Bartholdi zeta and $L$-functions of weighted digraphs, their coverings and products

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Abstract

Since a zeta function of a regular graph was introduced by Ihara [Y. Ihara, On discrete subgroups of the two by two projective linear group over $p$-adic fields, J. Math. Soc. Japan 19 (1966) 219–235], many kinds of zeta functions and $L$-functions of a graph or a digraph have been defined and investigated. Most of the works concerning zeta and $L$-functions of a graph contain the following: (1) defining a zeta function, (2) defining an $L$-function associated with a (regular) graph covering, (3) providing their determinant expressions, and (4) computing the zeta function of a graph covering and obtaining its decomposition formula as a product of $L$-functions. As a continuation of those works, we introduce a zeta function of a weighted digraph and an $L$-function associated with a weighted digraph bundle. A graph bundle is a notion containing a cartesian product of graphs and a (regular or irregular) graph covering. Also we provide determinant expressions of the zeta function and the $L$-function. Moreover, we compute the zeta function of a weighted digraph bundle and obtain its decomposition formula as a product of the $L$-functions.

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1. Introduction

1.1. Background

A zeta function is one of famous special functions in mathematics. It is likely that zeta functions have received the name as counting functions for mathematical objects—for instance, prime numbers, ideals of rings, periodic orbits, etc. Zeta functions of graphs which are our main objects are set up to count some special cycles in graphs and digraphs. The origin of all zeta functions is the Riemann zeta function and it has many generalizations including $L$-functions. In particular the Artin $L$-function combines zeta functions and group representations. It is also possible to combine zeta functions of (di)graphs and group representations, and to define $L$-functions of (di)graphs.

Zeta functions of graphs appeared first from the point of view of $p$-adic groups in work of Ihara [12] in 1966. He introduced a zeta function of a regular graph and showed that it is a rational function. And then, Bass [3] generalized the Ihara’s result on a regular graph to a general graph possibly irregular. In 1999, Bartholdi [2] introduced a zeta function with two variables, which can be regarded as a generalization of the Ihara zeta function, and proved that it is also a rational function.

The Ihara type $L$-function of a graph was developed by Sunada [31] and Hashimoto ([9–11]). And the Bartholdi type $L$-function of a graph was defined by Mizuno and Sato [19] in 2003. So far, many researchers have developed the theory of zeta functions of graphs [4,5,7,15–19,21,26,28–30].

In connection with digraphs, Mizuno and Sato [16] introduced the Ihara type zeta function and the $L$-function of a digraph in 2001, and also [17] those of a weighted digraph in 2002. They did the same work for the Bartholdi zeta function of a digraph in [18].

In this paper, we introduce a Bartholdi zeta function of a weighted digraph and an $L$-function associated with a weighted digraph bundle. In Section 2, determinant expressions of a Bartholdi zeta function of a weighted digraph are obtained. In Section 3, we introduce a weighted digraph covering and compute its Bartholdi zeta function. Actually we do our works with a weighted digraph bundle which is a notion containing weighted digraph coverings and a cartesian product of two weighted digraphs. In Section 4, a decomposition formula for the Bartholdi zeta function of a weighted digraph bundle is given when the fiber is regular with some special weights. As a special case, we compute the Bartholdi zeta function of a (regular or irregular) covering of a weighted digraph and the Bartholdi zeta function of a cartesian product of two weighted digraphs. In Section 5, we introduce a new $L$-function of a weighted digraph associated with a digraph bundle. In Section 6, the readers will see that most of the known results on zeta functions of graphs or digraphs can be deduced from our results as corollaries.

1.2. Weighted digraphs

Let $D = (V(D), A(D))$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. Let $\nu_D = |V(D)|$ and $\alpha_D = |A(D)|$. For an arc $a = (x, y)$ in $A(D)$, the vertices $x$ and $y$ are called the tail and head of $a$, and denoted by $t(a)$ and $h(a)$, respectively. An arc $a$ with $t(a) = h(a)$ is called a loop, and arcs which have the same tail and the same head are called multi-arcs. A digraph without loops and multi-arcs is called simple. In this paper, all digraphs are assumed to be finite simple unless stated otherwise. We say that an arc $a$ is adjacent to an arc $b$ (and also $b$ is adjacent from $a$) if $h(a) = t(b)$. If $h(a) = t(b)$ and $h(b) = t(a)$, then $b$ is denoted by $\bar{a}$ and vice versa. We say that
an arc \(a\) is symmetric if both \(a\) and \(a\) are in \(A(D)\), and asymmetric otherwise. For \(x \in V(D)\), the numbers \(d_D^+(x) = |\{a \in A(D) : h(a) = x\}|\) and \(d_D^-(x) = |\{a \in A(D) : t(a) = x\}|\) are called the in-degree and the out-degree of \(x\), respectively.

Let \(D_1\) and \(D_2\) be the spanning subdigraphs of \(D\) with \(A(D_1) = \{a \in A(D) \mid \bar{a} \not\in A(D)\}\) and \(A(D_2) = \{a \in A(D) \mid \bar{a} \in A(D)\}\), respectively. In this case, we write \(D = D_1 \cup D_2\), and call it the decomposition of \(D\) into the asymmetric part \(D_1\) and the symmetric part \(D_2\), where \(\cup\) denotes a disjoint union of digraphs. If \(D = D_2\), i.e., \(A(D_1) = \emptyset\), we say that \(D\) is symmetric. Note that a symmetric digraph can be viewed as an undirected graph by identifying each pair of symmetric arcs \(a\) and \(a\) as an edge.

A weight function of a digraph \(D\) is a complex-valued function \(\omega : A(D) \rightarrow \mathbb{C} \setminus \{0\}\) and \(\omega(a)\) is called the weight of an arc \(a \in A(D)\). In this case, a pair \((D, \omega)\) is called a weighted digraph. A weight function \(\omega\) is said to be reciprocal if \(\omega(\bar{a}) = \omega(a)^{-1}\) for all \(a \in A(D_2)\), and \(\omega\) is trivial if \(\omega(a) = 1\) for all \(a \in A(D)\). A weighted digraph with the trivial weight will be identified with an unweighted digraph \(D\).

1.3. Zeta functions

A path \(P = (a_1, \ldots, a_m)\) in a digraph \(D\) is a sequence of arcs such that \(a_i\) is adjacent to \(a_{i+1}\) for each \(i \in \{1, \ldots, m - 1\}\). A path \(C = (a_1, \ldots, a_m)\) such that \(a_m\) is adjacent to \(a_1\) is called a cycle. In this case, \(m\) is called the length of \(C\) and denoted by \(|C|\). We define a single vertex by a cycle of length 0. The cycle \(C'\) obtained by going \(r\) times around a cycle \(C\) is called a multiple of \(C\) and a cycle is said to be prime if it is not a multiple of a strictly smaller cycle. A cycle \(C = (a_1, \ldots, a_m)\) is said to have a bump at \(h(a_i)\) (or \(t(a_{i+1})\)) if \(a_{i+1} = \bar{a}_i\). We denote the number of all bumps in \(C\) by \(b(C)\). If a cycle has no bump, then it is said to be reduced. Two cycles \(C_1 = (a_1, \ldots, a_m)\) and \(C_2 = (b_1, \ldots, b_m)\) are called equivalent if there exists a positive integer \(k\) such that \(b_j = a_{j+k} \mod m\) for all \(j \in \{1, \ldots, m\}\). Let \([C]\) denote the equivalence class which contains a cycle \(C\).

For a cycle \(C = (a_1, \ldots, a_m)\), define the weight \(\omega(C)\) of \(C\) to be

\[
\omega(C) = \prod_{i=1}^{m} \omega(a_i).
\]

The Ihara zeta function [28] of an (undirected) graph \(G\) is defined to be a function of \(u \in \mathbb{C}\) with \(|u|\) sufficiently small, given by

\[
Z_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime reduced cycles in \(G\). It is the origin of all known zeta functions of graphs and digraphs.

As a generalization of the Ihara zeta function, Bartholdi [2] defined a zeta function of an (undirected) graph \(G\), called the Bartholdi zeta function of \(G\), as follows: for \(t, u \in \mathbb{C}\) with \(|t|, |u|\) sufficiently small,

\[
\zeta_G(t, u) = \prod_{[C]} (1 - t^{b(C)} u^{|C|})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime cycles in \(G\).
After that, Mizuno and Sato [18] defined the Bartholdi zeta function of an unweighted digraph and Sato [24] did the Bartholdi zeta function of a weighted undirected graph with a reciprocal weight function. As an extension of those zeta functions, we define the Bartholdi zeta function of a weighted digraph \((D, \omega)\) as the following function of two complex variables \(t\) and \(u\) with \(|t|\) and \(|u|\) sufficiently small:

\[
\zeta_{(D, \omega)}(t, u) = \prod_{[C]} (1 - \omega(C)t^{b(C)}u^{|C|})^{-1},
\]

where \([C]\) runs over all equivalence classes of prime cycles in \(D\).

All previously known types of zeta functions of (weighted or unweighted) graphs and digraphs are summarized in Table 1 in Section 6, as special cases of our zeta function.

### 2. Determinant expressions of Bartholdi zeta functions of weighted digraphs

It is not easy to compute the Bartholdi zeta function \(\zeta_{(D, \omega)}(t, u)\) from the definition because there are infinitely many prime cycles in most cases. However, the zeta function can be expressed in terms of the determinants of matrices.

To do this, we introduce some related matrices as follows. Let \(\alpha_1 = |A(D_1)|, \alpha_2 = |A(D_2)|\) and \(\varepsilon_D = \alpha_2/2\). We give an order of arcs in \(A(D)\) by

\[
A(D) = A(D_1) \cup \{e_1, \ldots, e_{\varepsilon_D}, \bar{e}_1, \ldots, \bar{e}_{\varepsilon_D}\},
\]

where \(\{e_1, \ldots, e_{\varepsilon_D}, \bar{e}_1, \ldots, \bar{e}_{\varepsilon_D}\} = A(D_2)\).

1. The (weighted) \((vertex-)adjacency matrix\) \(W_D\) is the \(\nu_D \times \nu_D\) matrix whose \((x, y)\)-entry is

\[
(W_D)_{xy} = \begin{cases} 
\omega(x, y) & \text{if } (x, y) \in A(D), \\
0 & \text{otherwise}.
\end{cases}
\]

Note that \(W_D = W_{D_1} + W_{D_2}\).

2. The \(head-incidence matrix\) \(B^+_D\) is the \(\alpha_D \times \nu_D\) matrix whose \((a, x)\)-entry is

\[
(B^+_D)_{ax} = \begin{cases} 
\omega(a) & \text{if } x = h(a), \\
0 & \text{otherwise}.
\end{cases}
\]

Similarly, the \(tail-incidence matrix\) \(B^-_D\) is defined as the \(\alpha_D \times \nu_D\) matrix whose \((a, x)\)-entry is 1 if \(x = t(a)\), and 0 otherwise. Note that \(B^+_D\) and \(B^-_D\) are of the form

\[
B^+_D = \begin{bmatrix} B^+_D_1 \\ B^+_D_2 \end{bmatrix} \quad \text{and} \quad B^-_D = \begin{bmatrix} B^-D_1 \\ B^-D_2 \end{bmatrix}.
\]

Furthermore, \((B^-_D)^T B^+_D = W_D\), where \(X^T\) denotes the transpose matrix of \(X\).

3. The \(arc-adjacency matrix\) \(B_D\) is the \(\alpha_D \times \alpha_D\) matrix whose \((a, b)\)-entry is

\[
(B_D)_{ab} = \begin{cases} 
\omega(a) & \text{if } h(a) = t(b), \\
0 & \text{otherwise},
\end{cases}
\]

where \([C]\) runs over all equivalence classes of prime cycles in \(D\).
and the bump matrix $C_D$ is the $\alpha_D \times \alpha_D$ matrix whose $(a, b)$-entry is

$$(C_D)_{ab} = \begin{cases} \omega(a) & \text{if } b = a, \\ 0 & \text{otherwise}. \end{cases}$$

Note that $B_D$ and $C_D$ are of the form

$$B_D = B_D^+(B_D^-)^T = \begin{bmatrix} B_{D1}^+ & B_{D1}^+ (B_{D2}^-)^T \\ B_{D2}^+ (B_{D1}^-)^T & B_{D2}^+ \end{bmatrix}$$

and

$$C_D = 0_{\alpha_1} \oplus C_{D2} = 0_{\alpha_1} \oplus \begin{bmatrix} 0_{\alpha_D} & N \\ N & 0_{\alpha_D} \end{bmatrix},$$

where $0_n$ is the zero matrix of size $n$, and $N$ ($\bar{N}$, respectively) is the $\varepsilon_D \times \varepsilon_D$ diagonal matrices whose diagonal entries are $\omega(e_i)$ ($\omega(\bar{e}_i)$, respectively) for $i = 1, \ldots, \varepsilon_D$.

(IV) The matrix $S_D$ is the $\nu_D \times \nu_D$ diagonal matrix whose diagonals are the in-degrees in $D_2$, that is,

$$(S_D)_{xx} = d_{D2}^+(x) = |\{a \in A(D_2): h(a) = x\}|.$$

And $I_n$ denotes the identity matrix of size $n$.

Foata and Zeilberger [7] used the algebra of Lyndon words and Amitsur’s identity in their combinatorial proof of the determinant expression for the Ihara zeta function of a graph. Given a totally ordered finite set $X$, consider the free monoid $X^*$ with the lexicographical order. A Lyndon word $\pi$ on $X$ is a nonempty word in $X^*$ which is prime and minimal in the class of its cyclic rearrangements, where prime means $\pi \neq (\pi')^r$ for any $\pi' \in X^*$ and any $r \geq 2$.

Now let $M_1, M_2, \ldots, M_k$ be square matrices of the same size, and let $L$ be the set of all Lyndon words on $\{1, 2, \ldots, k\}$. For each Lyndon word $\pi = i_1i_2\cdots i_p$ in $L$, let $M_\pi = M_{i_1}M_{i_2}\cdots M_{i_p}$. Then Amitsur identity [1] can be stated as

$$\det(I - (M_1 + \cdots + M_k)) = \prod_{\pi \in L} \det(I - M_\pi).$$

Let $M = (m_{ij})$ be an $n \times n$ matrix and $L$ the set of all Lyndon words on $\{1, 2, \ldots, n\}$. From Amitsur identity, one can get

$$\det(I - M) = \prod_{\pi \in L} (1 - m_\pi), \quad (1)$$

where $m_\pi = m_{i_1i_2}m_{i_2i_3}\cdots m_{i_{p-1}i_p}m_{i_pl_1}m_{l_1}\cdots m_{l_pl_1}$ for $\pi = i_1i_2\cdots i_p \in L$.

The following theorem can be found in [25] with an extra assumption but we present its proof for self-containment.
Theorem 1. Let \((D, \omega)\) be a weighted digraph with \(v_D\) vertices and \(\varepsilon_D = |A_2(D)|/2\). Then the reciprocal of the Bartholdi zeta function of \((D, \omega)\) is

\[
\zeta_{(D, \omega)}(t, u)^{-1} = \det[I_{\alpha_D} - u(B_D - (1-t)C_D)].
\]

(2)

In particular, if \(\omega\) is reciprocal then

\[
\zeta_{(D, \omega)}(t, u)^{-1} = \left(1 - (1-t)^2u^2\right)^{\varepsilon_D - v_D} \det \Phi,
\]

(3)

where

\[
\Phi = I_{v_D} - uW_D + (1-t)u^2[S_D - (1-t)I_{v_D} + (1-t)uW_{D_1}].
\]

Proof. We use an analogue of Bass’ method [3]. To show Eq. (2), set \(M = (m_{ij}) = B_D - (1-t)C_D\). Using Eq. (1), we obtain

\[
\det[I_{\alpha_D} - u(B_D - (1-t)C_D)] = \det(I_{\alpha_D} - uM) = \prod_{\pi \in L} (1 - u|\pi|m_\pi),
\]

where \(L\) denotes the set of Lyndon words \(\pi = i_1 \cdots i_p\) on \(X = \{1, 2, \ldots, \alpha_D\}\) and \(m_\pi = m_{i_1i_2}m_{i_2i_3} \cdots m_{i_pi}i\). Moreover, for each \(\pi \in L\) we have

\[
m_\pi = \begin{cases} \omega(C)t^{b(C)} & \text{if } C = (i_1, i_2, \ldots, i_p) \text{ is a prime cycle in } D, \\ 0 & \text{otherwise.} \end{cases}
\]

Thus Eq. (2) follows.

To show Eq. (3), let \(\mu = (1-t)u\), \(\overline{C} = \mu I_{\alpha_1} \oplus C_{D_2}\), and

\[
L = \begin{bmatrix} \mu B_{D_1}^- \\ K \end{bmatrix}_{\alpha_D \times v_D},
\]

where \(K\) is the \(\alpha_2 \times v_D\) matrix obtained from \(B_{D_2}^+\) by replacing each \(\omega(e)\) with \(\omega(\overline{e})\). Then we have

\[
L^T B_{D_2}^+ = S_D + \mu W_{D_1},
\]

\[
B_{D_2}^+ L^T = B_D \overline{C} = (B_D - C_D)\overline{C} + C_D \overline{C} = (B_D - C_D)\overline{C} + (0_{\alpha_1} \oplus I_{\alpha_2}).
\]

Consider two \((\alpha_D + v_D) \times (\alpha_D + v_D)\) matrices \(P\) and \(Q\) defined by

\[
P = \begin{bmatrix} (1 - \mu^2)I_{v_D} & -(B_D^-)^T + \mu L^T \\ 0 & I_{\alpha_D} \end{bmatrix}
\]

and

\[
Q = \begin{bmatrix} I_v & (B_D^-)^T - \mu L^T \\ uB_D^+ & (1 - \mu^2)I_{\alpha_D} \end{bmatrix}.
\]
Clearly, we have
\[
\begin{bmatrix}
(1 - \mu^2)I_{vD} - uW_D + \mu u(S_D + \mu W_{D_1}) & 0 \\
u B^+ D & (1 - \mu^2)I_{aD}
\end{bmatrix},
\]
\[
\begin{bmatrix}
(1 - \mu^2)I_{vD} & 0 \\
u(1 - \mu^2)B^+ D - uB_D + \mu uB_D C + (1 - \mu^2)I_{aD}
\end{bmatrix}.
\]

Hence,
\[
\det(PQ) = (1 - \mu^2)^{\alpha D} \det[(1 - \mu^2)I_{vD} - uW_D + \mu u(S_D + (1 - t)uW_{D_1})]
= (1 - \mu^2)^{\alpha D} \det[I_{vD} - uW_D + \mu u(S_D - (1 - t)I_{vD} + \mu W_{D_1})]
\]
and
\[
\det(QP) = (1 - \mu^2)^{\alpha P} \det[-uB_D + \mu uB_D C + (1 - \mu^2)I_{aD}]
= (1 - \mu^2)^{\alpha P} \det[I_{aD} - uB_D + \mu uB_D C - \mu^2(I_{a_1} \oplus 0_{a_2}) - \mu^2(0_{a_1} \oplus I_{a_2})]
= (1 - \mu^2)^{\alpha P} \det[I_{aD} - uB_D + \mu uB_D C - \mu(C - C_D) - \mu^2 C_D C]
= (1 - \mu^2)^{\alpha P} \det[(I_{aD} - u(B_D - (1 - t)C_D))(I_{aD} - \mu C)]
= (1 - \mu^2)^{\alpha P} \det(I_{aD} - u(B_D - (1 - t)C_D)) \det(I_{aD} - \mu C).
\]
Since \(\det(I_{aD} - \mu C) = (1 - \mu^2)^{\alpha_{1+\epsilon_D}}\), the equality \(\det(PQ) = \det(QP)\) gives Eq. (3). □

Note that the analogous results for unweighted digraphs in [18] and for undirected graphs in [24] follow by assigning the trivial weight \(\omega = 1\) and by considering a symmetric digraph, respectively.

3. Coverings and bundles of weighted digraphs

A digraph \(\Xe\) is called a covering of a digraph \(D\) with the covering projection \(p : \Xe \to D\) if \(p\) is a surjection from \(V(\Xe)\) to \(V(D)\) such that \(p\) maps the set of arcs initiating (and terminating, respectively) at \(\x \) bijectively to the set of arcs initiating (and terminating, respectively) at \(x\) for any vertex \(x \in V(D)\) and \(\x \in p^{-1}(x)\). When \(p\) is \(n\)-to-one, we call \(D\) an \(n\)-fold covering.

For a combinatorial construction of an \(n\)-fold covering of \(D\), let \(\phi\) be a function from \(A(D)\) to the symmetric group \(S_Y\) on an \(n\)-element set \(Y\). If \(\phi(\a) = (\phi(a))^{-1}\) for every \(a \in A(D_2)\), then \(\phi\) is called a permutation voltage assignment. We define a digraph \(D^\phi\) by

(i) \(V(D^\phi) = V(D) \times Y;\)
(ii) \((\x_1, y_1), (\x_2, y_2) \in A(D^\phi)\) if \((\x_1, x_2) \in A(D)\) and \(\phi(x_1, x_2)y_1 = y_2\).

Then the first coordinate projection is a covering projection \(p^\phi : D^\phi \to D\), and this covering \(D^\phi\) is said to be derived from the voltage assignment \(\phi\). As the case of the covering of an undirected graph (see [8]), one can show that every covering of a digraph \(D\) can be derived from a permutation voltage assignment on \(D\).
A covering $\tilde{D}$ of a digraph $D$ with the projection $p: \tilde{D} \rightarrow D$ is regular if for any $x \in V(D)$ and for any $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x)$, there exists a unique automorphism $\psi$ of $\tilde{D}$ such that $p \circ \psi = p$ and $\psi(\tilde{x}_1) = \tilde{x}_2$.

Now we introduce a bundle $D \times^\phi F$ of a digraph $D$ with a fiber $F$ as a general setting of a covering of $D$. Let $S_F$ denote the symmetric group on $V(F)$. Then the automorphism group $\text{Aut}(F)$ of $F$ can be considered as a subgroup of $S_F$. Let $\phi: A(D) \rightarrow \text{Aut}(F)$ ($\subseteq S_F$) be a voltage assignment, called an $\text{Aut}(F)$-valued voltage assignment. Let $v_D F$ denote the $v_D$ copies of the fiber $F$ indexed by the vertices of $D$ and let $D \times^\phi F$ be the arc disjoint union $D^\phi \dot{\cup} v_D F$ of the covering $D^\phi$ and $v_D F$ on the vertex set $V(D \times^\phi F) = V(D) \times V(F)$. This digraph $D \times^\phi F$ is called an $F$-bundle of $D$ derived from the voltage assignment $\phi$, or simply a digraph bundle.

The digraphs $D$ and $F$ are called the base and the fiber of $D \times^\phi F$, respectively. We call the arcs in $D^\phi$ the base-type arcs and the arcs in $v_D F$ the fiber-type arcs. Since, for any base-type arc $((x_1,y_1),(x_2,y_2))$, $y_2$ is determined by $\phi(x_1,x_2)$ and $y_1$, we denote it by $(a,y_1)$ where $a = (x_1,x_2)$. Likewise, the fiber-type arc $((x_1,y_1),(x_1,y_2))$ is denoted also by $(x_1,d)$ where $d = (y_1,y_2) \in A(F)$. The first coordinate projection induces the bundle projection $p^\phi: D \times^\phi F \rightarrow D$. Note that $p^\phi$ maps vertices to vertices, base-type arcs to arcs and fiber-type arcs to vertices of $D$, respectively.

If $F = \overline{K_n}$, the graph of $n$ isolated vertices, then the $\overline{K_n}$-bundle of $D$ is just an $n$-fold covering $D^\phi$ of $D$. If $\phi(a)$ is the identity of $\text{Aut}(F)$ for all $a \in A(D)$, then $D \times^\phi F$ is the cartesian product $D \times F$ of $D$ and $F$.

**Example 1.** Let $D$ be a digraph on three vertices $x, y, z$ as pictured in Fig. 1. Let $\phi_1$ and $\phi_2$ be two voltage assignments from $A(D)$ to $S_D$ defined by

$$\phi_1(z, x) = (x, y, z), \quad \phi_2(z, x) = (x, y),$$

and $\phi_1 = \phi_2 = \text{id}$ on all other arcs. Let $F = D$ as a fiber. Then the digraph bundle $D \times^{\phi_1} F$ and the covering $D^{\phi_1}$ of $D$ can be illustrated as (a) and (b) in Fig. 2, respectively. Figure 2(c) shows the covering $D^{\phi_2}$. Note that $D^{\phi_1}$ is regular, while $D^{\phi_2}$ is irregular. Figure 2(d) illustrates the cartesian product $D \times F$. Directions of all arcs in Fig. 2 are inherited from the directions of the arcs in $D$.

For (undirected) graph bundles, Kwak et al. [6,13,14] gave some computational formulae for the Ihara and the Bartholdi zeta functions of graph bundles with regular fibers.

To define a weighted version, let $(D, \omega_D)$ and $(F, \omega_F)$ be weighted digraphs. We define a weight function $\tilde{\omega}: A(D \times^\phi F) \rightarrow \mathbb{C} \setminus \{0\}$ of the digraph bundle $D \times^\phi F$ in a natural way: $\tilde{\omega}((a, y)) = \omega_D(a)$ for any base-type arc $(a, y)$ and $\tilde{\omega}((x, b)) = \omega_F(b)$ for any fiber-type arc $(x, b)$. 
Now we compute $\zeta(D \times \phi F, \tilde{\omega})(t, u)$ in two ways by using two equations given in Theorem 1. First we examine the arc-adjacency matrix $B_{D \times \phi F}$ and the bump matrix $C_{D \times \phi F}$. Suppose that the vertex and arc sets of $D$ and $F$ are totally ordered as $V(D) = \{x_1, \ldots, x_{\nu_D}\}$, $A(D) = \{a_1, \ldots, a_{\alpha_D}\}$, $V(F) = \{y_1, \ldots, y_{\nu_F}\}$, and $A(F) = \{b_1, \ldots, b_{\alpha_F}\}$. We give a total order of $A(D \times \phi F)$ and $V(D \times \phi F)$ in the following way:

(i) $(a, y) < (x, b)$ for any $(a, y) \in A(D^\phi)$ and $(x, b) \in A(\nu_D F)$,
(ii) both $A(D^\phi)$ and $A(\nu_D F)$ are ordered lexicographically,
(iii) $V(D \times \phi F)$ is ordered lexicographically.

Hence, the ordering on $A(D \times \phi F)$ gives a partition of the rows into the following blocks:

$$[(a_1, V(F)), (a_2, V(F)), \ldots, (a_{\alpha_D}, V(F)), (x_1, A(F)), (x_2, A(F)), \ldots, (x_{\nu_D}, A(F))]$$

where $(a_i, V(F)) := ((a_i, y_1), (a_i, y_2), \ldots, (a_i, y_{\nu_F}))$ as base-type arcs and $(x_j, A(F)) := ((x_j, b_1), \ldots, (x_j, b_{\alpha_F}))$ as fiber-type arcs. Similarly, $V(D \times \phi F)$ is also partitioned as

$$[(x_1, V(F)), (x_2, V(F)), \ldots, (x_{\nu_D}, V(F))]$$

The tensor product $X \otimes Y$ of the matrices $X$ and $Y$ is considered as the matrix $X = (x_{ij})$ having the entry $x_{ij}$ replaced by the matrix $x_{ij} Y$.

(I) Now we consider the arc-adjacency matrix $B_{D \times \phi F}$ of $(D \times \phi F, \tilde{\omega})$. Since $B_{D \times \phi F} = B_{D \times \phi F}^+ (B_{D \times \phi F}^-)^T$, we first observe $B_{D \times \phi F}^+$ and $B_{D \times \phi F}^-$. For any permutation $\sigma$ in $S_F$, we define the permutation matrix $P(\sigma)$ by the $\nu_F \times \nu_F$ matrix such that

$$(P(\sigma))_{ij} = \begin{cases} 1 & \text{if } \sigma(y_i) = y_j, \\ 0 & \text{otherwise}. \end{cases} \quad (4)$$

Consider a block $(a, V(F))$ of base-type arcs. Each arc $(a, y_k)$ in $(a, V(F))$ terminates at $(h(a), y_\ell)$ such that $y_\ell = \phi(a)(y_k)$ and hence the $((a, V(F)), (x, V(F)))$-block of $B_{D \times \phi F}^+$ is $(B_{D}^+(\sigma))_{ax} P(\sigma)$ where $\sigma = \phi(a)$. Let $B_{D}^+(\sigma)$ be the $\alpha_G \times \nu_G$ matrix such that

$$(B_{D}^+(\sigma))_{ax} = \begin{cases} (B_{D}^+)_{ax} & \text{if } \phi(a) = \sigma, \\ 0 & \text{otherwise}. \end{cases}$$

Then we have $B_{D^\phi}^+ = \sum_{\sigma \in Aut(F)} B_{D}^+(\sigma) \otimes P(\sigma)$. 

Fig. 2. Digraph bundles.
Consider a block \((x, A(F))\) of fiber-type arcs. Since every arc in \((x, A(F))\) has its head in \((x, V(F))\), the \(((x, A(F)), (x, V(F)))\)-block of \(B_{D \times \phi F}^+\) is equal to \(B_F^+\). Hence we have \(B_{vD}^+ = I_{vD} \otimes B_{vD}^+\). A similar argument is applied for \(B_{D \times \phi F}^-\). Since \(D \times \phi F = D \phi \cup vD F\), we have
\[
B_{D \times \phi F}^+ = \left[ \sum_{\sigma \in \text{Aut}(F)} B_D^+(\sigma) \otimes P(\sigma) \right]_{vD} \otimes B_F^+\]
and
\[
B_{D \times \phi F}^- = \left[ B_D^- \otimes I_{vF} \right]_{vD} \otimes B_F^-\]
Therefore, we obtain
\[
B_{D \times \phi F} = B_{D \times \phi F}^+(B_{D \times \phi F}^-)^T = \left[ \sum_{\sigma \in \text{Aut}(F)} B_D^+(\sigma) \otimes P(\sigma) \right]_{vD} \otimes B_F^+
\]
where the matrix \(B_D(\sigma)\) is the \(\alpha_D \times \alpha_D\) matrix defined by
\[
(B_D(\sigma))_{ab} = \begin{cases} (B_D)_{ab} & \text{if } \phi(a) = \sigma, \\ 0 & \text{otherwise}. \end{cases}
\] (5)

(II) Next we consider the bump matrix \(C_{D \times \phi F}\) of \((D \times \phi F, \tilde{\omega})\). First we define a \(\alpha_D \times \alpha_D\) matrix \(C_D(\sigma)\) by
\[
(C_D(\sigma))_{ab} = \begin{cases} (C_D)_{ab} & \text{if } \phi(a) = \sigma, \\ 0 & \text{otherwise}. \end{cases}
\] (6)
Then one can easily check that
\[
C_{D \times \phi F} = \left[ \sum_{\sigma \in \text{Aut}(F)} C_D(\sigma) \otimes P(\sigma) \right]_{vD} \otimes C_F
\]
Now, the first equation of Theorem 1 and the fact (see p. 32 in [32]) that, when \(X_1\) is invertible,
\[
\det \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \det X_1 \cdot \det(X_4 - X_3X_1^{-1}X_2)
\] give the following theorem.

**Theorem 2.** Let \((D, \omega_D)\) and \((F, \omega_F)\) be weighted digraphs. Let \(\phi\) be an \(\text{Aut}(F)\)-valued voltage assignment of \(D\). Then the reciprocal of the Bartholdi zeta function of the weighted digraph bundle \((D \times \phi F, \tilde{\omega})\) is
\[ \zeta_{(D \times \phi F, \tilde{\omega})}(t, u)^{-1} \]
\[ = \det \left[ I_{\sigma D} v_F - u \sum_{\sigma \in \text{Aut}(F)} \left( B_D(\sigma) - (1 - t)C_D(\sigma) \right) \otimes P(\sigma) \right] \]
\[ \times \det \left[ I_{v_D} \otimes \Phi_F - \left( (B_D^{-1})^T \otimes B_F^+ \Phi_D^{-1} \left( \sum_{\sigma \in \text{Aut}(F)} B_D^+(\sigma) \otimes P(\sigma)(B_F^{-1})^T \right) \right) \right], \tag{7} \]

where \( \Phi_G = I_{\alpha G} - u(B_G - (1 - t)C_G) \).

In particular, its first factor is the reciprocal of the Bartholdi zeta function of a covering \( D \phi \) of the weighted digraph \( (D \phi, \tilde{\omega}|_{D \phi}) \), that is,
\[ \zeta_{(D \phi, \tilde{\omega}|_{D \phi})}(t, u)^{-1} = \det \left[ I_{\sigma D} v_F - u \sum_{\sigma \in \text{Aut}(F)} \left( B_D(\sigma) - (1 - t)C_D(\sigma) \right) \otimes P(\sigma) \right]. \tag{8} \]

From Eqs. (7) and (8) in Theorem 2, one can see that \( \zeta_{(D \phi, \tilde{\omega}|_{D \phi})}(t, u)^{-1} \) is a divisor of \( \zeta_{(D \times \phi F, \tilde{\omega})}(t, u)^{-1} \). However, its quotient, that is the second factor in Eq. (7), may not be a polynomial but is a rational function. It will be illustrated in the following example. Furthermore, it will be shown that in general, \( \zeta_{(D, \omega_D)}(t, u)^{-1} \) or \( \zeta_{(F, \omega_F)}(t, u)^{-1} \) is not necessarily a divisor of \( \zeta_{(D \times \phi F, \tilde{\omega})}(t, u)^{-1} \), unlike the case of \( \zeta_{D \phi}(t, u)^{-1} \) which has \( \zeta_{D}(t, u)^{-1} \) as a factor with a polynomial quotient (see [13]).

**Example 2.** Consider the cartesian product \( D \times F \) with the trivial weight function \( 1 \) as in Fig. 2(d). By Theorem 2, we obtain
\[ \zeta_{(D \times F, 1)}(t, u)^{-1} = \left( 1 + u^3 \right)^2 \left( 1 - 8u^3 \right) \]
and
\[ \zeta_{(D \psi, 1)}(t, u)^{-1} = \left( 1 - u^3 \right)^3, \]
where \( \psi \) is the trivial voltage assignment. Clearly, \( \zeta_{(D \psi, 1)}(t, u)^{-1} \) is not a divisor of \( \zeta_{(D \times F, 1)}(t, u)^{-1} \). Also, \( \zeta_{(D, 1)}(t, u)^{-1} = 1 - u^3 \) is not a divisor of \( \zeta_{(D \times F, 1)}(t, u)^{-1} \).

Next, to use Eq. (3) in Theorem 1 for the bundle \( (D \times \phi F, \tilde{\omega}) \), we assume that \( \omega_D \) and \( \omega_F \) are reciprocal. To compute \( W_{D \times \phi F} \) and \( W_{(D \times \phi F)1} \), we introduce the following matrix. For each \( \sigma \in \Gamma \), \( W_D(\sigma) \) is the \( \nu_D \times \nu_D \) matrix such that
\[ (W_D(\sigma))_{xy} = \begin{cases} (W_D)_{xy} & \text{if } (x, y) \in A(D) \text{ and } \phi(x, y) = \sigma, \\ 0 & \text{otherwise}. \end{cases} \]

Since \( A(D \times \phi F) = A(D \phi) \cup A(\nu_D F) \), the weighted adjacency matrix \( W_{D \times \phi F} \) is the sum \( W_{D \phi} + W_{\nu_D F} \). But, \( W_{D \phi} = \sum_{\sigma \in \Gamma} W_D(\sigma) \otimes P(\sigma) \) and \( W_{\nu_D F} = I_{\nu_D} \otimes W_F \). Hence we have
\[ W_{D \times \phi F} = \sum_{\sigma \in \text{Aut}(F)} W_D(\sigma) \otimes P(\sigma) + I_{\nu_D} \otimes W_F \]
and

\[ W_{(D \times F)_1} = \sum_{\sigma \in \text{Aut}(F)} W_{D_1}(\sigma) \otimes P(\sigma) + I_{v_D} \otimes W_F. \]

For any vertex \((x, y) \in V(D \times F)\), \(|\{a \in A_2(D \times F): t(a) = (x, y)\}| = |\{a \in A_2(D): t(a) = x\}| + |\{a \in A_2(F): t(a) = y\}|. Therefore, \(S_{D \times F} = S_D \otimes I_{v_F} + I_{v_D} \otimes S_F\). Clearly, \(\varepsilon_{D \times F} = \varepsilon_{DVF} + \varepsilon_{DF} - \varepsilon_{VF}\). Hence the following comes immediately from Theorem 1.

**Theorem 3.** Let \((D, \omega_D)\) and \((F, \omega_F)\) be weighted digraphs with reciprocal weight functions \(\omega_D\) and \(\omega_F\), respectively. And let \(\phi\) be an \(\text{Aut}(F)\)-valued voltage assignment of \(D\). Then the reciprocal of the Bartholdi zeta function of the weighted digraph bundle \((D \times F, \tilde{\omega})\) is

\[
\zeta_{(D \times F, \tilde{\omega})}(t, u)^{-1} = \left(1 - (1 - t)^2 u^2\right)^{\theta} \text{det} \Psi,
\]

where \(\theta = \varepsilon_{DF} - \varepsilon_{DF} + \varepsilon_{VF}\), and

\[
\Psi = I_{v_D} \otimes I_{v_F} - u \left( \sum_{\sigma \in \text{Aut}(F)} W_{D_1}(\sigma) \otimes P(\sigma) + I_{v_D} \otimes W_F \right) + (1 - t)u^2 \left[ (S_D - (1 - t)I_{v_D}) \otimes I_{v_F} + I_{v_D} \otimes S_F \right]
\]

If both \(D\) and \(F\) are symmetric with the trivial weight, then the formula in Theorem 3 gives the Bartholdi zeta function of a graph bundle, as shown in [13].

**4. Decomposition formula for \(\zeta_{(D \times F, \tilde{\omega})}(t, u)\)**

In this section we give a decomposition formula for the Bartholdi zeta function of a weighted digraph bundle \((D \times F, \tilde{\omega})\). To do this, we first consider two determinant expressions of the Bartholdi zeta function of \((D \times F, \tilde{\omega})\) given in Theorem 1. Note that the matrix size in Eq. (2) is larger than that in Eq. (3). Thus Eq. (3) is more convenient in computing the Bartholdi zeta function of a weighted digraph. Hence we consider the determinant expression of \((D \times F, \tilde{\omega})\) given in Theorem 3. In this determinant expression, if all of the first or all of the second factors in the tensor products can be factorized simultaneously into smaller block matrices, then the reciprocal of the zeta function can be decomposed into polynomials of smaller degrees. With the first factors, it seems to be hard to find a method for their factorization. However, under some conditions on the fiber \((F, \omega_F)\), the second factors can be factorized simultaneously into smaller block matrices. To find those conditions, we need the following definitions.

A digraph \(F\) is said to be \(r\)-regular if \(d^+_F(x) = d^-_F(x) = r\) for all \(x \in V(F)\). A spanning subgraph of \(F\) which is \(k\)-regular is called a \(k\)-factor of \(F\). It is known [22] that every \(r\)-regular digraph \(F\) can be decomposed into \(r\) arc-disjoint 1-factors. Let \(F_1\) and \(F_2\) be the asymmetric and symmetric parts of \(F\), respectively. We say that \(F\) is \((r_1, r_2)\)-regular if \(F_i\) is \(r_i\)-regular for
i = 1, 2. Note that, if \( F \) is \((r_1, r_2)\)-regular, then it is also \((r_1 + r_2)\)-regular. For a subdigraph \( H \) in \( F \), we simply denote \( \omega_F|_H \) by \( \omega_H \). We say that a weighted digraph \((F, \omega_F)\) is \( r \)-regular if \( F \) is \( r \)-regular and it has a 1-factor decomposition \( F = C_1 \cup \cdots \cup C_r \) such that \( \omega_{C_i} \) is constant for each \( i = 1, \ldots, r \). Also we say that \((F, \omega_F)\) is \((r_1, r_2)\)-regular if \((F_i, \omega_{F_i})\) is \( r_i \)-regular for each \( i = 1, 2 \).

Now we assume that the fiber \((F, \omega_F)\) of a weighted digraph bundle \((D \times \phi, \tilde{\omega})\) is \((r_1, r_2)\)-regular. Let \( F_1 \) and \( F_2 \) have 1-factor decompositions \( F_1 = C_1 \cup \cdots \cup C_{r_1} \) and \( F_2 = C_{r_1+1} \cup \cdots \cup C_{r_1+r_2} \), so that

\[
F = C_1 \cup \cdots \cup C_{r_1} \cup C_{r_1+1} \cup \cdots \cup C_{r_1+r_2},
\]

and let \( \omega_F(C_j) = \omega_j \in \mathbb{C} \setminus \{0\} \) for \( j = 1, \ldots, r_1 + r_2 \). Note that, for each \( j \), each vertex in \( V(F) \) is the head of exactly one arc and the tail of exactly one arc in \( C_j \). This implies that \( W_{C_j} = \omega_j P(\sigma_j) \) for some permutation \( \sigma_j \in S_F \). Thus, we have

\[
W_F = \sum_{j=1}^{r_1+r_2} \omega_j P(\sigma_j) \quad \text{and} \quad W_{F_1} = \sum_{j=1}^{r_1} \omega_j P(\sigma_j).
\]

Moreover, the diagonal matrix \( S_F \) becomes a scalar matrix \( r_2 I_{v_F} \). Now, the matrix \( \Psi \) in Theorem 3 can be written as

\[
\Psi = I_{v_D} \otimes I_{v_F} - t \left( \sum_{\sigma \in \text{Aut}(F)} W_D(\sigma) \otimes P(\sigma) + I_{v_D} \otimes \sum_{j=1}^{r_1+r_2} \omega_j P(\sigma_j) \right)
\]

\[
\hspace{1cm} + (1-t)u^2 \left[ (S_D - (1-t-r_2)I_{v_D}) \otimes I_{v_F} \right.
\]

\[
\hspace{1cm} + \left. (1-t)u \left( \sum_{\sigma \in \text{Aut}(F)} W_{D_1}(\sigma) \otimes P(\sigma) + I_{v_D} \otimes \sum_{j=1}^{r_1} \omega_j P(\sigma_j) \right) \right].
\]

Let \( \Gamma \) be the subgroup of \( S_F \) generated by \( \{ \phi(a) \mid a \in A(D) \} \cup \{ \sigma_j \mid 1 \leq j \leq r_1 + r_2 \} \) and for \( \tau \in \Gamma \), let \( P(\tau) \) denote the permutation matrix corresponding to \( \tau \) defined in Eq. (4). A representation \( \rho \) of the group \( \Gamma \) over the complex field \( \mathbb{C} \) is a group homomorphism from \( \Gamma \) to the general linear group \( \text{GL}(d, \mathbb{C}) \) of invertible \( d \times d \) matrices over \( \mathbb{C} \). The number \( d \) is called the degree of the representation \( \rho \). The representation \( P : \Gamma \to \text{GL}(v_F, \mathbb{C}) \) defined by \( \tau \mapsto P(\tau) \) is the permutation representation of \( \Gamma \). Let \( \rho_1, \rho_2, \ldots, \rho_\ell \) be the irreducible representations of \( \Gamma \) with degree \( d_k \) for \( 1 \leq k \leq \ell \), so that \( \sum_{k=1}^\ell d_k^2 = |\Gamma| \). Then, the permutation representation \( P \) can be decomposed into a direct sum of irreducible representations: Say \( P = \bigoplus_{k=1}^\ell m_k \rho_k \), where \( m_k \) is the multiplicity of \( \rho_k \) (see, for example, [27]). Moreover, there exists an invertible matrix \( M \) such that

\[
M^{-1} P(\tau) M = \bigoplus_{k=1}^\ell (I_{m_k} \otimes \rho_k(\tau))
\]
for every $\tau \in \Gamma$. Then we have
\[(I_{\nu D} \otimes M)^{-1} \Psi(I_{\nu D} \otimes M) = \bigoplus_{k=1}^{\ell} (I_{m_k} \otimes \Psi_k),\]
where
\[\Psi_k = I_{\nu D} \otimes I_{d_k} - u \left( \sum_{\sigma \in \text{Aut}(F)} W_D(\sigma) \otimes \rho_k(\sigma) + I_{\nu D} \otimes \sum_{j=1}^{r_1+r_2} \omega_j \rho_k(\sigma_j) \right) + (1-t)u^2 \left( (S_D - (1-t-r_2)I_{\nu D}) \otimes I_{d_k} \right)
+ (1-t)u \left( \sum_{\sigma \in \text{Aut}(F)} W_D(\sigma) \otimes \rho_k(\sigma) + I_{\nu D} \otimes \sum_{j=1}^{r_1} \omega_j \rho_k(\sigma_j) \right) \right]. \tag{9}\]

Note that $\nu_F = \sum_{k=1}^{\ell} m_k d_k$ and $\varepsilon_F = r_2 \nu_F^2$. This proves the following theorem.

**Theorem 4.** Let $(D, \omega_D)$ be a weighted digraph with a reciprocal weight function $\omega_D$, and let $(F, \omega_F)$ be an $(r_1, r_2)$-regular weighted digraph with a reciprocal weight function $\omega_F$, $W_F = \sum_{j=1}^{r_1+r_2} \omega_j P(\sigma_j)$ and $W_{F_1} = \sum_{j=1}^{r_1} \omega_j P(\sigma_j)$ for some $\sigma_j \in S_F$. Let $\phi$ be an $\text{Aut}(F)$-valued voltage assignment of $D$ and $\Gamma$ the subgroup of $S_F$ generated by $\{\phi(a) | a \in A(D)\} \cup \{\sigma_j | 1 \leq j \leq r_1 + r_2\}$. Let $\rho_1 = 1, \rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\Gamma$ with degrees $d_1, d_2, \ldots, d_\ell$, respectively. Let $P$ be the permutation representation of $\Gamma$ and let $P = \bigoplus_{k=1}^{\ell} m_k \rho_k$, where $m_k$ is the multiplicity of $\rho_k$. Then the reciprocal of the Bartholdi zeta function of the weighted digraph bundle $(D \times^\phi F, \tilde{\omega})$ can be decomposed as follows:
\[\zeta_{(D \times^\phi F, \tilde{\omega})}(t,u)^{-1} = \prod_{k=1}^{\ell} \left[ (1 - (1-t)^2 u^2)^{\theta_k} \det \Psi_k \right]^{m_k},\]
where $\theta_k = (\varepsilon_D - \nu_D + \frac{1}{2} r_2 \nu_D) d_k$ and $\Psi_k$ is the matrix given in Eq. (9).

The following example shows that if the fiber $F$ in Theorem 4 is a Schreier digraph, then it satisfies all the hypotheses of the theorem.

**Example 3.** Let $\Lambda$ be a subgroup of a group $\Gamma$ and let $X = \{s_1, s_2, \ldots, s_r\}$ be a subset of $\Gamma$. The Schreier digraph $S(\Gamma : \Lambda, X)$ is the digraph whose vertex set is the right cosets of $\Lambda$ in $\Gamma$, and there is an arc from a vertex $\Lambda x$ to a vertex $\Lambda y$ if and only if there exists $\Lambda y = \Lambda s_j x$ for some $s_j \in X$. If $\Lambda = \{1\}$ as a special case, the Schreier digraph is just the Cayley digraph. Moreover, if $X$ is symmetric, i.e., every element $s_i$ has its inverse in $X$, then a Schreier digraph becomes an (undirected) Schreier graph. Observe that the group $\Gamma$ acts transitively on the right cosets of $\Lambda$ by right multiplication. By the permutation group theory, one can say that a simple digraph $F$ is a Schreier digraph if there exists a subset $X$ of the symmetric group $S_F$ such that any two vertices $x$ and $y$ are adjacent if and only if $y = x^s$ for some $s \in X$ (see Section 2.4.4 in [8]). We call such an $X$ the connecting set of the Schreier digraph $F$. Notice that a Schreier digraph with
connecting set $X$ is $(r_1, r_2)$-regular with $r_1 = |\{s \in X: s^{-1} \notin X\}|$ and $r_2 = |\{s \in X: s^{-1} \in X\}|$. Moreover, it is known that most regular graphs are Schreier graphs (see Section 2.3.4 in [8]).

Now we consider a digraph bundle $D \times^\phi F$ whose fiber $F$ is a Schreier digraph with connecting set $X = \{s_1, s_2, \ldots, s_r\}$. Then one can define a weight on $F$ by assigning a weight on each element in $X$. Let $\omega_X : X \to \mathbb{C} \setminus \{0\}$ be a function defined by $\omega_X(s_j) = \omega_j$ for each $j = 1, \ldots, r$. Then $\omega_X$ induces a weight function $\omega_F : A(F) \to \mathbb{C} \setminus \{0\}$ of $F$ defined by $\omega_F(x, x^j) = \omega_j$ for all $x \in V(F)$ and $j = 1, \ldots, r$. Now let $X_1 = \{s \in X: s^{-1} \notin X\} = \{s_1, \ldots, s_{r_1}\}$ and $X_2 = \{s \in X: s^{-1} \in X\} = \{s_{r_1+1}, \ldots, s_r\}$. Then the adjacency matrices of $(F, \omega_F)$ and $(F_1, \omega_{F_1})$ are

$$W_F = \sum_{j=1}^{r} \omega_j P(s_j)$$

and

$$W_{F_1} = \sum_{j=1}^{r_1} \omega_j P(s_j),$$

respectively. Let $\Gamma$ be the subgroup of the symmetric group $\text{Aut}(F)$ generated by $\{\phi(a): a \in A(F)\} \cup \{s: s \in X\}$, and let $\rho_1 = 1$, $\rho_2, \ldots, \rho_\ell$ be the irreducible representations of $\Gamma$ having degree $d_1 = 1$, $d_2, \ldots, d_\ell$, respectively. Then the reciprocal of the Bartholdi zeta function of the weighted digraph bundle $(D \times^\phi F, \omega)$ can be expressed as the following decomposition:

$$\zeta_{(D \times^\phi F, \omega)}(t, u)^{-1} = \prod_{k=1}^{\ell} [\sum_{\sigma \in \text{Aut}(F)} \omega_j \rho_k(\sigma_j)]^{m_k} \det \Psi_k,$$

where

$$\Psi_k = I_{V_D} \otimes I_{d_k} - u \left( \sum_{\sigma \in \text{Aut}(F)} W_D(\sigma) \otimes \rho_k(\sigma) + I_{V_D} \otimes \sum_{j=1}^{r} \omega_j \rho_k(\sigma_j) \right) + (1-t)u^2 \left( S_D - (1-t-(r-r_1))I_{V_D} \right) \otimes I_{d_k}$$

and

$$+ (1-t)u \left( \sum_{\sigma \in \text{Aut}(F)} W_{D_1}(\sigma) \otimes \rho_k(\sigma) + I_{V_D} \otimes \sum_{j=1}^{r_1} \omega_j \rho_k(\sigma_j) \right),$$

$\theta_k = (\varepsilon_D - v_D + \frac{1}{2}(r-r_1)v_D)d_k$, and $m_k$ is the multiplicity of $\rho_k$ in the permutation representation $P$ of $\Gamma$.

If $X$ is symmetric and also $D$ is symmetric, then one can obtain the decomposition of the Bartholdi zeta function of the weighted undirected graph bundle $(D \times^\phi F, \omega)$. In this case, the asymmetric parts $D_1$ and $F_1$ of $D$ and $F$ are both empty. Thus $W_{D_1}(\sigma) = 0$ for all $\sigma \in \text{Aut}(F)$ and $r_1 = 0$. Hence the matrix $\Psi_k$ is simplified as

$$\Psi_k = I_{V_D} \otimes I_{d_k} - u \left( \sum_{\sigma \in \text{Aut}(F)} W_D(\sigma) \otimes \rho_k(\sigma) + I_{V_D} \otimes \sum_{j=1}^{\lfloor |X| \rfloor} \omega_j \rho_k(\sigma_j) \right) + (1-t)u^2 \left( S_D - (1-t-|X|)I_{V_D} \right) \otimes I_{d_k}. $$

This induces Theorem 5 in [13] as an unweighted version.
If $F = \overline{K}_n$, then Theorem 4 gives a decomposition formula for the Barholtz zeta function of a weighted digraph covering $(D_φ, \tilde{ω})$ as follows:

$$
\zeta(D_φ, \tilde{ω})(t, u)^{-1} = \zeta(D, ω)(t, u)^{-1} \prod_{k=2}^\ell \left(1 - (1 - t)^2 u^2\right)^{θ_k} \det \Psi_k^m k,
$$

where $θ_k = (ε_D - v_D) d_k$ and

$$
\Psi_k = I_{v_D} \otimes I_{d_k} - u \sum_{σ \in Aut(F)} W_σ \otimes ρ_k(σ)
$$

$$
+ (1 - t) u^2 \left( (S_D - (1 - t) I_{v_D}) \otimes I_{d_k} + (1 - t) u \sum_{σ \in Aut(F)} W_{D_1} \otimes ρ_k(σ) \right).
$$

As another special case of Theorem 4, let the group $Γ$ generated by $\{φ(a) | a ∈ A(D)\}$ be abelian. For $k = 1, \ldots, v_F$, let $λ_k^{(σ)} = ρ_k(σ)$, $μ_k = \sum_{j=1}^{r_1+r_2} ω_j ρ_k(σ_j)$ and $μ_k' = \sum_{j=1}^{r_1} ω_j ρ_k(σ_j)$. Then the following corollary can be obtained from Theorem 4.

**Corollary 5.** Under the same hypothesis as those of Theorem 4, assume that the group $Γ$ generated by $\{φ(a) | a ∈ A(D)\}$ is abelian. For $k = 1, \ldots, v_F$, let $λ_k^{(σ)} = ρ_k(σ)$ be the eigenvalues of $P(σ)$, $μ_k = \sum_{j=1}^{r_1+r_2} ω_j ρ_k(σ_j)$ be the eigenvalues of $W_F$ and $μ_k' = \sum_{j=1}^{r_1} ω_j ρ_k(σ_j)$ be the eigenvalues of $W_{F_1}$. Then the reciprocal of the Bartholdi zeta function of the weighted digraph bundle $(D × D_φ F, \tilde{ω})$ can be decomposed as

$$
\zeta(D × D_φ F, \tilde{ω})(t, u)^{-1} = \prod_{k=1}^{v_F} \left(1 - (1 - t)^2 u^2\right)^{ε_D - v_D + \frac{r_1 + r_2}{2}} \det \Psi_k
$$

where

$$
\Psi_k = I_{v_D} - u \left( \sum_{σ \in Aut(F)} λ_k^{(σ)} W_σ(σ) + μ_k I_{v_D} \right)
$$

$$
+ (1 - t) u^2 \left[ S_D - (1 - t - r_2) I_{v_D} + (1 - t) u \left( \sum_{σ \in Aut(F)} λ_k^{(σ)} W_{D_1}(σ) + μ_k' I_{v_D} \right) \right].
$$

In Theorem 4, let the voltage assignment $φ$ be trivial, i.e., $φ(a) = 1$ for all $a ∈ A(D)$. Then, the digraph bundle $D × D_φ F$ is just the cartesian product $D × F$, and $\sum_{σ ∈ Γ} W_σ(σ) \otimes ρ_k(σ) = W_D(id) \otimes ρ_k(id) = W_D ⊗ I_{v_F}$ and $\sum_{σ ∈ Γ} W_{D_1}(σ) \otimes ρ_k(σ) = W_{D_1} ⊗ I_{v_F}$. Thus we have the following corollary from Theorem 4.
Corollary 6. Under the same hypothesis as those of Theorem 4, the reciprocal of the Bartholdi zeta function of the cartesian product \((D \times F, \tilde{\omega})\) can be decomposed as

\[
\zeta(D \times F, \tilde{\omega})(t, u)^{-1} = \prod_{k=1}^{\ell} \left[ \left(1 - (1-t)^2 u^2 \right)^{\varrho_k} \det \Psi_k \right]^{m_k},
\]

where \(\varrho_k = (\epsilon_D - \nu_D + \frac{r_1 \nu_D}{2}) d_k\), and

\[
\Psi_k = I_{v_D} \otimes I_{d_k} - u \left( W_D \otimes I_{v_F} + I_{v_D} \otimes \sum_{j=1}^{r_1 + r_2} \omega_j \rho_k(\sigma_j) \right) + (1-t)u^2
\]

\[
\times \left[ \left( S_D - (1-t-r_2)I_{v_D} \right) \otimes I_{d_k} + (1-t)u \left( W_{D_1} \otimes I_{v_F} + I_{v_D} \otimes \sum_{j=1}^{r_1} \omega_j \rho_k(\sigma_j) \right) \right].
\]

If the group \(\Gamma\) generated by \(\{\sigma_j \mid 1 \leq j \leq \ell\}\) is abelian in Corollary 6, then the decomposition formula can be expressed as

\[
\zeta(D \times F, \tilde{\omega})(t, u)^{-1} = \prod_{k=1}^{v_F} \left(1 - (1-t)^2 u^2 \right)^{\epsilon_D - \nu_D + \frac{r_1 \nu_D}{2}} \det \Theta_k,
\]

where

\[
\Theta_k = I_{v_D} - u(W_D + \mu_k I_{v_D}) + (1-t)u^2 \left( S_D - (1-t-r_2)I_{v_D} \right) + (1-t)u(W_{D_1} + \mu'_k I_{v_D})
\]

Here \(\mu_k\) and \(\mu'_k\) are the eigenvalues of \(W_F\) and \(W_{F_1}\), respectively.

5. \(L\)-functions of coverings and bundles

Many authors like Stark and Terras [30], and Sato [24] have investigated \(L\)-functions of graphs as graph theoretical analogues of the Artin \(L\)-function in number theory. They consider a regular graph covering as an analogue of a Galois extension of algebraic number fields. In this section we extend the established \(L\)-functions of a regular graph covering to an \(L\)-function of a weighted digraph bundle.

Let \((D \times F, \tilde{\omega})\) be a weighted digraph bundle with the base \((D, \omega_D)\) and the fiber \((F, \omega_F)\). For our convenience we assume that the fiber \((F, \omega_F)\) is a weighted Schreier graph with connecting set \(S = \{s_1, s_2, \ldots, s_r\}\). Let \(W_F = \sum_{j=1}^{r} \omega_j P(s_j)\) and \(W_{F_1} = \sum_{j=1}^{r_1} \omega_j P(s_j)\) as defined in Example 3. To define an \(L\)-function of a bundle \((D \times F, \tilde{\omega})\), we express the bundle \(D \times F\) of \(D\) as a covering of a new digraph \(D^*\) which is obtained from \(D\) by attaching to each vertex in \(D\) the \(r\) directed loops labeled by the connecting set \(S\). That is, \(A(D^*) = A(D) \cup \{s_j(x) \mid x \in V(D), j = 1, \ldots, r\}\), where \(s_j(x)\) denotes the loop on the vertex \(x\) labeled by \(s_j\). To do this we extend the voltage assignment \(\phi\) to a map \(\tilde{\phi} : A(D^*) \to \text{Aut}(F)(\leq S_F)\) by

\[
\tilde{\phi}(a) = \begin{cases} 
\phi(a) & \text{if } a \in A(D), \\
\tilde{s}_j & \text{if } a = s_j(x) \text{ for some } x \in V(D).
\end{cases}
\]

One can show that the bundle \(D \times F\) becomes a covering of \(D^*\) derived from \(\tilde{\phi}\).
For a cycle $C^* = (a_1, \ldots, a_m)$ in $D^*$, we say that it has a bump at $h(a_i)$ if either $a_{i+1} = a_i$ in $D$, or $a_i = s(x)$ and $a_{i+1} = s^{-1}(x)$ for some $x \in V(D)$, $s \in S$. Also for $C^* = (a_1, \ldots, a_m)$, we define $\overline{\phi}(C^*) = \overline{\phi}(a_1) \cdots \overline{\phi}(a_m)$. With the weight $\omega_j$ of $s_j$, the weight function $\omega_D$ of $D$ extends to a weight function $\omega^* : A(D^*) \to \mathbb{C} \setminus \{0\}$ of $D^*$ as

$$\omega^*(a) = \begin{cases} \omega_D(a) & \text{if } a \in A(D), \\ \omega_j & \text{if } a = s_j(x) \text{ for some } x \in V(D). \end{cases}$$

For a cycle $C^* = (a_1, \ldots, a_m)$ in $D^*$, we define $\omega^*(C^*) = \omega^*(a_1) \cdots \omega^*(a_m)$.

Let $\rho$ be a representation of $\Gamma$ with degree $d_\rho$. We define the $L$-function of a weighted digraph bundle $(D \times \phi F, \overline{\omega})$ associated with $\rho$ as follows: For complex variables $t$ and $u$ with $|t|$ and $|u|$ sufficiently small,

$$L(D \times \phi F, \overline{\omega})(t, u, \rho) = \prod_{[C^*]} \det (I_{d_\rho} - \omega^*(C^*) \rho(\overline{\phi}(C^*)) t^{|C^*|} u^{|C^*|} )^{-1},$$

where $[C^*]$ runs over all equivalence classes of prime cycles in $D^*$.

When $D$ is symmetric and $F = K_n$, this is the $L$-function of an undirected graph covering defined by Sato in [24].

In the following we see that the $L$-function has determinant expressions which are analogues of those in Theorem 1. To see this, we need the following matrices. Let $\sigma \in \Gamma$. Under the ordering of arcs in $A(D^*)$ given by

$$\{a_1, \ldots, a_{\alpha_D}; s_1(x_1), \ldots, s_1(x_1); \ldots; s_1(x_{\nu_D}), \ldots, s_1(x_{\nu_D})\},$$

where $A(D) = \{a_1, \ldots, a_{\alpha_D}\}$ and $V(D) = \{x_1, \ldots, x_{\nu_D}\}$, let $B_D^*(\sigma)$ and $C_D^*(\sigma)$ be the $\alpha_D^* \times \alpha_D^*$ matrices defined as in Eqs. (5) and (6), respectively. And let

$$B(\rho) = \sum_{\sigma \in \Gamma} B_D^*(\sigma) \otimes \rho(\sigma) \quad \text{and} \quad C(\rho) = \sum_{\sigma \in \Gamma} C_D^*(\sigma) \otimes \rho(\sigma).$$

Also we set

$$W(\rho) = \sum_{\sigma \in \Gamma} W_D \otimes \rho(\sigma) + I_{v_D} \otimes \sum_{j=1}^{r} \omega_j \rho(\sigma_j),$$

$$W_1(\rho) = \sum_{\sigma \in \Gamma} W_{D_1} \otimes \rho(\sigma) + I_{v_D} \otimes \sum_{j=1}^{r_1} \omega_j \rho(\sigma_j),$$

and

$$S(\rho) = (S_D + r_2 I_{v_D}) \otimes I_{d_\rho},$$

where $r_2 = r - r_1 = |\{s \in S; s^{-1} \in S\}|$.

An argument similar to the proof of Theorem 1 gives the following theorem.
Theorem 7. Let \((D, \omega_D)\) be a weighted digraph with \(v_D\) vertices and \(\varepsilon_D = |A_2(D)|/2\). Let \((F, \omega_F)\) be a weighted Schreier graph with connecting set \(S = \{s_1, s_2, \ldots, s_r\}\), \(W_F = \sum_{j=1}^{r} \omega_j P(s_j)\) and \(W_{F_1} = \sum_{j=1}^{r} \omega_j P(s_j)\). Let \(\phi\) be an \(\text{Aut}(F)\)-valued voltage assignment of \(D\). Let \(\Gamma\) be the subgroup of \(\text{Aut}(F)\) generated by \(\{\phi(a) | a \in A(D)\} \cup \{s | s \in S\}\) and let \(\rho\) be a representation of \(\Gamma\) with degree \(d_\rho\). Then the reciprocal of the \(L\)-function of a weighted digraph bundle \((D \times \phi F, \tilde{\omega})\) associated with \(\rho\) is

\[
L((D \times \phi F, \tilde{\omega}))(t, u, \rho)^{-1} = \det[\mathbf{I}_{d_\rho \alpha_D} - u \left( B(\rho) - (1-t)C(\rho) \right)].
\]

In particular, if \(\omega\) is reciprocal, then

\[
L((D \times \phi F, \tilde{\omega}))(t, u, \rho)^{-1} = (1 - (1-t)^2 u^2)^{\theta_\rho} \det \Omega,
\]

where \(\theta_\rho = (\varepsilon_D - v_D + \frac{1}{2} r_2 v_D) d_\rho\) and

\[
\Omega = \left[ \mathbf{I}_{d_\rho v_D} - u W(\rho) + (1-t) u^2 \left( S(\rho) - (1-t) I_{v_D} + (1-t) u W_1(\rho) \right) \right].
\]

If \(F = \overline{K_n}\), we have \(D^* = D\) and

\[
W(\rho) = \sum_{\sigma \in \Gamma} W_D \otimes \rho(\sigma), \quad W_1(\rho) = \sum_{\sigma \in \Gamma} W_D \otimes \rho(\sigma) \quad \text{and} \quad S(\rho) = S_D \otimes \mathbf{I}_{d_\rho}.
\]

Thus a (regular or irregular) covering version of Theorem 7 can be stated as follows.

Corollary 8. Let \((D, \omega_D)\) be a weighted digraph with \(v_D\) vertices and \(\varepsilon_D = |A_2(D)|/2\). Let \(\phi: A(D) \rightarrow SY\) be a voltage assignment with an \(n\)-element set \(Y\). Let \(\Gamma\) be the subgroup of \(SY\) generated by \(\{\phi(a) | a \in A(D)\}\) and let \(\rho\) be a representation of \(\Gamma\) with degree \(d_\rho\). Then the reciprocal of the \(L\)-function of a weighted \(n\)-fold covering \((D^\phi, \tilde{\omega})\) associated with \(\rho\) is

\[
L((D^\phi, \tilde{\omega}))(t, u, \rho)^{-1} = \det \left[ \mathbf{I}_{d_\rho \alpha_D} - u \left( \sum_{\sigma \in \Gamma} B_D(\sigma) - (1-t)C_D(\sigma) \otimes \rho(\sigma) \right) \right].
\]

In particular, if \(\omega\) is reciprocal, then

\[
L((D^\phi, \tilde{\omega}))(t, u, \rho)^{-1} = (1 - (1-t)^2 u^2)^{\theta_\rho} \det \Omega,
\]

where \(\theta_\rho = (\varepsilon_D - v_D) d_\rho\) and

\[
\Omega = \left( \mathbf{I}_{d_\rho v_D} - u \sum_{\sigma \in \Gamma} W_D \otimes \rho(\sigma) + (1-t) u^2 \left( S_D - (1-t) I_{v_D} \otimes \mathbf{I}_{d_\rho} + (1-t) u \sum_{\sigma \in \Gamma} W_D \otimes \rho(\sigma) \right) \right).
\]
Note that Theorem 6 in [25] states a regular covering version of Corollary 8.

Let $\Gamma$ be the subgroup of Aut($F$) generated by $\{\phi(a) \mid a \in A(D)\} \cup \{s \mid s \in S\}$. Then, by the representation theory of finite groups (see, for example, [27]), one can easily obtain the following properties.

If $F = K_n$ and $\rho$ is the trivial representation $\mathbf{1}$ of $\Gamma$, then

$$L_{(D \times \phi F, \tilde{\omega})}(t, u, \mathbf{1}) = \zeta_{(D, \omega_D)}(t, u).$$  \hspace{1cm} (10)

If $\rho_1, \ldots, \rho_\ell$ are inequivalent irreducible representations of $\Gamma$ such that $\rho = \bigoplus_{i=1}^\ell m_i \rho_i$, then

$$L_{(D \times \phi F, \tilde{\omega})}(t, u, \rho) = \prod_{i=1}^\ell L_{(D \times \phi F, \tilde{\omega})}(t, u, \rho_i)^{m_i}. \hspace{1cm} (11)$$

If $\rho = \mathbf{P}$, the permutation representation of $\Gamma$, then one can see that $B(P)$ and $C(P)$ are exactly the arc-adjacency matrix $B(D \times \phi F, \tilde{\omega})$ and the bump matrix $C(D \times \phi F, \tilde{\omega})$ of a weighted digraph bundle $(D \times \phi F, \tilde{\omega})$, respectively. Thus Theorem 2 implies that

$$L_{(D \times \phi F, \tilde{\omega})}(t, u, \mathbf{P}) = \zeta_{(D \times \phi F, \tilde{\omega})}(t, u).$$  \hspace{1cm} (12)

Hence the following decomposition theorem is established by Eqs. (11) and (12).

**Theorem 9.** Let $(D \times \phi F, \tilde{\omega})$ be a weighted digraph bundle. Assume that a fiber $(F, \omega_F)$ is a weighted Schreier graph with connecting set $S$ with $W_F = \sum_{j=1}^r \omega_j P(s_j)$ and $W_{F_1} = \sum_{j=1}^{r_1} \omega_j P(s_j)$. Let $\Gamma$ be the subgroup of Aut($F$) generated by $\{\phi(a) \mid a \in A(D)\} \cup \{s \mid s \in S\}$. Let $\rho_1, \ldots, \rho_\ell$ be the inequivalent irreducible representations of $\Gamma$ and let the permutation representation $\mathbf{P}$ of $\Gamma$ be decomposed as $\mathbf{P} = \bigoplus_{i=1}^\ell m_i \rho_i$. Then

$$\zeta_{(D \times \phi F, \tilde{\omega})}(t, u) = \prod_{i=1}^\ell L_{(D \times \phi F, \tilde{\omega})}(t, u, \rho_i)^{m_i}. \hspace{1cm} (13)$$

A digraph bundle $D \times \phi F$ is a digraph covering when $F$ is $K_n$. Also $D \times \phi F$ is an (undirected) graph covering when $F$ is $K_n$ and $D$ is symmetric. Thus Theorem 9 gives decomposition theorems into $L$-functions of regular or irregular coverings of a graph or a digraph (see Corollary 5 in [17] and Corollary 1 in [24]).

**6. Conclusion**

In this last section, we review several known zeta functions of graphs or digraphs as special cases of the Bartholdi zeta function of a weighted digraph. As shown in Table 1, there are two types of zeta functions of (di)graphs; one is an Ihara type and the other is a Bartholdi type. In the Ihara type of (undirected) graphs, i.e., in $Z_G(u)$ and $Z_{(G, \omega)}(u)$, the product runs over the equivalence classes $[C]$ of prime reduced cycles $C$ in a graph $G$, while in the other zeta functions the product runs over the equivalence classes $[C]$ of prime cycles $C$ in $G$.

We observe that the Ihara type zeta functions can be deduced from the Bartholdi type zeta functions by putting $t = 0$ for undirected graphs and $t = 1$ for digraphs. Also notice that the
Table 1
Zeta functions of various graphs

<table>
<thead>
<tr>
<th></th>
<th>Ihara type</th>
<th>Bartholdi type</th>
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<td>(Undirected)</td>
<td>unweighted</td>
<td></td>
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<tr>
<td>Graphs</td>
<td>$Z_G(u) = \prod_{</td>
<td>C</td>
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<tr>
<td></td>
<td>weighted</td>
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</tr>
<tr>
<td>Graphs</td>
<td>$Z_{(G,\omega)}(u) = \prod_{</td>
<td>C</td>
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<tr>
<td>Digraphs</td>
<td>unweighted</td>
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<tr>
<td></td>
<td>weighted</td>
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</tr>
<tr>
<td>Digraphs</td>
<td>$Z_{(D,\omega)}(u) = \prod_{</td>
<td>C</td>
</tr>
</tbody>
</table>

Zeta functions of unweighted (di)graphs can be deduced from the zeta functions of weighted (di)graphs by taking $\omega = 1$, the trivial weight function. Let $\hat{G}$ denote the symmetric digraph obtained from an undirected graph $G$ by replacing each edge $xy = \{x, y\}$ with the two opposing arcs $(x, y)$ and $(y, x)$. Then the zeta functions of an undirected graph $G$ coincide with the zeta functions of the symmetric digraph $\hat{G}$. Thus many known zeta functions can be deduced from the Bartholdi zeta function of a weighted digraph as follows:

$$
\zeta_D(t, u) = \zeta_{(D,1)}(t, u), \quad Z_{(D,\omega)}(u) = \zeta_{(D,\omega)}(1, u), \quad Z_D(u) = \zeta_{(D,1)}(1, u),
$$

$$
\zeta_{(G,\omega)}(t, u) = \zeta_{(G,\omega)}(t, u), \quad \zeta_G(t, u) = \zeta_{(\hat{G},1)}(t, u), \quad Z_G(u) = \zeta_{(\hat{G},1)}(0, u),
$$

and $Z_{(G,\omega)}(u) = \zeta_{(\hat{G},\omega)}(0, u)$.

Moreover, a graph bundle $G \times^\phi F$ has two important special cases: (1) when $F$ is $K_n$ and (2) when $\phi$ is the trivial voltage assignment. In case (1), a graph bundle $G \times^\phi F$ is a graph covering $G^\phi$. Thus Theorems 2–4 include the results on graph coverings such as Theorem 2 in [15] and Theorem 3 in [19]. In case (2), a graph bundle $G \times^\phi F$ is a graph product $G \times F$. Thus Theorems 2–4 include the results on graph products such as Corollary 10 in [13] and Corollary 7 in [6].

Also the zeta and $L$–functions for graphs, digraphs, coverings, and bundles discussed in ([6, 13,15–21,23,25,28–30], etc.) can follow from our results presented in this paper.

References