Journal of Complexity 25 (2009) 343–361

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

An optimization of Chebyshev's method

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A R T I C L E   I N F O

Article history:
Received 15 October 2008
Accepted 1 April 2009
Available online 8 April 2009

Keywords:
Nonlinear equations in Banach spaces
Newton's method
Chebyshev's method
Multipoint iteration
Semilocal convergence
R-order of convergence
Region of accessibility
Attraction basin

A B S T R A C T

From Chebyshev's method, new third-order multipoint iterations are constructed with their efficiency close to that of Newton's method and the same region of accessibility.

1. Introduction

In 1669, more than three centuries ago, Isaac Newton described what is now called Newton's method for finding numerical and algebraic solutions. The main features of the method are the simplicity of its principle (based on linear approximation) and its efficiency. Newton explained his method through the use of numerical examples and did not use the iterative expression that is currently used. This latter was developed by Raphson in 1690 [1]. The method is now called Newton's method or the Newton–Raphson method.

Nowadays, Newton's method goes on being the most used one-point iterative method for approximating numerical solutions of nonlinear equations. This is due to the relation between several factors, such as the number of necessary values of the function involved and its derivatives, the computational cost and the speed of convergence (quadratic convergence).

When it comes to choosing a one-point iterative method for solving nonlinear equations, we have in particular to take into account the speed of convergence and the computational cost. To do this, we can use the efficiency index of an iterative method, which is a measure of its efficiency and is defined

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0885-064X/$ – see front matter
doi:10.1016/j.jco.2009.04.001

by the order of convergence to the inverse power of the number of computations of the function involved and its derivatives [2]. In particular, this index is usually considered in the analysis of scalar equations, where the computational costs of the successive derivatives are not very different.

For one-point iterative methods, it is known that the order of convergence is a natural number. Moreover, one-point iterations of the form $x_{n+1} = G(x_n)$, $n \geq 0$, with order of convergence q, depend explicitly on the first $q-1$ derivatives of F. This implies that their efficiency index is $EI = q^{1/q}$, $q \in \mathbb{N}$. The best situation for this index is then obtained when $q = 3$, so third-order iterative methods are frequently used to solve nonlinear scalar equations.

In this paper, we are interested in constructing, from the third-order Chebyshev method, some multipoint iterations with cubical convergence and a better efficiency index than Newton's method in the scalar case, taking into account that their extension to nonlinear systems or Banach spaces does not have the negative effects of third-order one-point iterative methods: the significant increase of computational cost and the reduction in the region of accessibility, which consists of every starting point from which iterative methods are convergent.

In Section 2, we study the efficiency of Newton's method when it is used to solve nonlinear systems, compare it with other iterative methods of order of convergence 3 and conclude that its application is a better choice. This conclusion is deduced from the analysis of the following three points: the evaluation of the function involved and its derivatives, the computational cost and the region of accessibility. After that, from a modification of the technique presented in [3], we construct in Section 3, from Chebyshev's method, third-order multipoint iterative methods with efficiency close to the efficiency of Newton's method. In Section 4, we establish the convergence in Banach spaces of the iterations constructed previously, so that a further generalization is then given. We present a local convergence result, where the cubical convergence of the iterations is proved, and a semilocal convergence result under conditions of Newton-Kantorovich type [4]. Once we have obtained iterations with efficiency close to that of Newton's method, we devote our attention to the region of accessibility. So, we provide in Section 5 a family of hybrid iterative methods [5] which combines Newton's method as a predictor with the multipoint iterations constructed in the last section as correctors. These iterations have the advantage of having the same region of accessibility as Newton's method. Related to this idea, in [6], Argyros is mixing the modified Newton method with Newton's method to expand the applicability of Newton's method.

Finally, in Section 6, we give a practical result and illustrate how the new iterations can be used to solve the following nonlinear integral equation of mixed Hammerstein type:

$$x(s) = 1 + \frac{1}{2}\int_0^1 G(s,t)\,x(t)^2\,\mathrm{d}t, \quad s \in [0,1], \tag{1}$$

where $x \in C[0,1]$, $t \in [0,1]$, and the kernel G is $G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$ Solving a nonlinear integral equation of mixed Hammerstein type is illustrated using the dynamic model of a chemical reactor (see [7]).

2. Preliminary analysis

When finding successive approximations to the numerical solution of an equation $F(x) = 0$, Newton's method is also called the tangent method and the successive approximations are given by the recurrence formula,

$$\begin{cases} x_0 \text{ given}, \\ F'(x_n)\,\delta_n = -F(x_n), \quad n \geq 0, \\ x_{n+1} = x_n + \delta_n. \end{cases} \tag{2}$$

In the scalar case, it is known that the efficiency index of iterative methods is $EI = q^{1/d}$, where q is the order of convergence and d the number of new computations of F and its derivatives per iteration, and represents a good measure of the efficiency of the iterative method, [2].

For one-point iterative methods of order d, there is imposed in [2] the restriction of depending explicitly on the first $d-1$ derivatives of F. Moreover, for these kinds of methods, we know that

$d = q$ ($q \in \mathbb{N}$) and $EI = q^{1/q}$, so the best situation is obtained for $q = 3$, namely, for third-order one-point iterative methods. The best known one-point iterative methods are Chebyshev's method [8], Halley's method [9] and the super-Halley method [10]. However, for nonlinear systems, third-order methods are not considered as the most favourable; rather Newton's method is, although its efficiency index $EI = 2^{1/2}$ is worse. This is due to the fact that the efficiency index does not consider other determinants.

For example, if we consider the case of solving nonlinear systems of dimension n, $F(x_1, x_2, \ldots, x_n) = 0$, where $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function and $F \equiv (F_1, F_2, \ldots, F_n)$ with $F_i : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, it is necessary to compute the n functions F_i $(i = 1, 2, \ldots, n)$ for computing F. Moreover, for $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the computation of F',

$$F'(\mathbf{x}) = \begin{pmatrix} (F_1)_1(\mathbf{x}) & (F_1)_2(\mathbf{x}) & \cdots & (F_1)_n(\mathbf{x}) \\ (F_2)_1(\mathbf{x}) & (F_2)_2(\mathbf{x}) & \cdots & (F_2)_n(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ (F_n)_1(\mathbf{x}) & (F_n)_2(\mathbf{x}) & \cdots & (F_n)_n(\mathbf{x}) \end{pmatrix},$$

requires the computations of the n^2 partial derivatives of first order, and the computation of F'',

$$F''(\mathbf{x}) = \left(\begin{matrix} (F_1)_{11}(\mathbf{x}) & (F_1)_{12}(\mathbf{x}) & \cdots & (F_1)_{1n}(\mathbf{x}) \\ (F_1)_{21}(\mathbf{x}) & (F_1)_{22}(\mathbf{x}) & \cdots & (F_1)_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ (F_1)_{n1}(\mathbf{x}) & (F_1)_{n2}(\mathbf{x}) & \cdots & (F_1)_{nn}(\mathbf{x}) \end{matrix} \right| \cdots \left| \begin{matrix} (F_n)_{11}(\mathbf{x}) & (F_n)_{12}(\mathbf{x}) & \cdots & (F_n)_{1n}(\mathbf{x}) \\ (F_n)_{21}(\mathbf{x}) & (F_n)_{22}(\mathbf{x}) & \cdots & (F_n)_{2n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ (F_n)_{n1}(\mathbf{x}) & (F_n)_{n2}(\mathbf{x}) & \cdots & (F_n)_{nn}(\mathbf{x}) \end{matrix} \right),$$

requires the computations of the $n^2(n + 1)/2$ partial derivatives of second order. In addition, the application of Newton's method to solve the nonlinear system of n equations

$$\begin{cases} F_1(x_1, x_2, \ldots, x_n) = 0, \\ F_2(x_1, x_2, \ldots, x_n) = 0, \\ \qquad\qquad \vdots \\ F_n(x_1, x_2, \ldots, x_n) = 0, \end{cases} \tag{3}$$

requires $n^2 + n$ evaluations of functions per iteration, whereas a one-point third-order method, for example Chebyshev's method (which is possibly the most used, since its algorithm is the simplest),

$$\begin{cases} x_0 \text{ given}, \\ F'(x_n)\,\delta_n = -F(x_n), \quad n \ge 0, \\ F'(x_n)\,\gamma_n = (-1/2)\,F''(x_n)\,\delta_n^2, \\ x_{n+1} = x_n + \delta_n + \gamma_n, \end{cases}$$

requires $n^2(n + 1)/2$ evaluations of functions per iteration more than Newton's method. Therefore, for solving (3) with $n \ge 2$, it is better to use Newton's method than Chebyshev's method; see Fig. 1.

Another important point to bear in mind when choosing an iterative method is the number of operations (products and divisions) needed to apply it, which we define in this paper as the computational cost of doing an iteration of the algorithm. So, Newton's method requires $(n^3 + 6n^2 - 4n)/3$ operations to do an iteration (see (2)), whereas Chebyshev's method requires us to do the same operations plus obtaining the products $(-1/2)\,F''(x_n)\,\delta_n^2$ ($n^3 + n^2 + n$ operations) and the solution of the linear system $F'(x_n)\,\gamma_n = (-1/2)\,F''(x_n)\,\delta_n^2$ ($2n^2 - n$ operations). Consequently, the computational cost per iteration of Chebyshev's method is $(4n^3 + 15n^2 - 4n)/3$, which is higher than that of Newton's method. In consequence, for solving (3), it is clear that the application of Newton's method is a better option than that of Chebyshev's method; see Table 1.

From the above, our interest is focused on constructing iterations from a modification of Chebyshev's method which reduces the number of evaluations of functions and the computational cost.

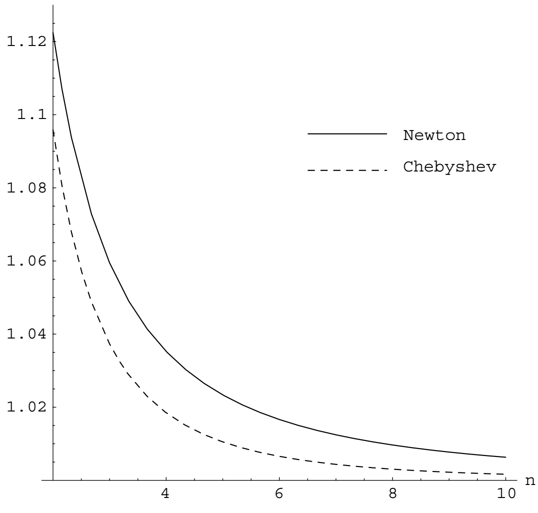

Fig. 1. Efficiency indices of Newton's and Chebyshev's methods for nonlinear systems, respectively $2^{1/(n^2+n)}$ and $3^{2/(n^3+3n^2+2n)}$.

Table 1
Number of evaluations of functions and computational cost per iteration when Newton's and Chebyshev's method are applied to solve nonlinear systems (10, 50 and 100 equations).

n	Newton's method		Chebyshev's method	
	n^2+n	$(n^3+6n^2-4n)/3$	$(n^3+3n^2+2n)/2$	$(4n^3+15n^2-4n)/3$
10	110	520	660	1 820
50	2 550	46 600	66 300	179 100
100	10 100	353 200	515 100	1383 200

Firstly, from Chebyshev's method, Hernández obtains in [3] the following family of third-order multipoint iterations which do not require the computation of F'':

$$\begin{cases} x_0 \text{ and } p \in (0,1] \text{ given}, \\ F'(x_n)\,\delta_n = -F(x_n), \quad n \geq 0, \\ z_n = x_n + p\,\delta_n, \\ F'(x_n)\,\hat{\gamma}_n = -\dfrac{1}{2p}(F'(z_n) - F'(x_n))\,\delta_n, \\ x_{n+1} = x_n + \delta_n + \hat{\gamma}_n. \end{cases} \tag{4}$$

To obtain (4), the expression $F''(x_n)\,\delta_n^2$ of Chebyshev's method is approximated by the expression $(1/p)(F'(z_n) - F'(x_n))\,\delta_n$, so the number of evaluations of functions and the computational cost per iteration are reduced respectively to $2n^2+n$ and $(n^3+15n^2-n)/3$. Therefore, the choice of iterations (4) to solve nonlinear system (3) is better than that of Chebyshev's method, although worse than that of Newton's method. See Table 2 and Fig. 2.

Secondly, by using a slight modification of the technique used in [3] by Hernández, we obtain in the following section a family of third-order iterations which reduces even more the number of evaluations of functions and the computational cost, such that these values are close to the ones of Newton's method.

On the other hand, when third-order methods are applied to solve nonlinear equations, it is important to note that the region of accessibility is reduced with respect to Newton's method. In practice, we can see this with the attraction basins (the set of points in the space such that initial conditions chosen in the set dynamically evolve to a particular attractor [11,12]) of iterative methods when they are applied to solve a complex equation $F(z) = 0$, where $F : \mathbb{C} \to \mathbb{C}$ and $z \in \mathbb{C}$,

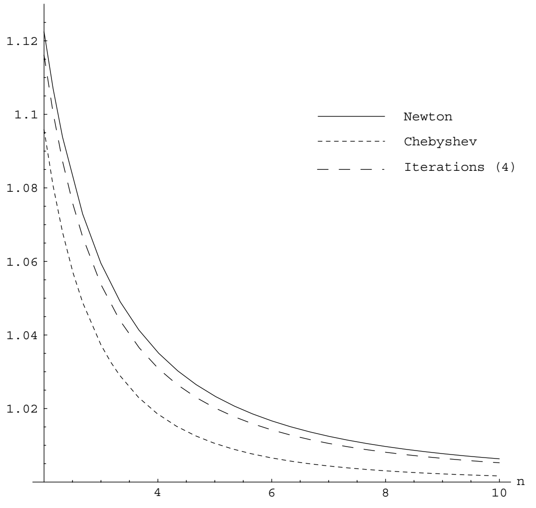

Fig. 2. Efficiency indices of Newton's and Chebyshev's methods and iterations (4) for nonlinear systems, respectively $2^{1/(n^2+n)}$, $3^{2/(n^3+3n^2+2n)}$ and $3^{1/(2n^2+n)}$.

Table 2
Number of evaluations of functions and computational cost per iteration when iterations (4) are applied to solve nonlinear systems (10, 50 and 100 equations).

n	Iterations (4)	
	$2n^2+n$	$(n^3+15n^2-n)/3$
10	210	830
50	5 050	54 150
100	20 100	383 300

and we are interested in identifying the attraction basin for two solutions z^* and z^{**} [12]. To do this, we choose for example Newton's method and a particular method (4) for solving the complex equation $F(z) = \sin z - 1/3 = 0$, and show the fractal pictures that they generate to approximate $z^* = \arctan(1/2\sqrt{2}) = 0.33983\ldots$ and $z^{**} = \pi - \arctan(1/2\sqrt{2}) = 2.80176\ldots$. This also allows us to compare the regions of accessibility of the two methods.

We take a rectangle $D \subseteq \mathbb{C}$ and iterations starting at "every" $z_0 \in D$. In practice, a grid of 512×512 points in D is considered and these points are chosen as z_0. The rectangle used is $[0, 3] \times [-2.5, 2.5]$, which contains the two zeros. The numerical methods starting at a point in the rectangle can converge to some of the zeros or, eventually, diverge.

In all the cases, the tolerance 10^{-3} and a maximum of 25 iterations are used. If we have not obtained the desired tolerance with 25 iterations, we do not continue and we decide that the iterative method starting at z_0 does not converge to any zero.

The rectangles mentioned above and corresponding to the two iterative methods when they are applied to approximate the solutions z^* and z^{**} of $F(z) = \sin z - 1/3 = 0$ are shown in Figs. 3 and 4. The strategy taken into account is the following. A colour is assigned to each basin of attraction of a zero. The colour is made lighter or darker according to the number of iterations needed to reach the root with the fixed precision required. Finally, if the iteration does not converge, the colour black is used. For more strategies, the reader can see [12] and the references appearing there. In particular, to obtain the pictures, the cyan and magenta colours have been assigned for the attraction basins of the two zeros. We mark with black the points of the rectangle for which the corresponding iterations starting at them do not reach any root with tolerance 10^{-3} in a maximum of 25 iterations. The graphics shown here have been generated with Mathematica 5.1 [13].

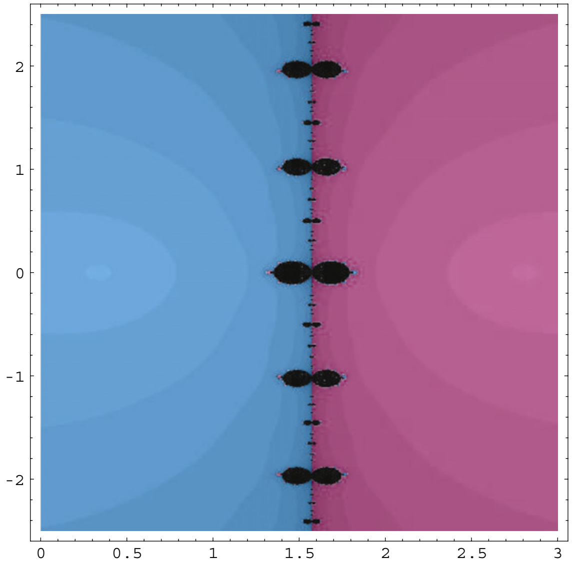

Fig. 3. Newton's method.

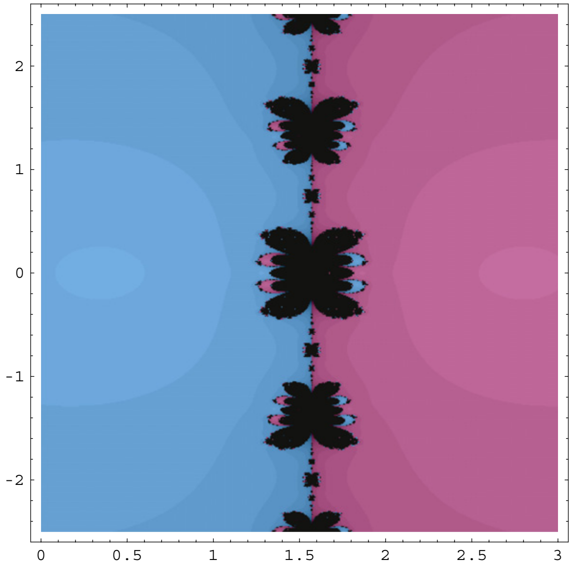

Fig. 4. Method (4) with $p = 1$.

If we observe the behaviour of the two methods, we see that method (4) with $p = 1$ is more demanding with respect to the starting point than Newton's method (see the black colour). We can also observe that there exist lighter areas for method (4) with $p = 1$. These observations are as a consequence of the higher speed of convergence of the last method (cubical convergence) as compared to Newton's method (quadratic convergence), and consequently, it is more difficult to locate starting points from which method (4) with $p = 1$ converges.

One goal will then be to construct, from iterations (4), new iterations that converge when they start at the same points as Newton's method. Firstly, we construct in the next section some iterations

Table 3
Number of evaluations of functions and computational cost per iteration when iterations (5) are applied to solve nonlinear systems (10, 50 and 100 equations).

n	Iterations (5)	
	n^2+2n	$(n^3+12n^2+2n)/3$
10	120	740
50	2 600	51 700
100	10 200	373 400

from Chebyshev's method that reduce the number of necessary values of the function involved and the computational cost, while preserving cubical convergence. Later, in Section 5, we define the new iterations as hybrid iterative methods, so that they have the same region of accessibility as Newton's method.

3. A modification of Chebyshev's method

To construct then iterations from Chebyshev's method we use a slight modification of the technique developed in [3] to obtain iterations (4). The idea is now to approximate the expression $F''(x_n)\delta_n^2$ in Chebyshev's algorithm by means of only combinations of F in different points, so that F'' is not used and F' is only evaluated in x_n. To do this, we consider $y_n = x_n - [F'(x_n)]^{-1}F(x_n)$, $z_n = x_n + p(y_n - x_n)$, $p \in (0, 1]$ and Taylor's formula in the following way:

$$F(z_n) = F(x_n) + pF'(x_n)(y_n - x_n) + \frac{p^2}{2}F''(x_n)(y_n - x_n)^2 + \frac{1}{2}\int_{x_n}^{z_n} F'''(x)(z_n - x)^2\,dx,$$

so that

$$F(z_n) - F(x_n) - pF'(x_n)(y_n - x_n) = \frac{p^2}{2}F''(x_n)(y_n - x_n)^2 + \frac{1}{2}\int_{x_n}^{z_n} F'''(x)(z_n - x)^2\,dx.$$

In consequence, since $y_n = x_n - [F'(x_n)]^{-1}F(x_n)$, we can consider the following approximation:

$$F''(x_n)(y_n - x_n)^2 \approx \frac{2}{p^2}((p-1)F(x_n) + F(z_n)),$$

and Chebyshev's method is now modified as

$$\begin{cases} x_0 \text{ and } p \in (0, 1] \text{ given}, \\ F'(x_n)\,\delta_n = -F(x_n), \quad n \geq 0, \\ z_n = x_n + p\,\delta_n, \\ F'(x_n)\,\tilde{\gamma}_n = -\dfrac{1}{p^2}\left((p-1)F(x_n) + F(z_n)\right), \\ x_{n+1} = x_n + \delta_n + \tilde{\gamma}_n. \end{cases} \tag{5}$$

With this modification of Chebyshev's method, we have reduced the computational cost from n^3+n^2+n operations for doing $(-1/2)\,F''(x_n)\,\delta_n^2$ to $2n$ operations for doing $(-1/p^2)((p-1)F(x_n)+F(z_n))$, which is a considerable reduction. Moreover, observe that the efficiency is also improved, since the number of evaluations of functions per iteration is also reduced from $(n^3+3n^2+2n)/2$ to n^2+2n; see Table 3.

Observe in Fig. 5 that the efficiency of Newton's method is now improved by iterations (5), even for high values of n. Consequently, to solve nonlinear system (3), method (5) is a better choice, since the number of computations of functions is similar.

Remark 1. If we consider iterative methods with memory, the efficiency index could be improved. Observe what happens in the one-dimensional case when the secant method, the most well known iterative method with memory, is applied for solving nonlinear equations. The secant method has better efficiency index than Newton's method and iterations (5), $(1+\sqrt{5})/2$ as opposed to $\sqrt{2}$

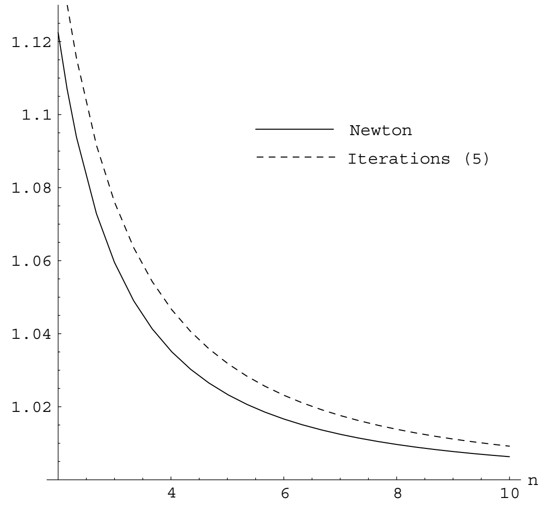

Fig. 5. Efficiency indices of Newton's method and iterations (5) for nonlinear systems, respectively $2^{1/(n^2+n)}$ and $3^{1/(n^2+2n)}$.

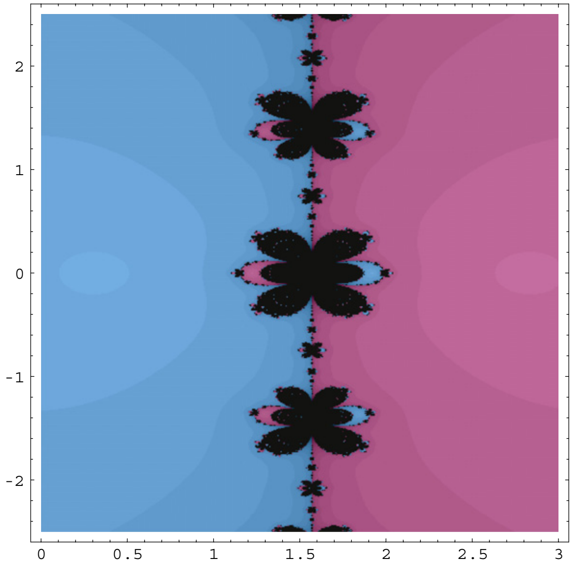

Fig. 6. Method (5) with $p = (\sqrt{5} - 1)/2$.

and $\sqrt[3]{3}$ respectively; so the efficiency index of iterations (5) lies between the efficiency indices of Newton's method and the secant method. Then, an interesting idea could be developed in future: we can approximate $F'(x_n)$ at each step of iterations (5) by a divided difference (exactly as we do in Newton's method to obtain the secant method) and construct new iterative methods with memory (as in the secant method) where only the evaluation of a new function is needed at each step.

On the other hand, if we now consider the problem of the region of accessibility for iterations (5), we can see in Figs. 3 and 6 that iteration (5) with $p = (\sqrt{5}-1)/2$ is still more demanding with respect to the starting points than Newton's method when we apply it to approximate the solutions z^* and z^{**} of the complex equation $F(z) = \sin z - 1/3 = 0$. This problem is studied and solved in Section 5.

4. Convergence analysis of iterations (5)

We establish in this section the convergence of iterations (5). In a more general situation, we consider

$$F(x) = 0, \tag{6}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator defined on a non-empty open convex subset Ω of a Banach space X with values in a Banach space Y, so that if certain conditions on the nonlinear operator F are required, different problems can be solved: integral equations, boundary value problems, systems of nonlinear equations, etc.

We begin with a local convergence result, where we prove that the order of convergence is at least 3. Next, we analyse the semilocal convergence of (5), which is now written as

$$\begin{cases} x_0 \in \Omega, \\ y_n = x_n - \Gamma_n F(x_n), \\ z_n = x_n + p\,(y_n - x_n), \quad p \in (0,1], \\ x_{n+1} = x_n - \dfrac{1}{p^2}\Gamma_n\left((p^2+p-1)F(x_n) + F(z_n)\right), \quad n \geq 0, \end{cases} \tag{7}$$

where $\Gamma_n = [F'(x_n)]^{-1}$, under mild differentiability conditions. In particular, we prove that iterations (7) converge under the same conditions as Newton's method:

$$\begin{cases} x_0 \in \Omega, \\ x_{n+1} = x_n - [F'(x_n)]^{-1}F(x_n), \quad n \geq 0. \end{cases}$$

4.1. Local convergence

We first see that iterations (7) have order of convergence at least 3. If $e_n = x_n - x^*$ is the error in the n-th iterate, the relation $e_{n+1} = Ce_n^q + \mathcal{O}\left(\|e_n\|^{q+1}\right)$, where $C \in \mathbb{R}$, is called the error equation [14]. By substituting $e_n = x_n - x^*$, for all n, in (7) and simplifying, we obtain the error equation for (7). The given value of q is called the order of method (7) [14].

Theorem 4.1. *Suppose that F is a sufficiently differentiable operator in Ω. If F has a simple root $x^* \in \Omega$, $[F'(x)]^{-1}$ exists in a neighborhood of x^* and x_0 is sufficiently close to x^*, then iterations (7) have order of convergence at least 3.*

Proof. From Taylor's formula

$$0 = F(x^*) = F(x_n) - F'(x_n)e_n + \frac{1}{2!}F''(x_n)e_n^2 - \frac{1}{3!}F'''(x_n)e_n^3 + \mathcal{O}\left(\|e_n\|^4\right),$$

where $e_n = x_n - x^*$, we obtain

$$\Gamma_n F(x_n) = e_n - \frac{1}{2}\Gamma_n F''(x_n)e_n^2 + \frac{1}{6}\Gamma_n F'''(x_n)e_n^3 + \mathcal{O}\left(\|e_n\|^4\right).$$

Moreover, since $z_n - x_n = -p\Gamma_n F(x_n)$, it follows that

$$z_n - x_n = -p\,e_n + \frac{p}{2}\Gamma_n F''(x_n)e_n^2 - \frac{p}{6}\Gamma_n F'''(x_n)e_n^3 + \mathcal{O}\left(\|e_n\|^4\right)$$

and, taking again into account Taylor's formula, we have

$$\begin{aligned} F(z_n) &= F(x_n) + F'(x_n)(z_n - x_n) + \frac{1}{2}F''(x_n)(z_n - x_n)^2 + \frac{1}{6}F'''(x_n)(z_n - x_n)^3 + \mathcal{O}\left(\|e_n\|^4\right) \\ &= (1-p)F'(x_n)e_n + \frac{1}{2}(p^2+p-1)F''(x_n)e_n^2 + \frac{1}{6}(1-p-p^3)F'''(x_n)e_n^3 \\ &\quad - \frac{p^2}{2}F''(x_n)\Gamma_n F''(x_n)e_n^3 + \mathcal{O}\left(\|e_n\|^4\right). \end{aligned}$$

Therefore,

$$\Gamma_n F(z_n) = (1-p)e_n + \frac{1}{2}(p^2+p-1)\Gamma_n F''(x_n)e_n^2 + \frac{1}{2}\left(\frac{1-p-p^3}{3}\Gamma_n F'''(x_n) - p^2(\Gamma_n F''(x_n))^2\right)e_n^3 + \mathcal{O}\left(\|e_n\|^4\right).$$

In consequence, from (7), it follows that

$$e_{n+1} = x_{n+1} - x^* = e_n - \Gamma_n\left(\frac{p^2+p-1}{p^2}F(x_n) + \frac{1}{p^2}F(z_n)\right) = \frac{1}{6}\left((p-1)\Gamma_n F'''(x_n) + 3(\Gamma_n F''(x_n))^2\right)e_n^3 + \mathcal{O}\left(\|e_n\|^4\right),$$

and iterations (7) have therefore order of convergence at least 3. ■

4.2. Semilocal convergence

When we study the convergence of an iterative method, there are three types of convergence that can be analysed: local, semilocal and global. The analysis of convergence presented here is focused on the semilocal convergence, where two kinds of conditions are required: conditions on the starting point and conditions on the operator involved.

Now, we give a semilocal convergence result for iterations (7), where mild differentiability conditions are required. In particular, we study the semilocal convergence of (7) under the same conditions as were used for Newton's method in [15], where F' is Lipschitz continuous in Ω. Note that third-order iterative methods are generally studied under more demanding semilocal convergence conditions (see, for example, [16,9,3]).

So, we suppose that there exists the operator $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y,X)$, for some $x_0 \in \Omega$, where $\mathcal{L}(Y,X)$ is the set of bounded linear operators from the Banach space Y into the Banach space X. We also suppose the following:

(i) $\|\Gamma_0\| \le \beta$,
(ii) $\|\Gamma_0 F(x_0)\| \le \eta$,
(iii) $\|F'(x) - F'(y)\| \le K\|x-y\|$, for all $x, y \in \Omega$.

And, from now on, we use the notation $\overline{B(x,\rho)} = \{y \in X;\ \|y-x\| \le \rho\}$ and $B(x,\rho) = \{y \in X;\ \|y-x\| < \rho\}$, where X is a Banach space.

Firstly, we guarantee the semilocal convergence of Newton's method under conditions **(i)–(iii)**.

Theorem 4.2 (*See [17]*). *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ an operator that is once Fréchet differentiable in an open convex domain Ω. Assume* **(i)–(iii)**. *If $B(x_0, r) \subseteq \Omega$, where $r = \frac{2(1-a)}{2-3a}\eta$ and $a = K\beta\eta < 1/2$, then Eq. (6) has a solution x^*, and Newton's method converges to x^* and has R-order of convergence at least 2.*

After that, we are interested in proving the semilocal convergence of iterations (7) under the same conditions, **(i)–(iii)**, as for Newton's method. In view of Theorem 4.2, we consider $R = \frac{(1+a_0/2)\,\eta}{1-f(a_0)g(a_0)}$, where

$$f(t) = \frac{2}{2-2t-t^2}, \qquad g(t) = \frac{t}{8}(8+4t+t^2), \tag{8}$$

such that $B(x_0,R) \subseteq \Omega$, and define $K\beta\eta = a_0$. Then $\|y_0 - x_0\| \le \eta$ and $\|z_0 - x_0\| \le p\eta$, so $y_0, z_0 \in \Omega$. Since

$$F(z_0) = (1-p)F(x_0) + p\int_0^1 \left[F'(x_0 + pt(y_0-x_0)) - F'(x_0)\right](y_0-x_0)\,\mathrm{d}t,$$

as a consequence of Taylor's formula, we have, provided that $x_1 \in \Omega$ and $a_0 < \sigma_1 = 0.4111\ldots$, where σ_1 is the root of the real equation $f(a_0)g(a_0) - 1 = 0$,

$$\|x_1 - y_0\| \le \frac{K}{2}\|\Gamma_0\|\|y_0 - x_0\|^2 \le \frac{a_0}{2}\|y_0 - x_0\|,$$
$$\|x_1 - x_0\| \le \|x_1 - y_0\| + \|y_0 - x_0\| \le (1 + a_0/2)\|y_0 - x_0\| < R.$$

Therefore $x_1 \in B(x_0, R)$ if $a_0 < \sigma_1$. Note that the value of R is later deduced.

On the other hand, if $a_0 < \sqrt{3} - 1$, it follows that $\|I - \Gamma_0 F'(x_1)\| < 1$, and consequently $\Gamma_1 = [F'(x_1)]^{-1}$ exists, by the Banach lemma on invertible operators [4], and $\|\Gamma_1\| \le f(a_0)\|\Gamma_0\|$. Therefore, y_1 and z_1 are well-defined. Moreover, $y_1, z_1 \in \Omega$.

Furthermore, from Taylor's formulas,

$$F(x_1) = \frac{1}{p}\int_0^1 \left[F'(x_0) - F'(x_0 + pt(y_0 - x_0))\right](y_0 - x_0)\,\mathrm{d}t + \int_0^1 \left[F'(x_0 + pt(x_1 - x_0)) - F'(x_0)\right](x_1 - x_0)\,\mathrm{d}t,$$

$$F(z_1) = (1 - p)F(x_1) + p\int_0^1 \left[F'(x_1 + pt(y_1 - x_1)) - F'(x_1)\right](y_1 - x_1)\,\mathrm{d}t,$$

we obtain

$$\|F(x_1)\| \le \frac{K}{8}(8 + 4a_0 + a_0^2)\|y_0 - x_0\|^2,$$
$$\|y_1 - x_1\| \le \|\Gamma_1\|\|F(x_1)\| \le f(a_0)g(a_0)\|y_0 - x_0\|,$$
$$K\|\Gamma_1\|\|y_1 - x_1\| \le a_0 f(a_0)^2 g(a_0),$$
$$\|x_2 - y_1\| \le \frac{a_0}{2} f(a_0)^2 g(a_0)\|y_1 - x_1\|,$$
$$\|x_2 - x_1\| \le \|x_2 - y_1\| + \|y_1 - x_1\| \le \left(1 + \frac{a_0}{2} f(a_0)^2 g(a_0)\right)\|y_1 - x_1\|,$$
$$\|x_2 - x_0\| \le \|x_2 - x_1\| + \|x_1 - x_0\| \le (1 + a_0/2)(1 + f(a_0)g(a_0))\|y_0 - x_0\| < R$$

and $x_2 \in \Omega$, provided that $a_0 < \sigma_2 = 0.3266\ldots$, where σ_2 is the smallest positive root of the real equation $f(a_0)^2 g(a_0) - 1 = 0$.

Besides, if $a_0 < \sigma_2$, then $\|I - \Gamma_1 F'(x_2)\| < 1$, $\Gamma_2 = [F'(x_2)]^{-1}$ exists, by the Banach lemma on invertible operators, and $\|\Gamma_2\| \le f(a_0 f(a_0)^2 g(a_0))\|\Gamma_1\|$. After that, we can deduce $y_2, z_2, x_3 \in \Omega$ from an analogous procedure.

Now, we define $a_0 f(a_0)^2 g(a_0) = a_1$ and define the real sequence

$$a_{n+1} = a_n f(a_n)^2 g(a_n), \quad n \ge 0, \tag{9}$$

which is decreasing and such that $a_n(1 + a_n/2) < 1$, for all $n \ge 0$, provided that $a_0 < \sigma_2$. Moreover, if $y_n, z_n, x_{n+1} \in \Omega$, this real sequence satisfies the following system of recurrence relations, from which we can guarantee that sequence (7) is well-defined. To prove them, we follow a similar method to the above and then invoke the induction hypothesis.

Lemma 4.3. *Let f and g be the two real functions defined in (8). If $a_0 < \sigma_2 = 0.3266\ldots$, the following items are satisfied for all $n \ge 1$:*

- **[I]** $\Gamma_n = [F'(x_n)]^{-1}$ *exists and* $\|\Gamma_n\| \le f(a_{n-1})\|\Gamma_{n-1}\|$,
- **[II]** $\|y_n - x_n\| \le f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \le (f(a_0)g(a_0))^n\|y_0 - x_0\| < \eta$,
- **[III]** $K\|\Gamma_n\|\|y_n - x_n\| \le a_n$,
- **[IV]** $\|x_{n+1} - y_n\| \le \frac{a_n}{2}\|y_n - x_n\|$,
- **[V]** $\|x_{n+1} - x_n\| \le \left(1 + \frac{a_n}{2}\right)\|y_n - x_n\|$,
- **[VI]** $\|x_{n+1} - x_0\| \le \left(1 + \frac{a_0}{2}\right)\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - f(a_0)g(a_0)}\|y_0 - x_0\| < R$, *where* $R = \frac{(1 + a_0/2)\,\eta}{1 - f(a_0)g(a_0)}$.

Next, the convergence of iterations (7) is easily guaranteed from (**i**)–(**iii**), as we can see in the following theorem.

Theorem 4.4. *Let $F : \Omega \subseteq X \to Y$ be a once-differentiable Fréchet operator on a non-empty open convex domain Ω of a Banach space X with values in a Banach space Y. We assume that there exists $\Gamma_0 \in \mathcal{L}(Y, X)$, $x_0 \in \Omega$, and* **(i)–(iii)**. *If $a_0 < \sigma_2 = 0.3266\ldots$ and $B(x_0, R) \subseteq \Omega$, where $R = \frac{(1+a_0/2)\,\eta}{1-f(a_0)g(a_0)}$; then sequence (7) is well-defined, is contained in $B(x_0, R)$ and converges to a solution x^* of (6) in the ball $\overline{B(x_0, R)}$. Besides, the solution x^* is unique in $B\left(x_0, \frac{2}{K\beta} - R\right) \cap \Omega$ if $R < 2/(K\beta)$.*

Proof. Firstly, we see that sequence (7) is well-defined. Observe

$$\begin{aligned}\|y_n - x_n\| &\le f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \le \cdots \le \left(\prod_{i=0}^{n-1} f(a_i)g(a_i)\right)\|y_0 - x_0\| \\ &\le (f(a_0)g(a_0))^n\,\|y_0 - x_0\|\end{aligned}$$

as a consequence of recurrence relation **[II]** of Lemma 4.3. Therefore, for $m \ge 1$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le \left(1 + \frac{a_{n+m-1}}{2}\right)\|y_{n+m-1} - x_{n+m-1}\| + \left(1 + \frac{a_{n+m-2}}{2}\right)\|y_{n+m-2} - x_{n+m-2}\| \\ &\quad + \cdots + \left(1 + \frac{a_n}{2}\right)\|y_n - x_n\| \\ &\le \left(1 + \frac{a_n}{2}\right)\sum_{j=n}^{n+m-1}\left(\prod_{i=0}^{j-1} f(a_i)g(a_i)\right)\|y_0 - x_0\| \\ &\le \left(1 + \frac{a_0}{2}\right)(f(a_0)g(a_0))^n\,\frac{1 - (f(a_0)g(a_0))^m}{1 - f(a_0)g(a_0)}\,\eta.\end{aligned} \tag{10}$$

If $n = 0$ in (10), it follows that

$$\|x_m - x_0\| \le \left(1 + \frac{a_0}{2}\right)\frac{1 - (f(a_0)g(a_0))^m}{1 - f(a_0)g(a_0)}\,\eta < R.$$

Then, $x_m \in B(x_0, R)$, for all $m \ge 1$. Similarly, $y_m, z_m \in B(x_0, R)$, for all $m \ge 0$. In consequence, $x_m, y_m, z_m \in \Omega$, for $m \ge 1$.

Note that $\{x_n\}$ is a Cauchy sequence, as a consequence of (10) and $a_0 < \sigma_2$. Then, $\{x_n\}$ converges to x^*, which is a solution of (6). Indeed, by letting $n \to \infty$, we have $\|\Gamma_n F(x_n)\| \to 0$ and, since $\|F(x_n)\| \le \|F'(x_n)\|\|\Gamma_n F(x_n)\|$ and the sequence $\{\|F'(x_n)\|\}$ is bounded, we have $\|F(x_n)\| \to 0$ and, by the continuity of F, it follows that $F(x^*) = 0$.

Finally, if we suppose that there exists another solution y^* of (6) in $B\left(x_0, \frac{2}{K\beta} - R\right) \cap \Omega$, we have

$$0 = \Gamma_0(F(y^*) - F(x^*)) = \int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\,\mathrm{d}t\,(y^* - x^*).$$

But, since

$$\begin{aligned}\|I - T\| &\le \|\Gamma_0\|\int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|\,\mathrm{d}t \le K\beta\int_0^1 \|x_0 - (x^* + t(y^* - x^*))\|\,\mathrm{d}t \\ &\le K\beta\int_0^1 \left(t\|y^* - x_0\| + (1 - t)\|x^* - x_0\|\right)\mathrm{d}t < 1,\end{aligned}$$

where $T = \Gamma_0\int_0^1 F'(x^* + t(y^* - x^*))\,\mathrm{d}t$, we obtain that the operator T is invertible, and consequently $y^* = x^*$. ■

5. Description of the new iterations

We now pay attention to the conditions that starting points of iteration (7) must satisfy to guarantee the convergence of (7) (see Theorem 4.4). We then observe the region of accessibility

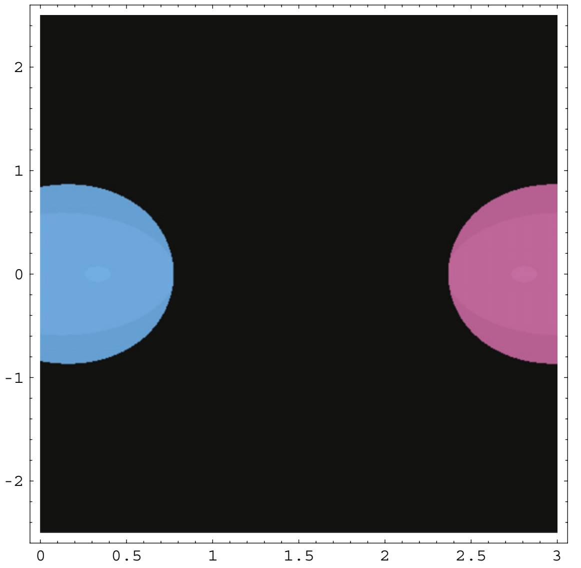

Fig. 7. Newton's method.

of (7). If we consider the complex equation $F(z) = \sin z - 1/3 = 0$ and the particular version of (7) given by (7) with $p = (\sqrt{5}-1)/2$,

$$\begin{cases} x_0 \in \Omega, \\ y_n = x_n - \Gamma_n F(x_n), \\ x_{n+1} = x_n - \dfrac{3+\sqrt{5}}{2}\, \Gamma_n F\left(x_n + \dfrac{\sqrt{5}-1}{2}(y_n - x_n)\right), \quad n \geq 0, \end{cases} \tag{11}$$

which is also presented in [18] for the scalar case, we can see the behaviour of (11) in Fig. 6. Observe that iteration (11) is also more demanding than Newton's method with respect to starting points (see Figs. 3 and 6), as a consequence of its higher speed of convergence. Consequently, it is more difficult to locate starting points for method (11) than for Newton's method.

On the other hand, it is clear that the condition $a_0 < \sigma_2 = 0.3266\ldots$ required to guarantee the convergence of iterations (7) in Theorem 4.4 is more demanding than the one required for Newton's method, $a_0 < 1/2$, under the same general convergence conditions **(i)–(iii)**. Therefore, the application of iterations (7) is more restrictive than the application of Newton's method. To illustrate this, we can respectively see in Figs. 7 and 8 the regions of accessibility of Newton's method and method (11), when they are applied to approximate the solutions z^* and z^{**} of $F(z) = \sin z - 1/3 = 0$. Observe that the domain of starting points for Newton's method is a little bigger than for method (11) (see the size of the regions of convergence).

Since the main goal is to construct iterative methods from iterations (7) that converge when they start at the same points as Newton's method, we define the following iterations:

$$\begin{cases} \begin{cases} x_0 \in \Omega, \\ x_n = x_{n-1} - \Gamma_{n-1} F(x_{n-1}), \quad n = 1, 2, \ldots, N_0, \end{cases} \\ \begin{cases} \bar{x}_0 = x_{N_0}, \\ \bar{y}_{k-1} = \bar{x}_{k-1} - [F'(\bar{x}_{k-1})]^{-1} F(\bar{x}_{k-1}), \\ \bar{z}_{k-1} = \bar{x}_{k-1} + p(\bar{y}_{k-1} - \bar{x}_{k-1}), \quad p \in (0, 1], \\ \bar{x}_k = \bar{x}_{k-1} - \dfrac{1}{p^2} [F'(\bar{x}_{k-1})]^{-1} \left((p^2 + p - 1) F(\bar{x}_{k-1}) + F(\bar{z}_{k-1})\right), \quad k \geq 1, \end{cases} \end{cases} \tag{12}$$

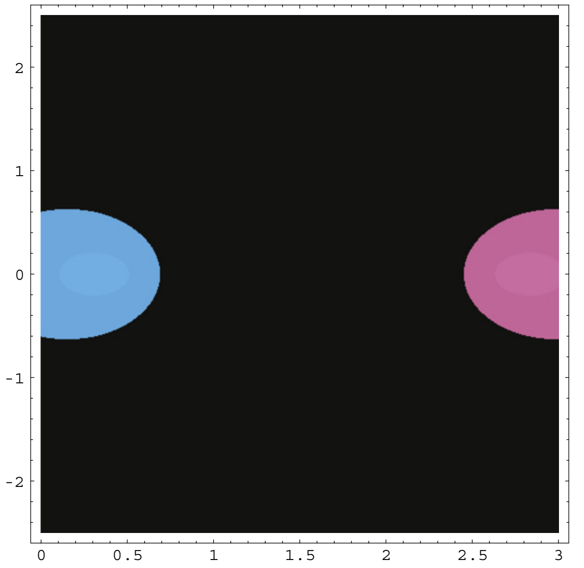

Fig. 8. Method (11).

where x_0 is such that $a = K\beta\eta < 1/2$ and $\bar{x}_0 = x_{N_0}$ such that $a_0 = K\tilde{\beta}\tilde{\eta} < \sigma_2 = 0.3266\ldots$ with $\tilde{\beta} \geq \|[F'(\bar{x}_0)]^{-1}\|$ and $\tilde{\eta} \geq \|[F'(\bar{x}_0)]^{-1}F(\bar{x}_0)\|$. In this case, we can apply Newton's method for a finite number of steps N_0, provided that the condition $a < 1/2$ is satisfied, until the condition $a_0 < \sigma_2$ is satisfied for $\bar{x}_0 = x_{N_0}$, and then apply iterations (7) to accelerate the convergence. To do this, we have to guarantee the existence of N_0.

5.1. Semilocal convergence of iterations (12)

We have seen that Newton's method and method (7) converge under the same conditions **(i)–(iii)**. The convergence of both methods is guaranteed as a consequence of the fact that both sequences are Cauchy sequences. We use the same argument to prove the semilocal convergence of iterations (12).

We suppose that the initial iterate x_0 is such that $a = K\beta\eta \in [\sigma_2, 1/2)$ and we look for the existence of $x_{N_0} = \bar{x}_0$, $N_0 \in \mathbb{N}$, such that $a_0 \in (0, \sigma_2)$, where $a_0 = K\tilde{\beta}\tilde{\eta}$, $\tilde{\beta} \geq \|[F'(\bar{x}_0)]^{-1}\|$ and $\tilde{\eta} \geq \|[F'(\bar{x}_0)]^{-1}F(\bar{x}_0)\|$.

Starting from $\alpha_0 = a$, we define the scalar sequence

$$\alpha_{n+1} = \frac{\alpha_n^2}{2(1-\alpha_n)^2}, \qquad n \geq 0,$$

and, for $n \geq 1$, we construct the system of recurrence relations (see [17])

$$\|\Gamma_n\| \leq \frac{1}{1-\alpha_{n-1}}\|\Gamma_{n-1}\|,$$

$$\|x_{n+1} - x_n\| \leq \frac{\alpha_{n-1}}{2(1-\alpha_{n-1})}\|x_n - x_{n-1}\| \leq \delta^n \|\Gamma_0 F(x_0)\|,$$

$$\|x_{n+1} - x_0\| \leq \frac{1-\delta^{n+1}}{1-\delta}\|\Gamma_0 F(x_0)\| < r,$$

where $\delta = \frac{a}{2(1-a)}$ and $r = \frac{2(1-a)}{2-3a}\,\eta$.

The strict decreasing of the positive real sequence $\{\alpha_n\}$ guarantees the existence of the term α_{N_0} such that $\alpha_{N_0} < \sigma_2$.

Now, from $\bar{x}_0 = x_{N_0}$,

$$K\|[F'(\bar{x}_0)]^{-1}\|\|[F'(\bar{x}_0)]^{-1}F(\bar{x}_0)\| \leq K\tilde{\beta}\tilde{\eta},$$

we define the initial parameter $a_0 = K\tilde{\beta}\tilde{\eta}$ for (7), which starts at $\bar{x}_0 = x_{N_0}$, where x_{N_0} is the last iteration obtained by Newton's method. Next, we define the real sequence (9) and construct the corresponding system of recurrence relations given in Lemma 4.3 so that the convergence of the sequence given by (7) is guaranteed from the strict decreasing of (9). In consequence, we can apply iterations (12) to approximate a solution of Eq. (6), starting at the same iterate x_0 as Newton's method, since

$$K\|\Gamma_{N_0}\|\|\Gamma_{N_0}F(x_{N_0})\| \leq \alpha_{N_0} = a_0 < \sigma_2,$$

so $x_{N_0} = \bar{x}_0$ can be chosen to start iterations (7) and the convergence of iterations (12) is then guaranteed by Theorem 4.4.

Since the sequence given by (12) is well-defined, we only have to prove that this sequence is a Cauchy sequence. To do this, we rewrite it as

$$w_n = \begin{cases} x_n, & \text{if } n \leq N_0, \\ \bar{x}_{n-N_0}, & \text{if } n > N_0. \end{cases}$$

Theorem 5.1. *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ an operator that is once Fréchet differentiable on a non-empty open convex domain Ω. Let $x_0 \in \Omega$ and* **(i)–(iii)** *be satisfied. Suppose that $a < 1/2$ and $B(x_0, r+R) \subseteq \Omega$. Then, the sequence $\{w_n\}$, which starts at w_0, converges to a solution x^* of (6). Moreover, $w_n, x^* \in \overline{B(x_0, r+R)}$. Furthermore, x^* is unique in $B\left(x_0, \frac{2}{K\beta} - (r+R)\right) \cap \Omega$ if $r + R < 2/(K\beta)$.*

Proof. From the above, it is clear that an N_0 exists. Besides, $w_i \in \Omega$, for $i = 0, 1, \ldots, N_0$. Indeed, since $w_i = x_i$ $(i = 0, 1, \ldots, N_0)$ are iterates of Newton's method, then $\|w_i - x_0\| < r \leq r + R$ $(i = 1, 2, \ldots, N_0)$ and $w_i \in \Omega$, for $i = 0, 1, \ldots, N_0$.

After that, we have that $w_{N_0} = \bar{x}_0 = x_{N_0}$ and w_i $(i > N_0)$ are iterates of (7), so $\|w_i - w_{N_0}\| < R$, for $i > N_0$. In consequence, $\|w_i - x_0\| \leq \|w_i - w_{N_0}\| + \|w_{N_0} - x_0\| < r + R$ $(i > N_0)$ and $w_i \in \Omega$, for $i > N_0$. Therefore, the sequence $\{w_n\}$ is well-defined.

The fact that $\{w_n\}$ is a Cauchy sequence in Ω follows immediately, since $\{w_n\}_{n\geq N_0}$ is given by (7), which is a Cauchy sequence (see Theorem 4.4). Consequently, $\lim_n w_n = x^*$, $x^* \in \overline{B(x_0, R)} \subset \overline{B(x_0, r+R)}$ and $F(x^*) = 0$.

Finally, the uniqueness of the solution x^* in $B\left(x_0, \frac{2}{K\beta} - (r+R)\right) \cap \Omega$ follows as in Theorem 4.4. ∎

Remark 2. Observe that the domain of starting points is extended in Theorem 5.1 compared to Theorem 4.4, so domains of existence and uniqueness of solutions can be given by Theorem 5.1 which cannot be given by Theorem 4.4.

Remark 3. Notice that iterations (12) have order of convergence at least 2 until iteration N_0 and order of convergence at least 3 from iteration $N_0 + 1$.

If we consider again the previous complex equation, $F(z) = \sin z - 1/3 = 0$, we can see in Figs. 9 and 10 the regions of accessibility of method (12) with $p = (\sqrt{5} - 1)/2$ when condition $a < 1/2$ is satisfied (Fig. 10) or not satisfied (Fig. 9). Observe that the domain of starting points is the same as for Newton's method, but the colour intensity is different, lighter or darker, according to the number of iterations needed to reach the roots. There are lighter areas for method (12) with $p = (\sqrt{5} - 1)/2$ as a consequence of the higher speed of convergence.

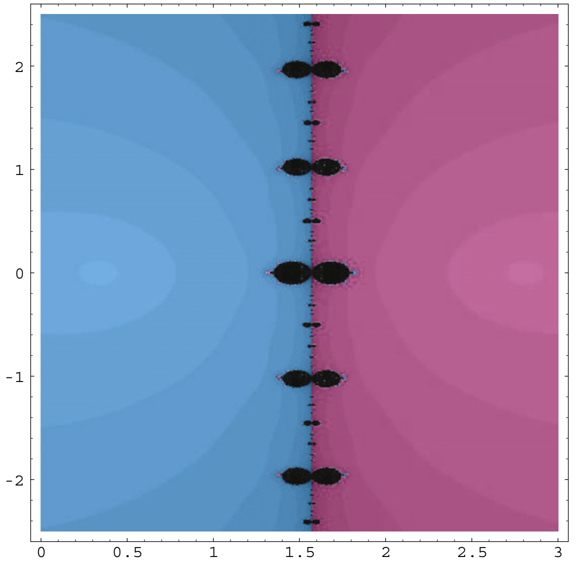

Fig. 9. Method (12) with $p = \frac{\sqrt{5}-1}{2}$.

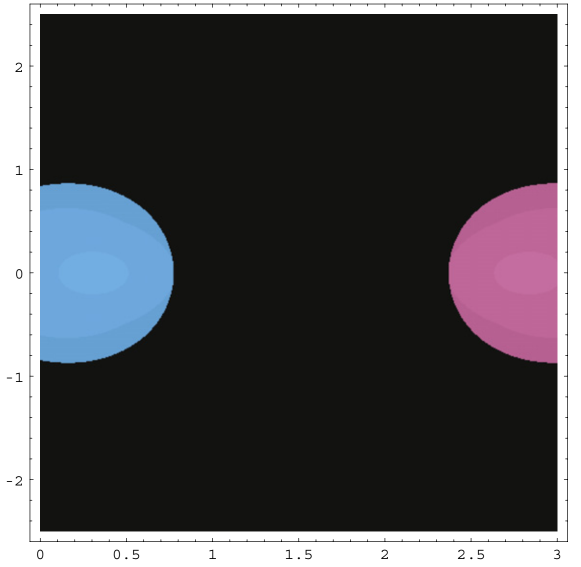

Fig. 10. Method (12) with $p = \frac{\sqrt{5}-1}{2}$.

6. Application of the new iterations

Note that if the conditions of Theorem 5.1 are verified, then iterations (12) can be applied, since N_0 always exists. The goal is now to estimate a priori the value of N_0, which improves the use of iterations (12), since the verification of $a_0 < \sigma_2 = 0.3266\ldots$ is saved in every step.

Theorem 6.1. *Under the general hypotheses of Theorem 5.1, we suppose $a \in [\sigma_2, 1/2)$ for some $x_0 \in \Omega$ which satisfies* **(i)** *and* **(ii)***. Set $\bar{x}_0 = x_{N_0}$ with $N_0 = 1 + \left[\frac{\ln \sigma_2 - \ln a}{\ln a - \ln(2(1-a)^2)}\right]$, and $[t]$ denoting the integer part of the real number t. Then, $\bar{x}_0$ is such that the condition $a_0 < \sigma_2$ holds.*

Table 4
Nodes and weights for the Gauss–Legendre formula.

i	t_i	w_i	i	t_i	w_i
1	0.019855...	0.050614...	5	0.591717...	0.181342...
2	0.101667...	0.111191...	6	0.762766...	0.156853...
3	0.237234...	0.156853...	7	0.898333...	0.111191...
4	0.408283...	0.181342...	8	0.980145...	0.050614...

Proof. We take into account that the above-mentioned ideas were carried out where we guarantee that iterations (12) are well-defined, since there always exists $N_0 \in \mathbb{N}$ such that iterations (7) can be applied starting at $\bar{x}_0 = x_{N_0}$. On the other hand,

$$\alpha_{N_0} = \frac{\alpha_{N_0-1}^2}{2(1-\alpha_{N_0-1})^2} = \cdots = \alpha_0 \prod_{i=0}^{N_0-1} \frac{\alpha_i}{2(1-\alpha_i)^2} < a\left(\frac{a}{2(1-a)^2}\right)^{N_0},$$

since the sequence $\{\alpha_n\}$ is decreasing and $\alpha_0 = a$. If $a\left(\frac{a}{2(1-a)^2}\right)^{N_0} < \sigma_2$, then x_{N_0} is a good starting point for iterations (7). In consequence, if

$$N_0 > \frac{\ln \sigma_2 - \ln a}{\ln a - \ln(2(1-a)^2)},$$

the theorem follows. ■

If we now take into account the nonlinear integral equation of mixed Hammerstein type (1), we see in the following that iterations (4) cannot be applied to solve Eq. (1), but iterations (12) can.

First, we discretize (1) to transform it into a finite dimensional problem. This procedure consists of approximating the integral appearing in (1) by a numerical quadrature formula. To obtain a numerical solution, we use the Gauss–Legendre formula to approximate an integral

$$\int_0^1 h(t)\,\mathrm{d}t \simeq \sum_{i=1}^{n} w_i h(t_i),$$

where the nodes t_i and the weights w_i are determined; in particular, see Table 4 for $n = 8$.

If we denote the approximation of $x(t_j)$ by x_j ($j = 1, 2, \ldots, 8$), (1) is now equivalent to the following nonlinear system of equations:

$$x_j = 1 + \frac{1}{2}\sum_{k=1}^{8} m_{jk} x_k^2, \quad j = 1, 2, \ldots, 8, \tag{13}$$

where

$$m_{jk} = \begin{cases} w_k t_k (1-t_j) & \text{if } k \le j, \\ w_k t_j (1-t_k) & \text{if } k < j. \end{cases}$$

System (13) can be now written in the form $\mathbf{x} = \mathbf{1} + \frac{1}{2}M\mathbf{x}^2$, or

$$F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8, \quad F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - \frac{1}{2}M\mathbf{x}^2 = 0,$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_8)^{\mathrm{T}}, \quad \mathbf{1} = (1, 1, \ldots, 1)^{\mathrm{T}}, \quad M = (m_{jk})_{j,k=1}^8, \quad \mathbf{x}^2 = (x_1^2, x_2^2, \ldots, x_8^2)^{\mathrm{T}}.$$

For this F, we have

$$F'(\mathbf{x})(\mathbf{u}) = \begin{pmatrix} 1 - m_{11}x_1 & -m_{12}x_2 & \cdots & -m_{18}x_8 \\ -m_{21}x_1 & 1 - m_{22}x_2 & \cdots & -m_{28}x_8 \\ \vdots & \vdots & \ddots & \vdots \\ -m_{81}x_1 & -m_{82}x_2 & \cdots & 1 - m_{88}x_8 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_8 \end{pmatrix},$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_8)^{\mathrm{T}}$.

Table 5
Numerical solution $\mathbf{x}^*$ of (13).

i	x_i^*	i	x_i^*	i	x_i^*	i	x_i^*
1	1.005450...	3	1.051629...	5	1.069365...	7	1.025815...
2	1.025815...	4	1.069365...	6	1.051629...	8	1.005450...

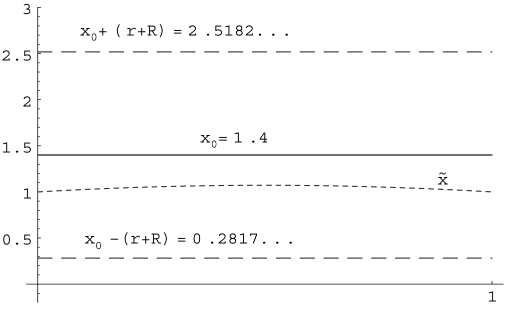

Fig. 11. Approximated solution $\tilde{\mathbf{x}}$ of Eq. (1).

If we choose $\mathbf{x_0} = (1.4, 1.4, \dots, 1.4)^{\mathrm{T}}$ and the max-norm, then

$$K = 1, \quad \beta = 1.1382\dots, \quad \eta = 0.0691\dots, \quad K\beta\eta = a = 0.4755\dots < 1/2.$$

Observe that we can apply Newton's method to solve (13), but we cannot use (5) because $K\beta\eta \geq \sigma_2 = 0.3266\dots$. However, by Theorem 6.1, we can use iteration (11) after the third approximation given by Newton's method, since $N_0 = 3$, and obtain the numerical solution $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_8^*)^{\mathrm{T}}$, which is shown in Table 5, after four more approximations.

Moreover, the existence of the solution is guaranteed in the ball $\overline{B(\mathbf{x_0}, 1.1182\dots)}$ and the unicity in $B(\mathbf{x_0}, 0.5440\dots)$ by Theorem 5.1.

Finally, we interpolate the points of Table 5 and taking into account that the solution of (1) satisfies $x(0) = x(1) = 1$, an approximation $\tilde{\mathbf{x}}$ of the numerical solution $\mathbf{x}^*$ is obtained (see Fig. 11). Notice that the interpolated approximation $\tilde{\mathbf{x}}$ lies within the existence domain of the solutions obtained above.

Acknowledgment

Preparation of this paper was partly supported by the Ministry of Science and Innovation (MTM 2008-01952/MTM).

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