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Algebraic approximation of analytic sets and mappings ☆

Marcin Bilski *

Institute of Mathematics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

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Abstract

Let $\{X_\nu\}$ be a sequence of analytic sets converging to some analytic set X in the sense of holomorphic chains. We introduce a condition which implies that every irreducible component of X is the limit of a sequence of irreducible components of the sets from $\{X_\nu\}$. Next we apply the condition to approximate a holomorphic solution $y = f(x)$ of a system $Q(x, y) = 0$ of Nash equations by Nash solutions. Presented methods allow to construct an algorithm of approximation of the holomorphic solutions.

Résumé

Soit $\{X_\nu\}$ une suite d'ensembles analytiques qui converge vers un ensemble analytique X au sens des cycles analytiques. Nous introduisons une condition qui implique que chaque composante irréductible de X est limite d'une suite des composantes irréductibles des ensembles de $\{X_\nu\}$. La condition est utilisée pour approcher des solutions analytiques $y = f(x)$ d'un système $Q(x, y) = 0$ d'équations de Nash par des solutions de Nash. Les méthodes que nous proposons permettent de construire un algorithme d'approximation de solutions analytiques.

Keywords: Analytic mapping; Analytic set; Nash set; Approximation

1. Introduction

Let $\mathbf{K}$ denote the field of complex or real numbers. The following approximation theorem is known to be true: every $\mathbf{K}$-analytic mapping $f : \Omega \to \mathbf{K}^k$ such that $Q(x, f(x)) = 0$ for $x \in \Omega$, where Q is a $\mathbf{K}$-Nash mapping (Ω described below), can be uniformly approximated by a $\mathbf{K}$-Nash mapping $F : \Omega \to \mathbf{K}^k$ such that $Q(x, F(x)) = 0$ for $x \in \Omega$.

In the complex case the theorem was proved by L. Lempert (see [16], Theorem 3.2) for every holomorphically convex compact subset Ω of an affine algebraic variety and in the real case it was proved by M. Coste, J. Ruiz and M. Shiota (see [12], Theorem 1.1) for every compact Nash manifold Ω. The approximation theorem turned out to be a very strong tool with many important applications (see [12,16]).

The proofs of the theorem presented in [12,16] rely on the solution to the M. Artin's conjecture: a deep result from commutative algebra for which the reader is referred to [1,17–20]. Such an approach enabled to reach the goal in an

☆ Research partially supported by the grant NN201 3352 33 of the Polish Ministry of Science and Higher Education.
* Tel.: +48 12 663 5230; fax: +48 12 632 4372.
E-mail address: marcin.bilski@im.uj.edu.pl.

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elegant and relatively short way. On the other hand, it seems to be very difficult to apply the proofs in order to find Nash approximations for concrete analytic mappings hence it is natural to ask whether the theorem can be obtained directly. The latter question is strongly motivated by the fact that approximating analytic objects by algebraic counterparts is one of central techniques used in numerical computations. From this point of view it is important to develop theory of approximation that could serve as a base for finding numerical algorithms.

In Section 3.2 of the present paper we give, using only some basic methods of analytic geometry, a proof of a semi-global version of the theorem in the complex case (see Theorem 3.8). The proof allows to construct an algorithm of approximation of the mapping f which is described in Section 3.2.4.

The following local version is an immediate consequence of Theorem 3.8.

Theorem 1.1. *Let U be an open subset of $\mathbf{C}^n$ and let $f: U \to \mathbf{C}^k$ be a holomorphic mapping that satisfies a system of equations $Q(x, f(x)) = 0$ for $x \in U$. Here Q is a Nash mapping from a neighborhood $\hat{U}$ in $\mathbf{C}^n \times \mathbf{C}^k$ of the graph of f into some $\mathbf{C}^q$. Then for every $x_0 \in U$ there are an open neighborhood $U_0 \subset U$ and a sequence $\{f^\nu : U_0 \to \mathbf{C}^k\}$ of Nash mappings converging uniformly to $f|_{U_0}$ such that $Q(x, f^\nu(x)) = 0$ for every $x \in U_0$ and $\nu \in \mathbf{N}$.*

In the local situation the problem of approximation of the solutions of algebraic or analytic equations was investigated by M. Artin in [2–4] and Theorem 1.1 can be derived from his results.

Our interest in Theorem 1.1 and its generalizations is partially motivated by applications in the theory of analytic sets. In particular, papers [6–10] contain a variety of results on approximation of complex analytic sets by complex Nash sets whose proofs can be divided into two stages: (i) preparation, where only direct geometric methods appear, (ii) switching Theorem 1.1. Thus the techniques of the present article allow to obtain many of these results in a purely geometric way. As an example let us mention the following main theorem of [9]. Let X be an analytic subset of pure dimension n of an open set $U \subset \mathbf{C}^m$ and let E be a Nash subset of U such that $E \subset X$. Then for every $a \in E$ there is an open neighborhood U_a of a in U and a sequence $\{X_\nu\}$ of complex Nash subsets of U_a of pure dimension n converging to $X \cap U_a$ in the sense of holomorphic chains such that the following holds for every $\nu \in \mathbf{N}$: $E \cap U_a \subset X_\nu$ and $\mu_x(X_\nu) = \mu_x(X)$ for $x \in (E \cap U_a) \setminus F_\nu$, where F_ν is a nowhere dense analytic subset of $E \cap U_a$. Here $\mu_x(X)$ denotes the multiplicity of X at x (see [11,13] for the properties and generalizations of this notion).

In the proof of Theorem 3.8 we apply Theorem 3.1 from Section 3.1 which, being of independent interest, is the first main result of this paper. The aim of Section 3.1 is to develop a method of controlling the behavior of irreducible components of analytic sets from a sequence $\{X_\nu\}$ converging in the sense of holomorphic chains to some analytic set X. More precisely, we formulate conditions which guarantee that the numbers of the irreducible components of X and of X_ν are equal for almost all ν which in the considered context implies that every irreducible component of X is the limit of a sequence of irreducible components of the sets from $\{X_\nu\}$.

Combining (the global version of) Theorem 1.1 with Theorem 3.1 one obtains a new method of algebraic approximation of analytic sets extending the approach of [6]. Namely, let X be an analytic subset of $U \times \mathbf{C}^k$ of pure dimension n with proper projection onto the Runge domain $U \subset \mathbf{C}^n$. It is well known [24] that X is a subset of another purely n-dimensional analytic set X' given by:

$$X' = \{(x, z_1, \ldots, z_k) \in U \times \mathbf{C}^k:\ p_1(x, z_1) = \cdots = p_k(x, z_k) = 0\},$$

where $p_i \in \mathcal{O}(U)[z_i]$ is unitary with non-zero discriminant, for $i = 1, \ldots, k$. After replacing the coefficients of p_i, for every i, by their Nash approximations on U we obtain the set $\tilde{X}'$ approximating X'. Clearly, this does not mean that some components of $\tilde{X}'$ automatically approximate X. Yet, by Theorem 3.1 there is a system of polynomial equations satisfied by the coefficients of p_i, $i = 1, \ldots, k$, with the following property. Let $\tilde{U}$ be any open relatively compact subset of U. If the Nash approximations of the coefficients (used to define $\tilde{X}'$) are close enough to the original coefficients and also satisfy the equations then $X \cap (\tilde{U} \times \mathbf{C}^k)$ is approximated by some components of $\tilde{X}' \cap (\tilde{U} \times \mathbf{C}^k)$. The existence of the Nash approximations of the coefficients satisfying the equations mentioned above in a neighborhood of $\overline{\tilde{U}}$ follows by the global version of Theorem 1.1.

Finally let us recall that the convergence of a sequence of analytic sets in the sense of chains, appearing in Theorem 3.1, is equivalent to the (introduced in [15]) convergence of currents of integration over these sets in the weak-$\star$ topology. The basic facts on holomorphic chains (and preliminaries on Nash sets and analytic sets with proper projection) are gathered in Section 2 below.

2. Preliminaries

2.1. Nash sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω. A holomorphic mapping defined on Ω with values in $\mathbf{C}^N$ is said to be a Nash mapping if each of its components is a Nash function.

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a Nash subset of Ω if and only if for every $y_0 \in \Omega$ there exists a neighborhood U of y_0 in Ω and there exist Nash functions $f_1, \dots, f_s$ on U such that

$$Y \cap U = \bigl\{x \in U:\ f_1(x) = \cdots = f_s(x) = 0\bigr\}.$$

We will use the following fact from [21], p. 239. Let $\pi : \Omega \times \mathbf{C}^k \to \Omega$ denote the natural projection.

Theorem 2.1. *Let X be a Nash subset of $\Omega \times \mathbf{C}^k$ such that $\pi|_X : X \to \Omega$ is a proper mapping. Then $\pi(X)$ is a Nash subset of Ω and* $\dim(X) = \dim(\pi(X))$.

The fact from [21] stated below explains the relation between Nash and algebraic sets.

Theorem 2.2. *Let X be a Nash subset of an open set $\Omega \subset \mathbf{C}^n$. Then every analytic irreducible component of X is an irreducible Nash subset of Ω. Moreover, if X is irreducible then there exists an algebraic subset Y of $\mathbf{C}^n$ such that X is an analytic irreducible component of $Y \cap \Omega$.*

2.2. Analytic sets

Let U, U' be domains in $\mathbf{C}^n, \mathbf{C}^k$ respectively and let $\pi : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}^n$ denote the natural projection. For any purely n-dimensional analytic subset Y of $U \times U'$ with proper projection onto U by $\mathcal{S}(Y, \pi)$ we denote the set of singular points of $\pi|_Y$:

$$\mathcal{S}(Y, \pi) = \operatorname{Sing}(Y) \cup \bigl\{x \in \operatorname{Reg}(Y):\ (\pi|_Y)'(x) \text{ is not an isomorphism}\bigr\}.$$

We often write $\mathcal{S}(Y)$ instead of $\mathcal{S}(Y, \pi)$ when it is clear which projection is taken into consideration.

It is well known that $\mathcal{S}(Y)$ is an analytic subset of $U \times U'$, $\dim(Y) < n$ (cf. [11], p. 50), hence by the Remmert theorem $\pi(\mathcal{S}(Y))$ is also analytic. Moreover, the following hold. The mapping $\pi|_Y$ is surjective and open and there exists an integer $s = s(\pi|_Y)$ such that

(1) $\sharp(\pi|_Y)^{-1}(\{a\}) < s$ for $a \in \pi(\mathcal{S}(Y))$,
(2) $\sharp(\pi|_Y)^{-1}(\{a\}) = s$ for $a \in U \setminus \pi(\mathcal{S}(Y))$,
(3) for every $a \in U \setminus \pi(\mathcal{S}(Y))$ there exists a neighborhood W of a and holomorphic mappings $f_1, \dots, f_s : W \to U'$ such that $f_i \cap f_j = \emptyset$ for $i \neq j$ and $f_1 \cup \cdots \cup f_s = (W \times U') \cap Y$.

Let E be a purely n-dimensional analytic subset of $U \times U'$ with proper projection onto a domain $U \subset \mathbf{C}^n$, where U' is a domain in $\mathbf{C}$. The unitary polynomial $p \in \mathcal{O}(U)[z]$ such that $E = \{(x, z) \in U \times \mathbf{C}:\ p(x, z) = 0\}$ and the discriminant of p is not identically zero will be called the optimal polynomial for E.

Finally, for any analytic subset X of an open set $\tilde{U} \subset \mathbf{C}^m$ let $X_{(k)} \subset \tilde{U}$ denote the union of all irreducible components of X of dimension k.

2.3. Convergence of closed sets and holomorphic chains

Let U be an open subset in $\mathbf{C}^m$. By a holomorphic chain in U we mean the formal sum $A = \sum_{j \in J} \alpha_j C_j$, where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{C_j\}_{j \in J}$ is a locally finite family of pairwise distinct irreducible analytic subsets of U (see [22], cf. also [5,11]). The set $\bigcup_{j \in J} C_j$ is called the support of A and is denoted by $|A|$ whereas the sets C_j

are called the components of A with multiplicities α_j. The chain A is called positive if $\alpha_j > 0$ for all $j \in J$. If all the components of A have the same dimension n then A will be called an n-chain.

Below we introduce the convergence of holomorphic chains in U. To do this we first need the notion of the local uniform convergence of closed sets (cf. [23]). Let Y, Y_ν be closed subsets of U for $\nu \in \mathbf{N}$. We say that $\{Y_\nu\}$ converges to Y locally uniformly if:

(1l) for every $a \in Y$ there exists a sequence $\{a_\nu\}$ such that $a_\nu \in Y_\nu$ and $a_\nu \to a$ in the standard topology of $\mathbf{C}^m$,
(2l) for every compact subset K of U such that $K \cap Y = \emptyset$ it holds $K \cap Y_\nu = \emptyset$ for almost all ν.

Then we write $Y_\nu \to Y$.

We say that a sequence $\{Z_\nu\}$ of positive n-chains converges to a positive n-chain Z if:

(1c) $|Z_\nu| \to |Z|$,
(2c) for each regular point a of $|Z|$ and each submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact and $|Z| \cap \overline{T} = \{a\}$, we have $\deg(Z_\nu \cdot T) = \deg(Z \cdot T)$ for almost all ν.

Then we write $Z_\nu \rightarrowtail Z$. By $Z \cdot T$ we denote the intersection product of Z and T (cf. [22]). Observe that the chains $Z_\nu \cdot T$ and $Z \cdot T$ for sufficiently large ν have finite supports and the degrees are well defined. Recall that for a chain $A = \sum_{j=1}^d \alpha_j \{a_j\}$, $\deg(A) = \sum_{j=1}^d \alpha_j$.

The following lemma from [22] will be useful to us.

Lemma 2.3. *Let $n \in \mathbf{N}$ and Z, Z_ν, for $\nu \in \mathbf{N}$, be positive n-chains. If $|Z_\nu| \to |Z|$ then the following conditions are equivalent*:

(1) $Z_\nu \rightarrowtail Z$,
(2) *for each point a from a given dense subset of* $\mathrm{Reg}(|Z|)$ *there is a submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact, $|Z| \cap \overline{T} = \{a\}$ and* $\deg(Z_\nu \cdot T) = \deg(Z \cdot T)$ *for almost all ν.*

Let $U \subset \mathbf{C}^n$ be a domain and let $\pi : U \times \mathbf{C}^k \to U$ be the natural projection. Theorem 2.4 below, taken from [6], will be applied in the proof of the main result ($s(\pi|_Y)$ is defined in Section 2.2).

Theorem 2.4. *Let Y, Y_ν, for $\nu \in \mathbf{N}$, be purely n-dimensional analytic subsets of $U \times \mathbf{C}^k$ with proper projection onto U such that $\{Y_\nu\}$ converges to Y locally uniformly and $s(\pi|_{Y_\nu}) = s(\pi|_Y)$ for every ν. Moreover, assume that for every ν the number of the irreducible components of Y does not exceed the number of the irreducible components of Y_ν. Then for each irreducible component A of Y there is a sequence $\{A_\nu\}$ converging to A locally uniformly such that every A_ν is an irreducible component of Y_ν and $s(\pi|_{A_\nu}) = s(\pi|_A)$ for almost all ν.*

3. Approximation

3.1. Approximation of components of analytic sets

Our first main result is the following theorem. Let $U \subset \mathbf{C}^n$ be a domain and let $\pi : U \times \mathbf{C}^k \to U$ denote the natural projection. Let $X \subset U \times \mathbf{C}^k$ be an analytic subset of pure dimension n with proper projection onto U. Recall that $s(\pi|_X)$ denotes the cardinality of the generic fiber in X over U.

Theorem 3.1. *Let $\{X_\nu\}$ be a sequence of purely n-dimensional analytic subsets of $U \times \mathbf{C}^k$ with proper projection onto U converging locally uniformly to X such that $s(\pi|_X) = s(\pi|_{X_\nu})$ for $\nu \in \mathbf{N}$. Assume that $\{(\pi(\mathcal{S}(X_\nu)))_{(n-1)}\}$ converges to $(\pi(\mathcal{S}(X)))_{(n-1)}$ in the sense of holomorphic chains. Then for every analytic subset Y of $U \times \mathbf{C}^k$ of pure dimension n such that $Y \subset X$ and for every open relatively compact subset $\tilde{U}$ of U there exists a sequence $\{Y_\nu\}$ of purely n-dimensional analytic subsets of $\tilde{U} \times \mathbf{C}^k$ converging to $Y \cap (\tilde{U} \times \mathbf{C}^k)$ in the sense of holomorphic chains such that $Y_\nu \subset X_\nu$ for every $\nu \in \mathbf{N}$.*

Remark 3.2. From the proof of Theorem 3.1 it follows that if $Y \cap (\tilde{U} \times \mathbf{C}^k)$ is irreducible than Y_ν is irreducible as well for almost all ν.

Proof of Theorem 3.1. First the theorem will be proved under the following extra hypotheses:

(1) $U = U_1 \times U_2 \subset \mathbf{C}^{n-1} \times \mathbf{C}$ where U_1, U_2 are open balls,
(2) $\pi(\mathcal{S}(X))$ is with proper projection onto U_1,
(3) there is a compact ball B in U_2 such that $(U_1 \times B) \cap \pi(\mathcal{S}(X)) = \emptyset$.

This will be done in two steps. Step 1 is of preparatory nature. Here we specify several technical conditions which may be assumed satisfied by X and X_ν (for large ν) without loss of generality. These conditions are used in Step 2 the idea of which is to show that for almost all ν the number of the irreducible components of $X \cap (\tilde{U} \times \mathbf{C}^k)$ does not exceed the number of the irreducible components of $X_\nu \cap (\tilde{U} \times \mathbf{C}^k)$ (in fact the numbers are equal). This is done by constructing an injective mapping which assigns to every irreducible component of $X \cap (\tilde{U} \times \mathbf{C}^k)$ an irreducible component of $X_\nu \cap (\tilde{U} \times \mathbf{C}^k)$. Then Theorem 2.4 from Section 2.3 may be applied which completes the proof (under the hypotheses (1)–(3) above).

Finally we show (Step 3) that (1)–(3) are not necessary.

Step 1. Without loss of generality, we assume that X and X_ν (for large ν) satisfy the conditions specified below.

Denote,

$$\hat{k} := \max\bigl\{\sharp\bigl(\bigl(\{x'\} \times U_2\bigr) \cap \bigl(\pi\bigl(\mathcal{S}(X)\bigr)\bigr)_{(n-1)}\bigr)\colon\ x' \in U_1\bigr\}.$$

Let $\Sigma'(X)$ be the subset of U_1 of points x' for which

$$\sharp\bigl(\bigl(\{x'\} \times U_2\bigr) \cap \bigl(\pi\bigl(\mathcal{S}(X)\bigr)\bigr)_{(n-1)}\bigr) < \hat{k}.$$

Put $\Sigma(X) = \Sigma'(X) \cup \rho(\overline{\pi(\mathcal{S}(X)) \setminus \pi(\mathcal{S}(X))_{(n-1)}})$, where $\rho : U_1 \times U_2 \to U_1$ is the natural projection. (The closure is taken in $U_1 \times U_2$. Generally, in this paper, the topological structure on any analytic set is induced by the standard topology of $\mathbf{C}^m$ in which the set is contained.)

Observe that $\Sigma(X)$ is a nowhere dense analytic subset of U_1 hence there are $x'_0 \in U_1 \setminus \Sigma(X)$ and compact balls $B_1, \ldots, B_{\hat{k}} \subset U_2$ such that $B \cap (\bigcup_{i=1}^{\hat{k}} B_i) = \emptyset$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Moreover, each B_i contains precisely one y such that $(x'_0, y) \in \pi(\mathcal{S}(X))_{(n-1)}$. Furthermore, since U may be replaced by its relatively compact subset containing the fixed $\tilde{U}$, we may assume that there is $r > 0$ such that for every

$$x \in (U_1 \times B) \cup \bigl(\{x'_0\} \times \bigl(U_2 \setminus (B_1 \cup \cdots \cup B_{\hat{k}})\bigr)\bigr)$$

if $(x, u), (x, v) \in X$ and $u \neq v$, then $\|u - v\|_{\mathbf{C}^k} > r$.

Next, by the fact that $\{\pi(\mathcal{S}(X_\nu))_{(n-1)}\}$ converges to $\pi(\mathcal{S}(X))_{(n-1)}$ in the sense of chains and again by the fact that one may pass on to a relatively compact subset of U, the following is assumed for large ν: $\pi(\mathcal{S}(X_\nu))$ is with proper projection onto U_1 and the cardinality of the generic fiber of $\pi(\mathcal{S}(X_\nu))_{(n-1)}$ over U_1 equals $\hat{k}$. Moreover, in every B_i there is precisely one y such that $(x'_0, y) \in \pi(\mathcal{S}(X_\nu))_{(n-1)}$.

Fix $x_0 \in U_1 \times B$. Let $A \subset U_1 \times B \times \mathbf{C}^k$ denote the fiber in X over x_0. For every irreducible component Y of X define $A_Y := Y \cap A$. For every such Y, there is an arc

$$\gamma_Y : [0, 1] \to \bigl(\bigl(U \setminus \bigl(\pi\bigl(\mathcal{S}(X)\bigr)\bigr)\bigr) \times \mathbf{C}^k\bigr) \cap Y$$

connecting all the points in A_Y. Let

$$r_0 = \inf\left\{\|u - v\|\colon\ (x, v), (x, u) \in X, u \neq v, x \in \bigcup_Y \pi\bigl(\gamma_Y\bigl([0, 1]\bigr)\bigr)\right\}.$$

Then $r_0 > 0$.

Pick any $0 < \delta < \min(\frac{r}{3}, \frac{r_0}{3})$. We complete Step 1 by observing that for large ν the following may be assumed:

$$\left(\bigcup_Y \pi\bigl(\gamma_Y\bigl([0, 1]\bigr)\bigr)\right) \cap \pi\bigl(\mathcal{S}(X_\nu)\bigr) = \emptyset$$

and

$$\operatorname{dist}\big(\big(\{x\} \times \mathbf{C}^k\big) \cap X, \big(\{x\} \times \mathbf{C}^k\big) \cap X_\nu\big) < \delta$$

for every $x \in U$ (the latter due to the fact that U may be replaced by its relatively compact subset). (Here dist denotes the Hausdorff distance.)

Step 2. We show that if X and X_ν satisfy the assumptions made in Step 1 (which holds for large ν) then the number of the irreducible components of X does not exceed the number of the irreducible components of X_ν. Therefore by Theorem 2.4 for every irreducible component Y of X, there is a sequence of purely n-dimensional analytic sets $\{Y_\nu\}$ converging to Y locally uniformly such that $Y_\nu \subset X_\nu$ and $s(\pi|_Y) = s(\pi|_{Y_\nu})$ for almost all ν. Consequently, by Lemma 2.3, $\{Y_\nu\}$ converges to Y in the sense of holomorphic chains as required.

To do this we need the following claim. Let $F \subset (U_1 \times B \times \mathbf{C}^k) \cap X$ be the graph of a holomorphic mapping defined on $U_1 \times B$. (Note that, by (1) and (3), $(U_1 \times B \times \mathbf{C}^k) \cap X$ is the union of such graphs.) Put $\Sigma = \Sigma(X)$.

Claim 3.3. *Let $\tilde{X} = \bigcup_{a \in (U_1 \setminus \Sigma)} X_a$, where for every $a \in (U_1 \setminus \Sigma)$ by X_a we denote the irreducible component of $(\{a\} \times U_2 \times \mathbf{C}^k) \cap X$ containing $(\{a\} \times B \times \mathbf{C}^k) \cap F$. Then $\tilde{X}$ is an analytic subset of $((U_1 \setminus \Sigma) \times U_2 \times \mathbf{C}^k)$.*

Proof. It is sufficient to check that for every $a \in U_1 \setminus \Sigma$ there is a ball $B' \subset U_1 \setminus \Sigma$ centered at a such that $(B' \times U_2 \times \mathbf{C}^k) \cap \tilde{X}$ is an analytic subset of $B' \times U_2 \times \mathbf{C}^k$.

Fix $a_0 \in U_1 \setminus \Sigma$ and take a ball $B' \subset U_1 \setminus \Sigma$ centered at a_0. We check that $(B' \times U_2 \times \mathbf{C}^k) \cap \tilde{X}$ equals the irreducible component (denoted by X') of $(B' \times U_2 \times \mathbf{C}^k) \cap X$ containing $(B' \times B \times \mathbf{C}^k) \cap F$. First note that $(B' \times U_2 \times \mathbf{C}^k) \cap \tilde{X} \subseteq X'$ (an immediate consequence of the fact that for every $a \in B'$ the analytic set $(\{a\} \times U_2 \times \mathbf{C}^k) \cap X'$ contains $(\{a\} \times B \times \mathbf{C}^k) \cap F$ so it must contain X_a as well).

For the converse inclusion, suppose for a moment that $X' \nsubseteq (B' \times U_2 \times \mathbf{C}^k) \cap \tilde{X}$. Then there is $(a, b) \in (B' \times B)$ such that the number of points in $(\{(a, b)\} \times \mathbf{C}^k) \cap X'$ is strictly greater than the number of points in $(\{(a, b)\} \times \mathbf{C}^k) \cap \tilde{X}$. Since X' is irreducible, there is an arc

$$\gamma : [0, 1] \to \big(\big((B' \times U_2) \setminus \pi\big(\mathcal{S}(X)\big)\big) \times \mathbf{C}^k\big) \cap X'$$

connecting all the points in $(\{(a, b)\} \times \mathbf{C}^k) \cap X'$.

It is easy to see (at least when B' is small which we may assume) that there is a homeomorphic deformation $H : (B' \times U_2) \to (B' \times U_2)$, such that $H(\{a'\} \times U_2) \subset \{a'\} \times U_2$ for every $a' \in B'$, after which the set $\pi(\mathcal{S}(X)) \cap (B' \times U_2)$ becomes the union of graphs of constant functions defined on B'. Then the arc $\tilde{H} \circ \gamma$, where $\tilde{H} = (H, id_{\mathbf{C}^k}) : B' \times U_2 \times \mathbf{C}^k \to B' \times U_2 \times \mathbf{C}^k$, can be deformed by shifting along $\tilde{E} = H(\pi(\mathcal{S}(X)) \cap (B' \times U_2))$ to the arc

$$\tau : [0, 1] \to \big(\big(\big(\{a\} \times U_2\big) \setminus \tilde{E}\big) \times \mathbf{C}^k\big) \cap \tilde{H}(X').$$

Consequently, $\tilde{H}^{-1} \circ \tau$ is an arc connecting all the points of $(\{(a, b)\} \times \mathbf{C}^k) \cap X'$ whose image is contained in $(((\{a\} \times U_2) \setminus \pi(\mathcal{S}(X))) \times \mathbf{C}^k) \cap X'$. This means that $(\{a\} \times U_2 \times \mathbf{C}^k) \cap X'$ is irreducible, hence

$$\big(\{a\} \times U_2 \times \mathbf{C}^k\big) \cap \tilde{X} = \big(\{a\} \times U_2 \times \mathbf{C}^k\big) \cap X',$$

because $(\{a\} \times B \times \mathbf{C}^k) \cap F$ is contained in both sets of the latter equation. On the other hand, these sets have different number of points in generic fibers over $\{a\} \times U_2$, a contradiction.

Thus we have checked that $(B' \times U_2 \times \mathbf{C}^k) \cap \tilde{X} = X'$ which implies the analyticity and the proof is complete. □

Let us return to the proof of Theorem 3.1. For every irreducible component Y of X select one graph F_Y of a holomorphic mapping defined on $U_1 \times B$ such that $F_Y \subset Y$. Then, by Step 1, there is the uniquely determined graph $F_{Y,\nu}$ of the holomorphic mapping defined on $U_1 \times B$ such that $F_{Y,\nu} \subset X_\nu$ and such that

$$\operatorname{dist}\big(\big(\{x\} \times \mathbf{C}^k\big) \cap F_Y, \big(\{x\} \times \mathbf{C}^k\big) \cap F_{Y,\nu}\big) < \delta$$

for every $x \in U_1 \times B$ (for δ picked in Step 1).

Now put $\Sigma_\nu = \Sigma(X_\nu)$ and define $\tilde{Y}_\nu = \bigcup_{a \in U_1 \setminus \Sigma_\nu} Y_{a,\nu}$ where $Y_{a,\nu}$ is the irreducible component of $(\{a\} \times U_2 \times \mathbf{C}^k) \cap X_\nu$ containing $(\{a\} \times B \times \mathbf{C}^k) \cap F_{Y,\nu}$. By Claim 3.3, applied to X_ν, $\tilde{Y}_\nu$ is an analytic subset of $(U_1 \setminus \Sigma_\nu) \times U_2 \times \mathbf{C}^k$

(clearly, of pure dimension n with proper projection onto $(U_1 \setminus \Sigma_\nu) \times U_2$). Let Y_ν be the closure of $\tilde{Y}_\nu$ in $U \times \mathbf{C}^k$. Note that Y_ν is analytic as $\Sigma_\nu \times U_2$ is a nowhere dense analytic subset of U and $\tilde{Y}_\nu \subset X_\nu$ and X_ν is with proper projection onto U (so its fibers over U are locally bounded at every $x \in \Sigma_\nu \times U_2$). It is easy to see that Y_ν is an irreducible component of X_ν (otherwise $(\{a\} \times U_2 \times \mathbf{C}^k) \cap Y_\nu$ would be reducible for some $a \in U_1 \setminus \Sigma_\nu$, a contradiction with the definition of $\tilde{Y}_\nu$).

We show that the mapping which assigns to every irreducible component Y of X the set Y_ν described above is injective, which completes the proof. To do this it is sufficient to check the following two facts for every fixed irreducible component Y of X:

(a) $s(\pi|_Y) = s(\pi|_{Y_\nu})$,
(b) for $x_0 \in U_1 \times B$ fixed in Step 1 and for every $(x_0, v) \in Y$ there is $(x_0, v_\nu) \in Y_\nu$ such that $\|v - v_\nu\| < \delta$.

Clearly, for every irreducible component Y_ν of X_ν there is at most one Y satisfying (a) and (b).

Let us handle (a). Take x_0' picked in Step 1. We may assume that $x_0' \notin \Sigma_\nu$ (otherwise it can be replaced by a point from an arbitrarily small neighborhood of x_0' in $U_1 \setminus (\Sigma \cup \Sigma_\nu)$ satisfying all the conditions specified for x_0' in Step 1).

Let $C_Y, C_{Y,\nu}$ denote the irreducible components of $(\{x_0'\} \times U_2 \times \mathbf{C}^k) \cap Y$, $(\{x_0'\} \times U_2 \times \mathbf{C}^k) \cap X_\nu$ respectively containing $(\{x_0'\} \times B \times \mathbf{C}^k) \cap F_Y$, $(\{x_0'\} \times B \times \mathbf{C}^k) \cap F_{Y,\nu}$, respectively.

Claim 3.4. *The cardinalities of the generic fibers in C_Y and $C_{Y,\nu}$ over $\{x_0'\} \times U_2$ are equal.*

Proof. Observe that the mapping,

$$G : \big(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k\big) \cap Y \to \big(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k\big) \cap X_\nu,$$

where $\tilde{B} = B_1 \cup \cdots \cup B_{\hat{k}}$, given by $G(x_0', y, u) = (x_0', y, v)$, where v is the unique vector in $\mathbf{C}^k$ such that $\|u - v\| < \delta$, is a biholomorphism onto its image. Moreover, since, by the choice of x_0', the interior of every B_i contains precisely one element from $\pi(\mathcal{S}(X))$ and one from $\pi(\mathcal{S}(X_\nu))$ and the intersection of every pair of distinct B_i's is empty, we may apply the following claim.

Claim 3.5. *Let $D \subset \mathbf{C}$ be a domain, $\rho : D \times \mathbf{C}^k \to D$ be the natural projection and let $E \subset D \times \mathbf{C}^k$ be an irreducible analytic curve such that $\rho|_E$ is proper. Finally, let $K_1, \ldots, K_s \subset D$ be compact balls, $K_i \cap K_j = \emptyset$ for $i \neq j$ and let $\sharp K_i \cap \rho(\mathcal{S}(E)) = 1$ for every $i = 1, \ldots, s$. Then $E \cap ((D \setminus \bigcup_{i=1}^s K_i) \times \mathbf{C}^k)$ is irreducible.*

Proof. Put:

$$\tilde{E} = E \cap \left(\left(D \setminus \bigcup_{i=1}^{s} K_i\right) \times \mathbf{C}^k\right).$$

It is sufficient to show that for every $a, b \in \tilde{E} \setminus (\rho(\mathcal{S}(E)) \times \mathbf{C}^k)$ there is an arc $\tau : [0,1] \to \tilde{E} \setminus (\rho(\mathcal{S}(E)) \times \mathbf{C}^k)$ such that $\tau(0) = a$, $\tau(1) = b$.

Fix $a, b \in \tilde{E} \setminus (\rho(\mathcal{S}(E)) \times \mathbf{C}^k)$. Let $\hat{K}_1, \ldots, \hat{K}_s$ be compact balls in D, $K_i \subsetneq \hat{K}_i$ for $i = 1, \ldots, s$, satisfying the hypotheses of the claim with $a, b \notin \bigcup_{i=1}^s \hat{K}_i \times \mathbf{C}^k$. Irreducibility of E implies that there is an arc $\gamma : [0,1] \to E \setminus (\rho(\mathcal{S}(E)) \times \mathbf{C}^k)$ such that $\gamma(0) = a$, $\gamma(1) = b$. Then τ can be obtained from γ as follows. For every $t \in [0,1]$ such that $\gamma(t) \notin (\bigcup_{i=1}^s \hat{K}_i \times \mathbf{C}^k)$ put $\tau(t) = \gamma(t)$.

Take $t \in [0,1]$ such that $\gamma(t) \in \hat{K}_i \times \mathbf{C}^k$ for some i. By the hypothesis $\rho(\mathcal{S}(E)) \cap \hat{K}_i = \{g\}$ for some $g \in D$. Let σ be the segment passing through $\rho(\gamma(t))$, connecting g with the uniquely determined $h(t) \in \partial \hat{K}_i$. Then $(\sigma \times \mathbf{C}^k) \cap E$ is the union of graphs of continuous functions defined on σ. Let $f \subset (\sigma \times \mathbf{C}^k)$ be the one of these graphs for which $\gamma(t) \in f$. Set $\tau(t) = (h(t), f(h(t)))$.

It is easy to check that τ defined as above is an arc connecting a and b in $\tilde{E} \setminus (\rho(\mathcal{S}(E)) \times \mathbf{C}^k)$. $\square$

Proof of Claim 3.4 (end). We show that

$$G\big(\big(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k\big) \cap C_Y\big) = \big(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k\big) \cap C_{Y,\nu},$$

which clearly implies the assertion of the claim. To do this observe that, by Claim 3.5, $(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k) \cap C_Y$ is irreducible. Then $G((\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k) \cap C_Y)$ is irreducible as well because G is a biholomorphism onto its image. This implies that

$$G\big(\big(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k\big) \cap C_Y\big) = \big(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k\big) \cap C_{Y,\nu}$$

because $(\{x_0'\} \times (U_2 \setminus \tilde{B}) \times \mathbf{C}^k) \cap C_{Y,\nu}$ is irreducible by Claim 3.5 and both sets contain $(\{x_0'\} \times B \times \mathbf{C}^k) \cap F_{Y,\nu}$ (recall that, by Step 1, $B \cap \bigcup_{i=1}^{\hat{k}} B_i = \emptyset$). □

Proof of Theorem 3.1 (end). Now define $\tilde{Y} = \bigcup_{a \in U_1 \setminus \Sigma} Y_a$ where Y_a is the irreducible component of $(\{a\} \times U_2 \times \mathbf{C}^k) \cap Y$ containing $(\{a\} \times B \times \mathbf{C}^k) \cap F_Y$. By Claim 3.3, $\tilde{Y}$ is an analytic subset of $(U_1 \setminus \Sigma) \times U_2 \times \mathbf{C}^k$ (of pure dimension n with proper projection onto $(U_1 \setminus \Sigma) \times U_2$). Since $\tilde{Y}$ is contained in $Y \cap ((U_1 \setminus \Sigma) \times U_2 \times \mathbf{C}^k)$ which is irreducible and n-dimensional, it holds

$$\tilde{Y} = Y \cap \big((U_1 \setminus \Sigma) \times U_2 \times \mathbf{C}^k\big).$$

The latter fact implies that

$$Y \cap \big(\{x_0'\} \times U_2 \times \mathbf{C}^k\big) = C_Y$$

and, consequently, that the cardinalities of the generic fibers in Y over U and in C_Y over $\{x_0'\} \times U_2$ are equal (because $\{x_0'\} \times U_2 \not\subseteq \pi(\mathcal{S}(X))$).

Finally, observe that (in view of the fact that $\{x_0'\} \times U_2 \not\subseteq \pi(\mathcal{S}(X_\nu))$) the cardinalities of the generic fibers in Y_ν over U and in $C_{Y,\nu}$ over $\{x_0'\} \times U_2$ are equal because

$$Y_\nu \cap \big(\{x_0'\} \times U_2 \times \mathbf{C}^k\big) = C_{Y,\nu}$$

by the definition of Y_ν. Thus, in view of Claim 3.4, $s(\pi|_Y) = s(\pi|_{Y_\nu})$ as required in (a).

Let us turn to (b). Consider the arc γ_Y defined in Step 1. Note that for every $\gamma_Y(t) \in X$, where $t \in [0,1]$, there is precisely one element $(e(t), f(t)) \in X_\nu \subset U \times \mathbf{C}^k$ such that $\pi(\gamma_Y(t)) = e(t)$ and the distance between $(e(t), f(t))$ and $\gamma_Y(t)$ is smaller than δ. Since $(\pi(\gamma_Y([0,1]))) \cap \pi(\mathcal{S}(X_\nu)) = \emptyset$, then

$$\tau : [0,1] \ni t \mapsto \big(e(t), f(t)\big) \in X_\nu$$

is an arc whose image is contained in one irreducible component of X_ν. On the other hand, there is t_0 such that $\gamma_Y(t_0) \in F_Y$ so $\tau(t_0) \in F_{Y,\nu}$, which implies that the irreducible component containing $\tau([0,1])$ contains $F_{Y,\nu}$ as well. Thus this irreducible component must be Y_ν. To complete the proof observe that for every $(x_0, v) \in Y$ there is $t' \in [0,1]$ such that $(x_0, v) = \gamma_Y(t')$, $\tau(t') = (x_0, f(t')) \in Y_\nu$, $\|v - f(t')\| < \delta$.

Thus we have proved the theorem under the extra hypotheses (1)–(3) formulated at the beginning.

Step 3. Let us show that (1)–(3) need not be assumed. Let $\{X_\nu\}$ be a sequence of analytic sets satisfying the hypotheses of Theorem 3.1 and let Y be an analytic subset of $U \times \mathbf{C}^k$ of pure dimension n with $Y \subset X$. Fix an open set $\tilde{U} \Subset U$.

Cover $\overline{\tilde{U}}$ by a finite number of domains $E_1, \ldots, E_s \subset U$, $E_i \Subset \tilde{E}_i \subset U$ such that for every $i \in \{1, \ldots, s\}$ there is a polynomial automorphism $L_i : \mathbf{C}^n \to \mathbf{C}^n$ with the following property. The conditions (1)–(3) are satisfied with U, X replaced by $L_i(\tilde{E}_i)$, $\beta_{L_i}(X \cap (\tilde{E}_i \times \mathbf{C}^k))$ respectively, where $\beta_{L_i} : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}^n \times \mathbf{C}^k$ is given by the formula $\beta_{L_i}(x,z) = (L_i(x), z)$.

By Step 2 for every $i \in \{1, \ldots, s\}$ there is a sequence $\{Y_{i,\nu}\}$ of purely n-dimensional analytic subsets of $E_i \times \mathbf{C}^k$, $Y_{i,\nu} \subset (E_i \times \mathbf{C}^k) \cap X_\nu$, converging to $(E_i \times \mathbf{C}^k) \cap Y$ in the sense of chains. Let us check that $Y_\nu = (\bigcup_{i=1}^s Y_{i,\nu}) \cap (\tilde{U} \times \mathbf{C}^k)$ is, for large ν, an analytic subset of $\tilde{U} \times \mathbf{C}^k$. One easily observes that this is the case as the convergence in the sense of chains of $\{X_\nu\}$ to X imply that for almost all ν and for every i, j it holds:

$$Y_{i,\nu} \cap \big((E_i \cap E_j) \times \mathbf{C}^k\big) = Y_{j,\nu} \cap \big((E_i \cap E_j) \times \mathbf{C}^k\big).$$

The latter equation also implies that $\{Y_\nu\}$ converges to $Y \cap (\tilde{U} \times \mathbf{C}^k)$ in the sense of chains. Thus the proof of Theorem 3.1 is complete. □

3.2. Approximation of mappings

First let us note that in Theorem 1.1 (and in its generalizations) the space $\mathbf{C}^n$ containing U may be replaced by an affine algebraic variety. In fact, in the global version of the approximation theorem (see [16], Theorem 3.2) the domain of the approximated mapping is admitted to have singularities. Since this case is reduced to the one where the mapping is defined on an open subset of some $\mathbf{C}^n$ and the reduction is of purely analytic geometric nature, we assume here that U is an open subset of $\mathbf{C}^n$.

Our aim is to give a direct geometric proof of Theorem 1.1, or more precisely, its semi-global version (Theorem 3.8). The proof is organized as follows. First using Theorem 3.1 we prove in Section 3.2.1 the following:

Proposition 3.6. *Let U be a domain in $\mathbf{C}^n$ and let $f : U \to \mathbf{C}^k$ be a holomorphic mapping that satisfies a system of equations $Q(x, f(x)) = 0$ for $x \in U$. Here Q is a Nash mapping from a neighborhood in $\mathbf{C}^n \times \mathbf{C}^k$ of the graph of f into some $\mathbf{C}^q$. Then there is $R \in \mathbf{C}[x, z_1, \dots, z_k]$ with $R(x, f(x))$ not identically zero such that the following holds. If for some open $\tilde{U}, U_0 \subset U$ with $U_0 \Subset \tilde{U}$ there is a sequence $\{g^\nu : \tilde{U} \to \mathbf{C}^k\}$ of Nash mappings converging locally uniformly to $f|_{\tilde{U}}$ such that $\{\{x \in \tilde{U}\colon\ R(x, g^\nu(x)) = 0\}\}$ converges to $\{x \in \tilde{U}\colon\ R(x, f(x)) = 0\}$ in the sense of chains then there is a sequence $\{f^\nu : U_0 \to \mathbf{C}^k\}$ of Nash mappings converging uniformly to $f|_{U_0}$ such that $Q(x, f^\nu(x)) = 0$ for $x \in U_0$, $\nu \in \mathbf{N}$.*

Next in Section 3.2.2 for any holomorphic mapping $f : U \to \mathbf{C}^k$, $f = f(x)$, and any $R \in \mathbf{C}[x, z_1, \dots, z_k]$ such that $R(x, f(x))$ is not identically zero, we construct a family $\mathcal{U}_{f,R}$ of open subsets of U such that the following holds.

Proposition 3.7. *Let $f : U \to \mathbf{C}^k$ be a holomorphic mapping, where U is a domain in $\mathbf{C}^n$, let $R \in \mathbf{C}[x, z_1, \dots, z_k]$ be such that $R(x, f(x))$ is not identically zero on U and let $U_0 \in \mathcal{U}_{f,R}$. Then there are an open $\tilde{U} \subset \mathbf{C}^n$ with $U_0 \Subset \tilde{U}$ and a sequence $\{f^\nu : \tilde{U} \to \mathbf{C}^k\}$ of Nash mappings converging uniformly to $f|_{\tilde{U}}$ such that $\{\{x \in \tilde{U}\colon\ R(x, f^\nu(x)) = 0\}\}$ converges to $\{x \in \tilde{U}\colon\ R(x, f(x)) = 0\}$ in the sense of chains.*

Proposition 3.7 is proved in Section 3.2.3. One of the main results of this paper is the following semi-global version of the approximation theorem.

Theorem 3.8. *Let U be a domain in $\mathbf{C}^n$ and let $f : U \to \mathbf{C}^k$ be a holomorphic mapping that satisfies a system of equations $Q(x, f(x)) = 0$ for $x \in U$. Here Q is a Nash mapping from a neighborhood in $\mathbf{C}^n \times \mathbf{C}^k$ of the graph of f into some $\mathbf{C}^q$. Let R be any polynomial obtained by applying Proposition* 3.6 *with f, Q. Then for every $U_0 \in \mathcal{U}_{f,R}$ there is a sequence $\{f^\nu : U_0 \to \mathbf{C}^k\}$ of Nash mappings converging uniformly to $f|_{U_0}$ such that $Q(x, f^\nu(x)) = 0$ for $x \in U_0$ and $\nu \in \mathbf{N}$.*

Proof. In view of the fact that R satisfies the assertion of Proposition 3.6, it is sufficient to apply Proposition 3.7. □

In order to characterize those U_0 for which the presented methods work we need an insight into $\mathcal{U}_{f,R}$. Here let us just mention two special cases which directly follow by Section 3.2.2

Corollary 3.9.

(a) *Let U be a domain in $\mathbf{C}$. Then for every holomorphic mapping $f : U \to \mathbf{C}^k$ and every $R \in \mathbf{C}[x, z_1, \dots, z_k]$, $R(x, f(x))$ not identically zero, the family $\mathcal{U}_{f,R}$ contains every open set U_0 for which there is a Runge domain $\tilde{U}$ with $U_0 \Subset \tilde{U} \subset U$. Consequently, if f depends on one variable we have the global version of Theorem* 1.1*, first proved by van den Dries in* [14].

(b) *Let U be a domain in $\mathbf{C}^n$. Then for every holomorphic mapping $f : U \to \mathbf{C}^k$, every $R \in \mathbf{C}[x, z_1, \dots, z_k]$, $R(x, f(x))$ not identically zero, and $x_0 \in U$ there is an open neighborhood $\tilde{U}$ of x_0 such that $\tilde{U} \in \mathcal{U}_{f,R}$, which implies Theorem* 1.1.

3.2.1. Proof of Proposition 3.6

Proposition 3.6 is a consequence of the following:

Proposition 3.10. *Let U, V be a domain in $\mathbf{C}^n$ and an algebraic subvariety of $\mathbf{C}^{\hat{m}}$, respectively. Let $F : U \to V$ be a holomorphic mapping. Then there is a polynomial R in $\hat{m}$ variables with $R \circ F$ not identically zero such that the following holds. If for some open $\tilde{U}, U_0 \subset U$ with $U_0 \Subset \tilde{U}$ there is a sequence $\{G^\nu : \tilde{U} \to \mathbf{C}^{\hat{m}}\}$ of Nash mappings converging locally uniformly to $F|_{\tilde{U}}$ such that $\{\{x \in \tilde{U}:\ (R \circ G^\nu)(x) = 0\}\}$ converges to $\{x \in \tilde{U}:\ (R \circ F)(x) = 0\}$ in the sense of chains then there is a sequence $\{F^\nu : U_0 \to V\}$ of Nash mappings converging uniformly to $F|_{U_0}$.*

First let us check that Proposition 3.10 implies Proposition 3.6. Let $f : U \to \mathbf{C}^k$ be the holomorphic mapping from Proposition 3.6. Put $F(x) = (x, f(x))$ and $\hat{m} = n + k$. Let V be the intersection of all algebraic subvarieties of $\mathbf{C}^{\hat{m}}$ containing $F(U)$. Then by Proposition 3.10 there is $R \in \mathbf{C}[x_1, \ldots, x_n, z_1, \ldots, z_k]$ satisfying the assertion of this proposition. Next fix $U_0, \tilde{U}$ as in Proposition 3.6, assume (without loss of generality) that U_0 is connected and take an open connected U_1 with $U_0 \Subset U_1 \Subset \tilde{U}$. Now let $g^\nu : \tilde{U} \to \mathbf{C}^k$ be a sequence of Nash mappings converging locally uniformly to $f|_{\tilde{U}}$ such that $\{\{x \in \tilde{U}:\ R(x, g^\nu(x)) = 0\}\}$ converges to $\{x \in \tilde{U}:\ R(x, f(x)) = 0\}$ in the sense of chains. Set $G^\nu(x) = (x, g^\nu(x))$. Then by Proposition 3.10 there is a sequence $\{F^\nu : U_1 \to V\}$ of Nash mappings converging uniformly to $F|_{U_1}$.

We need to show that the first n components of F^ν may be assumed to constitute the identity and that $Q \circ F^\nu = 0$ for sufficiently large ν. To this end denote $Y = \{(x, v) \in \hat{U} \subset \mathbf{C}^n \times \mathbf{C}^k:\ Q(x, v) = 0\}$, where $\hat{U}$ is the domain of Q. Clearly, we may assume that $F^\nu(U_1) \subset \hat{U}$ for almost all ν. Next observe that $F^\nu(U_1) \subset Y$ for almost all ν. Indeed, take $\hat{z} \in F(U_1) \cap \mathrm{Reg}(V)$ (the intersection is nonempty as $F(U_1) \subset \mathrm{Sing}(V)$ implies, by the connectedness of U, that $F(U) \subset \mathrm{Sing}(V) \subsetneq V$). Let B be an open neighborhood of $\hat{z}$ in $\mathbf{C}^n \times \mathbf{C}^k$ such that $B \cap V$ is a connected manifold and let U_2 be a nonempty open subset of U_1 such that $F(U_2), F^\nu(U_2) \subset B$ for almost all ν. Then $B \cap V \subset Y$ (otherwise $F(U_2) \subset \tilde{V}$ where $\tilde{V}$ is an algebraic subvariety of $\mathbf{C}^n \times \mathbf{C}^k$ with $\dim(\tilde{V}) < \dim(V)$). This implies that $F^\nu(U_2) \subset Y$ for almost all ν hence $F^\nu(U_1) \subset Y$ because U_1 is connected.

Let $\tilde{F}^\nu : U_1 \to \mathbf{C}^n$, for $\nu \in \mathbf{N}$, be the mapping whose components are the first n components of F^ν. Since $\{\tilde{F}^\nu\}$ converges uniformly to the identity on U_1 and $U_0 \Subset U_1$ there is a sequence $H^\nu : U_0 \to U_1$ of Nash mappings such that $\tilde{F}^\nu \circ H^\nu = id_{U_0}$ if ν is large enough. Consequently, $F^\nu \circ H^\nu(x) = (x, f^\nu(x))$ for $x \in U_0$ and $\{f^\nu : U_0 \to \mathbf{C}^k\}$ satisfies the assertion of Proposition 3.6.

Proof of Proposition 3.10. First observe that we may assume $F(U) \nsubseteq \mathrm{Sing}(V)$ as otherwise V may be replaced by $\mathrm{Sing}(V)$. Next, since U is connected, $F(U)$ is contained in one irreducible component of V so we may assume that V is of pure dimension, say m.

We may also assume that $V \subset \mathbf{C}^{\hat{m}} \approx \mathbf{C}^m \times \mathbf{C}^s$ is with proper projection onto $\mathbf{C}^m$. Indeed, there is a $\mathbf{C}$-linear isomorphism $J : \mathbf{C}^{m+s} \to \mathbf{C}^{m+s}$ such that $J(V)$ is with proper projection onto $\mathbf{C}^m$. Thus if there exists a sequence $H^\nu : U_0 \to J(V)$ of Nash mappings converging to $J \circ F|_{U_0}$ then the sequence $\{J^{-1} \circ H^\nu\}$ satisfies the assertion of the proposition.

To complete the preparations, by $\rho : \mathbf{C}^m \times \mathbf{C}^s \to \mathbf{C}^m$, $\tilde{\rho} : \mathbf{C}^m \times \mathbf{C} \to \mathbf{C}^m$ denote the natural projections. Passing to the image of V by a linear isomorphism arbitrarily close to the identity, if necessary, we assume (in view of $F(U) \nsubseteq \mathrm{Sing}(V)$) that $\rho(F(U)) \nsubseteq \rho(\mathcal{S}(V))$. Now the polynomial R is constructed as follows. Any $\mathbf{C}$-linear form $L : \mathbf{C}^s \to \mathbf{C}$ determines the mapping $\Phi_L : \mathbf{C}^m \times \mathbf{C}^s \to \mathbf{C}^m \times \mathbf{C}$ by $\Phi_L(u, v) = (u, L(v))$. Since V is an algebraic subset of $\mathbf{C}^m \times \mathbf{C}^s$ with proper projection onto $\mathbf{C}^m$ then $\Phi_L(V)$ is an algebraic subset of $\mathbf{C}^m \times \mathbf{C}$ also with proper projection onto $\mathbf{C}^m$ for every form L. Take a form L such that the fibers of the projections of $\Phi_L(V)$ and V onto $\mathbf{C}^m$ have generically the same cardinality and $\rho(F(U)) \nsubseteq \tilde{\rho}(\mathcal{S}(\Phi_L(V)))$. The set $\Phi_L(V)$ is described by the unitary polynomial in one variable (corresponding to the last coordinate of $\mathbf{C}^m \times \mathbf{C}$) whose coefficients are polynomials in m variables and whose discriminant is non-zero. The discriminant, denoted by R, is the polynomial we look for. In fact, after the preparations, R depends only on $m \leqslant \hat{m}$ variables (the last $s = \hat{m} - m$ variables are dummy).

Let us show that R indeed has all the required properties. First $R \circ F$ is not identically zero as $\rho(F(U)) \nsubseteq \tilde{\rho}(\mathcal{S}(\Phi_L(V)))$. Next take $U_0, \tilde{U}$ and G^ν as in Proposition 3.10. We need the following notation. For any holomorphic mapping $H : E \to \mathbf{C}^m$, where E, E' are open subsets of $\mathbf{C}^n$, $E' \subset E$, and any algebraic subvariety X of $\mathbf{C}^m \times \mathbf{C}^s$ denote:

$$\mathcal{V}(X, E', H) = \{(x, v) \in E' \times \mathbf{C}^s:\ (H(x), v) \in X\}.$$

The mappings F, G^ν are of the form $F = (\tilde{F}, \hat{F})$, $G^\nu = (\tilde{G}^\nu, \hat{G}^\nu)$, for some holomorphic $\tilde{F}: U \to \mathbf{C}^m$, $\tilde{G}^\nu : \tilde{U} \to \mathbf{C}^m$, $\hat{F}: U \to \mathbf{C}^s$, $\hat{G}^\nu : \tilde{U} \to \mathbf{C}^s$.

In order to prove Proposition 3.10 it is sufficient to prove the following:

Claim 3.11. *For every irreducible component Y of $\mathcal{V}(V, U_0, \tilde{F})$ there is a sequence $\{Y_\nu\}$ of analytic subsets of $U_0 \times \mathbf{C}^s$ converging to Y in the sense of chains such that Y_ν is an irreducible component of $\mathcal{V}(V, U_0, \tilde{G}^\nu)$ for every ν.*

For the notion of the convergence of holomorphic chains see Section 2.3. Let us check that Proposition 3.10 indeed follows by Claim 3.11. To this end note that $\operatorname{graph}(\hat{F}) \subset \mathcal{V}(V, U_0, \tilde{F})$. Since there is a sequence $\{Y_\nu\}$ of analytic sets converging to $\operatorname{graph}(\hat{F})$ in the sense of chains then $Y_\nu = \operatorname{graph}(H^\nu)$ for almost all ν, where $H^\nu : U_0 \to \mathbf{C}^s$ is a holomorphic mapping. In fact, since Y_ν is an irreducible component of $\mathcal{V}(V, U_0, \tilde{G}^\nu)$ which is a Nash set then Y_ν is a Nash set as well. Consequently H^ν is a Nash mapping. Obviously, H^ν converges to $\hat{F}$ so $(\tilde{G}^\nu, H^\nu): U_0 \to V$ converges to $F|_{U_0}$ as required.

Let us turn to the proof of Claim 3.11. First let us show that it is sufficient to prove this claim in the case where V is replaced by $\Phi_L(V)$ where $L: \mathbf{C}^s \to \mathbf{C}$ is the linear form which has been used to define R. By analogy to the definition of Φ_L put $\Psi_L(x, v) = (x, L(v))$ for any $x \in \mathbf{C}^n$, $v \in \mathbf{C}^s$. Let $\tilde{\pi}: \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}^n$, $\pi: \mathbf{C}^n \times \mathbf{C}^s \to \mathbf{C}^n$ denote the natural projections. We need the following obvious:

Remark 3.12. Let $Z \subset E \times \mathbf{C}^s$ be an analytic subset of pure dimension n with proper projection onto a domain $E \subset \mathbf{C}^n$ such that $s(\pi|_Z) = s(\tilde{\pi}|_{\Psi_L(Z)})$. Then for every irreducible analytic component Σ of $\Psi_L(Z)$ there exists an irreducible analytic component Γ of Z such that $\Psi_L(\Gamma) = \Sigma$ and $s(\pi|_\Gamma) = s(\tilde{\pi}|_\Sigma)$.

Assume that Claim 3.11 holds with $\Phi_L(V)$ taken instead of V ($s = 1$). We check that it also holds with V. First observe that $\mathcal{V}(\Phi_L(V), U_0, \tilde{F}) = \Psi_L(\mathcal{V}(V, U_0, \tilde{F}))$ and $\mathcal{V}(\Phi_L(V), U_0, \tilde{G}^\nu) = \Psi_L(\mathcal{V}(V, U_0, \tilde{G}^\nu))$ for $\nu \in \mathbf{N}$ and fix an irreducible component Y of $\mathcal{V}(V, U_0, \tilde{F})$. Then there are irreducible components Θ_ν of $\Psi_L(\mathcal{V}(V, U_0, \tilde{G}^\nu))$, for $\nu \in \mathbf{N}$, such that $\{\Theta_\nu\}$ converges to $\Psi_L(Y)$ in the sense of holomorphic chains.

Next note that the fact that $\tilde{F}(U_0) \nsubseteq \tilde{\rho}(\mathcal{S}(\Phi_L(V)))$ and the way L has been chosen imply that the cardinalities of the generic fibers in $\Psi_L(\mathcal{V}(V, U_0, \tilde{F}))$, $\mathcal{V}(V, U_0, \tilde{F})$, $\Psi_L(\mathcal{V}(V, U_0, \tilde{G}^\nu))$ and in $\mathcal{V}(V, U_0, \tilde{G}^\nu)$ over U_0 are equal for large ν. Therefore, by Remark 3.12, for almost all ν there is an irreducible component Y_ν of $\mathcal{V}(V, U_0, \tilde{G}^\nu)$ such that $\Psi_L(Y_\nu) = \Theta_\nu$ and $s(\pi|_{Y_\nu}) = s(\pi|_Y)$. Thus it remains to check, in view of Lemma 2.3, that $\{Y_\nu\}$ converges to Y locally uniformly. Observe that otherwise there would be a subsequence $\{Y_{\nu_\mu}\}$ of $\{Y_\nu\}$ converging to a purely n-dimensional analytic set $Z \neq Y$. But then, by the fact that Ψ_L preserves the cardinality of the generic fiber in $\mathcal{V}(V, U_0, \tilde{F})$, it holds $\Psi_L(Z) \neq \Psi_L(Y)$ which contradicts the fact that $\{\Psi_L(Y_\nu)\}$ converges to $\Psi_L(Y)$.

Now we turn to the proof of Claim 3.11 with V replaced by $\Phi_L(V)$. It holds

$$\Phi_L(V) = \bigl\{(y, z) \in \mathbf{C}^m \times \mathbf{C}\colon\ P(y, z) = 0\bigr\},$$

where

$$P(y, z) = z^t + z^{t-1} c_1(y) + \cdots + c_t(y) \in \bigl(\mathbf{C}[y]\bigr)[z],$$

for some $t \in \mathbf{N}$. We may assume that P treated as a polynomial in z has a non-zero discriminant. Let us recall that the polynomial R is, by definition, this discriminant.

To complete the proof put:

$$X_\nu = \mathcal{V}\bigl(\Phi_L(V), \tilde{U}, \tilde{G}^\nu\bigr) = \bigl\{(x, z) \in \tilde{U} \times \mathbf{C}\colon\ P\bigl(\tilde{G}^\nu(x), z\bigr) = 0\bigr\}$$

and

$$X = \mathcal{V}\bigl(\Phi_L(V), \tilde{U}, \tilde{F}\bigr) = \bigl\{(x, z) \in \tilde{U} \times \mathbf{C}\colon\ P\bigl(\tilde{F}(x), z\bigr) = 0\bigr\}.$$

Then

$$\tilde{\pi}\bigl(\mathcal{S}(X_\nu)\bigr) = \bigl\{x \in \tilde{U}\colon\ R\bigl(\tilde{G}^\nu(x)\bigr) = 0\bigr\}$$

and

$$\tilde{\pi}\bigl(\mathcal{S}(X)\bigr) = \bigl\{x \in \tilde{U}\colon\ R\bigl(\tilde{F}(x)\bigr) = 0\bigr\},$$

hence by the hypothesis the sequence $\{\tilde{\pi}(\mathcal{S}(X_\nu))\}$ converges to $\tilde{\pi}(\mathcal{S}(X))$ in the sense of holomorphic chains. Now it is sufficient to apply Theorem 3.1 and the proof of Claim 3.11 is complete. Consequently, we have also proved Propositions 3.10 and 3.6. □

3.2.2. Construction of $\mathcal{U}_{f,R}$

Put $x=(x_1,\dots,x_n)$, $x'=(x_1,\dots,x_{n-1})$ and $\pi(x)=x'$. Let $f:U\to\mathbf{C}^k$, $f=f(x)$, be a holomorphic mapping where U is a domain in $\mathbf{C}^n$ and let $R\in\mathbf{C}[x,z_1,\dots,z_k]$ be such that $R(x,f(x))$ is not identically zero. We construct the family $\mathcal{U}_{f,R}$. The construction is recursive with respect to the number n of the variables f depends on.

Let U_0 be an open subset of $\mathbf{C}$, $n=1$. Then $U_0\in\mathcal{U}_{f,R}$ iff U_0 is a relatively compact subset of some open simply connected subset of U (hence in this case $\mathcal{U}_{f,R}$ depends only on U).

Now assume that U_0 is an open subset of $\mathbf{C}^n$ for $n>1$. Then $U_0\in\mathcal{U}_{f,R}$ iff there is a biholomorphism $\phi:\check{U}\to\hat{U}\subset U$, where $\hat{U},\check{U}$ are a domain and a Runge domain respectively in $\mathbf{C}^n$ with $U_0\Subset\hat{U}$, and there is a domain $\check{U}_1\subset\mathbf{C}^{n-1}$ with $\check{U}\subset\check{U}_1\times\mathbf{C}$ such that the following hold:

(1) $R(\phi(x),f(\phi(x)))=\tilde{H}(x)W(x)$, $x\in\check{U}$, for some $\tilde{H}\in\mathcal{O}(\check{U})$ non-vanishing on $\overline{\phi^{-1}(U_0)}$ and some unitary polynomial $W\in\mathcal{O}(\check{U}_1)[x_n]$ such that $W^{-1}(0)\subset\check{U}$,
(2) $\pi(\phi^{-1}(U_0))\in\mathcal{U}_{g,S}$ for some holomorphic mapping $g:\check{U}_1\to\mathbf{C}^s$, $g=g(x')$, and some $S\in\mathbf{C}[x',z_1,\dots,z_s]$ determined by f,R,ϕ,W below.

Given f,R,ϕ,W we obtain g,S as follows. Put $\tilde{f}=f\circ\phi$. Then $\tilde{f},\phi$ are of the form $\tilde{f}=(f_1,\dots,f_k)$, $\phi=(\phi_1,\dots,\phi_n)$ for some $f_j,\phi_i\in\mathcal{O}(\check{U})$ for $j=1,\dots,k$, $i=1,\dots,n$. By (1) we have:

$$f_j(x)=W(x)H_j(x)+r_j(x),$$

$$\phi_i(x)=W(x)\check{H}_i(x)+\check{r}_i(x),$$

for $x\in\check{U}$, where $r_j(x),\check{r}_i(x)\in\mathcal{O}(\check{U}_1)[x_n]$ satisfy $\deg(r_j),\deg(\check{r}_i)<\deg(W)$ and $H_j,\check{H}_i\in\mathcal{O}(\check{U})$ for $j=1,\dots,k$, $i=1,\dots,n$.

Next, there are optimal polynomials (for the definition consult Section 2.2) $W_1,\dots,W_{\hat{s}}\in\mathcal{O}(\check{U}_1)[x_n]$ such that $W=W_1^{k_1}\cdot\dots\cdot W_{\hat{s}}^{k_{\hat{s}}}$ and $\dim(W_i^{-1}(0)\cap W_j^{-1}(0))<n-1$ for every $i\neq j$. Put $d=\deg(W)$. For $l=1,\dots,\hat{s}$, $j=1,\dots,k$, $i=1,\dots,n$, the polynomials $W_l,r_j,\check{r}_i$ are of the form:

$$\begin{aligned}W_l(x)&=x_n^{p_l}+x_n^{p_l-1}a_{l,1}(x')+\dots+a_{l,p_l}(x'),\\ r_j(x)&=x_n^{d-1}b_{j,0}(x')+x_n^{d-2}b_{j,1}(x')+\dots+b_{j,d-1}(x'),\\ \check{r}_i(x)&=x_n^{d-1}c_{i,0}(x')+x_n^{d-2}c_{i,1}(x')+\dots+c_{i,d-1}(x').\end{aligned}$$

Let s denote the number of all the coefficients of $W_l,r_j,\check{r}_i$ for all admissible l,j,i. The mapping $g:\check{U}_1\to\mathbf{C}^s$ is defined by:

$$g=(A_1,\dots,A_{\hat{s}},B_1,\dots,B_k,C_1,\dots,C_n),$$

where $A_l=(a_{l,1},\dots,a_{l,p_l})$, $B_j=(b_{j,0},\dots,b_{j,d-1})$, $C_i=(c_{i,0},\dots,c_{i,d-1})$ again for all admissible l,j,i.

Let us turn to determining S. Replacing the holomorphic coefficients

$$a_{l,1},\dots,a_{l,p_l},b_{j,0},\dots,b_{j,d-1},c_{i,0},\dots,c_{i,d-1},$$

for all l,j,i in $W_l,r_j,\check{r}_i$ by new variables denoted by the same letters we obtain polynomials $P_l,w_j,\check{w}_i$, respectively. Put $P=P_1^{k_1}\cdot\dots\cdot P_{\hat{s}}^{k_{\hat{s}}}$ and define:

$$\alpha_j=PS_j+w_j,\qquad\beta_i=P\check{S}_i+\check{w}_i$$

for $j=1,\dots,k$ and $i=1,\dots,n$, where $S_j,\check{S}_i$ are new variables. Now divide $R(\beta_1,\dots,\beta_n,\alpha_1,\dots,\alpha_k)$ by P (treated as a polynomial in x_n with polynomial coefficients) to obtain:

$$R(\beta_1,\dots,\beta_n,\alpha_1,\dots,\alpha_k)=\tilde{W}P+x_n^{d-1}T_1+x_n^{d-2}T_2+\dots+T_d,\tag{$*$}$$

where $\tilde{W}, T_1, \ldots, T_d$ are polynomials such that $T_1, \ldots, T_d$ depend only on the tuple of variables u, where

$$u = (A_1, \ldots, A_{\hat{s}}, B_1, \ldots, B_k, C_1, \ldots, C_n)$$

and $A_l = (a_{l,1}, \ldots, a_{l,p_l})$, $B_j = (b_{j,0}, \ldots, b_{j,d-1})$, $C_i = (c_{i,0}, \ldots, c_{i,d-1})$ for all admissible l, j, i.

Finally, put $T(u) = (T_1(u), \ldots, T_d(u))$ and observe that $T(g(x')) = 0$ for $x' \in \check{U}_1$. By Proposition 3.6 there is $S \in \mathbf{C}[x', z_1, \ldots, z_s]$ satisfying the assertion of Proposition 3.6 with g, T, S taken in place of f, Q, R. Any such S is suitable for our recursive definition.

Remark 3.13. For every holomorphic mapping $f : \mathbf{C}^n \supset U \to \mathbf{C}^k$, polynomial $R \in \mathbf{C}[x, z_1, \ldots, z_k]$ with $R(x, f(x))$ not identically zero and every $x_0 \in U$ the following holds. There is a neighborhood E of x_0 in U such that $\{x \in E\colon\ R(x, f(x)) = 0\}$ is either empty or with proper projection onto an open subset of some affine $(n-1)$-dimensional subspace of $\mathbf{C}^n$. This fact, applied recursively in the construction above, immediately implies that there is an open neighborhood $\tilde{U}$ of x_0 in U such that $\tilde{U} \in \mathcal{U}_{f,R}$.

3.2.3. Proof of Proposition 3.7

The proposition is proved by induction on n (the number of the variables f depends on). First suppose that $U \subset \mathbf{C}$ (i.e. $n = 1$) and fix f, R satisfying the assumptions of the proposition. Let U_0 be any open relatively compact subset of some open simply connected $\tilde{U} \subset U$. Then

$$R\big(x, f(x)\big) = (x - x_0)^{\alpha_0} \cdot \cdots \cdot (x - x_m)^{\alpha_m} g(x),$$

for some $m, \alpha_0, \ldots, \alpha_m \in \mathbf{N}$, $g \in \mathcal{O}(U)$ such that $g(x) \neq 0$ for $x \in \overline{U_0}$.

Put $W(x) = (x - x_0)^{\alpha_0} \cdot \cdots \cdot (x - x_m)^{\alpha_m}$. The mapping f is of the form $f = (f_1, \ldots, f_k)$ for some $f_j \in \mathcal{O}(U)$, $j = 1, \ldots, k$. It holds

$$f_j(x) = W(x)H_j(x) + r_j(x), \quad x \in U,$$

where $H_j \in \mathcal{O}(U)$, $r_j \in \mathbf{C}[x]$ for $j = 1, \ldots, k$. Now define $f^\nu = (f_1^\nu, \ldots, f_k^\nu)$ on $\tilde{U}$ by $f_j^\nu(x) = W(x)H_{j,\nu}(x) + r_j(x)$, $\nu \in \mathbf{N}$. Here $\{H_{j,\nu}\}$ is a sequence of polynomials converging locally uniformly to H_j on $\tilde{U}$. It is clear that $R(x, f^\nu(x)) = W(x)g_\nu(x)$, for some $g_\nu \in \mathcal{O}(\tilde{U})$. The function g is non-vanishing on $\overline{U_0}$ therefore shrinking $\tilde{U}$, if necessary, we complete the proof for $n = 1$.

Now suppose that $n > 1$. Let f, R be a holomorphic mapping and a polynomial respectively satisfying the hypotheses of the proposition. Fix $U_0 \in \mathcal{U}_{f,R}$. By the definition of $\mathcal{U}_{f,R}$ there exists a biholomorphism $\phi : \check{U} \to \hat{U}$, where $\check{U} \subset \mathbf{C}^n$ is a Runge domain, $U_0 \Subset \hat{U}$ and $\check{U} \subset \check{U}_1 \times \mathbf{C}$ for some open connected $\check{U}_1 \subset \mathbf{C}^{n-1}$, such that (1) and (2) of Section 3.2.2 are satisfied.

Next observe that to complete the proof it is sufficient to show that there is an open E with $\phi^{-1}(U_0) \Subset E \subset \check{U}$ and there are sequences $\{\tilde{f}^\nu\}$, $\{\phi^\nu\}$ of Nash mappings converging locally uniformly on E to $\tilde{f} = f \circ \phi, \phi$ respectively in such a way that $\{\{x \in E\colon\ R(\phi^\nu(x), \tilde{f}^\nu(x)) = 0\}\}$ converges in the sense of chains to $\{x \in E\colon\ R(\phi(x), \tilde{f}(x)) = 0\}$. Indeed, given such sequences we may assume, shrinking E if necessary, that $\phi^\nu|_E$ is invertible for almost all ν. Moreover, there is an open $\tilde{U} \subset \phi(E)$ such that $U_0 \Subset \tilde{U} \subset \phi^\nu(E)$ for almost all ν. Consequently, $\{\{x \in \tilde{U}\colon\ R(x, \tilde{f}^\nu \circ (\phi^\nu)^{-1}(x)) = 0\}\}$ converges to $\{x \in \tilde{U}\colon\ R(x, f(x)) = 0\}$ and we may set $f^\nu = \tilde{f}^\nu \circ (\phi^\nu)^{-1}$.

Before approximating $\tilde{f}, \phi$ we show that there are Nash mappings

$$A_l^\nu = \big(a_{l,1}^\nu, \ldots, a_{l,p_l}^\nu\big), \qquad B_j^\nu = \big(b_{j,0}^\nu, \ldots, b_{j,d-1}^\nu\big), \qquad C_i^\nu = \big(c_{i,0}^\nu, \ldots, c_{i,d-1}^\nu\big),$$

for $l = 1, \ldots, \hat{s}$, $j = 1, \ldots, k$, $i = 1, \ldots, n$, $\nu \in \mathbf{N}$, defined on some open set $E_1 \subset \mathbf{C}^{n-1}$ with $\pi(\phi^{-1}(U_0)) \Subset E_1 \subset \check{U}_1$ such that the following hold. The sequence $\{g^\nu : E_1 \to \mathbf{C}^s\}$, where $g^\nu = (A_1^\nu, \ldots, A_{\hat{s}}^\nu, B_1^\nu, \ldots, B_k^\nu, C_1^\nu, \ldots, C_n^\nu)$, converges uniformly to $g|_{E_1}$ and $T_1 \circ g^\nu = \cdots = T_d \circ g^\nu = 0$ for $\nu \in \mathbf{N}$. Here g is the mapping from the condition (2) and $T_1, \ldots, T_d$ are polynomials given by Eq. $(*)$ of Section 3.2.2.

To this end, observe that by (2) it holds $\pi(\phi^{-1}(U_0)) \in \mathcal{U}_{g,S}$, where the polynomial S is described in the previous subsection. By the properties of S it is sufficient to show that there is a sequence $\{h^\nu : \tilde{E}_1 \to \mathbf{C}^s\}$ of Nash mappings converging locally uniformly to $g|_{\tilde{E}_1}$, where $\pi(\phi^{-1}(U_0)) \Subset \tilde{E}_1 \subset \check{U}_1$, such that $\{\{x' \in \tilde{E}_1\colon\ S(x', h^\nu(x')) = 0\}\}$ converges to $\{x' \in \tilde{E}_1\colon\ S(x', g(x')) = 0\}$ in the sense of chains (then E_1 may be taken to be any open set with $\pi(\phi^{-1}(U_0)) \Subset E_1 \Subset \tilde{E}_1$). This in turn is immediate by the induction hypothesis.

Using the components of $A_l^\nu, B_j^\nu, C_i^\nu$ define on $E = (E_1 \times \mathbf{C}) \cap \check{U}$ the following functions:

$$W_l^\nu(x) = x_n^{p_l} + x_n^{p_l-1} a_{l,1}^\nu(x') + \cdots + a_{l,p_l}^\nu(x'),$$
$$r_j^\nu(x) = x_n^{d-1} b_{j,0}^\nu(x') + x_n^{d-2} b_{j,1}^\nu(x') + \cdots + b_{j,d-1}^\nu(x'),$$
$$\check{r}_i^\nu(x) = x_n^{d-1} c_{i,0}^\nu(x') + x_n^{d-2} c_{i,1}^\nu(x') + \cdots + c_{i,d-1}^\nu(x'),$$

for $l = 1, \ldots, \hat{s}$, $j = 1, \ldots, k$, $i = 1 \ldots, n$. Next put $W^\nu = (W_1^\nu)^{k_1} \cdot \cdots \cdot (W_{\hat{s}}^\nu)^{k_{\hat{s}}}$, where k_j is the multiplicity of the factor W_j of W (see Section 3.2.2). Now define $\tilde{f}^\nu = (f_1^\nu, \ldots, f_k^\nu)$, $\phi^\nu = (\phi_1^\nu, \ldots, \phi_n^\nu)$ by:

$$f_j^\nu = W^\nu H_j^\nu + r_j^\nu, \qquad \phi_i^\nu = W^\nu \check{H}_i^\nu + \check{r}_i^\nu$$

for $j = 1, \ldots, k$, $i = 1, \ldots, n$. Here $\{H_j^\nu\}, \{\check{H}_i^\nu\}$, are any sequences of polynomials converging locally uniformly on E to $H_j, \check{H}_i$, respectively. (Recall that $H_j, \check{H}_i$ are obtained in Section 3.2.2 dividing f_j, ϕ_i by W. The existence of $\{H_j^\nu\}, \{\check{H}_i^\nu\}$ follows by the fact that $\check{U}$ is a Runge domain.) Clearly, $\{\tilde{f}^\nu\}, \{\phi^\nu\}$ converge locally uniformly to $\tilde{f}|_E, \phi|_E$, respectively.

Finally, Eq. $(*)$ from Section 3.2.2, in view of the fact that $T_1 \circ g^\nu = \cdots = T_d \circ g^\nu = 0$, implies $R(\phi^\nu(x), \tilde{f}^\nu(x)) = \tilde{H}^\nu(x) W^\nu(x)$ for every $x \in E$, where $\tilde{H}^\nu \in \mathcal{O}(E)$. Since $\{W_l^\nu|_E\}$ converges to $W_l|_E$ locally uniformly, for $l = 1, \ldots, \hat{s}$ (where $W_1, \ldots, W_{\hat{s}}$ are optimal polynomials such that $W = W_1^{k_1} \cdot \cdots \cdot W_{\hat{s}}^{k_{\hat{s}}}$ and $\dim(W_i^{-1}(0) \cap W_j^{-1}(0)) < n-1$ for every $i \neq j$) it holds: $\{\{x \in E\colon \overline{W^\nu(x) = 0}\}\}$ converges to $\{x \in E\colon W(x) = 0\}$ in the sense of chains. The function $\tilde{H}$ given by (1) is non-vanishing on $\phi^{-1}(U_0)$ therefore shrinking E if necessary we obtain the required claim. □

3.2.4. Algorithm

Based on the proof of Theorem 3.8, we present a recursive algorithm of Nash approximation of a holomorphic mapping $f : U \to V \subset \mathbf{C}^{\hat{m}}$, where U is a domain in $\mathbf{C}^n$ and V is an algebraic variety. For $\nu \in \mathbf{N}$, the approximating mapping $f^\nu = (f_1^\nu, \ldots, f_{\hat{m}}^\nu) : U_0 \to V$, returned as the output of the algorithm, is represented by $\hat{m}$ non-zero polynomials $P_i^\nu(x, z_i) \in (\mathbf{C}[x])[z_i]$, $i = 1, \ldots, \hat{m}$, such that $P_i^\nu(x, f_i^\nu(x)) = 0$ for $x \in U_0$. For simplicity we restrict attention to the local case, i.e. U_0 is an open neighborhood of a fixed $x_0 \in U$. More precisely, we work with the following data:

Input: a holomorphic mapping $f = (f_1, \ldots, f_{\hat{m}}) : U \to V \subset \mathbf{C}^{\hat{m}}$, $f = f(x)$, where U is an open neighborhood of $0 \in \mathbf{C}^n$ and V is an algebraic variety.

Output: $P_i^\nu(x, z_i) \in (\mathbf{C}[x])[z_i]$, $P_i^\nu \neq 0$ for $i = 1, \ldots, \hat{m}$ and $\nu \in \mathbf{N}$, with the following properties:

(a) $P_i^\nu(x, f_i^\nu(x)) = 0$ for every $x \in U_0$, where $f^\nu = (f_1^\nu, \ldots, f_{\hat{m}}^\nu) : U_0 \to V$ is a holomorphic mapping such that $\{f^\nu\}$ converges uniformly to f on an open neighborhood U_0 of $0 \in \mathbf{C}^n$,

(b) P_i^ν is a unitary polynomial in z_i of degree independent of ν whose coefficients (belonging to $\mathbf{C}[x]$) converge uniformly to holomorphic functions on U_0 as ν tends to infinity.

Before going into detail let us comment on the notation and the idea of the algorithm. First, the meaning of the symbols $V_{(m)}, \mathcal{S}(V)$ and the notion of the optimal polynomial used below can be found in Section 2.2. Next, in Steps 2 and 5 we apply linear changes of the coordinates. Having approximated the mapping $\hat{J} \circ f \circ J|_{J^{-1}(U)} : J^{-1}(U) \to \hat{J}(V)$, where $\hat{J} : \mathbf{C}^{\hat{m}} \to \mathbf{C}^{\hat{m}}$, $J : \mathbf{C}^n \to \mathbf{C}^n$ are linear isomorphisms, one can obtain the output data for f following standard arguments. (Composing f and J does not lead to any difficulties. As for $\hat{J}$, it is sufficient to use the fact that the integral closure of a commutative ring in another commutative ring is again a ring.) Therefore, when the coordinates are changed, we write what (as a result) may be assumed about the mapping f, but the notation is left unchanged.

The aim of Steps 1–3 is to prepare the variety V so that the polynomial R calculated in Step 4 satisfies the assertion of Proposition 3.10 (cf. the proof of Proposition 3.10). Steps 5–9 are responsible for the fact that for $f_1^\nu, \ldots, f_m^\nu$ defined in Step 10 the sequence $\{\{R(f_1^\nu(x), \ldots, f_m^\nu(x)) = 0\}\}$ converges to $\{R(f_1(x), \ldots, f_m(x)) = 0\}$ in the sense of chains, in a neighborhood of $0 \in \mathbf{C}^n$, as ν tends to infinity. This property implies (cf. the proof of Proposition 3.10) that there is an open neighborhood U_0 of $0 \in \mathbf{C}^n$ such that for ν large enough the set $\{(x, z_{m+1}, \ldots, z_{m+s}) \in U_0 \times \mathbf{C}^s\colon (f_1^\nu(x), \ldots, f_m^\nu(x), z_{m+1}, \ldots, z_{m+s}) \in V\}$ contains a graph of the mapping $x \mapsto (f_{m+1}^\nu(x), \ldots, f_{m+s}^\nu(x))$

approximating the mapping $x \mapsto (f_{m+1}(x), \dots, f_{m+s}(x))$ (here $\hat{m} = m + s$). The latter fact is used in Step 11 to calculate $P^\nu_{m+1}, \dots, P^\nu_{m+s}$. As for $P^\nu_1, \dots, P^\nu_m$, these polynomials are obtained in Step 10 by applying the results of the algorithm switched for the lower dimensional case in Step 9.

Algorithm 1.

1. If $f(U) \subseteq \mathrm{Sing}(V)$ then repeat replacing V by $\mathrm{Sing}(V)$ until $f(U) \nsubseteq \mathrm{Sing}(V)$. Next replace V by $V_{(m)}$ such that $f(U) \subset V_{(m)}$.
2. Apply a linear change of the coordinates in $\mathbf{C}^{\hat{m}}$ after which $\rho|_V$ is a proper mapping and $\rho(f(U)) \nsubseteq \rho(\mathcal{S}(V))$, where $\rho : \mathbf{C}^m \times \mathbf{C}^s \approx \mathbf{C}^{\hat{m}} \to \mathbf{C}^m$ is the natural projection.
3. Choose a $\mathbf{C}$-linear form $L : \mathbf{C}^s \to \mathbf{C}$ such that the generic fibers of $\rho|_V$ and $\tilde{\rho}|_{\Phi_L(V)}$ over $\mathbf{C}^m$ have the same cardinalities and $\rho(f(U)) \nsubseteq \tilde{\rho}(\mathcal{S}(\Phi_L(V)))$. Here $\tilde{\rho} : \mathbf{C}^m \times \mathbf{C} \to \mathbf{C}^m$ is the natural projection and $\Phi_L(y, v) = (y, L(v))$ for $(y, v) \in \mathbf{C}^m \times \mathbf{C}^s$.
4. Calculate the discriminant $R \in \mathbf{C}[y]$ of the optimal polynomial $P(y, z) \in (\mathbf{C}[y])[z]$ describing $\Phi_L(V) \subset \mathbf{C}^m_y \times \mathbf{C}_z$.
5. Apply a linear change of the coordinates in $\mathbf{C}^n$ after which $R(\rho(f(x))) = \tilde{H}(x)W(x)$ in some neighborhood of $0 \in \mathbf{C}^n$, where $\tilde{H}$ is a holomorphic function, $\tilde{H}(0) \neq 0$ and W is a unitary polynomial in x_n with holomorphic coefficients depending on $x' = (x_1, \dots, x_{n-1})$ each of which vanishes at $0 \in \mathbf{C}^{n-1}$. Put $d = \deg(W)$.
6. Divide f_i by W to obtain $f_i(x) = W(x)H_i(x) + r_i(x)$ in some neighborhood of $0 \in \mathbf{C}^n$, $i = 1, \dots, m$. Here H_i is a holomorphic function and r_i is a polynomial in x_n, $\deg(r_i) < d$, with holomorphic coefficients depending on x'.
7. Find optimal polynomials $W_1, \dots, W_{\hat{s}}$ in x_n with holomorphic coefficients depending on x' such that $W = W_1^{k_1} \cdot \dots \cdot W_{\hat{s}}^{k_{\hat{s}}}$ and $\dim(W_i^{-1}(0) \cap W_j^{-1}(0)) < n - 1$ for every $i \neq j$.
8. Treating H_i, $i = 1, \dots, m$, and all the coefficients of $W_1, \dots, W_{\hat{s}}, r_1, \dots, r_m$ as new variables (except for the coefficient 1 standing at the leading terms of $W_1, \dots, W_{\hat{s}}$) apply the division procedure for polynomials to obtain: $R(WH_1 + r_1, \dots, WH_m + r_m) = \tilde{W}W + x_n^{d-1}T_1 + x_n^{d-2}T_2 + \dots + T_d$. Here $T_1, \dots, T_d$ are polynomials depending only on the variables standing for the coefficients of $W_1, \dots, W_{\hat{s}}, r_1, \dots, r_m$. Moreover, $T_1(g) = \dots = T_d(g) = 0$, where g is the holomorphic mapping whose components are these coefficients (cf. Section 3.2.2).
9. If g is not constant (i.e. it depends on $n - 1 \geqslant 1$ variables) then apply the algorithm with f, V replaced by g and $\{u \in \mathbf{C}^{\hat{d}} : T_1(u) = \dots = T_d(u) = 0\}$ respectively, where $\hat{d}$ is the number of the components of g. As a result, for every $c(x')$ which is a coefficient of some of $W_1, \dots, W_{\hat{s}}, r_1, \dots, r_m$ one obtains a sequence $\{Q^\nu_c(x', t_c)\}$ of unitary polynomials satisfying (a) and (b) above with $x, z_i, \{f^\nu\}$ replaced by $x', t_c, \{g^\nu\}$, respectively. Here $\{g^\nu\}$ is a sequence of Nash mappings converging to g in some neighborhood of $0 \in \mathbf{C}^{n-1}$ such that $T_1 \circ g^\nu = \dots = T_d \circ g^\nu = 0$ for every $\nu \in \mathbf{N}$ (cf. Section 3.2.3). If g is constant then it is its own approximation yielding the Q^ν_c's immediately.
10. Approximate H_i, for $i = 1, \dots, m$, by a sequence $\{H^\nu_i\}$ of polynomials. Let $W^\nu_1, \dots, W^\nu_{\hat{s}}, r^\nu_1, \dots, r^\nu_m$, for every $\nu \in \mathbf{N}$, be the polynomials in x_n defined by replacing the coefficients of $W_1, \dots, W_{\hat{s}}, r_1, \dots, r_m$ by their Nash approximations (i.e. the components of g^ν) determined in Step 9. Using Q^ν_c (for all c) and H^ν_i one can calculate $P^\nu_i \in (\mathbf{C}[x])[z_i]$, for $i = 1, \dots, m$, satisfying (b) and (a) with $f^\nu_i = H^\nu_i (W^\nu_1)^{k_1} \cdot \dots \cdot (W^\nu_{\hat{s}})^{k_{\hat{s}}} + r^\nu_i$ being the ith component of the mapping f^ν (whose last $\hat{m} - m$ components are determined by $P^\nu_{m+1}, \dots, P^\nu_{\hat{m}}$ obtained in the next step). To calculate $P^\nu_1, \dots, P^\nu_m$ one can follow the standard proof of the fact that the integral closure of a commutative ring in another commutative ring is again a ring.
11. Put $V^\nu = \{(x, z) \in \mathbf{C}^n_x \times \mathbf{C}^{m+s}_z : z \in V, P^\nu_i(x, z_i) = 0 \text{ for } i = 1, \dots, m\}$, where $z = (z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s})$. For $i = 1, \dots, s$ and $\nu \in \mathbf{N}$ take $P^\nu_{m+i} \in (\mathbf{C}[x])[z_{m+i}]$ to be the optimal polynomial describing the image of the projection of V^ν onto $\mathbf{C}^n_x \times \mathbf{C}_{z_{m+i}}$.

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