Discrete Mathematics 309 (2009) 4017–4026

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On 2-walks in chordal planar graphs☆

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ARTICLE INFO

Article history:
Received 14 November 2007
Received in revised form 14 November 2008
Accepted 18 November 2008
Available online 6 January 2009

Keywords:
2-walk
Toughness
Chordal planar graph

ABSTRACT

A 2-walk is a closed spanning trail which uses every vertex at most twice. A graph is said to be chordal if each cycle different from a 3-cycle has a chord. We prove that every chordal planar graph G with toughness $t(G) > \frac{3}{4}$ has a 2-walk.

1. Introduction

In this paper we will consider simple, undirected graphs without loops. For concepts and notations not defined here we refer the reader to the book [2]. A k-walk is a closed spanning trail which uses every vertex at most k times. The subgraph of a graph G induced by a set of vertices M is denoted by $\langle M\rangle_G$, and $N_G(x)$ denotes the set of all the neighbours of a vertex x in G. The *toughness* of a non-complete graph is $t(G) = \min(\frac{|S|}{c(G-S)})$, where the minimum is taken over all nonempty vertex sets S such that $c(G-S) \geq 2$, where $c(G-S)$ denotes the number of components of the graph $G-S$. For a complete graph K_n we set $t(K_n) = \infty$. The concept of toughness was introduced by Chvátal [4] in 1973. It is obvious that a Hamiltonian graph is 1-tough. This can be easily generalized as follows: Every graph containing a k-walk is $\frac{1}{k}$-tough.

One of the most famous conjectures concerning Hamiltonian cycles is due to Chvátal.

Conjecture 1.1 ([4]). *There exists an integer t_0 such that if $t \geq t_0$, then every t-tough graph is Hamiltonian.*

Chvátal's conjecture is known to be true for several special classes of graphs. We mention two results on chordal graphs. Recall that a graph is *chordal* if it does not contain an induced cycle of length four or more.

Theorem 1.1 ([3]). *Every 18-tough chordal graph is Hamiltonian.*

It is conjectured that 18 can be reduced to two in this statement [3]. Better bounds are known for chordal planar graphs.

Theorem 1.2 ([1]). *Every chordal planar graph with toughness more than 1 is Hamiltonian.*

☆ The research was partially supported by grant No. 1M0545 of the Czech Ministry of Education.
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doi:10.1016/j.disc.2008.11.027

This result is the best possible. Another approach to Chvátal's conjecture is to show the existence of weaker substructures than Hamilton cycles. A *k-walk* of G is a closed walk that visits each vertex of G at least once and at most k times. There is the following easy necessary condition for the existence of a k-walk in a graph G: Every graph containing a k-walk is $1/k$-tough.

Jackson and Wormald [8] conjectured that every 1-tough graph has a 2-walk. The conjecture is still open. The best known result is:

Theorem 1.3 ([7]). *Every 4-tough graph has a 2-walk.*

Motivated by these results, first we state the following conjecture which we find interesting:

Conjecture 1.2. *Every 2-tough chordal graph has a 2-walk.*

In this paper we prove the following theorem:

Theorem 1.4. *Every chordal planar graph G with toughness greater $\frac{3}{4}$ has a 2-walk.*

The best known lower bound on $t(G)$ for 2-connected chordal planar graphs G with a 2-walk is in [6]. The authors found chordal planar graphs with toughness 4/7 having no 2-walk.

2. Proof of Theorem 1.4

Throughout the rest of the paper, whenever we consider a planar graph G, we always mean a fixed embedding of G into the plane. Such a graph is called a *plane graph*. A vertex v of a graph G is called a *simplicial vertex* if the subgraph of G induced by the neighbours of v is complete.

Before proceeding with the proof of the theorem, several useful results about chordal planar graphs are given below. The following theorem is due to Dirac [5].

Theorem 2.1 ([5]). *Every chordal graph G has a simplicial vertex v.*

We will also need the following result from [1].

Theorem 2.2 ([1]). *If G is an ℓ-connected chordal graph and v is a simplicial vertex in G, then the graph $G-v$ is either ℓ-connected or complete.*

We will prove Theorem 1.4 by induction. Before we start the induction, we prove the following two useful lemmas. Lemma 2.1 shows that our induction will be well defined, and Lemma 2.2 shows the way the toughness changes during the induction.

Lemma 2.1. *If G is a 2-connected chordal planar graph, then there is a sequence of graphs $G_0, \ldots, G_k$ and a sequence of sets $S_0, \ldots, S_{k-1}$ such that:*

(i) $G_0 = K_3$,
(ii) $V(G_{i+1}) = V(G_i) \cup S_i$, *where* $S_i \cap V(G_i) = \emptyset$, $\langle V(G_i)\rangle_{G_{i+1}} = G_i$, $N_{G_{i+1}}(S_i) \subset V(G_i)$ *and, for every* $x \in S_i$, $\langle N_{G_{i+1}}(x)\rangle_{G_{i+1}}$ *is complete,* $i = 0, \ldots, k-1$,
(iii) $G_k = G$.
(iv) *The integer k, the graphs G_i ($i = 0, \ldots, k$) and the sets S_i ($i = 0, \ldots, k-1$) can be chosen so that, for every $i = 0, \ldots, k-1$,*
   (A) *there is a vertex $v_i \in V(G_i)$ such that v_i is simplicial in G_i and $S_i \subset N_{G_{i+1}}(v_i)$;*
   (B) *if $x \in S_i$ is of degree $d_{G_{i+1}}(x) = 3$, then x lies in the inner face of the triangle $\langle N_{G_{i+1}}(x)\rangle_{G_{i+1}}$.*

Proof. First we show that there is a sequence $G_0, \ldots, G_k$ satisfying the statements (i), (ii), (iii) and (iv-A). By Theorem 2.1, $G = G_k$ has at least one simplicial vertex. Let S be the set of all simplicial vertices in G_k. By Theorem 2.1, the graph $G - S$ is a chordal graph and there exists a simplicial vertex x in $G - S$. Let S'_{k-1} be the set of all simplicial vertices in G_k adjacent to x. If all the vertices in S'_{k-1} are independent in G_k, then we set $S'_{k-1} = S_{k-1}$ and $v_{k-1} = x$. Otherwise there exist in S'_{k-1} two vertices u_1 and u_2 that are adjacent in G_k, and we set $S_{k-1} = \{u_1\}$ and $v_{k-1} = u_2$. By Theorem 2.1, the graph $G - S_{k-1} = G_{k-1}$ is again chordal, and the vertex v_{k-1} is simplicial in G_{k-1}. We can repeat this procedure until we obtain K_3. If we reverse this procedure we can construct an arbitrary chordal graph from K_3 such that the statements (i), (ii), (iii) and (iv-A) hold.

(B) Suppose that statement (iv-B) holds for every G_j, for $1 \le j \le i$. We prove that the statement also holds for G_{i+1}. If not, then there is a vertex $u_2 \in S_i$ of degree 3, which lies in the outer face of the triangle $\langle N_{G_{i+1}}(u_2)\rangle_{G_{i+1}}$. Note that $d_{G_i}(v_i) = 2$. Otherwise, we would get a contradiction with the planarity of G. Let v', v'' be the neighbours of v_i in G_i. If u_2 is the only vertex in S_i of degree 3 then we can place u_1 into the inner face of the triangle $\langle N_{G_{i+1}}(u_i)\rangle_{G_{i+1}}$. There cannot be three vertices of degree 3 in S_i, otherwise $K_{3,3}$ is a subgraph of G_{i+1}.

Next assume that there are two vertices of degree 3, namely, $u_1, u_2 \in S_{i+1}$. Since u_1 and u_2 are simplicial vertices in G_{i+1}, we conclude that $N(u_1)_{G_{i+1}} = N(u_2)_{G_{i+1}} = \{v_i, v', v''\}$. We may assume that u_1 lies in the inner face of the triangle $\langle\{v_i, v', v''\}\rangle_{G_i}$ and u_2 lies in its outer face. We separate the construction step from G_i to G_{i+1} into two steps G_i to G'_i and G'_i to G'_{i+1}, in such a way that the statement (iv-B) will hold. That is, we define $S'_i = \{u_2\}$ and $G'_{i+1} = \langle V(G_i) \cup \{u_2\}\rangle_{G_{i+1}}$. Additionally, we define $S'_{i+1} = S_i \setminus \{u_2\}$. We connect vertices from S'_{i+1} with vertices in G'_{i+1} as in G_{i+1}, but every vertex will be incident with u_1 instead of v_i. Now $G_{i+1} \cong G'_{i+2}$. See Fig. 1. □

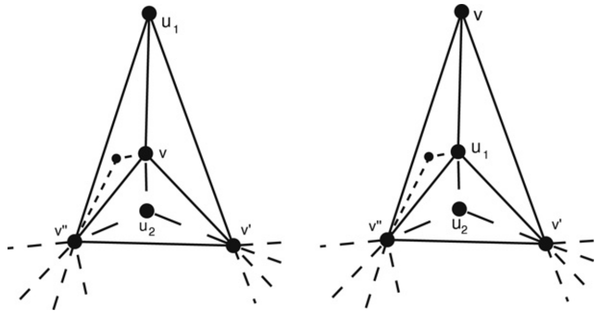

Fig. 1. Modification of construction.

Let G be a 2-connected chordal planar graph, and let $G_0, \ldots, G_k$ and $S_0, \ldots, S_{k-1}$ be any sequences of graphs and sets, respectively, satisfying the conditions of Lemma 2.1. We say that the sequence $(G_0, \ldots, G_k; S_0, \ldots, S_{k-1})$ is a *convenient construction of* G and, for any $x \in S_i$, $i = 0, \ldots, k-1$, a vertex v_i with the properties given in part (iv-A) of Lemma 2.1 will be said to be a *parent* of the vertex x, denoted $v_i = p(x)$. From Lemma 2.1, it is obvious that every vertex, except vertices in G_0, has exactly one parent. Furthermore, the vertex v_i is the parent of all vertices in S_i and there are no other vertices in G such that v_i is their parent. If $p(x) \notin V(G_0)$ then we write $p^2(x)$ to denote the parent of the parent of a vertex x. More generally, if $p^{j-1} \notin V(G_0)$ for some $j \geq 2$, then we write $p^j(x) = p(p^{j-1}(x))$.

In the rest of the paper we use the following notation: For an arbitrary nonsimplicial vertex u in a graph G_i from a convenient construction, we define an integer $\varphi(u)$, $0 \leq \varphi(u) < k$, as the integer such that u is simplicial in $G_{\varphi(u)}$ and is not simplicial in $G_{\varphi(u)+1}$ (i.e., we added some new simplicial vertices into the neighbourhood of u).

It is clear that for every vertex u from G, except the three vertices in G_0, the construction step $\varphi(p(u))$ is exactly the step in which vertex u was added into the graph $G_{\varphi(p(u))}$. Therefore, $u \in S_{\varphi(p(u))}$, for every $u \in V(G) \setminus V(G_0)$.

Lemma 2.2. *If* $(G_0, \ldots, G_k; S_0, \ldots, S_{k-1})$ *is a convenient construction of a t-tough chordal planar graph G, then each graph G_i for* $0 \leq i \leq k-1$, *is also t-tough.*

Proof. Let G_j be a graph from the convenient construction where $0 < j \leq k$. If there is a set of vertices P such that $\omega(G_j - P) < \omega(G_{j-1} - P)$, then there are two components C_1, C_2 of $G_{j-1} - P$ such that both C_1 and C_2 are in the same component of $G_j - P$. We get G_j by adding new simplicial vertices to G_{j-1}. There must be a simplicial vertex v in G_j which has two neighbours v_1, v_2, such that $v_1 \in C_1$ and $v_2 \in C_2$. This is a contradiction because v_1 and v_2 are not adjacent, which contradicts the fact that v is a simplicial vertex.

Hence for any subset of vertices P, $\omega(G_j - P) \geq \omega(G_{j-1} - P)$. Therefore, if G_j is t-tough, G_{j-1} is also t-tough. □

The following definitions will be useful in the proof of Theorem 1.4. If a graph G has a 2-walk T, then we can define, for every vertex v in G, the *multiplicity* of v in T as: $m_T(v) = 1$ if v is used once in the 2-walk T, and $m_T(v) = 2$ if v is used twice in the 2-walk T. For every vertex v with multiplicity $m_T(v) = 1$, the predecessor of the vertex v in the 2-walk T will be denoted v_T^- and the successor of v in T will be denoted v_T^+. Note that possibly $v_T^+ = v_T^-$. Also, for every vertex v with multiplicity $m_T(v) = 1$, we define $e_T(v) = |\{v_T^+, v_T^-\}|$.

Let G be a 2-connected chordal planar graph, and let $(G_0, \ldots, G_k; S_0, \ldots, S_{k-1})$ be its convenient construction. We say that a 2-walk T_i in a graph G_i $(0 \leq i \leq k)$ is a *good* 2-walk if there exists a sequence of 2-walks $T_0, \ldots T_i$ such that, T_j is a 2-walk in G_j, $0 \leq j \leq i$, with the following properties.

For every simplicial vertex x in G_i, different from vertices in G_0, we have

(i) $m_{T_i}(x) = 1$.
(ii) If $|S_{\varphi(x)}| < 4$ then $x_{T_i}^+ = p(x)$ or $x_{T_i}^- = p(x)$.
(iii) If $d_{G_i}(x) = 3$ and $e_{T_i}(x) = 1$ then $d_{G_{\varphi(p(x))}}(p(x)) = 3$ and
  (A) $e_{T_{\varphi(p(x))}}(p(x)) = 1$ and $p^2(x) \notin N_{G_i}(x)$ or
  (B) $e_{T_{\varphi(p(x))}}(p(x)) = 2$ and in the set $S_{\varphi(p(x))}$ there are three vertices of degree 3 in the graph $G_{\varphi(p(x))+1}$ (x is one of them) or
  (C) $e_{T_{\varphi(p(x))}}(p(x)) = 2$ and $x_{T_i}^+ \neq p(x)$ and $x_{T_i}^- \neq p(x)$.
(iv) If $d_{G_i}(x) = 3$, $e_{T_i}(x) = 2$, $m_{T_i}(p(x)) = 2$ and $x_{T_i}^- x_{T_i}^+ \notin E(T_i)$, then either:
  (A) $d_{G_{\varphi(p(x))}}(p(x)) = 2$ and $|S_{\varphi(p(x))}| = 2$, or
  (B) $d_{G_{\varphi(p(x))}}(p(x)) = 3$ and either
    • $|S_{\varphi(p(x))}| \geq 3$ or
    • $|S_{\varphi(p(x))}| = 2$ and there is a vertex $x' \in S_{\varphi(p(x))}$, $x' \neq x$, such that either
      – $|S_{\varphi(x')}| = 4$ or
      – $N_{G_{\varphi(x')}}(x') \subseteq N_{G_i}(x)$.
(v) Subject to the properties (i)–(iv), the number of simplicial vertices of degree 3 with $e_{T_i} = 2$ is maximal.

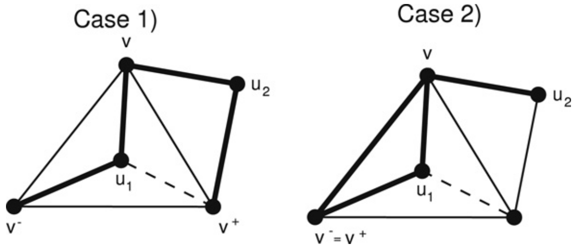

Fig. 2. Construction of a 2-walk in Cases 1 and 2.

Next we proceed with a crucial lemma. We prove this lemma using a induction and all the properties of a good 2-walk will be needed. In most of the cases we get a 2-walk in G_{i+1} only by local extensions of a good 2-walk in G_i. This would not be possible if we assumed that G_i had only a 2-walk.

Lemma 2.3. *Let G be a chordal planar graph with toughness greater than $\frac{3}{4}$. Let $(G_0, \dots, G_r; S_0, \dots, S_{r-1})$ be a convenient construction of G. If for some i with $0 \le i \le r-1$, all graphs G_ℓ have good 2-walks T_ℓ, for $\ell = \{0, 1, \dots, i\}$, then the graph G_{i+1} has a good 2-walk T_{i+1}.*

Proof. Since G_i is a chordal planar graph with toughness greater than $\frac{3}{4}$, all simplicial vertices in G_i have degree 2 or 3. Let v be a simplicial vertex in G_i such that all vertices $u_j \in S_i$ are incident with v in G_{i+1}.

Case 1. $d(v) = 2$, $|S_i| = 2$ and $e_{T_i}(v) = 2$.

Let $u_1, u_2 \in S_i$. Suppose first that $N_{G_{i+1}}(u_1) = N_{G_{i+1}}(u_2)$. Due to Lemma 2.1, statement (iv-B), and planarity of G_{i+1}, $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = 2$. Let $N_{G_{i+1}}(u_1) = N_{G_{i+1}}(u_2) = X$. Then the graph $G_{i+1} - X$ must have at least three components — a contradiction with the toughness of G_{i+1}. Hence $N_{G_{i+1}}(u_1) \ne N_{G_{i+1}}(u_2)$.

Since $e_{T_i}(v) = 2$, we may assume that $v, v^-_{T_i} \in N_{G_{i+1}}(u_1)$ and $v, v^+_{T_i} \in N_{G_{i+1}}(u_2)$. Hence the subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 2. We get T_{i+1} as follows: we remove from T_i the walk $v^-_{T_i}\ v\ v^+_{T_i}$ and replace it with the walk $v^-_{T_i}\ u_1\ v\ u_2\ v^+_{T_i}$.

Clearly the 2-walk T_{i+1} meets conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$. Since $e_{T_{i+1}}(u_1) = e_{T_{i+1}}(u_2) = 2$ and $m_{T_{i+1}}(v) = 1$, the 2-walk T_{i+1} also trivially satisfies conditions (iii) and (iv) of a good 2-walk. Hence T_{i+1} is a good 2-walk in G_{i+1} (see Fig. 2).

Case 2. $d(v) = 2$, $|S_i| = 2$ and $e_{T_i}(v) = 1$.

Set $S_i = \{u_1, u_2\}$. As in the proof of Case 1, $N_{G_{i+1}}(u_1) \ne N_{G_{i+1}}(u_2)$. Since $e_{T_i}(v) = 1$, we may assume that $v, v^-_{T_i} \in N_{G_{i+1}}(u_1)$ and $d_{G_{i+1}}(u_2) = 2$. Hence the subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 2. We get T_{i+1} as follows: we remove $v^-_{T_i}\ v\ v^+_{T_i}$ from T_i and replace it with $v^-_{T_i}\ u_1\ v\ u_2\ v\ v^+_{T_i}$.

Clearly, the 2-walk T_{i+1} meets conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$. Since $e_{T_{i+1}}(u_1) = 2$ and $d_{G_{i+1}}(u_2) = 2$, the 2-walk T_{i+1} also trivially satisfies condition (iii) of a good 2-walk. Furthermore, $m_{T_{i+1}}(v) = 2$ but $d_{G_{\varphi(v)}}(v) = 2$ and $|S_{\varphi(v)}| = 2$, therefore T_{i+1} meets condition (iv) as well. Hence T_{i+1} is a good 2-walk in G_{i+1} (see Fig. 2).

Case 3. $d(v) = 2$ and $|S_i| \ne 2$.

Then $|S_i| = 1$, otherwise we would get a contradiction with the toughness of G_{i+1}. We get a 2-walk T_{i+1} in a similar way as in Case 1 or 2. Observe that if the vertex $u_1 \in S_i$ has degree 3 in G_{i+1}, there always exists a 2-walk T_{i+1} in G_{i+1} such that $e_{T_{i+1}}(u_1) = 2$. Hence, there always exists a good 2-walk T_{i+1}.

Case 4. $d(v) = 3$ and $|S_i| \le 3$.

Similarly as in Case 1, for every $u_a \ne u_b$ from the set S_i, $N_{G_{i+1}}(u_a) \ne N_{G_{i+1}}(u_b)$.

Subcase 4.1. There is at most one vertex $u \in S_i$ such that $\{v^-_{T_i}, v^+_{T_i}\} \cap N_{G_{i+1}}(u) = \emptyset$.

We prove the existence of a good 2-walk T_{i+1} in G_{i+1} separately for $|S_i| = 3$, $|S_i| = 2$ and $|S_i| = 1$.

<u>*Subcase* 4.1.1:</u> $|S_i| = 1$.

Let $S_i = \{u_1\}$. Note that the vertex u_1 is adjacent in G_{i+1} to v and one or two vertices in $N_{G_i}(v)$.

- u_1 is adjacent to $v^-_{T_i}$ or $v^+_{T_i}$ in G_{i+1}.

  Without loss of generality, we may assume that u_1 is adjacent to $v^-_{T_i}$. If not, then change the orientation of T_i. We get T_{i+1} as follows: we remove $v^-_{T_i}\ v$ from T_i and we replace it with $v^-_{T_i}\ u_1\ v$. Observe that $e_{T_{i+1}}(u_1) = 2$ and $m_{T_{i+1}}(v) = 1$. Clearly, T_{i+1} is a good 2-walk in G_{i+1}.

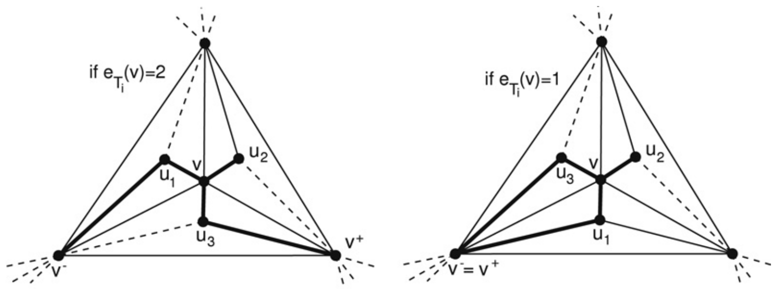

Fig. 3. Construction of a 2-walk in Case 4.1.3.

- u_1 is not adjacent to $v^-_{T_i}$ and $v^+_{T_i}$ in G_{i+1}

  We get T_{i+1} as follows: we remove $v^-_{T_i}\ v\ v^+_{T_i}$ from T_i and we replace it with $v^-_{T_i}\ v\ u_1\ v\ v^+_{T_i}$. Observe that $e_{T_{i+1}}(u_1) = 1$ and $m_{T_{i+1}}(v) = 2$. Clearly, the 2-walk T_{i+1} meets conditions (i), (ii), (iv) and (v) of a good 2-walk. If $d_{G_{i+1}}(u_1) = 3$ then $v^-_{T_i} = v^+_{T_i}$. Hence $e_{T_i}(v) = 1$ and T_{i+1} meets condition (iii) as well.

<u>*Subcase* 4.1.2:</u> $|S_i| = 2$.

Let $S_i = \{u_1, u_2\}$. Due to the assumption of Subcase 4.1, u_1 or u_2 is adjacent to $v^-_{T_i}$ or $v^+_{T_i}$ in G_{i+1}. Without loss of generality, we may assume that u_1 is adjacent to $v^-_{T_i}$. Otherwise, we change the orientation of T_i. We distinguish two cases.

- u_2 is adjacent in G_{i+1} to $v^+_{T_i}$.

  We get T_{i+1} as follows: we remove $v^-_{T_i}\ v\ v^+_{T_i}$ from T_i and we replace it with $v^-_{T_i}\ u_1\ v\ u_2\ v^+_{T_i}$. Observe that $e_{T_{i+1}}(u_1) = 2$, $e_{T_{i+1}}(u_2) = 2$ and $m_{T_{i+1}}(v) = 1$. Clearly, T_{i+1} is a good 2-walk in G_{i+1}.
- u_2 is not adjacent to $v^+_{T_i}$.

  We may assume that, if u_2 is adjacent to $v^-_{T_i}$ then $d_{G_{i+1}}(u_1) = 3$, otherwise we relabel vertices in S_i. Note that, in this case, both vertices u_1 and u_2 cannot have degree 2 in G_{i+1}, otherwise we would get a contradiction with the toughness of G_{i+1}. We get T_{i+1} as follows: we remove $v^-_{T_i}\ v\ v^+_{T_i}$ from T_i and we replace it with $v^-_{T_i}\ u_1\ v\ u_2\ v\ v^+_{T_i}$. Observe that $e_{T_{i+1}}(u_1) = 2$, $e_{T_{i+1}}(u_2) = 1$ and $m_{T_{i+1}}(v) = 2$. Clearly, the 2-walk T_{i+1} meets conditions (i), (ii) and (v) of a good 2-walk.

  Note that $d_{G_{i+1}}(u_2) = 3$ only if $v^-_{T_i} = v^+_{T_i}$. Hence $e_{T_i}(v) = 1$ and T_{i+1} meets condition (iii) of a good 2-walk.

  Recall that $e_{T_{i+1}}(u_1) = 2$ and $m_{T_{i+1}}(p(u_1)) = 2$. Now, we distinguish two cases.

  (A) $v^-_{T_i} = v^+_{T_i}$.

  Observe that $u_1{}^+_{T_{i+1}} = v$ and $u_1{}^-_{T_{i+1}} = v^+_{T_i} = v^-_{T_i}$. Since the edge $v\ v^+_{T_i} \in E(T_{i+1})$, T_{i+1} meets condition (iv) of a good 2-walk.

  (B) $v^-_{T_i} \neq v^+_{T_i}$.

  Since $d_{G_{i+1}}(u_1) = 3$ and vertex u_2 is not adjacent to $v^+_{T_i}$, $d_{G_{i+1}}(u_2) = 2$. Moreover, $N_{G_{i+1}}(u_2) \subseteq N_{G_{i+1}}(u_1)$. Hence T_{i+1} meets condition (iv) of a good 2-walk.

<u>*Subcase* 4.1.3:</u> $|S_i| = 3$.

Let $S_i = \{u_1, u_2, u_3\}$. There are no vertices $v_a, v_b \in N_{G_i}(v)$ such that $N_{G_{i+1}}(S_i) = \{v_a, v_b, v\}$ since otherwise the graph $G_{i+1} - \{v, v_a, v_b\}$ has exactly four components — a contradiction with the toughness of G_{i+1}. In other words, it means that $|N_{G_{i+1}}(S_i)| = 4$.

Now we can rename the vertices in S_i such that $v^-_{T_i} \in N_{G_{i+1}}(u_1)$ and $v^+_{T_i} \in N_{G_{i+1}}(u_3)$. Moreover, if $e_{T_i}(v) = 2$, we may assume that if there is a vertex in S_i of degree 2 in G_{i+1} then it is the vertex u_2. If $e_{T_i}(v) = 1$, then we rename vertices in such a way that $d_{G_{i+1}}(u_2) = 3$ if and only if u_2 is not adjacent to $v^-_{T_i} = v^+_{T_i}$ in G_{i+1}.

Now the subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 3. We get T_{i+1} as follows: we remove $v^-_{T_i}\ v\ v^+_{T_i}$ from T_i and we replace it with $v^-_{T_i}\ u_1\ v\ u_2\ v\ u_3\ v^+_{T_i}$.

Clearly, the 2-walk T_{i+1} meets conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$ and $e_{T_{i+1}}(u_2) = 1$. Vertex u_2 has degree 3 in G_{i+1} if and only if either

(A) $e_{T_i}(v) = 1$ and u_2 is not adjacent to $p^2(x) = v^-_{T_i} = v^+_{T_i}$ in G_{i+1} or
(B) $e_{T_i}(v) = 2$ and $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = 3$.

Hence the 2-walk T_{i+1} also satisfies condition (iii) of a good 2-walk. Furthermore, $m_{T_{i+1}}(v) = 2$ but $d_{G_{\varphi(v)}}(v) = 3$ and $|S_{\varphi(v)}| = 3$, therefore T_{i+1} meets (iv) as well. Hence T_{i+1} is a good 2-walk in G_{i+1} (see Fig. 3).

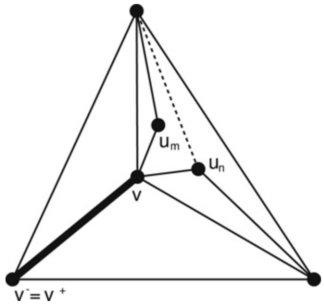

Fig. 4. Subcase 4.2.

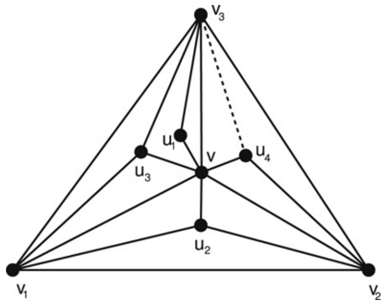

Fig. 5. Case 5.

Note that we proved a slightly stronger statement. One vertex, let us say u_2, from S_i has $e_{T_{i+1}}(u_2) = 1$. If $e_{T_i}(v) = 2$ and all the vertices in S_i have degree 3 in G_{i+1}, then we can choose the vertex u_2 from S_i arbitrarily. Hence we can get three different good 2-walks in G_{i+1}. Since we use this observation later, we state it as a claim.

Claim 2.1. *Under the assumption of Subcase 4.1.3, if* $e_{T_i}(v) = 2$ *and* $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = 3$, *there exist three different good 2-walks* T_{i+1}, T'_{i+1} *and* T''_{i+1} *in* G_{i+1} *such that* $e_{T_{i+1}}(u_2) = 1$, $e_{T'_{i+1}}(u_1) = 1$ *and* $e_{T''_{i+1}}(u_3) = 1$.

Subcase 4.2. There are $u_m, u_n \in S_i$ such that $\{v^-_{T_i}, v^+_{T_i}\} \cap N_{G_{i+1}}(u_m) = \emptyset$ and $\{v^-_{T_i}, v^+_{T_i}\} \cap N_{G_{i+1}}(u_n) = \emptyset$.

Now $e_{T_i}(v) = 1$ and the subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 4. In this case we cannot simply extend the good 2-walk T_i on the new vertices in S_i. We postpone the proof of this subcase. Later, together with Subcase 5.2, and after Subcase 5.1, we show that there exists a good 2-walk T^*_i in G_i, such that $e_{T^*_i}(v) = 2$. Then we transform Subcase 4.2 back to Subcase 4.1.

Case 5. $d(v) = 3$ and $|S_i| \geq 4$.

Now $|S_i| = 4$, otherwise we would get a contradiction with the toughness of G_{i+1}. As in Case 1, $N_{G_i}(u_a) \neq N_{G_i}(u_b)$, for every $u_a, u_b \in S_i$.

If S'_i is an arbitrary subset of S_i, such that $|S'_i| = 3$, then there is no vertex $v' \in N_{G_i}(v)$, such that $\{v'\} \cap N_{G_{i+1}}(S_i) = \emptyset$. If not, then for the set $X = \{v\} \cup N_{G_i}(v) \setminus \{v'\}$ we have $|X| = 3$ and the graph $G_{i+1} - X$ has exactly four components — a contradiction with the toughness of G_{i+1}. The subgraph $\langle N_{G_{i+1}}(v)\rangle_{G_{i+1}}$ has the structure shown in Fig. 5 (up to a symmetry).

Subcase 5.1. $e_{T_i}(v) = 2$.

Let $S_i = \{u_1, u_2, u_3, u_4\}$. We first prove that $m_{T_i}(p(v)) = 1$ or the edge $v^-_{T_i} v^+_{T_i}$ is in the 2-walk T_i in G_i.

Suppose to the contrary that $m_{T_i}(p(v)) = 2$ and $v^-_{T_i}\ v^+_{T_i} \notin E(T_i)$. Due to the properties of a good 2-walk T_i (property (iv)), we have the following cases:

(A) $d_{G_{\varphi(p(v))}}(p(v)) = 2$ and $|S_{\varphi(p(v))}| = 2$.

If $X = N_{G_i}(v) \cup \{v\}$, then $|X| = 4$ and the graph $G_{i+1} - X$ has at least six components, namely, four isolated vertices from S_i, one component with the other vertex from $S_{\varphi(p(v))}$ and the rest of the graph. This contradicts the toughness of G_{i+1}.

(B) $d_{G_{\varphi(p(v))}}(p(v)) = 3$ and

- $|S_{\varphi(p(v))}| \geq 3$.

  If $X = N_{G_{\varphi(p(v))}}(p(v)) \cup \{v, p(v)\}$. $|X| = 5$, then the graph $G_{i+1} - X$ has at least seven components, namely, four isolated vertices from S_i, two components each containing a vertex from $S_{\varphi(p(v))}$ different from v, and the rest of the graph. This contradicts the toughness of G_{i+1}.

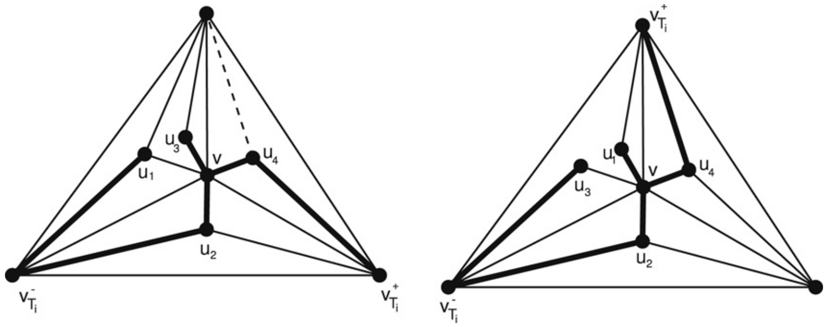

Fig. 6. Construction of a 2-walk when $|S_i| = 4$ and $m_{T_i}(p(v)) = 1$.

- $|S_{\varphi(p(v))}| = 2$ and there is a vertex $v' \in S_{\varphi(p(v))}$, $v' \neq v$, such that
  - $|S_{\varphi(v')}| = 4$.
    If $X = N_{G_{\varphi(p(v))}}(p(v)) \cup \{v, v', p(v)\}$, then $|X| = 6$ and the graph $G - X$ has at least nine components, namely, four isolated vertices from S_i, four components each containing a vertex from $S_{\varphi(v')}$, and the rest of the graph. This contradicts the toughness of G_{i+1}.
  - $N_{G_{\varphi(v')}}(v') \subseteq N_{G_i}(v)$.
    If $X = N_{G_{i+1}}(v) \cup \{v\}$, then $|X| = 4$ and the graph $G - X$ has at least six components, namely, four isolated vertices from S_i, one component with v' and the rest of the graph. This contradicts the toughness of G.

Therefore, $m_{T_i}(p(v)) = 1$ or the edge $v^-_{T_i}v^+_{T_i}$ is in the 2-walk T_i in G_i. We obtain a good 2-walk T_{i+1} as follows:

- If $m_{T_i}(p(v)) = 1$, then we choose orientation of T_i, such that $p(v) = v^-_{T_i}$ (see the property of a good 2-walk (ii)). We label vertices in S_i such that, if there is a vertex of degree 2 adjacent to $v^-_{T_i}$ in G_{i+1}, then we name this vertex u_1, and then we rename the rest of S_i such that $v^-_{T_i} \in N_{G_{i+1}}(u_2)$, $v^+_{T_i} \in N_{G_{i+1}}(u_4)$ and u_3 is the remaining vertex. We may assume that $d_{G_{i+1}}(u_2) = 3$ and $d_{G_{i+1}}(u_4) = 3$.
  If there is no vertex of degree 2 adjacent to $v^-_{T_i}$ in G_{i+1}, then we take an arbitrary vertex from S_i of degree 3 in G_{i+1}, which is incident with $v^-_{T_i}$ in G_{i+1}, and we label this vertex u_1. Rename the rest of S_i such that $v^-_{T_i} \in N_{G_{i+1}}(u_2)$, $v^+_{T_i} \in N_{G_{i+1}}(u_4)$ and u_3 is the remaining vertex. Note that the degree of u_3 is 2 in G_{i+1}.
  We get T_{i+1} as follows : we remove $v^-_{T_i}vv^+_{T_i}$ from T_i and we replace it with $v^-_{T_i}u_1v^-_{T_i}u_2vu_3vu_4v^+_{T_i}$.
  Since $|S_i| = 4$, T_{i+1} meets conditions (i), (ii) and (v) of a good 2-walk. Note that $p(u_1) = p(u_2) = v$ and $e_{T_{i+1}}(u_1) = e_{T_{i+1}}(u_3) = 1$. If $d_{G_{i+1}}(u_1) = 2$, then $d_{G_{i+1}}(u_3) = 3$ if and only if $d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = d_{G_{i+1}}(u_4) = 3$. If $d_{G_{i+1}}(u_1) = 3$, then $d_{G_{i+1}}(u_3) = 2$. Therefore, the 2-walk T_{i+1} satisfies either condition (iii-B) or condition (iii-C) of a good 2-walk. Furthermore, $m_{T_{i+1}}(v) = 2$ and $d_{G_{\varphi(v)}}(v) = 3$, but $|S_{\varphi(v)}| = 4$. Therefore T_{i+1} meets (iv) as well. Hence T_{i+1} is a good 2-walk in G_{i+1}.
- If $m_{T_i}(p(v)) = 2$ and the edge $v^-_{T_i}v^+_{T_i}$ is in the 2-walk T_i in G_i, then we have the following cases:
  (A) If there is a vertex u_a in S_i such that $\{v^-_{T_i}, v^+_{T_i}\} \subset N_{G_{i+1}}(u_a)$, then we relabel vertices in S_i in the following way: $u_1 = u_a$, $v^-_{T_i} \in N_{G_{i+1}}(u_2)$, $v^+_{T_i} \in N_{G_{i+1}}(u_4)$ and u_3 is the remaining vertex. Clearly, the degree of u_1 is 3 in G_{i+1} and we may assume that $d_{G_{i+1}}(u_3) = 2$. We obtain T_{i+1} as follows: we remove $v^-_{T_i}v^+_{T_i}$ and $v^+_{T_i}vv^-_{T_i}$ from T_i and we replace it by $v^-_{T_i}u_1v^+_{T_i}$ and $v^+_{T_i}u_2vu_3vu_4v^-_{T_i}$.
  Clearly, the 2-walk T_{i+1} meets conditions (i), (ii) (iv) and (v) of a good 2-walk. Note that $e_{T_{i+1}}(u_3) = 1$ but $d_{G_{i+1}}(u_3) = 2$. Therefore the 2-walk T_{i+1} satisfies condition (iii) of a good 2-walk. Hence T_{i+1} is a good 2-walk in G_{i+1}.
  (B) If there is no vertex u_a in S_i such that $\{v^-_{T_i}, v^+_{T_i}\} \subset N_{G_{i+1}}(u_a)$, then every vertex in S_i is adjacent to either $v^+_{T_i}$ or $v^-_{T_i}$ in G_{i+1}. Moreover, due to the toughness condition, there are exactly two vertices from S_i adjacent to $v^+_{T_i}$ in G_{i+1} and the other two vertices in S_i are adjacent to $v^-_{T_i}$ in G_{i+1}. Relabel vertices from S_i in the following way: $v^-_{T_i} \in N_{G_{i+1}}(u_1)$, $v^+_{T_i} \in N_{G_{i+1}}(u_2)$, $v^+_{T_i} \in N_{G_{i+1}}(u_3)$ and $v^-_{T_i} \in N_{G_{i+1}}(u_4)$. We obtain T_{i+1} as follows: we remove $v^-_{T_i}v^+_{T_i}$ and $v^+_{T_i}, v, v^-_{T_i}$ from T_i and we replace it by $v^-_{T_i}u_1vu_2v^+_{T_i}$ and $v^+_{T_i}u_3vu_4v^-_{T_i}$.
  Observe that $e_{T_{i+1}}(u_1) = 2$, $e_{T_{i+1}}(u_2) = 2$, $e_{T_{i+1}}(u_3) = 2$, $e_{T_{i+1}}(u_4) = 2$ and $m_{T_{i+1}}(v) = 2$. Since $|S_i| = 4$, the 2-walk T_{i+1} satisfies all the conditions (i)–(v) of a good 2-walk.

See examples of 2-walk T_{i+1} in G_{i+1}, for $m_{T_i}(p(v)) = 1$, in Fig. 6 and, for $m_{T_i}(p(v)) = 2$, in Fig. 7.

Before we move to another subcase, we summarize when a vertex from S_i, let us say u_3, has $d_{G_{i+1}}(u_3) = 3$ and $e_{T_{i+1}}(u_3) = 1$. It happens only if $m_{T_i}(p(v)) = 1$ in the two following cases.

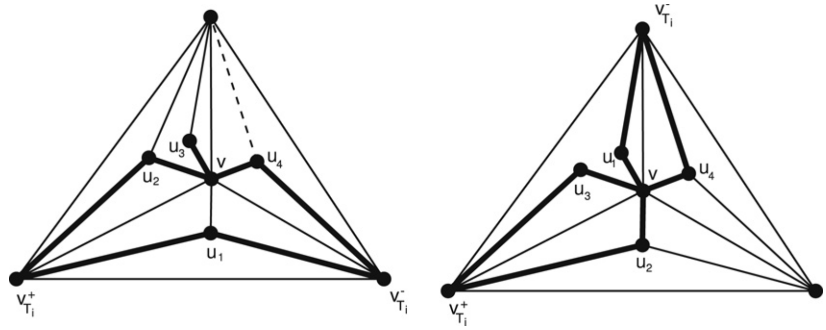

Fig. 7. Construction of a 2-walk when $|S_i| = 4$ and $m_{T_i}(p(v)) = 2$.

- There is a vertex $u_1 \in S_i$ of degree 2 in G_{i+1} adjacent to $v^-_{T_i} = p(v)$.
  See that, $e_{T_{i+1}}(u_3) = 1$ if and only if all the vertices in S_i, except for u_1, have degree 3 in G_{i+1}. Now we can choose the vertex u_3, with $e_{T_{i+1}}(u_3) = 1$, arbitrarily from $S_i \setminus \{u_1\}$. Hence we can get three different good 2-walks in G_{i+1}. Since we use this observation later, we state it as a claim.

Claim 2.2. *Under the assumption of Subcase 5.1, if* $m_{T_i}(p(v)) = 1$ *and there is a vertex* $u_1 \in S_i$ *of degree 2 in* G_{i+1} *adjacent to* $v^-_{T_i} = p(v)$ *and* $d_{G_{i+1}}(u_2) = d_{G_{i+1}}(u_3) = d_{G_{i+1}}(u_4) = 3$, *there exist three different good 2-walks* T_{i+1}, T'_{i+1} *and* T''_{i+1} *in* G_{i+1} *such that* $e_{T_{i+1}}(u_3) = 1$, $e_{T'_{i+1}}(u_2) = 1$ *and* $e_{T''_{i+1}}(u_4) = 1$.

- There is no such vertex (i.e., vertex from S_i of degree 2 in G_{i+1} and adjacent to $v^-_{T_i} = p(v)$).
  See that, there are two vertices, let us say $u_1, u_2 \in S_i$, adjacent to $v^-_{T_i} = p(v)$ in G_{i+1} and $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = 3$. Clearly, either $e_{T_{i+1}}(u_1) = 1$ and $e_{T_{i+1}}(u_2) = 2$, or $e_{T_{i+1}}(u_1) = 2$ and $e_{T_{i+1}}(u_2) = 1$. Hence we can get two different good 2-walks in G_{i+1}. We also use this observation later.

Claim 2.3. *Under the assumption of Subcase 5.1, if* $m_{T_i}(p(v)) = 1$ *and there is no vertex of degree 2 in* G_{i+1} *from* S_i *adjacent to* $v^-_{T_i} = p(v)$, *then there are two vertices* $u_1, u_2 \in S_i$, $d_{G_{i+1}}(u_1) = d_{G_{i+1}}(u_2) = 3$, *adjacent to* $v^-_{T_i} = p(v)$ *in* G_{i+1}. *Then there exist two different good 2-walks* T_{i+1} *and* T'_{i+1} *in* G_{i+1} *such that* $e_{T_{i+1}}(u_1) = 1$ *and* $e_{T'_{i+1}}(u_2) = 1$.

Subcase 5.2. $e_{T_i}(v) = 1$.

In this case we cannot simply extend the good 2-walk T_i on the new vertices in S_i. Due to property (i) of a good 2-walk, we should use vertex v more than twice, which is impossible. So we need to show that there exists a good 2-walk T^*_i in G_i, such that $e_{T^*_i}(v) = 2$. Then we transform Subcase 5.2 to Subcase 5.1. This will be done together with Subcase 4.2.

Claim 2.4. *Let W be the class of all good 2-walks in G_i. Under the assumptions of Subcase 4.2 or 5.2 there exists a good 2-walk* $T^*_i \in W$, *such that* $e_{T^*_i}(v) = 2$.

Proof. Suppose, to the contrary, that for every good 2-walk T_i from $We_{T_i}(v) = 1$. In the graph G_0, any vertex x has $e_{T_0}(x) = 2$. Thus there is an integer k such that the vertex $p^k(v)$ exists and satisfies

$$e_{T_{\varphi(p^k(v))}}(p^k(v)) = 2.$$

Suppose that the good 2-walk $T_i \in W$ is chosen such that the integer k is the smallest possible.

Denote the vertices $p^j(v)$ as w_j, denote the graphs $G_{\varphi(p^j(v))}$ as G'_j, denote the sets $S_{\varphi(p^j(v))}$ as S'_j, and denote the walks $T_{\varphi(p^j(v))}$ as T'_j, for $j = \{1, \dots, k\}$.

Due to property (iii) of a good 2-walk, we have:

$$d_{G'_j}(w_j) = 3, \quad j = \{1, \dots, k\}$$
$$e_{T'_j}(w_j) = 1, \quad j = \{1, \dots, k-1\}$$
$$w_{j+2} \notin N_{G'_j}(w_j), \quad j = \{1, \dots, k-2\}.$$

Since $e_{T'_k}(w_k) = 2$ and $e_{T'_{k-1}}(w_{k-1}) = 1$, there are three vertices in S'_k of degree 3 in G'_{k-1}. We will call the path $v, w_1, \dots, w_k$ a *critical path* (i.e., a critical path is a path starting at a vertex v, v satisfying the assumption of Subcase 4.2 or 5.2, and ending at a vertex w_k, $e_{T_{\varphi(w_k)}}(w_k) = 2$, where $p(w_i) = w_{i+1}$). Now we consider two cases: (A) $|S'_k| = 3$ and (B) $|S'_k| = 4$.

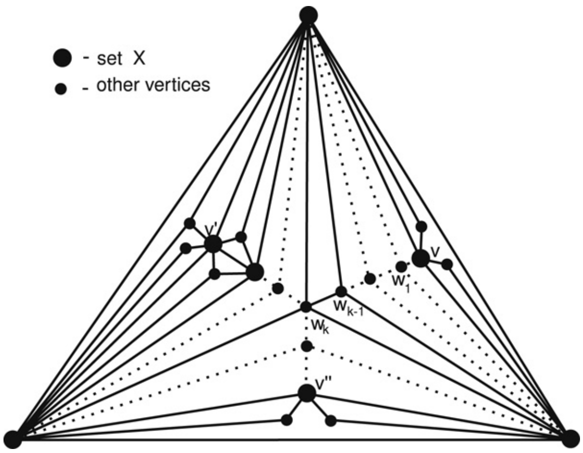

Fig. 8. Example of three critical paths ending at vertex w_k; Case (A).

(A) If $|S'_k| = 3$, then all the vertices in S'_k have degree 3 in G'_{k-1}. Let $S'_k = \{w_{k-1}, w'_{k-1}, w''_{k-1}\}$ and let $N_{G'_k}(w_k) = \{x_1, x_2, x_3\}$, $N_{G'_{k-1}}(w_{k-1}) = \{w_k, x_1, x_2\}$, $N_{G'_{k-1}}(w'_{k-1}) = \{w_k, x_1, x_3\}$, $N_{G'_{k-1}}(w''_{k-1}) = \{w_k, x_2, x_3\}$. Using Claim 2.1, we can choose another good 2-walk T^*_{k-1} such that either $e_{T^*_{k-1}}(w'_{k-1}) = 1$ or $e_{T^*_{k-1}}(w''_{k-1}) = 1$. Clearly $e_{T^*_{k-1}}(w_{k-1}) = 2$ and therefore, we cannot obtain any critical path starting at the vertex v in G_{i+1}. If not, then such a critical path would end at the vertex w_{k-1} at the latest, which is impossible due to our choice of T_i. We need to show that for such a T^*_{k-1} there exists a good 2-walk T^*_i in G_i, i.e. we will not get any critical path ending at vertex w_k in some graph G_ℓ, for $1 < \ell \le i$. Recall that we can choose T^*_{k-1} in G'_{k-1} such that either $e_{T^*_{k-1}}(w'_{k-1}) = 1$ or $e_{T^*_{k-1}}(w''_{k-1}) = 1$.

Assume otherwise, i.e., for both choices of T^*_{k-1} in G'_{k-1} we obtain a critical path ending at the vertex w_k. If $e_{T^*_{k-1}}(w'_{k-1}) = 1$, then we denote this critical path $v', w'_1, \ldots, w'_a$, where $w'_a = w_k$ and $w'_{a-1} = w'_{k-1}$. If $e_{T^*_{k-1}}(w''_{k-1}) = 1$, then we denote the critical path $v'', w''_1, \ldots, w''_b$, where $w''_b = w_k$ and $w''_{b-1} = w''_{k-1}$. Observe that all the vertices $v', w'_1, \ldots, w'_{a-1}$ lie inside the triangle w_k, x_1, x_3 and the vertex v' is adjacent to w'_1, x_1 and x_3 in $G_{\varphi(v')}$. Similarly, all the vertices $v'', w''_1, \ldots, w''_{b-1}$ lie inside the triangle w_k, x_2, x_3 and the vertex v'' is adjacent to w''_1, x_2 and x_3 in $G_{\varphi(v'')}$. Recall that the original critical path $v, w_1, \ldots, w_k$ lies inside the triangle w_k, x_1, x_2 and the vertex v is adjacent to w_1, x_1 and x_2 in G_i.

Now we show that G_{i+1} must have toughness less than or equal to $\frac{3}{4}$. We define a set of vertices X as follows. $X = N_{G'_k}(w_k) \cup \{v, v', v''\}$. If v satisfies the assumptions of Subcase 5.2, add the vertex w_1 into the set X. If v' or v'' satisfy the assumptions of Subcase 5.2, add the vertex w'_1 or w''_1 into the set X. If we remove X from G, then the number of components of $G - X$ will be greater than $\frac{4}{3}|X|$, which contradicts the toughness of G (see Fig. 8). Hence, we can obtain at most two critical paths ending at the vertex w_k. Therefore we can choose T^*_{k-1} in G'_{k-1} such that in the graph G_i there exists a good 2-walk T^*_i with $e_{T^*_i}(v) = 2$.

(B) If $|S'_k| = 4$. Let $N_{G'_k}(w_k) = \{p(w_k), x_2, x_3\}$ and $N_{G'_{k-1}}(w_{k-1}) = \{p(w_k), w_k, x_2\}$. First we show that v satisfies only the assumption of Subcase 4.2.

Assume otherwise, i.e., v satisfies the assumption of Subcase 5.2. We define a set X as follows : $X = N_{G'_k}(w_k) \cup \{w_k, v, w_1\}$. Then $|X| = 6$ because w_k is a simplicial vertex of degree 3 in G'_k. Graph $G_{i+1} - X$ must have at least eight components, namely isolated vertices from S_i, three components, each containing a vertex from S'_k, and the rest of the graph. This contradicts the toughness assumption.

There are two possible ends of the critical path at vertex w_k. One possible end is that $(w_{k-1})^-_{T'_{k-1}} = w_k$. The second is that $(w_{k-1})^-_{T'_{k-1}} = p(w_k)$ (see Fig. 9).

The first case is similar to case (A) (i.e., $|S'_k| = 3$), hence the proof is also similar (just instead of using Claim 2.1 we use Claim 2.2). Consider the second case. Since $d_{G'_{k-1}}(w_k) = 3$ and $(w_{k-1})^-_{T'_{k-1}} = p(w_k)$, we can use Claim 2.3. Therefore, there exists another good 2-walk T^*_{k-1} in G'_{k+1}, different from T'_{k-1}. Clearly $e_{T^*_{k-1}}(w_{k-1}) = 2$ and therefore, we cannot obtain any critical path starting at the vertex v in G_{i+1}. Otherwise, such a critical path would end at the vertex w_{k-1} at the latest, which is impossible due to our choice of T_i. We need to show that for such T^*_{k-1} there exists a good 2-walk T^*_i in G_i, i.e., that we will not get any critical path ending at vertex w_k in some graph G_ℓ, for $1 < \ell \le i$.

Assume otherwise, i.e., for the good 2-walk T^*_{k-1} in G'_{k-1} we obtain a critical path ending at w_k in the graph G_ℓ. Let $v', w'_1, \ldots, w'_a$ be this critical path, where $w'_a = w_k$, and $w'_{a-1} = w'_{k-1}$. Similarly as for v, v' satisfies only the assumption of Subcase 4.2. Observe that all the vertices $v', w'_1, \ldots, w'_{a-1}$ lie inside the triangle $w_k, p(w_k), x_3$ and the vertex v' is adjacent to w'_1, w_k and x_3 in $G_{\varphi(v')}$. Recall that the original critical path $v, w_1, \ldots, w_k$ lies inside the triangle $w_k, p(w_k), x_2$ and the vertex v is adjacent to w_1, w_k and x_2 in G_i.

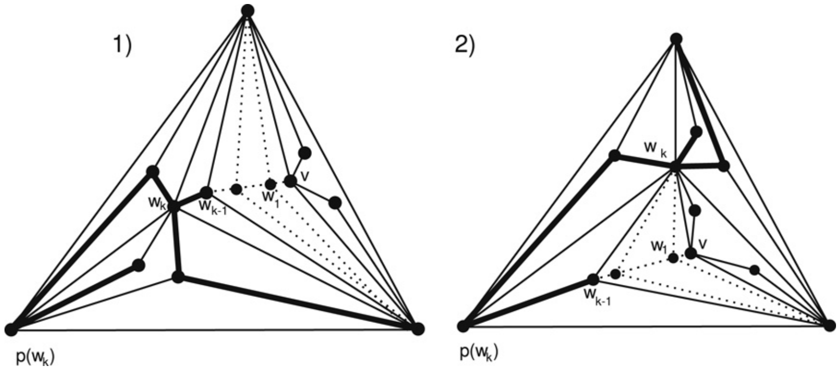

Fig. 9. Examples of two different critical paths in Case (B).

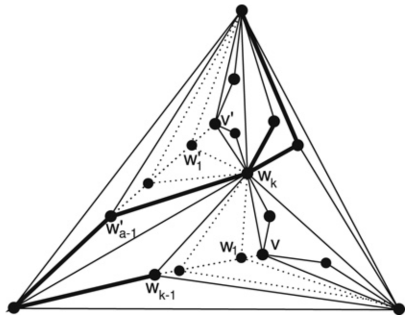

Fig. 10. Two critical paths ending at vertex w_k; Case (B).

Note that the graph $G - \{v', w_k, x_3\}$ has four components, namely two isolated vertices from the set S_ℓ, one isolated vertex from the set S'_k and the rest of the graph — a contradiction with the toughness assumption (see Fig. 10). Hence, we cannot obtain a critical path ending at the vertex w_k. Therefore, for the good 2-walk T^*_{k-1} in G'_{k-1}, there exists a good 2-walk T^*_i in G_i with $e_{T^*_i}(v) = 2$. □

At this stage we have finished the proof of Subcases 4.2 and Subcase 5.2. It follows that we have finished the proof of Lemma 2.3, since we have discussed all possible sets S_i. □

Since the graph K_3 has a 2-walk, proof of Theorem 1.4 follows immediately from Lemma 2.3.

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