\ &\geq \eta_s(\mathbf{u},\mathbf{m})-\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{m})+\sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right)+\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{m})-\mu(\mathbf{m})\\ &\geq s+1+\eta_s(\mathbf{u},\mathbf{m})-\mu(\mathbf{m}). \end{aligned}$$

If $s=0$, then we have $\mathbf{u}=\mathbf{0}$ and $\eta_0(\mathbf{0},\mathbf{m})=\sum_{\ell=1}^{\infty}\Delta(\{\frac{\mathbf{m}}{p^\ell}\})\geq\sum_{\ell=1}^{\infty}\mathbf{1}_{\mathcal{D}}(\{\frac{\mathbf{m}}{p^\ell}\})=\mu(\mathbf{m})$ and we have (6.14). On the other hand, if $s\geq 1$ then, applying Lemma 14 with $s-1$ instead of s and $\mathbf{a}=\mathbf{u}$, we get $\eta_s(\mathbf{u},\mathbf{m})\geq\mu(\mathbf{m})$, which implies (6.14). This finishes the proof of Eq. (6.13) assuming the truth of the various lemmas.

6.4.4. *Proof of Lemmas 12–15*

Proof of Lemma 12. We have to prove that $X_s(\mathbf{v},\mathbf{u},\mathbf{m})\in Y_s(\mathbf{v},\mathbf{u},\mathbf{m})(1+p^{s+1}\mathbb{Z}_p)$.

We have

$$X_s(\mathbf{v},\mathbf{u},\mathbf{m})=\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\mathcal{Q}(\mathbf{u}p)}\frac{\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1})}{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}\cdot\frac{\mathcal{Q}(\mathbf{u}p)}{\mathcal{Q}(\mathbf{u})}\frac{\mathcal{Q}(\mathbf{u}+\mathbf{m}p^{s})}{\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1})}.$$

Applying Lemma 7 with $\mathbf{c}=\mathbf{u}$ we obtain $\frac{\mathcal{Q}(\mathbf{u}p)}{\mathcal{Q}(\mathbf{u})}\frac{\mathcal{Q}(\mathbf{u}+\mathbf{m}p^{s})}{\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1})}\in 1+p^{s+1}\mathbb{Z}_p$, so that

$$X_s(\mathbf{v},\mathbf{u},\mathbf{m})\in\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\mathcal{Q}(\mathbf{u}p)}\frac{\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1})}{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}(1+p^{s+1}\mathbb{Z}_p). \tag{6.20}$$

Furthermore, we have

$$\begin{aligned} &\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\mathcal{Q}(\mathbf{u}p)}\cdot\frac{\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1})}{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}\\ &=\frac{\left(\prod_{i=1}^{q_1}\prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{v}}(\mathbf{e}_i\cdot\mathbf{u}p+k)\right)\left(\prod_{i=1}^{q_2}\prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{v}}(\mathbf{f}_i\cdot(\mathbf{u}p+\mathbf{m}p^{s+1})+k)\right)}{\left(\prod_{i=1}^{q_2}\prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{v}}(\mathbf{f}_i\cdot\mathbf{u}p+k)\right)\left(\prod_{i=1}^{q_1}\prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{v}}(\mathbf{e}_i\cdot(\mathbf{u}p+\mathbf{m}p^{s+1})+k)\right)}\\ &=\frac{\prod_{i=1}^{q_2}\prod_{k=1}^{\mathbf{f}_i\cdot\mathbf{v}}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{m}p^{s+1}}{\mathbf{f}_i\cdot\mathbf{u}p+k}\right)}{\prod_{i=1}^{q_1}\prod_{k=1}^{\mathbf{e}_i\cdot\mathbf{v}}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{m}p^{s+1}}{\mathbf{e}_i\cdot\mathbf{u}p+k}\right)}. \end{aligned}$$

If $\mathbf{d}\in\{\mathbf{e}_1,\dots,\mathbf{e}_{q_1},\mathbf{f}_1,\dots,\mathbf{f}_{q_2}\}$ and $k\in\{1,\dots,\mathbf{d}\cdot\mathbf{v}\}$, then p divides $\mathbf{d}\cdot\mathbf{u}p+k$ if and only if there exists $j\in\{1,\dots,\lfloor\mathbf{d}\cdot\mathbf{v}/p\rfloor\}$ such that $k=jp$. Thus we have

$$\prod_{k=1}^{\mathbf{d}\cdot\mathbf{v}}\left(1+\frac{\mathbf{d}\cdot\mathbf{m}p^{s+1}}{\mathbf{d}\cdot\mathbf{u}p+k}\right)=\prod_{j=1}^{\lfloor\mathbf{d}\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{d}\cdot\mathbf{m}p^{s}}{\mathbf{d}\cdot\mathbf{u}+j}\right)(1+O(p^{s+1})).$$

Therefore

$$\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\mathcal{Q}(\mathbf{u}p)}\cdot\frac{\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1})}{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}=\frac{\prod_{i=1}^{q_2}\prod_{j=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{m}p^s}{\mathbf{f}_i\cdot\mathbf{u}+j}\right)}{\prod_{i=1}^{q_1}\prod_{j=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{m}p^s}{\mathbf{e}_i\cdot\mathbf{u}+j}\right)}(1+O(p^{s+1}))$$
$$=Y_s(\mathbf{v},\mathbf{u},\mathbf{m})(1+O(p^{s+1}))$$

and so $X_s(\mathbf{v},\mathbf{u},\mathbf{m})\in Y_s(\mathbf{v},\mathbf{u},\mathbf{m})(1+p^{s+1}\mathbb{Z}_p)$, as expected.

We will now prove that we also have $v_p(Y_s(\mathbf{v},\mathbf{u},\mathbf{m}))\geq\eta_s(\mathbf{u},\mathbf{m})-\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{m})$. We have seen above that $v_p(Y_s(\mathbf{v},\mathbf{u},\mathbf{m}))=v_p(X_s(\mathbf{v},\mathbf{u},\mathbf{m}))$. Furthermore, according to (6.20), we also have

$$\begin{aligned}v_p(X_s(\mathbf{v},\mathbf{u},\mathbf{m}))&=v_p\left(\frac{\mathcal{Q}(\mathbf{v}+\mathbf{u}p)}{\mathcal{Q}(\mathbf{u}p)}\cdot\frac{\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1})}{\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1})}\right)\\&=v_p(\mathcal{Q}(\mathbf{v}+\mathbf{u}p))-v_p(\mathcal{Q}(\mathbf{u}p))+v_p(\mathcal{Q}(\mathbf{u}p+\mathbf{m}p^{s+1}))\\&\quad-v_p(\mathcal{Q}(\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1}))\\&=\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right)-\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}p}{p^\ell}\right\}\right)\\&\quad+\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}p+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)-\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right).\end{aligned}$$

We have

$$\begin{aligned}&\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right)-\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\\&\quad=\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right)-\sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right)-\sum_{\ell=s+2}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\\&\quad=\sum_{\ell=s+2}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^\ell}\right\}\right)-\sum_{\ell=s+2}^{\infty}\Delta\left(\left\{\frac{\mathbf{v}+\mathbf{u}p+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\\&\quad=\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{0})-\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{m}),\end{aligned}$$

and

$$\begin{aligned}&\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}p}{p^\ell}\right\}\right)-\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}p+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\\&\quad=\sum_{\ell=s+2}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}p}{p^\ell}\right\}\right)-\sum_{\ell=s+2}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}p+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\\&\quad=\sum_{\ell=s+1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}}{p^\ell}\right\}\right)-\sum_{\ell=s+1}^{\infty}\Delta\left(\left\{\frac{\mathbf{u}+\mathbf{m}p^{s}}{p^\ell}\right\}\right)\\&\quad=\eta_s(\mathbf{u},\mathbf{0})-\eta_s(\mathbf{u},\mathbf{m}).\end{aligned}$$

Thus, $v_p(Y_s(\mathbf{v},\mathbf{u},\mathbf{m}))=\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{0})-\eta_s(\mathbf{u},\mathbf{0})+\eta_s(\mathbf{u},\mathbf{m})-\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{m})$. We now have to prove that if $\mathbf{u}\in\Psi_s(\mathcal{N})$, then we have $\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{0})-\eta_s(\mathbf{u},\mathbf{0})\geq 0$. As $\mathbf{u}\in\Psi_s(\mathcal{N})$, we have $\{\mathbf{u}/p^s\} \notin \mathcal{D}$. Hence, for all $\ell \geq s+1$ and all $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$, we obtain $\mathbf{L}\cdot\left\{\mathbf{u}/p^\ell\right\} = \mathbf{L}\cdot\mathbf{u}/p^\ell \leq \mathbf{L}\cdot\mathbf{u}/p^s = \mathbf{L}\cdot\{\mathbf{u}/p^s\} < 1$, *i.e.*, for all $\ell \geq s+1$, $\{\mathbf{u}/p^\ell\} \notin \mathcal{D}$. Then we have $\eta_s(\mathbf{u},\mathbf{0}) = \sum_{\ell=s+1}^{\infty} \Delta\left(\left\{\mathbf{u}/p^\ell\right\}\right) = 0$ and $\eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{0}) - \eta_s(\mathbf{u},\mathbf{0}) = \eta_{s+1}(\mathbf{v}+\mathbf{u}p,\mathbf{0}) \geq 0$, which completes the proof of Lemma 12. □

Proof of Lemma 13. Given $s \in \mathbb{N}$, $\mathbf{v} \in \{0,\dots,p-1\}^d$ and $\mathbf{u} \in \{0,\dots,p^s-1\}^d$, we write $\mathbf{u} = \sum_{k=0}^{\infty} \mathbf{u}_k p^k$, where $\mathbf{u}_k \in \{0,\dots,p-1\}^d$. Given $\mathbf{L} \in \{\mathbf{e}_1,\dots,\mathbf{e}_{q_1},\mathbf{f}_1,\dots,\mathbf{f}_{q_2}\}$, we define $s+1$ non-negative integers by the formulas $b_{\mathbf{L},0} := \lfloor \mathbf{L}\cdot\mathbf{v}/p\rfloor$ and $b_{\mathbf{L},k+1} := \lfloor(\mathbf{L}\cdot\mathbf{u}_k + b_{\mathbf{L},k})/p\rfloor$ for $k \in \{0,\dots,s-1\}$. For all $x \in \mathbb{R}$, we write $\lceil x\rceil$ for the smallest integer greater than or equal to x and we define $s+1$ non-negative integers by the formulas $a_{\mathbf{L},0} := 1$ and $a_{\mathbf{L},k+1} := \lceil(\mathbf{L}\cdot\mathbf{u}_k + a_{\mathbf{L},k})/p\rceil$. First, we will prove by induction on r that assertion $\mathcal{A}_r$:

$$\prod_{n=1}^{\lfloor \mathbf{L}\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{L}\cdot\mathbf{m}p^s}{\mathbf{L}\cdot\mathbf{u}+n}\right) = \prod_{n=a_{\mathbf{L},r}}^{b_{\mathbf{L},r}}\left(1+\frac{\mathbf{L}\cdot\mathbf{m}p^{s-r}}{\mathbf{L}\cdot\left(\sum_{k=r}^{\infty}\mathbf{u}_k p^{k-r}\right)+n}\right)\left(1+O(p^{s-r+1})\right)$$

is true for all $r \in \{0,\dots,s\}$.

We have $b_{\mathbf{L},0} = \lfloor\mathbf{L}\cdot\mathbf{v}/p\rfloor$ and $a_{\mathbf{L},0} = 1$, thus $\mathcal{A}_0$ is true.

Given $r \geq 0$, let us assume that $\mathcal{A}_r$ is true and prove $\mathcal{A}_{r+1}$. If $a_{\mathbf{L},r} > b_{\mathbf{L},r}$ then $a_{\mathbf{L},r+1} > b_{\mathbf{L},r+1}$ and $\mathcal{A}_r$ implies $\mathcal{A}_{r+1}$. Thus we can assume that $a_{\mathbf{L},r} \leq b_{\mathbf{L},r}$. If $n \in \{a_{\mathbf{L},r},\dots,b_{\mathbf{L},r}\}$, then p divides $\mathbf{L}\cdot\left(\sum_{k=r}^{\infty}\mathbf{u}_k p^{k-r}\right)+n$ if and only if p divides $\mathbf{L}\cdot\mathbf{u}_r + n$, *i.e.* if and only if an $i \in \{\lceil(\mathbf{L}\cdot\mathbf{u}_r + a_{\mathbf{L},r})/p\rceil,\dots,\lfloor(\mathbf{L}\cdot\mathbf{u}_r + b_{\mathbf{L},r})/p\rfloor\}$ exists such that $\mathbf{L}\cdot\mathbf{u}_r + n = ip$. So we get

$$\begin{aligned}
&\prod_{n=a_{\mathbf{L},r}}^{b_{\mathbf{L},r}}\left(1+\frac{\mathbf{L}\cdot\mathbf{m}p^{s-r}}{\mathbf{L}\cdot\left(\sum_{k=r}^{\infty}\mathbf{u}_k p^{k-r}\right)+n}\right)\\
&\quad=\prod_{i=a_{\mathbf{L},r+1}}^{b_{\mathbf{L},r+1}}\left(1+\frac{\mathbf{L}\cdot\mathbf{m}p^{s-r}}{\mathbf{L}\cdot\left(\sum_{k=r+1}^{\infty}\mathbf{u}_k p^{k-r}\right)+ip}\right)(1+O(p^{s-r}))\\
&\quad=\prod_{i=a_{\mathbf{L},r+1}}^{b_{\mathbf{L},r+1}}\left(1+\frac{\mathbf{L}\cdot\mathbf{m}p^{s-r-1}}{\mathbf{L}\cdot\left(\sum_{k=r+1}^{\infty}\mathbf{u}_k p^{k-r-1}\right)+i}\right)(1+O(p^{s-r})). \qquad (6.21)
\end{aligned}$$

According to $\mathcal{A}_r$ and (6.21), we have $\mathcal{A}_{r+1}$, which finishes the induction on r.

Given $\mathbf{L} \in \{\mathbf{e}_1,\dots,\mathbf{e}_{q_1},\mathbf{f}_1,\dots,\mathbf{f}_{q_2}\}$, we will prove by induction on k that assertion $\mathcal{B}_k$: $a_{\mathbf{L},k} \geq 1$ and $b_{\mathbf{L},k} \leq \lfloor\mathbf{L}\cdot\{(\mathbf{v}+\mathbf{u}p)/p^{k+1}\}\rfloor$ is true for all $k \in \{0,\dots,s\}$.

We have $a_{\mathbf{L},0} = 1$ and $b_{\mathbf{L},0} = \lfloor\mathbf{L}\cdot\mathbf{v}/p\rfloor = \lfloor\mathbf{L}\cdot\{(\mathbf{v}+\mathbf{u}p)/p\}\rfloor$, so $\mathcal{B}_0$ is true.

Given $k \geq 0$, let us assume that $\mathcal{B}_k$ is true and let us prove $\mathcal{B}_{k+1}$. We have $a_{\mathbf{L},k+1} = \lceil(\mathbf{L}\cdot\mathbf{u}_k + a_{\mathbf{L},k})/p\rceil$ and $b_{\mathbf{L},k+1} = \lfloor(\mathbf{L}\cdot\mathbf{u}_k + b_{\mathbf{L},k})/p\rfloor$, thus $a_{\mathbf{L},k+1} \geq \lceil(\mathbf{L}\cdot\mathbf{u}_k + 1)/p\rceil \geq 1$ and

$$b_{\mathbf{L},k+1} \leq \left\lfloor \frac{\mathbf{L}\cdot\mathbf{u}_k}{p} + \frac{\mathbf{L}}{p}\cdot\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^{k+1}}\right\}\right\rfloor = \left\lfloor \mathbf{L}\cdot\left(\frac{\mathbf{u}_k p^{k+1}}{p^{k+2}} + \frac{\mathbf{v} + p\sum_{i=0}^{k-1}\mathbf{u}_i p^i}{p^{k+2}}\right)\right\rfloor$$

$$= \left\lfloor \mathbf{L}\cdot\left\{\frac{\mathbf{v}+\mathbf{u}p}{p^{k+2}}\right\}\right\rfloor,$$

which completes the induction on k.

Given $j \in \{1,\dots,s+1\}$ such that $\{(\mathbf{v}+\mathbf{u}p)/p^j\} \notin \mathcal{D}$, for all $\mathbf{L} \in \{\mathbf{e}_1,\dots,\mathbf{e}_{q_1},\mathbf{f}_1,\dots,\mathbf{f}_{q_2}\}$, we obtain, *via* $\mathcal{B}_{j-1}$, that $a_{\mathbf{L},j-1} \geq 1$ and $b_{\mathbf{L},j-1} \leq \lfloor \mathbf{L}\cdot\{(\mathbf{v}+\mathbf{u}p)/p^j\}\rfloor = 0$. Hence, according to $\mathcal{A}_{j-1}$, we get $\prod_{n=1}^{\lfloor \mathbf{L}\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{L}\cdot\mathbf{m}p^s}{\mathbf{L}\cdot\mathbf{u}+n}\right) = 1 + O(p^{s-j+2})$ and thus

$$Y_s(\mathbf{v},\mathbf{u},\mathbf{m}) = \frac{\prod_{i=1}^{q_2}\prod_{n=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{f}_i\cdot\mathbf{m}p^s}{\mathbf{f}_i\cdot\mathbf{u}+n}\right)}{\prod_{i=1}^{q_1}\prod_{n=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{v}/p\rfloor}\left(1+\frac{\mathbf{e}_i\cdot\mathbf{m}p^s}{\mathbf{e}_i\cdot\mathbf{u}+n}\right)} = \frac{1+O(p^{s-j+2})}{1+O(p^{s-j+2})} = 1 + O(p^{s-j+2}),$$

which finishes the proof of Lemma 13. □

Proof of Lemma 14. First, we will prove that we have (6.15). Let us write $\mathbf{m} = \sum_{k=0}^{q}\mathbf{m}_k p^k$, where $\mathbf{m}_k \in \{0,\dots,p-1\}^d$. We have

$$\eta_{s+1}(\mathbf{a},\mathbf{m}) - \mu(\mathbf{m}) = \sum_{\ell=s+2}^{\infty}\Delta\left(\left\{\frac{\mathbf{a}+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right) - \sum_{\ell=1}^{\infty}1_{\mathcal{D}}\left(\left\{\frac{\mathbf{m}}{p^\ell}\right\}\right)$$

$$= \sum_{\ell=s+2}^{\infty}\left(\Delta\left(\left\{\frac{\mathbf{a}+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right) - 1_{\mathcal{D}}\left(\left\{\frac{\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)\right)$$

$$= \sum_{\ell=s+2}^{\infty}\left(\Delta\left(\frac{\mathbf{a}+\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell}\right) - 1_{\mathcal{D}}\left(\frac{\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell}\right)\right).$$

Furthermore, for all $\ell \geq s+2$, we have

$$\mathbf{0} \leq \frac{\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell} \leq \frac{\mathbf{a}+\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell} \leq \frac{(p^\ell-1)\mathbf{1}}{p^\ell} \in [0,1[^d.$$

Thus

$$1_{\mathcal{D}}\left(\frac{\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell}\right) = 1 \Longrightarrow \frac{\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell} \in \mathcal{D}$$

$$\Longrightarrow \frac{\mathbf{a}+\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell}\in\mathcal{D}$$

$$\Longrightarrow \Delta\left(\frac{\mathbf{a}+\sum_{k=0}^{\ell-s-2}\mathbf{m}_k p^{k+s+1}}{p^\ell}\right)\geq 1$$

and so $\eta_{s+1}(\mathbf{a},\mathbf{m})-\mu(\mathbf{m})\geq 0$. This completes the proof of (6.15).

Let us now prove (6.16). We have

$$\begin{aligned}
v_p\left(\frac{\mathcal{Q}(\mathbf{a}+\mathbf{m}p^{s+1})}{g_p(\mathbf{m})}\right) &= \sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{a}+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)-\mu(\mathbf{m})\\
&= \sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{a}}{p^\ell}\right\}\right)+\sum_{\ell=s+2}^{\infty}\Delta\left(\left\{\frac{\mathbf{a}+\mathbf{m}p^{s+1}}{p^\ell}\right\}\right)-\mu(\mathbf{m})\\
&= \sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{a}}{p^\ell}\right\}\right)+\eta_{s+1}(\mathbf{a},\mathbf{m})-\mu(\mathbf{m})\\
&\geq \sum_{\ell=1}^{s+1}\Delta\left(\left\{\frac{\mathbf{a}}{p^\ell}\right\}\right) \qquad (6.22)
\end{aligned}$$

where we used inequality (6.15) for (6.22). □

Proof of Lemma 15. We have $v_p(\mathcal{Q}(\mathbf{a}))=\sum_{\ell=1}^{\infty}\Delta\left(\left\{\frac{\mathbf{a}}{p^\ell}\right\}\right)$. As $\mathbf{a}\in\Psi_s(\mathcal{N})$, we have $\{\mathbf{a}/p^s\}\notin\mathcal{D}$ and, for all $\ell\geq s+1$ and all $\mathbf{L}\in\{\mathbf{e}_1,\dots,\mathbf{e}_{q_1},\mathbf{f}_1,\dots,\mathbf{f}_{q_2}\}$, we get

$$\mathbf{L}\cdot\left\{\frac{\mathbf{a}}{p^\ell}\right\}=\mathbf{L}\cdot\frac{\mathbf{a}}{p^\ell}\leq\mathbf{L}\cdot\frac{\mathbf{a}}{p^s}=\mathbf{L}\cdot\left\{\frac{\mathbf{a}}{p^s}\right\}<1,$$

i.e. $\{\mathbf{a}/p^\ell\}\notin\mathcal{D}$. Thus, for all $\ell\geq s+1$, we have $\Delta\left(\{\mathbf{a}/p^\ell\}\right)=0$. This gives us the expected result. □

7. Proofs of assertion (ii) of Theorems 1 and 2

We assume the hypothesis of Theorems 1 and 2. Furthermore, we assume that $\mathbf{x}_0\in\mathcal{D}_{e,f}$ is a zero of $\Delta_{e,f}$. In Section 7.1, we prove an elementary result of analysis which we will use for the proofs of assertion (ii) of Theorems 1 and 2. We prove assertion (ii) of Theorem 1 in Section 7.2. We will use certain results from Section 7.2 for the proof of assertion (ii) of Theorem 2 which we present in Section 7.3.

7.1. Preliminary

The aim of this section is to prove that there exists a nonempty open subset $\mathcal{U}$ of $\mathcal{D}_{e,f}$ such that, for all $\mathbf{x}\in\mathcal{U}$, $i\in\{1,\dots,q_1\}$ and $j\in\{1,\dots,q_2\}$, we have $\lfloor\mathbf{e}_i\cdot\mathbf{x}\rfloor=\lfloor\mathbf{e}_i\cdot\mathbf{x}_0\rfloor$, $\mathbf{e}_i\cdot\mathbf{x}\neq 0$, $\lfloor\mathbf{f}_j\cdot\mathbf{x}\rfloor=\lfloor\mathbf{f}_j\cdot\mathbf{x}_0\rfloor$, $\mathbf{f}_j\cdot\mathbf{x}\neq 0$ and $\mathbf{e}_i\cdot\mathbf{x}\neq\mathbf{f}_j\cdot\mathbf{x}$.

In particular, for all $\mathbf{x}\in\mathcal{U}$, we would have $\Delta_{e,f}(\mathbf{x})=\Delta_{e,f}(\mathbf{x}_0)=0$. We will use this open set $\mathcal{U}$ throughout the rest of the proof.

Applying Lemma 1 with, instead of u, the sequence constituted by the elements of e and f, we obtain that there exists $\mu > 0$ such that, for all $\mathbf{x} \in [0, \mu]^d$ and all $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots \mathbf{f}_{q_2}\}$, we have $\lfloor \mathbf{L} \cdot (\mathbf{x}_0 + \mathbf{x}) \rfloor = \lfloor \mathbf{L} \cdot \mathbf{x}_0 \rfloor$. As $\mathbf{x}_0 \in [0, 1[^d$, there exists $\mu_1 > 0$, $\mu_1 \le \mu$, such that, for all $\mathbf{x} \in [0, \mu_1]^d$, we have $\mathbf{x}_0 + \mathbf{x} \in [0, 1[^d$. Since $\mathbf{x}_0 \in \mathcal{D}_{e,f}$, a $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$ exists such that $\mathbf{L} \cdot \mathbf{x}_0 \ge 1$, which gives us the result that, for all $\mathbf{x} \in [0, \mu_1]^d$, we have $\mathbf{L} \cdot (\mathbf{x}_0 + \mathbf{x}) \ge \mathbf{L} \cdot \mathbf{x}_0 \ge 1$ and thus, since $\mathbf{x}_0 + \mathbf{x} \in [0, 1[^d$, we get that $\mathbf{x}_0 + \mathbf{x} \in \mathcal{D}_{e,f}$. Thereby, there exists a nonempty open subset $\mathcal{U}_1$ of $\mathcal{D}_{e,f}$ such that, for all $\mathbf{x} \in \mathcal{U}_1$ and $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$, we have $\lfloor \mathbf{L} \cdot \mathbf{x} \rfloor = \lfloor \mathbf{L} \cdot \mathbf{x}_0 \rfloor$.

For all $i \in \{1, \dots, q_1\}$ and $j \in \{1, \dots, q_2\}$, we define the sets $\mathcal{H}_{\mathbf{e}_i} := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{e}_i \cdot \mathbf{x} = 0\}$, $\mathcal{H}_{\mathbf{f}_j} := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{f}_j \cdot \mathbf{x} = 0\}$ and $\mathcal{H}_{\mathbf{e}_i,\mathbf{f}_j} := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{e}_i \cdot \mathbf{x} = \mathbf{f}_j \cdot \mathbf{x}\}$. Since e and f are two disjoint sequences constituted by nonzero vectors, we obtain that the $\mathcal{H}_{\mathbf{e}_i}$, $\mathcal{H}_{\mathbf{f}_j}$ and $\mathcal{H}_{\mathbf{e}_i,\mathbf{f}_j}$ are hyperplanes in $\mathbb{R}^d$ and are therefore closed subsets of $\mathbb{R}^d$ with empty interiors. Therefore, their complements are dense open subsets of $\mathbb{R}^d$ and the complement $\mathcal{U}_2$ of the union of $\mathcal{H}_{\mathbf{e}_i}$, $\mathcal{H}_{\mathbf{f}_j}$ and $\mathcal{H}_{\mathbf{e}_i,\mathbf{f}_j}$ is a dense open subset of $\mathbb{R}^d$. As a result, $\mathcal{U} := \mathcal{U}_1 \cap \mathcal{U}_2$ is a nonempty open subset of $\mathcal{D}_{e,f}$ and, for all $\mathbf{x} \in \mathcal{U}$, $i \in \{1, \dots, q_1\}$ and $j \in \{1, \dots, q_2\}$, we have $\mathbf{e}_i \cdot \mathbf{x} \neq 0$, $\mathbf{f}_j \cdot \mathbf{x} \neq 0$, $\mathbf{e}_i \cdot \mathbf{x} \neq \mathbf{f}_j \cdot \mathbf{x}$, $\lfloor \mathbf{e}_i \cdot \mathbf{x} \rfloor = \lfloor \mathbf{e}_i \cdot \mathbf{x}_0 \rfloor$ and $\lfloor \mathbf{f}_j \cdot \mathbf{x} \rfloor = \lfloor \mathbf{f}_j \cdot \mathbf{x}_0 \rfloor$.

7.2. *Proof of assertion (ii) of Theorem 1*

The aim of this section is to prove that there exists $k \in \{1, \dots, d\}$ such that there are only finitely many prime numbers p such that $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}_p[[\mathbf{z}]]$. Following Section 4, we only have to prove that there exists $k \in \{1, \dots, d\}$ such that, for all large enough prime numbers p, there exist $\mathbf{a} \in \{0, \dots, p-1\}^d$ and $\mathbf{K} \in \mathbb{N}^d$ such that $\Phi_{p,k}(\mathbf{a} + p\mathbf{K}) \notin p\mathbb{Z}_p$. We will actually prove that there exists $k \in \{1, \dots, d\}$ such that, for all large enough prime numbers p, there is an $\mathbf{a} \in \{0, \dots, p-1\}^d$ such that $\Phi_{p,k}(\mathbf{a}) \notin p\mathbb{Z}_p$. In this case, we have

$$\Phi_{p,k}(\mathbf{a}) = -p\mathcal{Q}(\mathbf{a}) \left(\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)} H_{\mathbf{a}\cdot\mathbf{e}_i} - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)} H_{\mathbf{a}\cdot\mathbf{f}_i} \right). \tag{7.1}$$

For all $\mathbf{d} \in \mathbb{N}^d$, we have $pH_{\mathbf{d}\cdot\mathbf{a}} = p\sum_{i=1}^{\mathbf{d}\cdot\mathbf{a}} \frac{1}{i} \equiv \sum_{j=1}^{\lfloor \mathbf{d}\cdot\mathbf{a}/p \rfloor} \frac{1}{j} \mod p\mathbb{Z}_p$. For all $k \in \{1, \dots, d\}$ and $\mathbf{x} \in [0, 1]^d$, we set

$$\Psi_k(\mathbf{x}) := \sum_{i=1}^{q_1} \sum_{j=1}^{\lfloor \mathbf{e}_i\cdot\mathbf{x} \rfloor} \frac{\mathbf{e}_i^{(k)}}{j} - \sum_{i=1}^{q_2} \sum_{j=1}^{\lfloor \mathbf{f}_i\cdot\mathbf{x} \rfloor} \frac{\mathbf{f}_i^{(k)}}{j}.$$

Thus, for all $k \in \{1, \dots, d\}$ and $\mathbf{a} \in \{0, \dots, p-1\}^d$, we have $\Phi_{p,k}(\mathbf{a}) \equiv -\mathcal{Q}(\mathbf{a})\Psi_k(\mathbf{a}/p) \mod p\mathbb{Z}_p$. Therefore we now have to prove that there exists $k \in \{1, \dots, d\}$ such that, for all large enough prime numbers p, there exists $\mathbf{a} \in \{0, \dots, p-1\}^d$ such that $v_p(\mathcal{Q}(\mathbf{a})) = v_p(\Psi_k(\mathbf{a}/p)) = 0$. We set $\mathcal{M} := \max\{|\mathbf{d}| : \mathbf{d} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}\}$.

A constant $\mathcal{P}_1 \ge \mathcal{M}$ exists such that, for all prime numbers $p \ge \mathcal{P}_1$, there exists $\mathbf{a}_p \in \{0, \dots, p-1\}^d$ such that $\mathbf{a}_p/p \in \mathcal{U}$. For all $\ell \ge 2$, we have $\mathbf{a}_p/p^\ell \le \mathbf{a}_p/p^2 < \mathbf{1}/p$ and thus, for all $\mathbf{L} \in \{\mathbf{e}_1, \dots, \mathbf{e}_{q_1}, \mathbf{f}_1, \dots, \mathbf{f}_{q_2}\}$, we have $\mathbf{L} \cdot \mathbf{a}_p/p^\ell < \mathbf{L} \cdot \mathbf{1}/p \le \mathcal{M}/p \le 1$. Hence, for all prime numbers $p \ge \mathcal{P}_1$ and all $\ell \ge 2$, we have $\mathbf{a}_p/p^\ell \notin \mathcal{D}_{e,f}$, which implies that $v_p(\mathcal{Q}(\mathbf{a}_p)) = \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\mathbf{a}_p/p^\ell\right) = \Delta_{e,f}\left(\mathbf{a}_p/p\right) = 0$, because $\Delta_{e,f}$ vanishes on $\mathcal{U}$ and on $[0, 1[^d \setminus \mathcal{D}_{e,f}$.

So we now have to prove that there exist $k \in \{1, \dots, d\}$ and a constant $\mathcal{P} \ge \mathcal{P}_1$ such that, for all prime numbers $p \ge \mathcal{P}$, we have $v_p(\Psi_k(\mathbf{a}_p/p)) = 0$.

For all prime numbers $p \ge \mathcal{P}_1$, all $i \in \{1, \dots, q_1\}$ and $j \in \{1, \dots, q_2\}$, we write $\alpha_i := \lfloor \mathbf{e}_i \cdot \mathbf{a}_p/p \rfloor$ and $\beta_j := \lfloor \mathbf{f}_j \cdot \mathbf{a}_p/p \rfloor$. According to the construction of $\mathcal{U}$ and since $\mathbf{a}_p/p \in \mathcal{U}$, we have $\lfloor \mathbf{e}_i \cdot \mathbf{a}_p/p \rfloor = \lfloor \mathbf{e}_i \cdot \mathbf{x}_0 \rfloor$ and $\lfloor \mathbf{f}_j \cdot \mathbf{a}_p/p \rfloor = \lfloor \mathbf{f}_j \cdot \mathbf{x}_0 \rfloor$. Therefore, the α_i and β_j do not depend on p. Thus there exists a constant $\mathcal{P} \geq \mathcal{P}_1$ such that, for all prime numbers $p \geq \mathcal{P}$ and all $k \in \{1, \ldots, d\}$, we have

$$\Psi_k(\mathbf{a}_p/p) = \sum_{i=1}^{q_1} \sum_{j=1}^{\alpha_i} \frac{\mathbf{e}_i^{(k)}}{j} - \sum_{i=1}^{q_2} \sum_{j=1}^{\beta_j} \frac{\mathbf{f}_i^{(k)}}{j} \in \mathbb{Z}_p^\times \cup \{0\}.$$

Therefore we only have to prove that there exists $k \in \{1, \ldots, d\}$ such that $\Psi_k(\mathbf{a}_p/p) \neq 0$. For this purpose, we will use Lemma 16 of [4] which reads as follows.

Lemma 16. *Let* $\mathbf{E} := (E_1, \ldots, E_{q_1})$ *and* $\mathbf{F} := (F_1, \ldots, F_{q_2})$ *be two disjoint sequences of positive integers. We write* $\mathcal{A} := \{E_1, \ldots, E_{q_1}, F_1, \ldots, F_{q_2}\}$ *and* $\gamma_1 < \cdots < \gamma_t = 1$ *for the rational numbers which satisfy* $\{\gamma_1, \ldots, \gamma_t\} = \bigcup_{a \in \mathcal{A}} \{\frac{1}{a}, \frac{2}{a}, \ldots, \frac{a-1}{a}, 1\}$ *and* $m_i \in \mathbb{Z}$ *the amplitude of the jump of* $\Delta_{\mathbf{E},\mathbf{F}}$ *at* γ_i. *If there exists* $i_0 \in \{1, \ldots, t\}$ *such that* $\Delta_{\mathbf{E},\mathbf{F}} \geq 0$ *on* $[\gamma_1, \gamma_{i_0}]$, *then we have* $\sum_{k=1}^{i_0} \frac{m_k}{\gamma_k} > 0$ *and* $\prod_{k=1}^{i_0} \left(1 + \frac{1}{\gamma_k}\right)^{m_k} > 1$.

We will use Lemma 16 with $\mathbf{E}_p := (\mathbf{e}_1 \cdot \mathbf{a}_p, \ldots, \mathbf{e}_{q_1} \cdot \mathbf{a}_p)$ instead of $\mathbf{E}$ and $\mathbf{F}_p := (\mathbf{f}_1 \cdot \mathbf{a}_p, \ldots, \mathbf{f}_{q_2} \cdot \mathbf{a}_p)$ instead of $\mathbf{F}$.

First, we have to prove that $\mathbf{E}_p$ and $\mathbf{F}_p$ are two disjoint sequences of positive integers. Indeed, according to the construction of $\mathcal{U}$, for all $i \in \{1, \ldots, q_1\}$ and all $j \in \{1, \ldots, q_2\}$, we have $\mathbf{e}_i \cdot \mathbf{a}_p/p \neq 0, \mathbf{f}_j \cdot \mathbf{a}_p/p \neq 0$ and $\mathbf{e}_i \cdot \mathbf{a}_p/p \neq \mathbf{f}_j \cdot \mathbf{a}_p/p$, thus $\mathbf{e}_i \cdot \mathbf{a}_p \neq 0, \mathbf{f}_j \cdot \mathbf{a}_p \neq 0$ and $\mathbf{e}_i \cdot \mathbf{a}_p \neq \mathbf{f}_j \cdot \mathbf{a}_p$, which gives us that $\mathbf{E}_p$ and $\mathbf{F}_p$ are two disjoint sequences of positive integers.

We write $\mathcal{A} := \{\mathbf{e}_1 \cdot \mathbf{a}_p, \ldots, \mathbf{e}_{q_1} \cdot \mathbf{a}_p, \mathbf{f}_1 \cdot \mathbf{a}_p, \ldots, \mathbf{f}_{q_2} \cdot \mathbf{a}_p\}$ and $\gamma_1 < \cdots < \gamma_t = 1$ for the rational numbers which satisfy $\{\gamma_1, \ldots, \gamma_t\} = \bigcup_{a \in \mathcal{A}} \{\frac{1}{a}, \frac{2}{a}, \ldots, \frac{a-1}{a}, 1\}$ and $m_i \in \mathbb{Z}$ the amplitude of the jump of $\Delta_{\mathbf{E}_p,\mathbf{F}_p}$ in γ_i. As $\mathbf{a}_p/p \in \mathcal{D}_{\mathbf{e},\mathbf{f}}$, there exists $a \in \mathcal{A}$ such that $a \geq p$ and so $\max(\mathcal{A}) \geq p$. Hence, we have $\gamma_1 = 1/\max(\mathcal{A}) \leq 1/p$. Thus there exists $i_0 \in \{1, \ldots, t-1\}$ such that $\gamma_{i_0} \leq 1/p < \gamma_{i_0+1}$. Furthermore, for all $x \in [0, 1]$, we have

$$\Delta_{\mathbf{E}_p,\mathbf{F}_p}(x) = \sum_{i=1}^{q_1} \lfloor (\mathbf{e}_i \cdot \mathbf{a}_p)x \rfloor - \sum_{j=1}^{q_2} \lfloor (\mathbf{f}_j \cdot \mathbf{a}_p)x \rfloor = \Delta_{e,f}(x\mathbf{a}_p) \geq 0,$$

because $\Delta_{\mathbf{e},\mathbf{f}} \geq 0$ on $[0, 1]^d$. In particular, $\Delta_{\mathbf{E}_p,\mathbf{F}_p} \geq 0$ on $[\gamma_1, \gamma_{i_0}]$.

We can therefore apply Lemma 16 which results in

$$\begin{aligned} 0 < \sum_{i=1}^{i_0} \frac{m_i}{\gamma_i} &= \sum_{c \in \mathbf{E}_p} \sum_{j=1}^{\lfloor c/p \rfloor} \frac{c}{j} - \sum_{d \in \mathbf{F}_p} \sum_{j=1}^{\lfloor d/p \rfloor} \frac{d}{j} = \sum_{i=1}^{q_1} \sum_{j=1}^{\lfloor \mathbf{a}_p \cdot \mathbf{e}_i/p \rfloor} \frac{\mathbf{a}_p \cdot \mathbf{e}_i}{j} - \sum_{i=1}^{q_2} \sum_{j=1}^{\lfloor \mathbf{a}_p \cdot \mathbf{f}_i/p \rfloor} \frac{\mathbf{a}_p \cdot \mathbf{f}_i}{j} \\ &= \sum_{k=1}^{d} \mathbf{a}_p^{(k)} \left(\sum_{i=1}^{q_1} \sum_{j=1}^{\alpha_i} \frac{\mathbf{e}_i^{(k)}}{j} - \sum_{i=1}^{q_2} \sum_{j=1}^{\beta_i} \frac{\mathbf{f}_i^{(k)}}{j} \right) = \sum_{k=1}^{d} \mathbf{a}_p^{(k)} \Psi_k(\mathbf{a}_p/p), \end{aligned} \tag{7.2}$$

where the first equality in (7.2) is valid because the abscissas of the jumps of $\Delta_{\mathbf{E}_p,\mathbf{F}_p}$ on $[0, 1/p]$ are exactly the rational numbers j/a with $a \in \mathcal{A}$ and $j \leq \lfloor a/p \rfloor$, and an abscissa j/a corresponds to a jump with positive amplitude when $a \in \mathbf{E}_p$ and to a jump with negative amplitude when $a \in \mathbf{F}_p$.

Thus there exists $k \in \{1, \ldots, d\}$ such that $\Psi_k(\mathbf{a}_p/p) \neq 0$, which finishes the proof of assertion (ii) of Theorem 1.

7.3. Proof of assertion (ii) of Theorem 2

According to Section 7.2, there exists $k_0 \in \{1, \dots, d\}$ such that there are only finitely many prime numbers p such that $q_{e,f,k_0}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$. In order to finish the proof of assertion (ii) of Theorem 2, we only have to prove that, for all $\mathbf{L} \in \mathcal{E}$ satisfying $\mathbf{L}^{(k_0)} \geq 1$, there are only finitely many prime numbers p such that $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$. During the proof, we fix $\mathbf{L} \in \mathcal{E}_{e,f}$ satisfying $\mathbf{L}^{(k_0)} \geq 1$. ([2]) We will divide the proof into two cases depending on whether $\lfloor \mathbf{L} \cdot \mathbf{x}_0 \rfloor = 0$ or $\lfloor \mathbf{L} \cdot \mathbf{x}_0 \rfloor \neq 0$.

According to Section 7.2, we know that there exists a constant $\mathcal{P}_1$ such that, for all prime numbers $p \geq \mathcal{P}_1$, there exists $\mathbf{a}_p \in \{0, \dots, p-1\}^d$ such that $\mathbf{a}_p/p \in \mathcal{U}$ and $v_p(\mathcal{Q}(\mathbf{a}_p)) = 0$.

7.3.1. First case

We assume that $\lfloor \mathbf{L} \cdot \mathbf{x}_0 \rfloor \neq 0$. The aim of this section is to prove that there exists a constant $\mathcal{P} \geq \mathcal{P}_1$ such that, for all prime numbers $p \geq \mathcal{P}$, we have $\Phi_{\mathbf{L},p}(\mathbf{a}_p) \notin p\mathbb{Z}_p$, which, according to Section 4, will prove that there are only finitely many prime numbers p such that $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$.

We recall that, for all $\mathbf{a} \in \{0, \dots, p-1\}^d$, we have

$$\Phi_{\mathbf{L},p}(\mathbf{a}) = -p\mathcal{Q}(\mathbf{a})H_{\mathbf{L}\cdot\mathbf{a}} \equiv -\mathcal{Q}(\mathbf{a})H_{\lfloor \mathbf{L}\cdot\mathbf{a}/p \rfloor} \bmod p\mathbb{Z}_p. \tag{7.3}$$

For all prime numbers $p \geq \mathcal{P}_1$, we have $\lfloor \mathbf{L} \cdot \mathbf{a}_p/p \rfloor = \lfloor \mathbf{L} \cdot \mathbf{x}_0 \rfloor \neq 0$ therefore $H_{\lfloor \mathbf{L}\cdot\mathbf{a}_p/p \rfloor} \in \{H_1, \dots, H_{|\mathbf{L}|}\}$. A constant $\mathcal{P} \geq \mathcal{P}_1$ exists such that, for all prime numbers $p \geq \mathcal{P}$, we have $\{H_1, \dots, H_{|\mathbf{L}|}\} \subset \mathbb{Z}_p^\times$. Thus, for all prime numbers $p \geq \mathcal{P}$, we have $\mathcal{Q}(\mathbf{a}_p)H_{\lfloor \mathbf{L}\cdot\mathbf{a}_p/p \rfloor} \in \mathbb{Z}_p^\times$ and, according to (7.3), we obtain $\Phi_{\mathbf{L},p}(\mathbf{a}_p) \notin p\mathbb{Z}_p$.

We observe that in this case, we did not use the hypothesis $\mathbf{L}^{(k_0)} \geq 1$.

7.3.2. Second case

We assume that $\lfloor \mathbf{L} \cdot \mathbf{x}_0 \rfloor = 0$. The aim of this section is to prove that there exist $r \in \mathbb{N}, r \geq 1$ and a constant $\mathcal{P}' \geq \mathcal{P}_1$ such that, for all prime numbers $p \geq \mathcal{P}'$, we have $\Phi_{\mathbf{L},p}(\mathbf{a}_p + pr\mathbf{1}_{k_0}) \notin p\mathbb{Z}_p$. According to Section 4, this will prove that there are only finitely many prime numbers p such that $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$.

In the sequel, for all $k \in \{1, \dots, d\}$, we write R_k for the rational function defined by

$$R_k(X) := \frac{\prod_{i=1}^{q_1} \prod_{j=1}^{\alpha_i} \left(1 + \frac{\mathbf{e}_i^{(k)}}{j} X\right)}{\prod_{i=1}^{q_2} \prod_{j=1}^{\beta_i} \left(1 + \frac{\mathbf{f}_i^{(k)}}{j} X\right)}. \tag{7.4}$$

We will use the following lemma, which we will prove at the end of this section.

Lemma 17. *For all $r \in \mathbb{N}, r \geq 1$, there exists a constant $\mathcal{P}_r \geq \mathcal{P}_1$ such that, for all prime numbers $p \geq \mathcal{P}_r$ and all $k \in \{1, \dots, d\}$, we have*

$$\Phi_{\mathbf{L},p}(\mathbf{a}_p + pr\mathbf{1}_k) \equiv -\sum_{j=1}^{r} H_{j\mathbf{L}^{(k)}}\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(j\mathbf{1}_k)\mathcal{Q}((r-j)\mathbf{1}_k) \times \big(R_k(j) - R_k(r-j)\big) \bmod p\mathbb{Z}_p.$$

[2] Such an $\mathbf{L}$ exists because $q_{e,f,k_0}(\mathbf{z}) \notin z_{k_0}\mathbb{Z}[[\mathbf{z}]]$.

According to the end of Section 7.2, we know that $\sum_{i=1}^{q_1}\sum_{j=1}^{\alpha_i}\frac{\mathbf{e}_i^{(k_0)}}{j}-\sum_{i=1}^{q_2}\sum_{j=1}^{\beta_i}\frac{\mathbf{f}_i^{(k_0)}}{j}\neq 0$ and therefore $R_{k_0}(X)$ is not a constant equal to 1. Thus there exists $r\in\mathbb{N}$ such that $R_{k_0}(r)\neq 1$. Let r_0 be the smallest positive integer satisfying $R_{k_0}(r_0)\neq 1$. Applying Lemma 17 with k_0 instead of k and r_0 instead of r, we obtain that a constant $\mathcal{P}_{r_0}\geq\mathcal{P}_1$ exists such that, for all prime numbers $p\geq\mathcal{P}_{r_0}$, we have

$$\begin{aligned}&\Phi_{\mathbf{L},p}(\mathbf{a}_p+pr_0\mathbf{1}_{k_0})\\&\quad\equiv-\sum_{j=1}^{r_0}H_{j\mathbf{L}^{(k_0)}}\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(j\mathbf{1}_{k_0})\mathcal{Q}((r_0-j)\mathbf{1}_{k_0})\big(R_{k_0}(j)-R_{k_0}(r_0-j)\big)\bmod p\mathbb{Z}_p\\&\quad\equiv-H_{r_0\mathbf{L}^{(k_0)}}\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(r_0\mathbf{1}_{k_0})\big(R_{k_0}(r_0)-1\big)\bmod p\mathbb{Z}_p,\end{aligned}\tag{7.5}$$

where (7.5) is valid because, for all $j\in\{1,\dots,r_0-1\}$, we have $R_{k_0}(j)=R_{k_0}(r_0-j)=1$. Since $R_{k_0}(r_0)\neq 1$, we obtain that if $\mathbf{L}^{(k_0)}\geq 1$, then there exists a constant $\mathcal{P}\geq\mathcal{P}_{r_0}$ such that, for all prime numbers $p\geq\mathcal{P}$, we have $H_{r_0\mathbf{L}^{(k_0)}}\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(r_0\mathbf{1}_{k_0})\left(R_{k_0}(r_0)-1\right)\in\mathbb{Z}_p^\times$ and therefore $\Phi_{\mathbf{L},p}(\mathbf{a}_p+pr_0\mathbf{1}_{k_0})\notin p\mathbb{Z}_p$, which completes the proof of assertion (ii) of Theorem 2 modulo the proof of Lemma 17.

Proof of Lemma 17. According to Section 4, for all prime numbers $p\geq\mathcal{P}_1$ and all $\mathbf{K}\in\mathbb{N}^d$, we have

$$\Phi_{\mathbf{L},p}(\mathbf{a}_p+p\mathbf{K})=\sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}}\mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}_p+p\mathbf{j})\left(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{j})}-pH_{\mathbf{L}\cdot(\mathbf{a}_p+p\mathbf{j})}\right).\tag{7.6}$$

Furthermore, we have $pH_{\mathbf{L}\cdot(\mathbf{a}_p+p\mathbf{j})}\equiv H_{\left\lfloor\frac{\mathbf{L}\cdot\mathbf{a}_p+p\mathbf{L}\cdot\mathbf{j}}{p}\right\rfloor}\bmod p\mathbb{Z}_p$ with $\left\lfloor\frac{\mathbf{L}\cdot\mathbf{a}_p+p\mathbf{L}\cdot\mathbf{j}}{p}\right\rfloor=\lfloor\mathbf{L}\cdot\mathbf{a}_p/p\rfloor+\mathbf{L}\cdot\mathbf{j}=\mathbf{L}\cdot\mathbf{j}$ because $\lfloor\mathbf{L}\cdot\mathbf{a}_p/p\rfloor=\lfloor\mathbf{L}\cdot\mathbf{x}_0\rfloor=0$. Thereby, for all $\mathbf{K},\mathbf{j}\in\mathbb{N}^d$, $\mathbf{j}\leq\mathbf{K}$, we obtain

$$\mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}_p+p\mathbf{j})pH_{\mathbf{L}\cdot(\mathbf{a}_p+p\mathbf{j})}\equiv\mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}_p+p\mathbf{j})H_{\mathbf{L}\cdot\mathbf{j}}\bmod p\mathbb{Z}_p.\tag{7.7}$$

Applying (7.7) to (7.6), we obtain that, for all $\mathbf{K}\in\mathbb{N}^d$, we have

$$\begin{aligned}&\Phi_{\mathbf{L},p}(\mathbf{a}_p+p\mathbf{K})\\&\quad\equiv\sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}}\mathcal{Q}(\mathbf{K}-\mathbf{j})\mathcal{Q}(\mathbf{a}_p+p\mathbf{j})\left(H_{\mathbf{L}\cdot(\mathbf{K}-\mathbf{j})}-H_{\mathbf{L}\cdot\mathbf{j}}\right)\bmod p\mathbb{Z}_p\\&\quad\equiv-\sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}}H_{\mathbf{L}\cdot\mathbf{j}}\big(\mathcal{Q}(\mathbf{a}_p+p\mathbf{j})\mathcal{Q}(\mathbf{K}-\mathbf{j})-\mathcal{Q}(\mathbf{j})\mathcal{Q}(\mathbf{a}_p+p(\mathbf{K}-\mathbf{j}))\big)\bmod p\mathbb{Z}_p\\&\quad\equiv-\sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{K}}H_{\mathbf{L}\cdot\mathbf{j}}\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(\mathbf{j})\mathcal{Q}(\mathbf{K}-\mathbf{j})\\&\qquad\times\left(\frac{\mathcal{Q}(\mathbf{a}_p+p\mathbf{j})}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(\mathbf{j})}-\frac{\mathcal{Q}(\mathbf{a}_p+p(\mathbf{K}-\mathbf{j}))}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(\mathbf{K}-\mathbf{j})}\right)\bmod p\mathbb{Z}_p.\end{aligned}\tag{7.8}$$

Applying (7.8) with $r\mathbf{1}_k$ instead of $\mathbf{K}$, we finally obtain

$$\begin{aligned}\Phi_{\mathbf{L},p}(\mathbf{a}_p+pr\mathbf{1}_k)\equiv&-\sum_{j=0}^{r}H_{j\mathbf{L}^{(k)}}\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(j\mathbf{1}_k)\mathcal{Q}((r-j)\mathbf{1}_k)\\&\times\left(\frac{\mathcal{Q}(\mathbf{a}_p+pj\mathbf{1}_k)}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(j\mathbf{1}_k)}-\frac{\mathcal{Q}(\mathbf{a}_p+p(r-j)\mathbf{1}_k)}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}((r-j)\mathbf{1}_k)}\right)\bmod p\mathbb{Z}_p.\end{aligned}\tag{7.9}$$

We will now prove that, for all $n \in \mathbb{N}$ and $k \in \{1, \ldots, d\}$, we have

$$\frac{\mathcal{Q}(\mathbf{a}_p + pn\mathbf{1}_k)}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(n\mathbf{1}_k)} = R_k(n)(1 + O(p)), \tag{7.10}$$

which will enable us to conclude the proof. We have

$$\frac{\mathcal{Q}(\mathbf{a}_p + pn\mathbf{1}_k)}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(n\mathbf{1}_k)} = \frac{\mathcal{Q}(\mathbf{a}_p + pn\mathbf{1}_k)}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(pn\mathbf{1}_k)} \frac{\mathcal{Q}(pn\mathbf{1}_k)}{\mathcal{Q}(n\mathbf{1}_k)}$$
$$= \frac{1}{\mathcal{Q}(\mathbf{a}_p)} \frac{\prod_{i=1}^{q_1} \prod_{j=1}^{\mathbf{e}_i \cdot \mathbf{a}_p} (pn\mathbf{e}_i^{(k)} + j)}{\prod_{i=1}^{q_2} \prod_{j=1}^{\mathbf{f}_i \cdot \mathbf{a}_p} (pn\mathbf{f}_i^{(k)} + j)} \big(1 + O(p)\big),$$

where we obtain the last equality by applying Lemma 7 with $s = 0$, $\mathbf{c} = \mathbf{0}$ and $n\mathbf{1}_k$ instead of $\mathbf{m}$, which leads to $\mathcal{Q}(pn\mathbf{1}_k)/\mathcal{Q}(n\mathbf{1}_k) = 1 + O(p)$. Thus we get

$$\frac{\mathcal{Q}(\mathbf{a}_p + pn\mathbf{1}_k)}{\mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(n\mathbf{1}_k)} = \frac{\prod_{i=1}^{q_1} \prod_{j=1}^{\mathbf{e}_i \cdot \mathbf{a}_p} \left(1 + \frac{pn\mathbf{e}_i^{(k)}}{j}\right)}{\prod_{i=1}^{q_2} \prod_{j=1}^{\mathbf{f}_i \cdot \mathbf{a}_p} \left(1 + \frac{pn\mathbf{f}_i^{(k)}}{j}\right)} \big(1 + O(p)\big)$$
$$= \frac{\prod_{i=1}^{q_1} \prod_{j=1}^{\lfloor \mathbf{e}_i \cdot \mathbf{a}_p/p \rfloor} \left(1 + \frac{\mathbf{e}_i^{(k)}}{j} n\right)}{\prod_{i=1}^{q_2} \prod_{j=1}^{\lfloor \mathbf{f}_i \cdot \mathbf{a}_p/p \rfloor} \left(1 + \frac{\mathbf{f}_i^{(k)}}{j} n\right)} \big(1 + O(p)\big) = R_k(n)\big(1 + O(p)\big), \tag{7.11}$$

where Eq. (7.11) is valid because, for all $\mathbf{d} \in \{\mathbf{e}_1, \ldots, \mathbf{e}_{q_1}, \mathbf{f}_1, \ldots, \mathbf{f}_{q_2}\}$ and $j \in \{1, \ldots, \mathbf{d} \cdot \mathbf{a}_p\}$, if j is not divisible by p then we have $1 + \frac{pn\mathbf{d}^{(k)}}{j} = 1 + O(p)$.

There exists a constant $\mathcal{P}_r \geq \mathcal{P}_1$ such that, for all prime numbers $p \geq \mathcal{P}_r$ and all $n \in \{0, \ldots, r\}$, we have $R_k(n) \in \mathbb{Z}_p^\times$ and $H_{n\mathbf{L}^{(k)}} \in \mathbb{Z}_p$. Therefore, applying (7.10) to (7.9), we obtain that, for all prime numbers $p \geq \mathcal{P}_r$, we have

$$\Phi_{\mathbf{L},p}(\mathbf{a}_p + pr\mathbf{1}_k) \equiv -\sum_{j=1}^{r} H_{j\mathbf{L}^{(k)}} \mathcal{Q}(\mathbf{a}_p)\mathcal{Q}(j\mathbf{1}_k)\mathcal{Q}((r-j)\mathbf{1}_k)$$
$$\times (R_k(j) - R_k(r-j)) \mod p\mathbb{Z}_p,$$

which finishes the proof of Lemma 17. □

8. Proof of Theorem 3

We assume the hypothesis of Theorem 3. The aim of this section is to prove that there are only finitely many prime numbers p such that $q_{e,f,k}(\mathbf{z}) \in z_k\mathbb{Z}_p[[\mathbf{z}]]$ and that, for all $\mathbf{L} \in \mathcal{E}_{e,f}$ satisfying $\mathbf{L}^{(k)} \geq 1$, there are only finitely many prime numbers p such that $q_{\mathbf{L},e,f}(\mathbf{z}) \in \mathbb{Z}_p[[\mathbf{z}]]$. We fix a $\mathbf{L} \in \mathcal{E}_{e,f}$ satisfying $\mathbf{L}^{(k)} \geq 1$ throughout this section.

According to Section 4, we only have to prove that, for all large enough prime numbers p, there exist $\mathbf{a} \in \{0, \ldots, p-1\}^d$ and $\mathbf{K} \in \mathbb{N}^d$ such that $\Phi_{p,k}(\mathbf{a} + \mathbf{K}p) \notin p\mathbb{Z}_p$ and $\Phi_{\mathbf{L},p}(\mathbf{a} + \mathbf{K}p) \notin p\mathbb{Z}_p$. In fact, we will prove that, for all large enough prime numbers p, we have

$\Phi_{p,k}(p\mathbf{1}_k) \notin p\mathbb{Z}_p$ and $\Phi_{\mathbf{L},p}(p\mathbf{1}_k) \notin p\mathbb{Z}_p$. We have

$$\begin{aligned}\Phi_{p,k}(p\mathbf{1}_k) &= \sum_{j=0}^{1} \mathcal{Q}((1-j)\mathbf{1}_k)\mathcal{Q}(jp\mathbf{1}_k)\Bigg(\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)}(H_{\mathbf{e}_i^{(k)}(1-j)} - pH_{\mathbf{e}_i^{(k)}jp}) \\ &\quad - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)}(H_{\mathbf{f}_i^{(k)}(1-j)} - pH_{\mathbf{f}_i^{(k)}jp})\Bigg) \\ &= \mathcal{Q}(\mathbf{1}_k)\left(\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)} H_{\mathbf{e}_i^{(k)}} - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)} H_{\mathbf{f}_i^{(k)}}\right) \\ &\quad - p\mathcal{Q}(p\mathbf{1}_k)\left(\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)} H_{\mathbf{e}_i^{(k)}p} - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)} H_{\mathbf{f}_i^{(k)}p}\right) \end{aligned} \tag{8.1}$$

and

$$\begin{aligned}\Phi_{\mathbf{L},p}(p\mathbf{1}_k) &= \sum_{j=0}^{1} \mathcal{Q}((1-j)\mathbf{1}_k)\mathcal{Q}(jp\mathbf{1}_k)(H_{\mathbf{L}^{(k)}(1-j)} - pH_{\mathbf{L}^{(k)}jp}) \\ &= \mathcal{Q}(\mathbf{1}_k)H_{\mathbf{L}^{(k)}} - p\mathcal{Q}(p\mathbf{1}_k)H_{\mathbf{L}^{(k)}p}. \end{aligned} \tag{8.2}$$

There exists a constant $\mathcal{P}_1$ such that, for all prime numbers $p \geq \mathcal{P}_1$, we have

$$\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)} H_{\mathbf{e}_i^{(k)}} - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)} H_{\mathbf{f}_i^{(k)}} \in \mathbb{Z}_p^\times \cup \{0\}$$

and $H_{\mathbf{L}^{(k)}} \in \mathbb{Z}_p^\times$ because $\mathbf{L}^{(k)} \geq 1$. In the sequel, we write Δ_k for Landau's function associated with sequences $e^{(k)} := (\mathbf{e}_1^{(k)}, \ldots, \mathbf{e}_{q_1}^{(k)})$ and $f^{(k)} := (\mathbf{f}_1^{(k)}, \ldots, \mathbf{f}_{q_2}^{(k)})$. We also write M for the largest element of sequences $e^{(k)}$ and $f^{(k)}$. We note that M is nonzero because $|e|^{(k)} > |f|^{(k)}$, and that Δ_k vanishes on $[0, 1/M[$. If $p > M$, then, for all $\ell \geq 1$, we have $1/p^\ell < 1/M$ and thus $v_p(\mathcal{Q}(\mathbf{1}_k)) = \sum_{\ell=1}^{\infty} \Delta_{e,f}(\mathbf{1}_k/p^\ell) = \sum_{\ell=1}^{\infty} \Delta_k(1/p^\ell) = 0$. Hence, for all prime numbers $p > \max(\mathcal{P}_1, M) =: \mathcal{P}_2$, we have

$$\mathcal{Q}(\mathbf{1}_k)\left(\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)} H_{\mathbf{e}_i^{(k)}} - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)} H_{\mathbf{f}_i^{(k)}}\right) \in \mathbb{Z}_p^\times \cup \{0\} \quad \text{and} \quad \mathcal{Q}(\mathbf{1}_k)H_{\mathbf{L}^{(k)}} \in \mathbb{Z}_p^\times. \tag{8.3}$$

Furthermore, we have $pH_{\mathbf{L}^{(k)}p} \equiv H_{\mathbf{L}^{(k)}} \mod p\mathbb{Z}_p$, which gives us that, for all prime numbers $p > \mathcal{P}_2$, we have $pH_{\mathbf{L}^{(k)}p} \in \mathbb{Z}_p^\times$. Similarly, we get

$$p\left(\sum_{i=1}^{q_1} \mathbf{e}_i^{(k)} H_{\mathbf{e}_i^{(k)}p} - \sum_{i=1}^{q_2} \mathbf{f}_i^{(k)} H_{\mathbf{f}_i^{(k)}p}\right) \in \mathbb{Z}_p.$$

Finally, for all prime numbers $p > \mathcal{P}_2$, we have

$$\begin{aligned} v_p(\mathcal{Q}(p\mathbf{1}_k)) &= \sum_{\ell=1}^{\infty} \Delta_{e,f}\left(\frac{p\mathbf{1}_k}{p^\ell}\right) = \sum_{\ell=1}^{\infty} \Delta_k\left(\frac{p}{p^\ell}\right) \\ &= \Delta_k(1) + \sum_{\ell=1}^{\infty} \Delta_k\left(\frac{1}{p^\ell}\right) = |\mathbf{e}|^{(k)} - |\mathbf{f}|^{(k)} \geq 1, \end{aligned}$$

from which we obtain that, for all prime numbers $p > \mathcal{P}_2$, we have

$$pQ(p\mathbf{1}_k)\left(\sum_{i=1}^{q_1}\mathbf{e}_i^{(k)}H_{\mathbf{e}_i^{(k)}p}-\sum_{i=1}^{q_2}\mathbf{f}_i^{(k)}H_{\mathbf{f}_i^{(k)}p}\right)\in p\mathbb{Z}_p \quad\text{and}\quad pQ(p\mathbf{1}_k)H_{\mathbf{L}^{(k)}p}\in p\mathbb{Z}_p. \tag{8.4}$$

Applying (8.3) and (8.4) to (8.2), we obtain that, for all prime numbers $p > \mathcal{P}_2$, we have $\Phi_{\mathbf{L},p}(p\mathbf{1}_k) \notin p\mathbb{Z}_p$.

Congruences (8.3) and (8.4) in combination with (8.1) prove that it suffices to prove that $\sum_{i=1}^{q_1}\mathbf{e}_i^{(k)}H_{\mathbf{e}_i^{(k)}}-\sum_{i=1}^{q_2}\mathbf{f}_i^{(k)}H_{\mathbf{f}_i^{(k)}} \neq 0$ to conclude that, for all prime numbers $p > \mathcal{P}_2$, we have $\Phi_{p,k}(p\mathbf{1}_k) \notin p\mathbb{Z}_p$.

For this purpose, we write $\mathbf{E}$ and $\mathbf{F}$ for the respective subsequences of $e^{(k)}$ and $f^{(k)}$ obtained as follows. We remove the zero elements of $e^{(k)}$ and $f^{(k)}$ and, if $e^{(k)}$ and $f^{(k)}$ have an element in common, then we remove it from $e^{(k)}$ and $f^{(k)}$ once only. This last step is repeated until the obtained sequences are disjoint. The sequence $\mathbf{F}$ can be empty but the hypothesis $|e|^{(k)} > |f|^{(k)}$ ensures that the sequence $\mathbf{E}$ is nonempty. Thus we have

$$\sum_{i=1}^{q_1}\mathbf{e}_i^{(k)}H_{\mathbf{e}_i^{(k)}}-\sum_{i=1}^{q_2}\mathbf{f}_i^{(k)}H_{\mathbf{f}_i^{(k)}}=\sum_{c\in\mathbf{E}}cH_c-\sum_{d\in\mathbf{F}}dH_d \quad\text{and}\quad \Delta_k=\Delta_{\mathbf{E},\mathbf{F}}. \tag{8.5}$$

In particular, if $\mathbf{F}$ is empty then we have $\sum_{i=1}^{q_1}\mathbf{e}_i^{(k)}H_{\mathbf{e}_i^{(k)}}-\sum_{i=1}^{q_2}\mathbf{f}_i^{(k)}H_{\mathbf{f}_i^{(k)}}=\sum_{c\in\mathbf{E}}cH_c > 0$. In the sequel of the proof, we assume that $\mathbf{F}$ is nonempty.

Since $\mathbf{E}$ and $\mathbf{F}$ are two disjoint sequences of positive integers, we can apply Lemma 16 to the sequences $\mathbf{E}$ and $\mathbf{F}$. Using the notations of Lemma 16, we obtain

$$\sum_{c\in\mathbf{E}}cH_c-\sum_{d\in\mathbf{F}}dH_d=\sum_{c\in\mathbf{E}}\sum_{j=1}^{c}\frac{c}{j}-\sum_{d\in\mathbf{F}}\sum_{j=1}^{d}\frac{d}{j}=\sum_{i=1}^{t}\frac{m_i}{\gamma_i}. \tag{8.6}$$

Furthermore, for all $x \in \mathbb{R}$, we have $\Delta_{\mathbf{E},\mathbf{F}}(x) = \Delta_k(x) = \Delta_{e,f}(x\mathbf{1}_k) \geq 0$ so $\Delta_{\mathbf{E},\mathbf{F}} \geq 0$ on $[\gamma_1, \gamma_t]$ and Lemma 16 leads to $\sum_{i=1}^{t}\frac{m_i}{\gamma_i} > 0$. This inequality in combination with (8.5) and (8.6) proves that $\sum_{i=1}^{q_1}\mathbf{e}_i^{(k)}H_{\mathbf{e}_i^{(k)}}-\sum_{i=1}^{q_2}\mathbf{f}_i^{(k)}H_{\mathbf{f}_i^{(k)}} \neq 0$ and completes the proof of Theorem 3.

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