Degree sum and nowhere-zero 3-flows

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Abstract

Let $G$ be a simple graph on $n$ vertices. In this paper, we prove that if $G$ satisfies the condition that $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then $G$ has no nowhere-zero 3-flow if and only if $G$ is either one of the five graphs on at most 6 vertices or one of a very special class of graphs on at least 6 vertices.

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1. Introduction

The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. For $xy \in E(G)$, we call $y$ a neighbor of $x$, and the set of neighbors of $x$ in $G$ is denoted by $N_G(x)$, or simply $N(x)$. Let $H$ be a subgraph of $G$ and $v \in V(G)$, define that $d_H(v) = |N(x) \cap V(H)|$, the number of the neighbors of $v$ in $H$. When $H = G$, $d_G(v)$ is called the degree of $v$, and abbreviated to $d(v)$. Denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degree of $G$, respectively. For subgraphs $A$ and $B$, $e(A, B)$ denotes the number of edges with one end in $A$ and the other end in $B$.

An edge is contracted if it is deleted and its two ends are identified into a single vertex. Let $H$ be a connected subgraph of $G$. $G/H$ denotes the graph obtained from $G$ by contracting all the edges of $H$ and deleting all the resulting loops. For $S \subseteq V(G)$, $G - S$ denotes the graph obtained from $G$ by deleting all the vertices of $S$ together with all the edges with at least one end in $S$. When $S = \{v\}$, we simplify this notation to $G - v$. The complete graph on $n$ vertices is denoted by $K_n$. Denote by $K_n^-$ the graph obtained from $K_n$ by deleting an edge. $K_{3,n-3}^+$ denotes the simple graph obtained from the complete bipartite graph $K_{3,n-3}$ by adding an edge between two vertices of degree $n - 3$.

A $k$-circuit is a circuit of $k$ vertices. A wheel $W_k$ is the graph obtained from a $k$-circuit by adding a new vertex, called the center of the wheel, which is joined to every vertex of the $k$-circuit. $W_k$ is an odd (even) wheel if $k$ is odd (even). For a technical reason, a single edge is regarded as 1-circuit, and thus $W_1$ is a triangle, called the trivial wheel. For simplicity, a 3-circuit (triangle) on vertices $\{x, y, z\}$ is denoted by $xyz$.

Let $G$ be a graph with an orientation. For each vertex $v \in V(G)$, $E^+(v)$ is the set of non-loop edges with tail $v$, and $E^-(v)$ is the set of non-loop edges with head $v$. Let $\mathbb{Z}_k$ denote an abelian group of $k$ elements with identity 0. Let

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Proposition 1.5. None of the five graphs in Fig. 1 has nowhere-zero 3-flows.

Proof. $G_1$ is the complete graph $K_4$, which does not have nowhere-zero 3-flows, and neither does $G_2$ since it is obtained from $K_4$ by subdividing an edge. It is known that the odd wheel has no nowhere-zero 3-flow, and thus $G_3$ has no nowhere-zero 3-flow. It is known that a cubic graph has a nowhere-zero 3-flow if and only if it is bipartite. $G_4$ is cubic and non-bipartite (containing triangles), and thus has no nowhere-zero 3-flows. Consider the graph $G_5$. Suppose, to the contrary, that it has a nowhere-zero 3-flow, and so a nowhere-zero $\mathbb{Z}_3$-flow. By choosing orientations, we may obtain a $\mathbb{Z}_3$-flow in which each edge has flow value 1. Then, at each vertex of degree 5, all the three edges incident with the vertex have the same orientation, that is, either all are out of or all are into the vertex, which means that the edge between the two vertices of degree 5 can have only zero flow value, a contradiction. This shows that none of the five graphs in Fig. 1 has nowhere-zero 3-flows.

Proposition 1.5. For each $n \geq 6$, $K_{3n-3}^+$ has no nowhere-zero 3-flow.
Proof. The proof is similar to the one we just did to the graph $G_5$ in Fig. 1. Suppose, to the contrary, that $K_{3,n-3}^+$ has a nowhere-zero 3-flow. Then it has a nowhere-zero $\mathbb{Z}_3$-flow, in which each edge has flow value 1, and thus, at each vertex of degree 3, all the three edges incident with the vertex have the same orientation, that is, either all are out of or all are into the vertex. Let $x$ and $y$ be the two vertices of degree $n-2$. For each vertex $z$ of degree 3, $zx$ and $zy$ have the same orientation and same flow value. This means that the edge $xy$ can have only zero flow value, a contradiction. $\blacksquare$

Consider simple graphs on $n$ vertices in which $d(x) + d(y) \geq n + 2$ for each edge $xy$. For instance, $K_4$ is such a graph. A main result of this paper is Theorem 1.6 below, which shows that for simple graphs on $n$ vertices with $d(x) + d(y) \geq n + 2$ for each edge $xy$, $K_4$ is the only one that is not $\mathbb{Z}_3$-flow contractible.

**Theorem 1.6.** Let $G$ be a simple graph on $n$ vertices. If $d(x) + d(y) \geq n + 2$ for each $xy \in E(G)$, then $G$ is $\mathbb{Z}_3$-flow contractible if and only if $G$ is not $K_4$.

Another main result of this paper is the following theorem, which gives a complete characterization of those simple graphs on $n$ vertices with $d(x) + d(y) \geq n$ for each edge $xy$ and without nowhere-zero 3-flows.

**Theorem 1.7.** Let $G$ be a 2-edge-connected simple graph on $n$ vertices. If $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then $G$ has no nowhere-zero 3-flow if and only if $G$ is $K_{3,n-3}^+$ or one of the five graphs in Fig. 1.

2. Preliminaries

Let $G$ be a graph. A triangle-path in $G$ is a sequence of distinct triangles $T_1T_2\cdots T_m$ in $G$ such that for $1 \leq i \leq m - 1$,

$$|E(T_i) \cap E(T_{i+1})| = 1 \quad \text{and} \quad E(T_i) \cap E(T_j) = \emptyset \quad \text{if} \quad j > i + 1. \quad (2.1)$$

Furthermore, if $m \geq 3$ and (2.1) holds for all $i$, $1 \leq i \leq m$, with the additionally taken mod $m$, then the sequence is called a triangle-cycle. The number $m$ is the length of the triangle-path (triangle-cycle). A connected graph $G$ is triangularly connected if for any distinct $e, e' \in E(G)$, which are not parallel, there is a triangle-path $T_1T_2\cdots T_m$ such that $e \in E(T_1)$ and $e' \in E(T_m)$. The following result was proved in [3] (Theorem 4.1 in [3]), and is needed in the proof of Theorem 1.6.

**Lemma 2.1.** Let $G$ be a triangle-regular-connected graph with $|V(G)| \geq 3$. If $G$ is not $\mathbb{Z}_3$-flow contractible, then it contains either two vertices of degree 2 or at least three vertices of degree at most 3.

3. Proof of Theorem 1.6

**Proof of Theorem 1.6.** If $G$ is $K_4$, then $G$ is not $\mathbb{Z}_3$-flow contractible, in fact, it has no nowhere-zero 3-flow. Conversely, suppose that $G$ is not $K_4$, we shall prove that $G$ is $\mathbb{Z}_3$-flow contractible. By the given condition that $d(x) + d(y) \geq n + 2$ for each edge $xy \in E(G)$, we can easily see that $\delta(G) \geq 3$ and $n \geq 4$, and moreover, since $G$ is not $K_4$, $n \geq 5$. We use induction on $n$. When $n = 5$, if there is a vertex $v \in V(G)$ such that $d(v) = 3$, then the vertices adjacent with $v$ have degree 4, which means that $G$ is $K_5^3$; if all vertices of $G$ have degree 4, then $G$ is $K_5$. In either case, by Proposition 1.2, $G$ is $\mathbb{Z}_3$-flow contractible. Thus, suppose that $n \geq 6$ and the theorem holds for any graph $G'$ with $|V(G')| < n$.

**Claim 1.** If $G$ contains a $\mathbb{Z}_3$-flow contractible subgraph, then $G$ is $\mathbb{Z}_3$-flow contractible.

Let $H$ be a maximal, connected, $\mathbb{Z}_3$-flow contractible subgraph in $G$. Denote by $G'$ the graph obtained from $G$ by contracting $H$ and $u^*$ the vertex of $G'$ into which $H$ is contracted. If $G'$ is a single vertex graph, we are done. Assume that $H$ is a proper subgraph of $G$. Obviously, $G'$ is simple since 2-circuit is $\mathbb{Z}_3$-flow contractible. Note that $G'$ is a simple graph in which all vertices, except for $u^*$, have the same degree as in $G$. Let $|V(G')| = n^*$. For any edge $xy \in E(G')$, we shall prove that $d_{G'}(x) + d_{G'}(y) \geq n^* + 2$.

If $x \neq u^*$, $y \neq u^*$, then

$$d_G(x) + d_G(y) = d_{G'}(x) + d_{G'}(y) \geq n + 2 > n^* + 2.$$
Proposition 1.2

Lemma 2.1

Clearly, if \( K \) contains no odd circuit, then \( G \) is \( K_{3,3} \) or \( G \) is \( G_4 \) in Fig. 1.

**Proof.** Clearly, if \( G \) contains no odd circuit, then \( G \) is \( K_{3,3} \), Suppose that \( G \) has an odd circuit. Let \( C \) be the shortest odd circuit in \( G \). If \( |V(C)| = 3 \), then \( C \) has no chord, which means the only vertex not in \( C \) has degree 5, a contradiction. Thus, \( |V(C)| = 3 \), and it is easy to check that \( G \) is \( G_4 \) in Fig. 1.  

**Lemma 4.1.** \( K_{m,n} \) has a nowhere-zero 3-flow for \( m \geq 2, n \geq 2 \).

**Proof.** Without loss of generality, we may suppose that \( m \geq n \). We use induction on \( m + n \). When \( m = 2 \) or \( m = 3 \), it is not difficult to see that \( K_{m,n} \) has a nowhere-zero 3-flow. Then suppose that \( m \geq 4 \) and the lemma holds for any graphs with less than \( m + n \) vertices. Note that the edge of graph \( K_{m,n} \) can be decomposed into two subgraphs \( K_{2,n} \) and \( K_{m-2,n} \). Then, by the induction hypothesis, \( K_{2,n} \) has a nowhere-zero 3-flow \( f_1 \) and \( K_{m-2,n} \) has a nowhere-zero 3-flow \( f_2 \). By combining \( f_1 \) and \( f_2 \), we get a nowhere-zero 3-flow on \( K_{m,n} \).  

**Lemma 4.2.** If \( G \) is a cubic simple graph on 6 vertices, then either \( G \) is \( K_{3,3} \) or \( G \) is \( G_4 \) in Fig. 1.

**Proof.** The first lemma is a known one (see [7] and [8]). For completeness, we give a proof here.

At first, we present some lemmas which will be used in the proof of Theorem 1.7. The first lemma is a known one (see [7] and [8]). For completeness, we give a proof here.
Lemma 4.3. Let $G$ be a simple 2-edge-connected graph on $n$ vertices, where $n \geq 7$. If $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then $G$ has a nowhere-zero 3-flow if and only if $G$ is either $K_{3,3}^+$ or one of the five graphs $G_1$ in Fig. 1.

**Proof.** If $G$ is $K_{3,3}^+$ or $G$ is one of the five graphs described in Fig. 1, then by Propositions 1.4 and 1.5, $G$ has no nowhere-zero 3-flow. Conversely, suppose that $G$ has no nowhere-zero 3-flow. We shall prove that it must be $K_{3,3}^+$ or one of the five graphs in Fig. 1. Since $G$ is 2-edge-connected and $G$ has no nowhere-zero 3-flow, we have that $\delta(G) \geq 2$ and $n \geq 4$. If $n = 4$, by the condition that $d(x) + d(y) \geq n$ for each $xy \in E(G)$, $G$ must be $K_4$, the graph $G_1$ in Fig. 1. Suppose therefore that $n \geq 5$.

(i) $n = 5$. If $\delta(G) = 2$, let $w \in V(G)$ with $d(w) = 2$ and $N(w) = \{u_1, u_2\}$. Let $G'$ be the graph obtained from $G$ by deleting $w$ and adding $u_1u_2$. If $G'$ has a nowhere-zero 3-flow, then so does $G$, which is impossible. Therefore, $G'$ must be $K_4$, which implies that $G$ is $G_2$ in Fig. 1. If $\delta(G) \geq 3$, since $n$ is odd, there is a vertex $u \in V(G)$ such that $d(u) = 4$. If $d(v) = 3$ for each $v \in V(G) \setminus \{u\}$, then $G$ is the even wheel $W_4$ centered at $u$; if there is $v \in V(G) \setminus \{u\}$ such that $d(v) = 4$, then $G$ is $K_5$ or $K_5^*$. In either case, by Proposition 1.2, $G$ has a nowhere-zero 3-flow, a contradiction.

(ii) $n = 6$. If $\delta(G) = 2$, let $w$ be a vertex of degree 2 in $G$ and $N(w) = \{u_1, u_2\}$. Then, by the condition that $d(x) + d(y) \geq n$ for each $xy \in E(G)$, $u_1$ and $u_2$ have degree more than 3 in $G$. Let $G'$ be the graph obtained from $G$ by deleting $w$ and adding $u_1u_2$. If $G'$ has a nowhere-zero 3-flow, then so does $G$, which is impossible. Therefore, if $G'$ is simple, it must be $G_2$ in Fig. 1. Note that the vertices in $G'$ have the same degree as in $G$. Thus, there is at least one edge $xy \in E(G)$ such that $d(x) + d(y) \leq 5$. Contrary to the hypothesis that $d(x) + d(y) \geq n$ for each $xy \in E(G)$. Thus, suppose that $G'$ has a 2-circuit on $\{u_1, u_2\}$. Let $G^*$ be the graph obtained from $G'$ by contracting the 2-circuit. If $G^*$ has a nowhere-zero 3-flow, then so does $G'$, which is impossible. Note that $|V(G^*)| = 4$, then $G^*$ must be $K_4$. On the other hand, since $d(u_1) \geq 4$ and $d(u_2) \geq 4$, we have that $u_1$ and $u_2$ have a common neighbor other than $w$, which implies that $G^*$ is not simple, a contradiction. Thus we may assume that $\delta(G) \geq 3$.

If $\Delta(G) = 3$, then $G$ is cubic. By Lemma 4.2, $G$ is $G_4$ in Fig. 1.

If $\Delta(G) = 4$, let $u$ be a vertex of $G$ with degree 4 and $N(u) = \{u_1, u_2, u_3, u_4\}$. Let $H$ be the subgraph induced by $N(u)$. If $H$ contains two independent edges, say that $u_1u_2, u_3u_4 \in E(H)$, then let $G'$ be the graph obtained from $G$ by deleting $u$ and adding $u_1u_2$ and $u_3u_4$. By the condition that $d(x) + d(y) \geq n$ for each $xy \in E(G)$, and since $G$ is simple and $n = 6$, we see that $G'$ is 2-edge-connected. By contracting the resulting 2-circuits, we get a graph with 3 vertices, which has a nowhere-zero 3-flow. Thus, $G$ has a nowhere-zero 3-flow, a contradiction. Suppose therefore that $H$ does not contain two independent edges. Now, $\delta(G) \geq 3$ implies that $\delta(H) \geq 1$. It follows that $H$ is a star, and so $G$ is $K_{3,3}^+$. If $\Delta(G) = 5$, let $u \in V(G)$ with $d(u) = 5$ and let $N$ denote the subgraph induced by $N(u)$. We note that an even circuit in $N$ together with $u$ gives an even wheel centered at $u$, which implies, by Proposition 1.2 (i) and Proposition 1.3, that $G$ has a nowhere-zero 3-flow. Thus, we may assume that there is no even circuit in $N$. Then, each block of $N$ is either $K_1$ or $K_2$, or an odd circuit (see exercise 3.2.3 in [1]). But, $\delta(G) \geq 3$ and $|N(u)| = 5 = |V(G)| - 1$, and so, each block of $N$ is an odd circuit and $N$ has at most two blocks. If $N$ has exactly one block, then $N$ is a circuit of length 5, and hence $G$ is $G_3$ in Fig. 1; if $N$ has two blocks, then $N$ consists of two triangles with exactly one vertex in common, and hence $G$ is $G_5$ in Fig. 1. This completes the proof of Lemma 4.3. □

Lemma 4.4. Let $G$ be a simple 2-edge-connected graph on $n$ vertices, where $n \geq 7$. If $d(x) + d(y) \geq n$ for each $xy \in E(G)$, then either $G$ has a nowhere-zero 3-flow or $G$ contains $K_4^-$. 

**Proof.** Suppose first that $\delta(G) \leq \frac{n-1}{2}$. Since $G$ is 2-edge-connected, $\delta(G) \geq 2$. Let $u \in V(G)$ with $d(u) = \delta(G)$. If there is no edge in the subgraph induced by $N(u)$, then all vertices in $N(u)$ have the same neighbor set $V(G) \setminus N(u)$. Let $u_1, u_2 \in N(u)$. If there is one edge $xy \in V(G) \setminus N(u)$, then the union of the two triangles $xyu_1$ and $xyu_2$ is the $K_4^-$ required in the lemma. Suppose then there is no edge in $V(G) \setminus N(u)$. It is not difficult to see that $G$ is a complete bipartite graph $K_{m,n-m}$, where $m = \delta(G)$. By Lemma 4.1, $G$ has a nowhere-zero 3-flow. Therefore suppose that there is an edge in $N(u)$. Without loss of generality, assume that $u_1u_2 \in E(G)$. If $\delta(G) = 2$, then $N(u) = \{u_1, u_2\}$. By the given degree-sum condition, $d(u_1) \geq n - 2$ and $d(u_2) \geq n - 2$, which implies that there is another vertex other than $u$ in $N(u_1) \cap N(u_2)$. Thus, we get a $K_4^-$. Suppose then that $\delta(G) \geq 3$. If $u_1$ and $u_2$ have a common neighbor other than $u$, we also have a $K_4^-$. Thus, we assume that $u_1$ and $u_2$ have only one common neighbor $u$. But, by the given degree-sum condition, we have that $d(u_1) + d(u_2) \geq \frac{n-1}{2} + \frac{n-1}{2} = n + 1$. It follows that $d(u_1) + d(u_2) = n + 1$, which
means that every vertex of $G$ must be a neighbor of $u_1$ or $u_2$. Since $|N(u)| \geq 3$, there is a vertex $u_3 \in N(u)$. Then $u_3$ is the neighbor of $u_1$ or $u_2$, without loss of generality, suppose that $u_3 \in N(u_1)$. Then the union of two triangles $uu_1u_2$ and $uu_1u_3$ is the $K^-_3$ desired. In what follows, we assume that $\delta(G) \geq \frac{3}{2}$. Let $e = |E(G)|$. Then $e \geq \frac{3n}{2}$, and by Turan Theorem, either $G$ contains a triangle or $G$ is the complete bipartite graph $K_{m,m}$, where $m = \frac{n}{2}$. If $G$ is the complete bipartite graph $K_{m,m}$, then $G$ has a nowhere-zero 3-flow by Lemma 4.1. Suppose therefore $G$ contains a triangle $T = v_1v_2v_3$. For each $v \in V(G) \setminus V(T)$, if $v$ has two neighbors in $T$, then we have a $K^-_4$. Suppose therefore that each vertex in $V(G) \setminus V(T)$ has at most one neighbor in $T$. It follows that
\[
d(v_1) + d(v_2) + d(v_3) \leq (n - 3) + 6 = n + 3.
\]
But $\delta(G) \geq \frac{3}{2}$, we have that
\[
d(v_1) + d(v_2) + d(v_3) \geq \frac{3n}{2}.
\]
Combining the two inequalities yields that $n \leq 6$, a contradiction. This completes the proof of the lemma. 

**Proof of Theorem 1.7.** If $G$ is $K^+_{3,n-3}$ or one of the five graphs described in Fig. 1, then by Propositions 1.4 and 1.5, $G$ has no nowhere-zero 3-flow. Conversely, suppose that $G$ is neither $K^+_{3,n-3}$ nor any of the five graphs in Fig. 1. We shall prove that $G$ has a nowhere-zero 3-flow. Since $G$ is 2-edge-connected, we have that $\delta(G) \geq 2$.

We use induction on $n = |V(G)|$, when $n \leq 6$, the theorem holds by Lemma 4.3. Suppose thus that $n \geq 7$ and the theorem holds for any graph $G'$ with $|V(G')| < n$. By Lemma 4.4, we may assume that $G$ contains a $K^-_4$.

If $\delta(G) = 2$, then let $d(v) = 2$. Consider the graph $G'$ obtained from $G$ by suppressing the degree 2 vertex $v$. It is obvious that $d_G(x) + d_G(y) \geq |V(G')|$ and $G'$ admits a nowhere-zero 3-flow if and only if $G$ admits a nowhere-zero 3-flow. If $G'$ is simple, by the induction hypothesis, then $G'$ admits a nowhere-zero 3-flow, and so does $G$. If $G$ is not simple, let $N(v) = \{x, y\}$, then $d(x) \geq n - 2$ and $d(y) \geq n - 2$, which implies that $|N(x) \cap N(y)| \geq n - 4$. Let $G^*$ be the simple graph obtained from $G'$ by recursively contracting 2-circuits. Then $|V(G^*)| \leq 3$. By Lemma 4.3, $G^*$ admits a nowhere-zero 3-flow. Therefore, $G'$, and so $G$, admits a nowhere-zero 3-flow. We may then suppose that $\delta(G) \geq 3$.

**Claim 1.** There is a $K^-_4$, the union of the two triangles $xyz$ and $xyw$ with edge $xy$ in common, in $G$ such that $d(z) \geq 4$.

Suppose that the $K^-_4$ contained in $G$ is the union of $u_1u_2u_3$ and $u_1u_2u_4$. If $d(u_3) \geq 4$ or $d(u_4) \geq 4$, we are done. Suppose then $d(u_3) = d(u_4) = 3$. By the given degree-sum condition, we have that $u_3u_4 \notin E(G)$ and $d(u_1) \geq 4$, $d(u_2) \geq 4$. Let $v \in N(u)$. If $uu_1 \in E(G)$ or $uu_2 \in E(G)$, without loss of generality, assume that $uu_1 \in E(G)$, the union of $uu_1u$ and $uu_1u_2$ with $d(u_2) \geq 4$ is the $K^-_4$ required in the claim. Thus assume that $uu_1 \notin E(G)$ and $uu_2 \notin E(G)$. By the given degree-sum condition, $d(v) \geq n - 3$. Note that $u_1, u_2 \notin N(v)$, hence, $N(v) = V(G) \setminus \{u_1, u_2\}$. Since $d(u_1) + d(u_2) \geq n$ and $d(u_2) + d(u_3) \geq n$, we have that $|N(u_1) \cap N(u_2)| \geq n - 4$ and $|N(u_2) \cap N(u_3)| \geq n - 4$. Let $S = N(u_1) \cap N(u_2)$. Then $|S| \geq n - 5$ and $|N(u_1) \setminus N(u_2)| \leq 1$, $|N(u_2) \setminus N(u_1)| \leq 1$. 

(i) $|S| = n - 5$. Then $|N(u_1) \setminus N(u_2)| = 1$ and $|N(u_2) \setminus N(u_1)| = 1$. Let $v_1 \in N(u_1) \setminus N(u_2), v_2 \in N(u_2) \setminus N(u_1)$. If there is $s \in S$ such that $v_1s \in E(G)$, then we obtain a $K^+_4$, the union of $v_1sv_2$ and $v_1sv$, with $d(v) \geq 4$, as claimed. So, we assume that $v_1s \notin E(G)$ for any $s \in S$. Similarly, $v_2s \notin E(G)$ for any $s \in S$. It follows that $d(v_1) \leq 3$, $d(v_2) \leq 3$. Note that $v_1v_2 \notin E(G)$ by the given degree-sum condition. Thus, $d(v_1) = d(v_2) = 2$. The contradiction follows from $\delta(G) \geq 3$.

(ii) $|S| = n - 4$. Suppose that $|N(u_1) \setminus N(u_2)| = 1$. Let $v_1 \in N(u_1) \setminus N(u_2)$, we have that $d(v_1) = 2$ by the similar argument in case (i), a contradiction. Thus, $N(u_1) \setminus N(u_2) = \emptyset$. Similarly, $N(u_2) \setminus N(u_1) = \emptyset$. Therefore, $N(u_1) = N(u_2)$. Let $u \in N(v) \setminus N(u_1)$. If there are $s_1, s_2 \in S$ such that $s_1s_2 \in E(G)$, then $s_1, s_2$ together with $u, v$ induce a $K^-_4$, the union of $uvs_1$ and $uvs_2$, with $d(s) \geq 4$, as claimed. Suppose then $|N(u) \cap S| \leq 1$. It follows that $d(u) \leq 2$, a contradiction as before.

(iii) $|S| = n - 3$. Then $N(v) = N(u_1) = N(u_2)$. If there are $s_1, s_2 \in S$ such that $s_1s_2 \in E(G)$, then the union of $s_1s_2u_1$ and $s_1s_2u_2$ with $d(u_1) \geq 4$ gives the desired $K^-_4$. Thus, suppose that $S$ is an independent set. It is not difficult to see that $G$ is $K^+_{3,n-3}$, a contradiction. This completes the proof of Claim 1.

By Claim 1, we suppose that there is a $K^-_4$, the union of two triangles $xyz$ and $xyw$ with $d(z) \geq 4$. Let $G'$ be the graph obtained from $G$ by deleting $xz, yz$, and adding $xy$. Let $H$ be the maximal, connected, $Z_3$-flow contractible subgraph of $G'$ and $G^* = G'/H$. Denote by the $u^*$ the new vertex into which $H$ is contracted. Note that $G^*$ is a simple graph, in which all vertices, except for $u^*$ and $z$, have the same degree as in $G$. Since $G^*$ is obtained from $G'$
by consecutively contracting 2-circuits, if $G^*$ has a nowhere-zero 3-flow, then so does $G'$. Let $|V(G')| = n^*$. We have that $n^* \leq n - 2$. If $n^* \leq 3$, then $G^*$ has a nowhere-zero 3-flow, which implies that $G'$, and so $G$, has a nowhere-zero 3-flow. Thus, assume that $n^* \geq 4$.

Claim 2. For any distinct $u, v \in V(G^*)$ and $uv \in E(G^*)$, $d_{G^*}(u) + d_{G^*}(v) \geq n^*$.

If $u, v \in V(G^*) \setminus \{z, u^*\}$, then

$$d_{G^*}(u) + d_{G^*}(v) = d_G(u) + d_G(v) \geq n > n^*.$$ 

If $u \neq u^*$ and $v = z$, then, using that $d_{G^*}(z) = d_G(z) - 2$,

$$d_{G^*}(u) + d_{G^*}(v) = d_G(u) + d_G(v) - 2 \geq n - 2 \geq n^*.$$ 

If $u = u^*$ and $v \neq z$, then, there is $a \in V(H)$ such that $va \in E(G)$. Since $d_{G^*}(u^*) \geq d_G(a) - (|V(H)| - 1)$, we have that

$$d_{G^*}(u) + d_{G^*}(v) \geq d_G(a) - (|V(H)| - 1) + d_G(v) \geq n - (|V(H)| - 1) = n^*.$$ 

What remains is the case that $u = u^*$ and $v = z$. Let $R = G - V(H)$. Then there is $a \in V(H) \setminus \{x, y\}$ such that $az \in E(G)$. By the given condition, $d(z) + d(a) \geq n$. Note that

$$d_{G^*}(u^*) = e(a, R) + e(H - a, R - z).$$ 

Since $d_{G^*}(z) = d_G(z) - 2$ and $d_G(a) = e(a, H - a) + e(a, R)$, hence

$$d_{G^*}(z) + d_{G^*}(u^*) \geq d_G(z) - 2 + d_G(a) - e(a, H - a) + e(H - a, R - z).$$ 

If $d_G(z) + d_G(a) \geq n + 2$, then

$$d_{G^*}(z) + d_{G^*}(u^*) \geq n - e(a, H - a) + e(H - a, R - z).$$ 

Since $e(a, H - a) \leq |V(H)| - 1$ and $n - (|V(H)| - 1) = n^*$, we have that $d_{G^*}(z) + d_{G^*}(u^*) \geq n^* + e(H - a, R - z) \geq n^*$, as claimed. Thus, suppose that

$$n \leq d_G(z) + d_G(a) \leq n + 1. \quad (4.1)$$ 

Using that $d_G(z) + d_G(a) \geq n$, we obtain that

$$d_{G^*}(z) + d_{G^*}(u^*) \geq n - 2 - e(a, H - a) + e(H - a, R - z).$$ 

If $e(a, H - a) \leq |V(H)| - 3$ or $e(H - a, R - z) \geq 2$, we are done. Thus suppose that

$$e(a, H - a) \geq |V(H)| - 2 \quad (4.2)$$ 

and

$$e(H - a, R - z) \leq 1. \quad (4.3)$$ 

Note that $G^*$ is simple, thus, for any $r \in V(R - z)$, we have that

$$e(r, H) \leq 1. \quad (4.4)$$ 

If $|V(H)| = 3$, then $V(H) = \{x, y, w\}$. By (4.3), we have that $e(xy, R - z) \leq 1$, and then $d(x) + d(y) \leq 7$, which means that $n \leq 7$. By the hypothesis $n \geq 7$, we see that $n = 7$. Without loss of generality, we may assume that $d(x) = 4$ and $d(y) = 3$. If $d(w) \leq 3$, then $d(y) + d(w) \leq 6$, contrary to the given degree-sum condition. Then suppose that $d(w) \geq 4$, note that $d_G(z) \geq 4$, then $d_{G^*}(z) + d_{G^*}(u^*) \geq |V(H)| - 4 = 5 = n^*$, as claimed. Therefore suppose that $|V(H)| \geq 4$.

If there is $b \in V(H - \{x, y, a\})$ such that $bx \notin E(G)$ or $by \notin E(G)$, by (4.3), we have that $e(xy, R - z) \leq 1$, and then $n \leq d(x) + d(y) \leq 2|V(H)|$, if $bx \in E(G)$ and $by \in E(G)$, also by (4.3), $e(xy, R - z) = 0$, $e(xb, R - z) = 0$ or $e(yb, R - z) = 0$, as before, we get that $n \leq 2|V(H)|$. In either case, $2|V(H)| \geq n$, that is $|V(H)| \geq \frac{n}{2}$. Therefore, $|V(R)| \leq \frac{n}{2}$. If there is $r_1 \in V(R - z)$ such that $e(r_1, H) = 0$, then there is $r_2 \in V(R - z)$ such that $r_1r_2 \in E(G)$. By (4.4), we have that $e(r_2, H) \leq 1$, then $d(r_1) + d(r_2) \leq n - 1$, a contradiction. Thus,

$$e(r, H) = 1 \quad \text{for any } r \in V(R - z). \quad (4.5)$$
Since \(d(z) \geq 4\), there is \(r \in V(R-z)\) such that \(rz \in E(G)\). If \(d(r) = 2\), then \(d(z) \geq n-2\). Note that \(|V(H)| \geq 4\) and \(e(z, H) = 3\), we have \(d(z) = n-2\) and \(|V(H)| = 4\), which implies that \(d(a) \leq 3\) by (4.1). It follows from (4.3) that \(d(a) = 3\) and \(e(a, R-z) = 0\). Note that \(|V(H)| \geq \frac{2}{3}\), then \(n \leq 8\). Recall that \(d_H(a) = 2\), by the given degree-sum condition, the two vertices in \(H\) adjacent to \(a\) have degree more than 4. Thus, \(e(H-a, R-z) \geq 2\), contrary to (4.3). Therefore, assume that \(d(r) \geq 3\) for any \(r \in V(R-z)\) and \(rz \in E(G)\). Let \(r\) be a vertex of \(R\) with \(rz \in E(G)\). It follows from (4.5) and (4.3) that there is \(r' \in V(R-z)\) such that \(rr' \in E(G)\). By the given condition, \(d(r) + d(r') \geq n\). Then \(2|V(R)| \geq n\), which means that \(|V(R)| = \frac{n}{2} = |V(H)|\). Let \(b\) be a vertex of \(H - \{x, y, a\}\). If \(e(b, R) = 0\), then \(d(b) \leq |V(H)| - 1\). Let \(b' \in V(H-a)\) with \(bb' \in E(G)\). By the given degree-sum condition, we have that \(d(b') \geq |V(H)| + 1\). By (4.3), the possible vertex in \(V(H-a)\) which has degree more than \(|V(H)|\) is \(x\) or \(y\). Thus, \(b' = x\) or \(b' = y\). Without loss of generality, assume that \(b' = x\). Since \(d(x) \leq |V(H)| + 1\), we have that \(d(b) = |V(H)| - 1\) and \(d(x) = |V(H)| + 1\), which implies that \(d(y) \leq |V(H)|\). Note that \(N(b) = V(H)\), then \(by \in E(G)\). But, \(d(b) + d(y) < n\), impossible. Therefore, \(e(b, R) \geq 1\) for each \(b \in V(H - \{x, y, a\})\). By (4.3), \(|V(H)| \leq 4\), which means that \(n = 8\). Hence, by the given degree-sum condition and (4.3), we can easily get that \(d(x) = d(y) = d(b) = 4\). By (4.3) and (4.5), we have that \(d(a) \geq 3 + 3 = 6\). Then, \(d(z) + d(a) \geq 10 = n + 2\), contrary to (4.1). This completes the proof of Claim 2.

By Claim 2 and the induction hypothesis, either \(G^*\) has a nowhere-zero 3-flow or \(G^*\) is \(K^+_{3, n-3}\) or one of the five graphs in Fig. 1. In the former case, \(G'\), and so \(G\), has a nowhere-zero 3-flow, then we are done. In the latter case, excluding the case \(G^* = G_5\), we always have two vertices \(x, y \in V(G^*) - \{u^*, z\}\) such that \(d_G(x) + d_G(y) = d_G(x) + d_G(y) \leq n^* + 1 < n\), a contradiction. When \(G^* = G_5\), then \(d_G(u^*) = d_G(z) = 3\) and \(u^*z \notin E(G)\). Otherwise, there is an edge whose ends have degree-sum less than 6, contrary to the hypothesis. It is easy to see that there is an edge \(uv \in E(G^*)\), also \(uv \in E(G)\), such that \(d_G(u) + d_G(v) = 3 + 5 = 8\), which implies that \(n = 8\) and \(|V(H)| = 3\). Then \(V(H) = \{x, y, w\}\). By the given degree-sum condition, \(d(x) + d(y) + d(w) \geq \frac{3n}{2}\), which means that \(6 + d_G(u^*) + 2 \geq \frac{3n}{2}\). But \(d_G(u^*) = 3\), and so \(\frac{3n}{2} \leq 11\), which is impossible. This shows that \(G^*\) is neither \(K^+_{3, n-3}\) nor one of the five graphs in Fig. 1, and completes the proof of the theorem.

We note that all the exceptional graphs in Theorem 1.7, except for \(K_4\), contain an edge \(xy\) with \(d(x) + d(y) = n\). An immediate consequence of Theorem 1.7 is the following corollary.

**Corollary 4.5.** Let \(G\) be a simple graph on \(n\) vertices. If \(d(x) + d(y) \geq n + 1\) for each \(xy \in E(G)\), then \(G\) has a nowhere-zero 3-flow if and only if \(G\) is not \(K_4\).

**Remark.** In the corollary, the condition that \(d(x) + d(y) \geq n + 1\) for each edge \(xy\) implies that each edge of the graph is contained in a triangle, and thus, the graph can be partitioned into edge-disjoint subgraphs, each of which is triangularly connected, called \(T\)-blocks in [3]. Then, using Theorem 4.1 in [3], one can derive the corollary without using Theorem 1.7.

**References**


