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Nonuniform wavelets and wavelet sets related to one-dimensional spectral pairs

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Abstract

A generalization of Mallat's classical multiresolution analysis, based on the theory of spectral pairs, was considered in two articles by Gabardo and Nashed. In this setting, the associated translation set is no longer a discrete subgroup of $\mathbb{R}$ but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. As a generalization of Dai, Larson, and Speegle's theory of wavelet sets, we prove in this paper the existence of nonuniform wavelet sets associated with the same translation and dilation parameters.

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1. Introduction

In the two papers [7,8], Gabardo and Nashed considered a generalization of Mallat's [18] celebrated theory of multiresolution analysis (MRA), in which the translation set acting on the scaling function associated with the MRA to generate the subspace V_0 is no longer a group (or a translate of a group), but is the union of $\mathbb{Z}$ and a translate of $\mathbb{Z}$. Such constructions were called

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non-uniform multiresolution analysis (NUMRA). In this theory, the translation set Λ is chosen so that for some measurable set $A \subset \mathbb{R}$ with $0 < |A| < \infty$, (A, Λ) forms a spectral pair, i.e. the collection $\{|A|^{-1/2} e^{2\pi i \xi \cdot \lambda} \chi_A(\xi)\}_{\lambda \in \Lambda}$ forms an orthonormal basis for $L^2(A)$, where $|A|$ and $\chi_A(\xi)$ denote, respectively, the Lebesgue measure and the characteristic function of A. If $N \geqslant 1$ is an integer, we define the set $\Gamma_N = \{mN + j/2 : m \in \mathbb{Z}, j = 0, 1, \ldots, N-1\}$ and, if $r \in \{1, 3, \ldots, 2N-1\}$ is any odd integer, the set $\Lambda_{r,N} = \{0, r/N\} + 2\mathbb{Z}$.

The following theorem and its corollary characterize certain one-dimensional classes of spectral pairs.

Theorem 1.1 (*Gabardo and Nashed [7]*). *Let $\Lambda = \{0, a\} + 2\mathbb{Z}$, where $0 < a < 2$ and let A be a measurable subset of $\mathbb{R}$ with $0 < |A| < \infty$. Then (A, Λ) is a spectral pair if and only if there exist an integer $N \geqslant 1$ and an odd integer r, with $1 \leqslant r \leqslant 2N-1$ and r and N relatively prime, such that $a = r/N$, and*

$$\sum_{j=0}^{N-1} \delta_{j/2} * \sum_{n \in \mathbb{Z}} \delta_{nN} * \chi_A = 1, \tag{1.1}$$

where $*$ denotes the usual convolution product of Schwartz distributions and δ_c is the Dirac measure at c. In this case $|A| = 1$.

Corollary 1.2 (*Yu [25]*). *Let $N \geqslant 1$ be an integer and A be a measurable subset of $\mathbb{R}$ with $0 < |A| < \infty$. If the tiling equation* (1.1) *is true, then $(A, \Lambda_{r,N})$ is a spectral pair for each odd $r \in \{1, 3, \ldots, 2N-1\}$ (not necessarily prime with N) and, furthermore, $|A| = 1$.*

Additional results on spectral pairs can be found in [6–9,13–17,19–21,24]. We now define nonuniform wavelets associated with the translation sets $\Lambda_{r,N}$ and the dilation $2N$.

Definition 1.3. Let $N \geqslant 1$ be a positive integer, and $1 \leqslant r \leqslant 2N-1$ be a fixed odd integer. A collection of functions $\Psi = \{\psi^k : k = 1, \ldots, K\} \subset L^2(\mathbb{R})$ is called a set of wavelets (associated with the dilation $2N$ and the translation set $\Lambda_{r,N}$) if the family $\{\psi^k_{j,\lambda} : k = 1, \ldots, K, j \in \mathbb{Z}, \lambda \in \Lambda_{r,N}\}$ forms a complete orthonormal system for $L^2(\mathbb{R})$, where $\psi^k_{j,\lambda}(x) = (2N)^{\frac{j}{2}} \psi^k((2N)^j x - \lambda)$.

The characterization of nonuniform wavelets associated with a NUMRA was given in our recent paper [9]. Our main goal in this note is to characterize nonuniform wavelet sets (see Definition 2.3) and to prove their existence for any values of the parameters r and N with r odd and $1 \leqslant r \leqslant 2N-1$. We will also show that, for a fixed dilation $2N$, a wavelet set associated with the translation set $\Lambda_{r,N}$ for some odd integer r with $1 \leqslant r \leqslant 2N-1$ is also a wavelet set associated with the translation set $\Lambda_{r',N}$ for *any* odd integer r' with $1 \leqslant r' \leqslant 2N-1$. The Fourier transform will be defined by

$$\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}} e^{-2\pi i \xi x} f(x)\, dx \quad \text{for } f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). \tag{1.2}$$

2. Nonuniform wavelets and wavelet sets

The first part of the following lemma is a result of [7](see Eqs. (3.3) and (3.4)).

Lemma 2.1. *Let $N \geqslant 1$ be a positive integer, and $r \in \{1, 3, \ldots, 2N-1\}$ be an odd integer. Let $\phi \in L^2(\mathbb{R})$ with $\|\phi\|_{L^2} = 1$. Then,*

(i) *For a given odd r, the collection $\{\phi(x-\lambda)\}_{\lambda \in \Lambda_{r,N}}$ is an orthonormal system in $L^2(\mathbb{R})$ if and only if*

$$\sum_{p \in \mathbb{Z}} \left| \hat{\phi}\left(\xi + \frac{p}{2}\right) \right|^2 = 2 \quad \text{for a.e. } \xi \in \mathbb{R} \tag{2.1}$$

and

$$\sum_{p \in \mathbb{Z}} e^{-i\pi \frac{r}{N} p} \left| \hat{\phi}\left(\xi + \frac{p}{2}\right) \right|^2 = 0 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{2.2}$$

(ii) *The collection $\{\phi(x-\lambda)\}_{\lambda \in \Lambda_{r,N}}$ is an orthonormal system for every odd integer $r \in \{1, 3, \ldots, 2N-1\}$ if and only if*

$$\sum_{\gamma \in \Gamma_N} \left| \hat{\phi}(\xi - \gamma) \right|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{2.3}$$

Proof. The proof of (i) is given in [7, Lemma 3.2]. We will only prove (ii). By (i), the collection $\{\phi(x-\lambda)\}_{\lambda \in \Lambda_{r,N}}$ is an orthonormal system for every odd integer $r \in \{1, 3, \ldots, 2N-1\}$ if and only if (2.1) holds and (2.2) holds for every odd integer $r \in \{1, 3, \ldots, 2N-1\}$. Define, for a.e. $\xi \in \mathbb{R}$,

$$c_j(\xi) = \sum_{q \in \mathbb{Z}} \left| \hat{\phi}\left(\xi - qN - \frac{j}{2}\right) \right|^2, \quad 0 \leqslant j \leqslant 2N,$$

and note that $c_0(\xi) = c_{2N}(\xi)$. Suppose that (2.1) and (2.2) hold for every odd $r \in \{1, 3, \ldots, 2N-1\}$. By (2.1) we have

$$\sum_{j=0}^{2N-1} c_j(\xi) = \sum_{j=0}^{N-1} \left(c_j(\xi) + c_{N+j}(\xi)\right) = 2. \tag{2.4}$$

By (2.2), for every $r = 2s+1,\ s = 0, 1, \ldots, N-1$, we have

$$\begin{aligned} 0 &= \sum_{p \in \mathbb{Z}} e^{i\pi \frac{r}{N} p} \left| \hat{\phi}\left(\xi + \frac{p}{2}\right) \right|^2 = \sum_{q \in \mathbb{Z}} \sum_{j=0}^{2N-1} e^{-i\pi \frac{r}{N}(2qN+j)} \left| \hat{\phi}\left(\xi - qN - \frac{j}{2}\right) \right|^2 \\ &= \sum_{j=0}^{N-1} e^{-i\pi \frac{r}{N} j} \left(\sum_{q \in \mathbb{Z}} \left| \hat{\phi}\left(\xi - qN - \frac{j}{2}\right) \right|^2 - \sum_{q \in \mathbb{Z}} \left| \hat{\phi}\left(\xi - qN - \frac{j+N}{2}\right) \right|^2 \right) \\ &= \sum_{j=0}^{N-1} e^{-i2\pi \frac{s}{N} j} \left(c_j(\xi) - c_{N+j}(\xi)\right) e^{-i\pi \frac{j}{N}}. \end{aligned} \tag{2.5}$$

Considering the sum in the right-hand side of (2.5) as a discrete Fourier series on the group $\mathbb{Z}/N\mathbb{Z}$, we deduce that, for a.e. $\xi \in \mathbb{R}$, $c_j(\xi) = c_{N+j}(\xi)$, for $j = 0, 1, \ldots, N-1$. It follows from this last fact and (2.4) that, for a.e. $\xi \in \mathbb{R}$,

$$\sum_{j=0}^{N-1} c_j(\xi) = 1, \tag{2.6}$$

which is exactly (2.3). Conversely, suppose that (2.3) or, equivalently that (2.6) holds, for a.e. $\xi \in \mathbb{R}$. Replacing ξ by $\xi - \frac{N}{2}$ and $\xi - \frac{(N+1)}{2}$ in the previous equation, we obtain that

$$\sum_{j=N}^{2N-1} c_j(\xi) = \sum_{j=N+1}^{2N} c_j(\xi) = 1$$

and, in particular, $c_N(\xi) = c_{2N}(\xi) = c_0(\xi)$. Similarly, we see that $c_j(\xi) = c_{N+j}(\xi)$ for $j = 0, 1, \ldots, N-1$. Hence, (2.1) and (2.2) hold for each odd $r \in \{1, 3, \ldots, 2N-1\}$ by using (2.4) and (2.5). This completes the proof. □

The following theorem follows as a special case of Calogero's characterization of wavelets on general lattices with expansive matrix dilations ([2, Theorem 3.1]; see also [1]) applied to the function system $\Phi = \{\phi^k : k = 1, \ldots, 2K\} \subset L^2(\mathbb{R})$, where

$$\phi^k(\cdot) = \begin{cases} \psi^k(\cdot) & \text{for } 1 \leqslant k \leqslant K, \\ \psi^k\left(\cdot - \frac{r}{N}\right) & \text{for } K+1 \leqslant k \leqslant 2K, \end{cases}$$

together with the dilation $M = M^* = 2N$ and the translation lattice $\Sigma = 2\mathbb{Z}$ with dual lattice $\Sigma^* = \frac{1}{2}\mathbb{Z}$. This result can also be obtained from various more general results that have appeared recently in the literature characterizing certain normalized tight frame systems. In particular, it is also a special case of [4, Theorem 1; 3, Corollary 1; 22, Corollary 4.15 (ii)]. With quite a bit of extra work needed to verify the assumptions, this result can also be obtained from [11, Theorem 2.1].

Theorem 2.2. *Let $1 \leqslant r \leqslant 2N-1$ be a fixed odd integer. A collection of functions $\Psi = \{\psi^k : k = 1, \ldots, K\} \subset L^2(\mathbb{R})$ with $\|\psi^k\|_{L^2} = 1$, $k = 1, \ldots, K$, is a set of wavelets associated with the dilation $2N$ and the translation set $\Lambda_{r,N}$ if and only if for a.e. $\xi \in \mathbb{R}$,*

$$\sum_{k=1}^{K} \sum_{l \in \mathbb{Z}} \left| \hat{\psi}^k((2N)^l \xi) \right|^2 = 1, \tag{2.7}$$

and, for any $q \in \mathbb{Z} \setminus 2N\mathbb{Z}$,

$$\begin{aligned} t_q(\xi) = 2 \sum_{k=1}^{K} \sum_{l=1}^{\infty} \hat{\psi}^k\left((2N)^l \xi\right) \overline{\hat{\psi}^k\left((2N)^l (\xi + \frac{q}{2})\right)} \\ + (1 + e^{i\pi q \frac{r}{N}}) \sum_{k=1}^{K} \hat{\psi}^k(\xi) \overline{\hat{\psi}^k\left(\xi + \frac{q}{2}\right)} = 0. \end{aligned} \tag{2.8}$$

Note that, when $N = 1$, Eq. (2.8) reduces to

$$\sum_{k=1}^{K} \sum_{l=0}^{\infty} \hat{\psi}^k(2^l \xi) \overline{\hat{\psi}^k(2^l(\xi + q))} = 0, \quad q \in \mathbb{Z} \setminus 2\mathbb{Z},$$

and thus (2.7) and (2.8) give us the well-known necessary and sufficient conditions satisfied by a wavelet system in that case ([10,23]; see also [12, p. 332]).

Definition 2.3. Let $N \geqslant 1$ be a positive integer, and $r \in \{1, 3, \ldots, 2N-1\}$ be an odd integer. Let $A \subset \mathbb{R}$ be a measurable set with $|A| = 1$. We call A a wavelet set associated with the dilation $2N$ and translation set $\Lambda_{r,N}$ if $\mathcal{F}^{-1}(\chi_A(\xi))$ is an orthonormal wavelet in the sense of Definition 1.3 (with $K = 1$), where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Lemma 2.4. *Let $A \subset \mathbb{R}$ be a measurable set with finite measure, and $\psi(x)$ be an $L^2(\mathbb{R})$ function with $|\hat{\psi}(\xi)| = \chi_A(\xi)$. If ψ satisfies* (2.1) *and* (2.2) *for a fixed odd integer $r \in \{1, 3, \ldots, 2N-1\}$, then it also satisfies* (2.8) (*with $K = 1$ and $\psi^1 = \psi$*) *for the same r.*

Proof. Fix any $q \in \mathbb{Z} \setminus 2N\mathbb{Z}$. For almost every $\xi \in \mathbb{R}$, if $\hat{\psi}((2N)^l \xi) \neq 0$ for a fixed $l \geqslant 0$, then $|\hat{\psi}((2N)^l \xi)| = \chi_A((2N)^l \xi) = 1$, and thus, by (2.1),

$$\sum_{p \in \mathbb{Z} \setminus \{0\}} |\hat{\psi}((2N)^l \xi + p/2)|^2 = 1.$$

There exists an integer $p^{(l)}(\xi) \neq 0$ such that $|\hat{\psi}((2N)^l \xi + p^{(l)}(\xi)/2)| = 1$ and

$$\hat{\psi}((2N)^l \xi + p/2) = 0, \quad p \in \mathbb{Z} \setminus \{0, p^{(l)}(\xi)\}. \tag{2.9}$$

These equalities and (2.2) yield $e^{-i\pi \frac{r}{N} p^{(l)}(\xi)} = -1$. So $p^{(l)}(\xi) \frac{r}{N} \in 2\mathbb{Z} + 1$ and $p^{(l)}(\xi) \notin 2N\mathbb{Z}$. It follows from (2.9) that in the case $l \geqslant 1$

$$\hat{\psi}\left((2N)^l \left(\xi + \frac{q}{2}\right)\right) = \hat{\psi}\left((2N)^l \xi + \frac{1}{2}(2N)^l q\right) = 0,$$

since $p^{(l)}(\xi) \neq (2N)^l q$. Therefore, one has

$$\sum_{l=1}^{\infty} \hat{\psi}((2N)^l \xi) \overline{\hat{\psi}\left((2N)^l \left(\xi + \frac{q}{2}\right)\right)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}. \tag{2.10}$$

If $\hat{\psi}(\xi) \neq 0$ then, whether or not $q = p^{(0)}(\xi)$, the previous argument with $l = 0$ shows that $\hat{\psi}(\xi + \frac{q}{2})(1 + e^{i\pi \frac{r}{N} q}) = 0$. This together with (2.10) proves the equality (2.8) with $K = 1$. $\square$

Lemma 2.5. *Let $A \subset \mathbb{R}$ be a measurable set with finite measure. Then, the following two statements are equivalent*

(A) *For a.e. $\xi \in \mathbb{R}$, one has*

$$\sum_{p \in \mathbb{Z}} \chi_A\left(\xi + \frac{p}{2}\right) = 2 \quad \text{and} \quad \sum_{p \in \mathbb{Z}} e^{i\pi \frac{r}{N} p} \chi_A\left(\xi + \frac{p}{2}\right) = 0,$$

where r is a fixed odd integer coprime to N with $1 \leqslant r \leqslant 2N-1$.

(B) *For a.e. $\xi \in \mathbb{R}$, one has $\sum_{\gamma \in \Gamma_N} \chi_A(\xi - \gamma) = 1$.*

Proof. The implication (B)$\Rightarrow$(A) is an immediate consequence of Lemma 2.1 with $\hat{\phi} = \chi_A$. Conversely, assume that (A) holds. For a.e. $\xi \in \mathbb{R}$ there exist two distinct integers $p_1 = p_1(\xi)$

and $p_2 = p_2(\xi)$ such that

$$\xi + \frac{p_1}{2} \in A, \quad \xi + \frac{p_2}{2} \in A \quad \text{and} \quad \xi + \frac{p}{2} \notin A,\ p \in \mathbb{Z} \setminus \{p_1, p_2\}$$

by the first equality in (A). Then $e^{i\pi \frac{r}{N} p_1} + e^{i\pi \frac{r}{N} p_2} = 0$ by the second equality in (A). It follows that $p_1 - p_2 = (2l+1)N$ for some $l \in \mathbb{Z}$ since r and N are relatively prime. Write $-p_2 = 2Nm + j$ with $m \in \mathbb{Z}$ and $0 \leqslant j \leqslant 2N-1$. Then $\xi \in A - \frac{p_2}{2} = A + mN + \frac{j}{2}$, and $\xi \in A - \frac{p_1}{2} = A + (m-l)N + \frac{j-N}{2}$. Thus, if $N \leqslant j \leqslant 2N-1$, then $0 \leqslant j - N \leqslant N-1$. If $0 \leqslant j \leqslant N-1$, then $N + \frac{j-N}{2} = \frac{j+N}{2}$ and $N \leqslant j + N \leqslant 2N-1$. This means that one and only one of $A - \frac{p_2}{2}$ and $A - \frac{p_1}{2}$ must be a component of the union $\bigcup_{\gamma \in \Gamma_N} (A + \gamma)$. This last fact and the arbitrariness of ξ show that $\bigcup_{\gamma \in \Gamma_N} (A + \gamma) = \mathbb{R}$ where the union is disjoint, which is equivalent to (B). □

The following theorem provides a characterization for nonuniform wavelet sets and proves their existence. The corresponding result for the uniform case, which is also valid in higher dimensions and for arbitrary dilations, is due to Dai, Larson and Speegle ([5]).

Theorem 2.6. *Let $A \subset \mathbb{R}$ be a measurable set with $|A| = 1$. Then, the following three statements are equivalent.*

(A) *$\mathcal{F}^{-1}(\chi_A(\xi))$ is a wavelet associated with the dilation $2N$ and the translation set $\Lambda_{r,N}$ for one particular odd r prime to N with $1 \leqslant r \leqslant 2N-1$.*

(B) (i) $\bigcup_{l \in \mathbb{Z}} (2N)^l A = \mathbb{R}$ *and* (ii) $\bigcup_{\gamma \in \Gamma_N} (A + \gamma) = \mathbb{R}$, *where both unions are disjoint almost everywhere.*

(C) *For every odd integer $r \in \{1, 3, \ldots, 2N-1\}$, $\mathcal{F}^{-1}(\chi_A(\xi))$ is a wavelet associated with the dilation $2N$ and the translation set $\Lambda_{r,N}$.*

Moreover, a measurable set A satisfying the three previous equivalent statements always exists.

Proof. First of all, it is clear that when $\hat{\psi}(\xi) = \chi_A(\xi)$, the statement (B), (i) is equivalent to (2.7) with $K = 1$ while (B),(ii) is equivalent to (2.3) when ϕ is replaced by ψ.

If (A) holds, part (i) of (B) is a consequence of Theorem 2.2, while part (ii) of (B) follows from Lemmas 2.1 and 2.5. On the other hand, if (B) holds, (C) follows from Lemmas 2.1, 2.4, and Theorem 2.2. Since (C) obviously implies (A), this proves our claim.

We now prove the existence of such a set A. Consider the set

$$E = \left[-\frac{N}{2}, -\frac{N}{2} + \frac{1}{4}\right) \cup \left[-\frac{1}{4}, \frac{1}{4}\right) \cup \left[\frac{N}{2} - \frac{1}{4}, \frac{N}{2}\right).$$

Note that E satisfies the tiling condition

$$\sum_{m \in \mathbb{Z}} \sum_{j=0}^{N-1} \chi_E * \delta_{mN} * \delta_{j/2} = 1. \tag{2.11}$$

Let $F = [-2N, -1) \cup [1, 2N)$ and note that

$$\sum_{l \in \mathbb{Z}} \chi_{(2N)^l F} = 1. \tag{2.12}$$

Since 0 is an interior point of E and F is bounded away from 0 and has nonempty interior, we can use a key result of Dai, Larson and Speegle in [5] to construct a measurable set A which is both

$N\mathbb{Z}$-translation congruent to E and $2N$-dilation congruent to F. Equivalently, A satisfies

$$\sum_{m\in\mathbb{Z}} \chi_A * \delta_{mN} = \sum_{m\in\mathbb{Z}} \chi_E * \delta_{mN} \tag{2.13}$$

and

$$\sum_{l\in\mathbb{Z}} \chi_{(2N)^l A} = \sum_{l\in\mathbb{Z}} \chi_{(2N)^l F}. \tag{2.14}$$

It follows immediately from (2.11), (2.12), (2.13) and (2.14) that the conditions in (B) are all satisfied by set A, and this completes the proof. □

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