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The inclusion relation between Sobolev and modulation spaces

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Abstract

The inclusion relations between the L^p-Sobolev spaces and the modulation spaces is determined explicitly. As an application, mapping properties of unimodular Fourier multiplier $e^{i|D|^\alpha}$ between L^p-Sobolev spaces and modulation spaces are discussed.

Keywords: Modulation space; L^p-Sobolev space; Inclusion; Unimodular Fourier multiplier

1. Introduction

The modulation spaces $M_s^{p,q}$ are one of the function spaces introduced by Feichtinger [6] in 1980's to measure the decaying and regularity property of a function or distribution in a way different from L^p-Sobolev spaces L_s^p or Besov spaces $B_s^{p,q}$. The precise definitions of these function spaces will be given in Section 2, but the main idea of modulation spaces is to consider the space variable and the variable of its Fourier transform simultaneously, while they are treated independently in L^p-Sobolev spaces and Besov spaces.

Because of this special nature, modulation spaces are now considered to be suitable spaces in the analysis of pseudo-differential operators after a series of important works [4,9–11,24, 25] and so on. "Modulation spaces and pseudo-differential operators" is still an active fields

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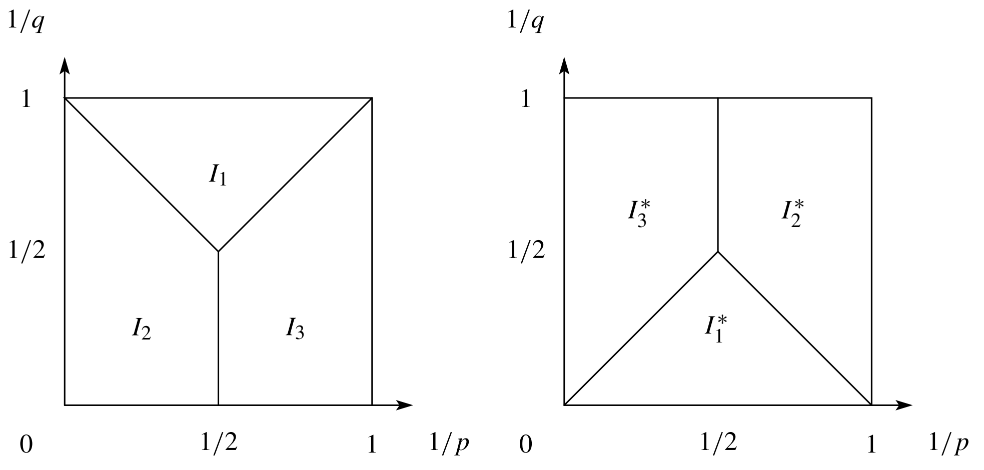
of research (see, for example, [5,12,13,17,23,28]). On the other hand, modulation spaces have also remarkable applications in the analysis of partial differential equations. For example, the Schrödinger and wave propagators, which are not bounded on neither L^p nor $B_s^{p,q}$, are bounded on $M_s^{p,q}$ [2]. Modulation spaces are also used as a regularity class of initial data of the Cauchy problem for nonlinear evolution equations, and in this way the existence of the solution is shown under very low regularity assumption for initial data (see [30–32]).

In the last several years, many basic properties of modulation spaces are established. In particular, the inclusion relation between Besov spaces and modulation spaces has been completely determined. Let us define the indices $\nu_1(p,q)$ and $\nu_2(p,q)$ for $1 \leqslant p, q \leqslant \infty$ in the following way:

$$\nu_1(p,q) = \begin{cases} 0 & \text{if } (1/p, 1/q) \in I_1^*\colon\ \min(1/p, 1/p') \geqslant 1/q, \\ 1/p + 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^*\colon\ \min(1/q, 1/2) \geqslant 1/p', \\ -1/p + 1/q & \text{if } (1/p, 1/q) \in I_3^*\colon\ \min(1/q, 1/2) \geqslant 1/p, \end{cases}$$

$$\nu_2(p,q) = \begin{cases} 0 & \text{if } (1/p, 1/q) \in I_1\colon\ \max(1/p, 1/p') \leqslant 1/q, \\ 1/p + 1/q - 1 & \text{if } (1/p, 1/q) \in I_2\colon\ \max(1/q, 1/2) \leqslant 1/p', \\ -1/p + 1/q & \text{if } (1/p, 1/q) \in I_3\colon\ \max(1/q, 1/2) \leqslant 1/p, \end{cases}$$

where $1/p + 1/p' = 1 = 1/q + 1/q'$. We remark $\nu_2(p,q) = -\nu_1(p', q')$.



Then the following result is known:

Theorem 1.1. *(See Sugimoto and Tomita [22], Toft [25].) Let* $1 \leqslant p, q \leqslant \infty$ *and* $s \in \mathbf{R}$. *Then we have*

(1) $B_s^{p,q}(\mathbf{R}^n) \hookrightarrow M^{p,q}(\mathbf{R}^n)$ *if and only if* $s \geqslant n\nu_1(p,q)$;
(2) $M^{p,q}(\mathbf{R}^n) \hookrightarrow B_s^{p,q}(\mathbf{R}^n)$ *if and only if* $s \leqslant n\nu_2(p,q)$.

As for the inclusion relation between L^p-Sobolev spaces and modulation spaces, the following result (see also [27]) is immediately obtained from Theorem 1.1 if we notice the inclusion property $L^p_{s+\varepsilon} \hookrightarrow B_s^{p,q} \hookrightarrow L^p_{s-\varepsilon}$ for $\varepsilon > 0$ (see, [29, p. 97]):

Corollary 1.2. *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s \in \mathbf{R}$. *Then we have*

(1) $L^p_s(\mathbf{R}^n) \hookrightarrow M^{p,q}(\mathbf{R}^n)$ *if* $s > n\nu_1(p,q)$. *Conversely, if* $L^p_s(\mathbf{R}^n) \hookrightarrow M^{p,q}(\mathbf{R}^n)$, *then* $s \geqslant n\nu_1(p,q)$;
(2) $M^{p,q}(\mathbf{R}^n) \hookrightarrow L^p_s(\mathbf{R}^n)$ *if* $s < n\nu_2(p,q)$. *Conversely, if* $M^{p,q}(\mathbf{R}^n) \hookrightarrow L^p_s(\mathbf{R}^n)$, *then* $s \leqslant n\nu_2(p,q)$.

But in Corollary 1.2, there still remains a question whether the critical case $s = n\nu_1(p,q)$ or $s = n\nu_2(p,q)$ is sufficient or not for the inclusion. The objective of this paper is to answer this basic question and complete the picture of inclusion relations between the L^p-Sobolev spaces and the modulation spaces. The following theorems are our main results:

Theorem 1.3. *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s \in \mathbf{R}$. *Then* $L^p_s(\mathbf{R}^n) \hookrightarrow M^{p,q}(\mathbf{R}^n)$ *if and only if one of the following conditions is satisfied*:

(1) $q \geqslant p > 1$ *and* $s \geqslant n\nu_1(p,q)$;
(2) $p > q$ *and* $s > n\nu_1(p,q)$;
(3) $p = 1$, $q = \infty$, *and* $s \geqslant n\nu_1(1,\infty)$;
(4) $p = 1$, $q \neq \infty$ *and* $s > n\nu_1(1,q)$.

Theorem 1.4. *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s \in \mathbf{R}$. *Then* $M^{p,q}(\mathbf{R}^n) \hookrightarrow L^p_s(\mathbf{R}^n)$ *if and only if one of the following conditions is satisfied*:

(1) $q \leqslant p < \infty$ *and* $s \leqslant n\nu_2(p,q)$;
(2) $p < q$ *and* $s < n\nu_2(p,q)$;
(3) $p = \infty$, $q = 1$, *and* $s \leqslant n\nu_2(\infty,1)$;
(4) $p = \infty$, $q \neq 1$, *and* $s < n\nu_2(\infty,q)$.

It should be mentioned that Kobayashi, Miyachi and Tomita [14] determines the inclusion relation between modulation spaces $M^{p,q}_s$ and local Hardy spaces h^p for $0 < p \leqslant 1$. Our main results extend this result to the case $p > 1$ since we have $h^p = L^p$ then. As a matter of fact, the proof of Theorems 1.3 and 1.4 heavily depends on the results and arguments established in [14].

As an application of our main theorems, we also consider mapping properties of unimodular Fourier multiplier $e^{i|D|^\alpha}$, $\alpha \geqslant 0$, which is a generalization of wave ($\alpha = 1$) and Schrödinger ($\alpha = 2$) propagators. See Corollaries 5.2 and 5.4 in Section 5. As Theorem A and Theorem B there say, the operator $e^{i|D|^\alpha}$ ($0 \leqslant \alpha \leqslant 2$) is bounded on modulation spaces while not on L^p-Sobolev spaces. Theorems 1.3 and 1.4 help us to understand what happen if we consider the operator between L^p-Sobolev spaces and modulation spaces.

We explain the organization of this paper. After the next preliminary section devoted to the definitions and basic properties of function spaces treated in this paper, we give a proof of Theorem 1.4 in Sections 3 and 4. We remark that Theorem 1.3 is just the dual statement of Theorem 1.4. In Section 5, we consider mapping properties of unimodular Fourier multipliers between L^p-Sobolev spaces and modulation spaces, as well as those of invertible pseudo-differential operators.

2. Preliminaries

2.1. Basic notation

The following notation will be used throughout this article. We write $\mathcal{S}(\mathbf{R}^n)$ to denote the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^n$ and $\mathcal{S}'(\mathbf{R}^n)$ to denote the space of tempered distributions on $\mathbf{R}^n$, i.e., the topological dual of $\mathcal{S}(\mathbf{R}^n)$. The Fourier transform is defined by $\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-ix\cdot\xi}\,dx$ and the inverse Fourier transform by $f^\vee(x) = (2\pi)^{-n}\hat{f}(-x)$. We define

$$\|f\|_{L^p} = \left(\int_{\mathbf{R}^n} |f(x)|^p\,dx\right)^{1/p}$$

for $1 \leqslant p < \infty$ and $\|f\|_{L^\infty} = \operatorname{ess.sup}_{x\in\mathbf{R}^n}|f(x)|$. We also define the L^p-Sobolev norm $\|\cdot\|_{L^p_s}$ by

$$\|f\|_{L^p_s} = \left\|\big(\langle\cdot\rangle^s \hat{f}(\cdot)\big)^\vee\right\|_{L^p} \quad \text{with } \langle\cdot\rangle = \big(1+|\cdot|^2\big)^{1/2}$$

following the notation of Sogge [20] and Stein [21]. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, which include $\mathcal{S}(\mathbf{R}^n)$, respectively. We say that an operator T from X to Y is bounded if there exists a constant $C > 0$ such that $\|Tf\|_Y \leqslant C\|f\|_X$ for all $f \in \mathcal{S}(\mathbf{R}^n)$, and we set

$$\|T\|_{X\to Y} = \sup\big\{\|Tf\|_Y \;\big|\; f \in \mathcal{S}\big(\mathbf{R}^n\big),\ \|f\|_X = 1\big\}.$$

We use the notation $I \lesssim J$ if I is bounded by a constant times J, and we denote $I \approx J$ if $I \lesssim J$ and $J \lesssim I$.

2.2. Modulation spaces

We recall the modulation spaces. Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbf{R}$ and $\varphi \in \mathcal{S}(\mathbf{R}^n)$ be such that

$$\operatorname{supp}\varphi \subset [-1,1]^n \quad \text{and} \quad \sum_{k\in\mathbf{Z}^n} \varphi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbf{R}^n. \tag{1}$$

Then the modulation space $M^{p,q}_s(\mathbf{R}^n)$ consists of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^n)$ such that the norm

$$\|f\|_{M^{p,q}_s} = \left(\sum_{k\in\mathbf{Z}^n} \langle k\rangle^{sq}\left(\int_{\mathbf{R}^n} \big|\varphi(D-k)f(x)\big|^p\,dx\right)^{q/p}\right)^{1/q}$$

is finite, with obvious modifications if p or $q = \infty$. Here we denote $\varphi(D-k)f(x) = (\varphi(\cdot - k)\hat{f}(\cdot))^\vee(x)$.

We simply write $M^{p,q}(\mathbf{R}^n)$ instead of $M^{p,q}_0(\mathbf{R}^n)$. The space $M^{p,q}_s(\mathbf{R}^n)$ is a Banach space which is independent of the choice of $\varphi \in \mathcal{S}(\mathbf{R}^n)$ satisfying (1) [6, Theorem 6.1]. If $1 \leqslant p, q < \infty$, then $\mathcal{S}(\mathbf{R}^n)$ is dense in $M^{p,q}_s(\mathbf{R}^n)$ [6, Theorem 6.1]. If $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$, $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$ and $s_1 \geqslant s_2$ then $M^{p_1,q_1}_{s_1}(\mathbf{R}^n) \hookrightarrow M^{p_2,q_2}_{s_2}(\mathbf{R}^n)$ [6, Proposition 6.5]. Let us define by $\mathcal{M}^{p,q}_s(\mathbf{R}^n)$ the completion of $\mathcal{S}(\mathbf{R}^n)$ under the norm $\|\cdot\|_{M^{p,q}_s}$. If $1 \leqslant p, q < \infty$, then

$\mathcal{M}_s^{p,q}(\mathbf{R}^n) = M_s^{p,q}(\mathbf{R}^n)$ [1, Lemma 2.2] and the dual of $\mathcal{M}_s^{p,q}(\mathbf{R}^n)$ can be identified with $M_{-s}^{p',q'}(\mathbf{R}^n)$, where $1/p + 1/p' = 1 = 1/q + 1/q'$. Moreover, the complex interpolation theory for these spaces reads as follows: Let $0 < \theta < 1$ and $1 \leqslant p_1, p_2, q_1, q_2 \leqslant \infty, s_1, s_2 \in \mathbf{R}$. Set $1/p = (1-\theta)/p_1 + \theta/p_2$, $1/q = (1-\theta)/q_1 + \theta/q_2$ and $s = (1-\theta)s_1 + \theta s_2$, then $(\mathcal{M}_{s_1}^{p_1,q_1}, \mathcal{M}_{s_2}^{p_2,q_2})_{[\theta]} = \mathcal{M}_s^{p,q}$ ([6, Theorem 6.1], [30, Theorem 2.3]).

We recall the following lemmas.

Lemma 2.1. *(See [6, Proposition 6.7].) Let* $1 \leqslant p \leqslant \infty$, $1/p + 1/p' = 1$ *and* $s \in \mathbf{R}$. *Then*

$$M_s^{p,\min(p,p')}(\mathbf{R}^n) \hookrightarrow L_s^p(\mathbf{R}^n) \hookrightarrow M_s^{p,\max(p,p')}(\mathbf{R}^n).$$

Let $U_\lambda : f(x) \mapsto f(\lambda x)$ be the dilation operator. Then the following dilation property of $M^{p,q}$ is known.

Lemma 2.2. *(See [22, Theorem 3.1].) Let* $1 \leqslant p, q \leqslant \infty$. *We have, for* $C_1, C_2 > 0$,

$$\|U_\lambda f\|_{M^{p,q}} \leqslant C_1 \lambda^{n\mu_1(p,q)} \|f\|_{M^{p,q}}, \quad \forall \lambda \geqslant 1,\ \forall f \in M^{p,q}(\mathbf{R}^n),$$
$$\|U_\lambda f\|_{M^{p,q}} \geqslant C_2 \lambda^{n\mu_2(p,q)} \|f\|_{M^{p,q}}, \quad \forall \lambda \geqslant 1,\ \forall f \in M^{p,q}(\mathbf{R}^n),$$

where

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1^*\text{: } \min(1/p, 1/p') \geqslant 1/q, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^*\text{: } \min(1/q, 1/2) \geqslant 1/p', \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3^*\text{: } \min(1/q, 1/2) \geqslant 1/p, \end{cases}$$

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1\text{: } \max(1/p, 1/p') \leqslant 1/q, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2\text{: } \max(1/q, 1/2) \leqslant 1/p', \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3\text{: } \max(1/q, 1/2) \leqslant 1/p. \end{cases}$$

Let $I_{s_0} : f \mapsto (\langle\cdot\rangle^{s_0} \hat{f}(\cdot))^\vee$, $s_0 \in \mathbf{R}$. Then following lifting property of $M_s^{p,q}$ is known.

Lemma 2.3. *(See [25].) Let* $1 \leqslant p, q \leqslant \infty$, $s \in \mathbf{R}$. *Then* I_{s_0} *maps* $M_s^{p,q}(\mathbf{R}^n)$ *isomorphically onto* $M_{s-s_0}^{p,q}(\mathbf{R}^n)$.

2.3. Besov spaces

We recall the Besov spaces. Let $1 \leqslant p, q \leqslant \infty$ and $s \in \mathbf{R}$. Suppose that $\psi_0, \psi \in \mathcal{S}(\mathbf{R}^n)$ satisfy $\operatorname{supp} \psi_0 \subset \{\xi \mid |\xi| \leqslant 2\}$, $\operatorname{supp} \psi \subset \{\xi \mid 1/2 \leqslant |\xi| \leqslant 2\}$ and $\psi_0(\xi) + \sum_{j=1}^\infty \psi(\xi/2^j) = 1$ for all $\xi \in \mathbf{R}^n$. Set $\psi_j(\cdot) = \psi(\cdot/2^j)$ if $j \geqslant 1$. Then the Besov space $B_s^{p,q}(\mathbf{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{B_s^{p,q}} = \left(\sum_{j=0}^\infty 2^{jsq} \left\| \left(\hat{f}(\cdot)\psi_j(\cdot) \right)^\vee \right\|_{L^p}^q \right)^{1/q} < \infty,$$

with usual modification again if $q = \infty$.

If $1 \leqslant p, q < \infty$, then the dual of $B_s^{p,q}(\mathbf{R}^n)$ can be identified with $B_{-s}^{p',q'}(\mathbf{R}^n)$, where $1/p + 1/p' = 1 = 1/q + 1/q'$.

2.4. Local Hardy spaces

We recall the local Hardy spaces. Let $0 < p < \infty$, and let $\Psi \in \mathcal{S}(\mathbf{R}^n)$ be such that $\int_{\mathbf{R}^n} \Psi(x)\,dx \neq 0$. Then the local Hardy space $h^p(\mathbf{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbf{R}^n)$ such that

$$\|f\|_{h^p} = \Big\| \sup_{0<t<1} |\Psi_t * f| \Big\|_{L^p} < \infty,$$

where $\Psi_t(x) = t^{-n}\Psi(x/t)$. We remark that $h^1(\mathbf{R}^n) \hookrightarrow L^1(\mathbf{R}^n)$ [7, Theorem 2], $h^p(\mathbf{R}^n) = L^p(\mathbf{R}^n)$ if $1 < p < \infty$ [7, p. 30], and the definition of $h^p(\mathbf{R}^n)$ is independent of the choice of $\Psi \in \mathcal{S}(\mathbf{R}^n)$ with $\int_{\mathbf{R}^n} \Psi(x)\,dx \neq 0$ [7, Theorem 1]. The complex interpolation theory for these spaces reads as follows: Let $1 \leqslant p_1, p_2 < \infty$ and $0 < \theta < 1$. Set $1/p = (1-\theta)/p_1 + \theta/p_2$, then $(h^{p_1}, h^{p_2})_{[\theta]} = h^p$ [29, p. 45].

Lemma 2.4. *(See [14].) Let $1 \leqslant q \leqslant \infty$ and $s \in \mathbf{R}$. Then $h^1(\mathbf{R}^n) \hookrightarrow M_s^{1,q}(\mathbf{R}^n)$ if and only if $s \leqslant -n/q$. However, in the case $q \neq \infty$, $L^1(\mathbf{R}^n) \hookrightarrow M_s^{1,q}(\mathbf{R}^n)$ only if $s < -n/q$.*

3. Sufficient conditions

We prove the *if* part of Theorem 1.4. First we remark the following fact:

Lemma 3.1. *Let $1 < p \leqslant 2$, $p \leqslant q \leqslant p'$ and $s \leqslant -n(1/p + 1/q - 1)$. Then $L^p(\mathbf{R}^n) \hookrightarrow M_s^{p,q}(\mathbf{R}^n)$.*

Proof of Lemma 3.1. We note that $L^2(\mathbf{R}^n) = M^{2,2}(\mathbf{R}^n)$ and, by Lemma 2.4,

$$h^1(\mathbf{R}^n) \hookrightarrow M_{-n/q}^{1,q}(\mathbf{R}^n)$$

for $1 \leqslant q \leqslant \infty$. The complex interpolation method yields

$$L^p(\mathbf{R}^n) \hookrightarrow M_{-n(1/p+1/q-1)}^{p,q}(\mathbf{R}^n),$$

which gives the desired result. □

Proof of Theorem 1.4 ("if" part). Suppose $q \leqslant p$ and $s \leqslant n\nu_2(p,q)$. If $q \leqslant \min(p,p')$, then $s \leqslant n\nu_2(p,q) = 0$ and we have

$$M^{p,q}(\mathbf{R}^n) \hookrightarrow M_s^{p,\min(p,p')}(\mathbf{R}^n) \hookrightarrow L_s^p(\mathbf{R}^n)$$

by Lemma 2.1. If $2 < p < \infty$ and $p' \leqslant q \leqslant p$, then $s \leqslant n\nu_2(p,q) = n(1/p + 1/q - 1)$ and we have $L^{p'}(\mathbf{R}^n) \hookrightarrow M_s^{p',q'}(\mathbf{R}^n)$, since p', q', s satisfy the conditions of Lemma 3.1. Hence we have $M_{-s}^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$ by duality and $M^{p,q}(\mathbf{R}^n) \hookrightarrow L_s^p(\mathbf{R}^n)$ by the lifting properties of modulation spaces (Lemma 2.3) and L^p-Sobolev spaces (trivial by definition). Thus we have the sufficiency of conditions (1) and (3). Conditions (2) and (4) are sufficient by Corollary 1.2. □

4. Necessary conditions

We prove the *only if* part of Theorem 1.4. For the purpose, we prepare Lemmas 4.1–4.4 whose proofs are repetitions of arguments in [14]:

Lemma 4.1. *Let* $1 \leqslant p, q \leqslant \infty$, $p < q$ *and* $s \in \mathbf{R}$. *If* $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$, *then* $s > n(1/p - 1/q)$.

Lemma 4.2. *Let* $1 \leqslant p, q \leqslant \infty$ *and* $s \in \mathbf{R}$. *If* $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$, *then*

$$\left\| \{c_k\} \right\|_{\ell^p} \lesssim \left\| \left\{ \left(1 + |k|\right)^s c_k \right\} \right\|_{\ell^q}$$

for all finitely supported sequences $\{c_k\}_{k \in \mathbf{Z}^n}$ *(that is,* $c_k = 0$ *except for a finite number of* k*'s).*

Proof of Lemma 4.2. Let $\eta \in \mathcal{S}(\mathbf{R}^n) \setminus \{0\}$ be such that $\operatorname{supp} \eta \subset [-1/2, 1/2]^n$. For a finitely supported sequence $\{c_\ell\}_{\ell \in \mathbf{Z}^n}$, we set

$$f(x) = \sum_{\ell \in \mathbf{Z}^n} c_\ell e^{i\ell \cdot x} \eta(x - \ell).$$

Let $\varphi \in \mathcal{S}(\mathbf{R}^n)$ be satisfying (1). Since

$$\hat{f}(\xi) = \sum_{\ell \in \mathbf{Z}^n} c_\ell e^{i|\ell|^2} e^{-i\ell \cdot \xi} \hat{\eta}(\xi - \ell),$$

we see that

$$\varphi(D - k) f(x) = \frac{1}{(2\pi)^n} \sum_{\ell \in \mathbf{Z}^n} c_\ell e^{i|\ell|^2} \int_{\mathbf{R}^n} e^{i(x-\ell)\cdot\xi} \varphi(\xi - k) \hat{\eta}(\xi - \ell)\, d\xi. \tag{2}$$

Using

$$\int_{\mathbf{R}^n} \left(1 + |x - y|\right)^{-M} \left(1 + |y|\right)^{-M} dy \lesssim \left(1 + |x|\right)^{-M},$$

where $M > n$, and

$$\begin{aligned} &(x - \ell)^\alpha \int_{\mathbf{R}^n} e^{i(x-\ell)\cdot\xi} \varphi(\xi - k) \hat{\eta}(\xi - \ell)\, d\xi \\ &\quad = \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2} \int_{\mathbf{R}^n} e^{i(x-\ell)\cdot\xi} \left(\partial^{\alpha_1} \varphi\right)(\xi - k) \left(\partial^{\alpha_2} \hat{\eta}\right)(\xi - \ell)\, d\xi, \end{aligned}$$

we have

$$\left| \int_{\mathbf{R}^n} e^{i(x-\ell)\cdot\xi} \varphi(\xi - k) \hat{\eta}(\xi - \ell)\, d\xi \right| \leqslant C_N \left(1 + |x - \ell|\right)^{-N} \left(1 + |k - \ell|\right)^{-N} \tag{3}$$

for all $N \geqslant 1$. Let N be a sufficiently large integer. Then, by (2) and (3),

$$\left|\varphi(D-k)f(x)\right| \lesssim \sum_{\ell\in\mathbf{Z}^n} \frac{|c_\ell|}{(1+|x-\ell|)^N(1+|k-\ell|)^N},$$

which provides

$$\begin{aligned}\left\|\varphi(D-k)f\right\|_{L^p} &\lesssim \sum_{\ell\in\mathbf{Z}^n} \frac{|c_\ell|}{(1+|k-\ell|)^N} \left\|\left(1+|\cdot-\ell|\right)^{-N}\right\|_{L^p} \\ &\approx \sum_{\ell\in\mathbf{Z}^n} \frac{|c_\ell|}{(1+|k-\ell|)^N}.\end{aligned}$$

Then, since

$$\begin{aligned}\|f\|_{M_s^{p,q}} &= \left\|\left\{\left(1+|k|\right)^s \left\|\varphi(D-k)f\right\|_{L^p}\right\}\right\|_{\ell^q} \\ &\lesssim \left\|\left\{\left(1+|k|\right)^s \sum_{\ell\in\mathbf{Z}^n} \frac{|c_\ell|}{(1+|k-\ell|)^N}\right\}\right\|_{\ell^q} \\ &\lesssim \left\|\left\{\sum_{\ell\in\mathbf{Z}^n} \frac{(1+|\ell|)^s|c_\ell|}{(1+|k-\ell|)^{N-|s|}}\right\}\right\|_{\ell^q},\end{aligned}$$

we have by Young's inequality

$$\|f\|_{M_s^{p,q}} \lesssim \left\|\left\{\left(1+|\ell|\right)^s c_\ell\right\}\right\|_{\ell^q}. \tag{4}$$

On the other hand, since $\operatorname{supp}\eta(\cdot-\ell) \subset \ell+[-1/2,1/2]^n$ for all $\ell\in\mathbf{R}^n$, we see that

$$\begin{aligned}\|f\|_{L^p} &= \left(\int_{\mathbf{R}^n}\left|\sum_{\ell\in\mathbf{Z}^n} c_\ell e^{i\ell\cdot x}\eta(x-\ell)\right|^p dx\right)^{1/p} \\ &= \left(\int_{\mathbf{R}^n}\sum_{\ell\in\mathbf{Z}^n}\left|c_\ell e^{i\ell\cdot x}\eta(x-\ell)\right|^p dx\right)^{1/p} = \|\eta\|_{L^p}\left\|\{c_\ell\}\right\|_{\ell^p}\end{aligned} \tag{5}$$

for $p\neq\infty$. We have easily the same conclusion for $p=\infty$. By our assumption $M_s^{p,q}(\mathbf{R}^n)\hookrightarrow L^p(\mathbf{R}^n)$ and (4)–(5), we have

$$\left\|\{c_\ell\}\right\|_{\ell^p} \lesssim \|f\|_{L^p} \lesssim \|f\|_{M_s^{p,q}} \lesssim \left\|\left\{\left(1+|\ell|\right)^s c_\ell\right\}\right\|_{\ell^q}.$$

The proof is complete. □

Proof of Lemma 4.1. Suppose $M_s^{p,q}(\mathbf{R}^n)\hookrightarrow L^p(\mathbf{R}^n)$. By Lemma 4.2, we have

$$\left(\sum_{k\in\mathbf{Z}^n}|c_k|^p\right)^{1/p} \lesssim \left\|\left\{\left(1+|k|\right)^s c_k\right\}\right\|_{\ell^q}$$

for all finitely supported sequences $\{c_k\}_{k\in\mathbf{Z}^n}$. Setting $c_k = (1+|k|)^{-s}|d_k|^{1/p}$, we see that it is equivalent to

$$\sum_{k\in\mathbf{Z}^n} \bigl(1+|k|\bigr)^{-sp}|d_k| \lesssim \bigl\|\{d_k\}\bigr\|_{\ell^{q/p}}$$

for all finitely supported sequences $\{d_k\}_{k\in\mathbf{Z}^n}$. Hence we have

$$\bigl\|\bigl\{\bigl(1+|k|\bigr)^{-sp}\bigr\}\bigr\|_{\ell^{(q/p)'}} = \sup\left|\sum_{k\in\mathbf{Z}^n}\bigl(1+|k|\bigr)^{-sp}d_k\right| \lesssim 1,$$

where the supremum is taken over all finitely supported sequences $\{d_k\}_{k\in\mathbf{Z}^n}$ such that $\|\{d_k\}\|_{\ell^{q/p}} = 1$. Note here that $(q/p)' < \infty$ from the assumption $p < q$. Hence p, q, s must satisfy $sp(q/p)' > n$, that is, $s > n(1/p - 1/q)$. $\square$

Lemma 4.3. *Let* $1 \leqslant q < p < \infty$ *and* $s \in \mathbf{R}$. *If* $L^p(\mathbf{R}^n) \hookrightarrow M_s^{p,q}(\mathbf{R}^n)$, *then* $s < -n(1/p + 1/q - 1)$.

Lemma 4.4. *Let* $1 \leqslant p, q < \infty$ *and* $s \in \mathbf{R}$. *If* $L^p(\mathbf{R}^n) \hookrightarrow M_s^{p,q}(\mathbf{R}^n)$, *then*

$$\left\{\sum_{k\neq 0}|k|^{(n(1/p-1)+s)q}\left(\sum_{|k|/2\leqslant|\ell|\leqslant 2|k|}|c_\ell|^p\right)^{q/p}\right\}^{1/q} \lesssim \left(\sum_{k\neq 0}|c_k|^p\right)^{1/p}$$

for all finitely supported sequences $\{c_k\}_{k\in\mathbf{Z}^n\setminus\{0\}}$.

Proof of Lemma 4.4.. Let $0 < \delta < 1$ and $a \in \mathcal{S}(\mathbf{R}^n)$ be such that

$$\operatorname{supp} a \subset [-\delta/8, \delta/8]^n, \quad \|a\|_{L^\infty} \leqslant 1, \quad \text{and} \quad \bigl|\hat{a}(\xi)\bigr| \geqslant C > 0 \quad \text{on } |\xi| \leqslant 2$$

(see, for example, [14, Lemma 4.3]). For a finitely supported sequence $\{c_\ell\}_{\ell\in\mathbf{Z}^n\setminus\{0\}}$, we define $f \in \mathcal{S}(\mathbf{R}^n)$ by

$$f(x) = \sum_{\ell\neq 0} c_\ell|\ell|^{n/p}a\bigl(|\ell|(x-\ell)\bigr).$$

We first estimate $\|f\|_{L^p}$. Since

$$\operatorname{supp} a\bigl(|\ell|(\cdot-\ell)\bigr) \subset \ell + \bigl[-\delta/\bigl(8|\ell|\bigr), \delta/\bigl(8|\ell|\bigr)\bigr]^n,$$

we have

$$\|f\|_{L^p}^p = \int_{\mathbf{R}^n}\left|\sum_{\ell\neq 0} c_\ell|\ell|^{n/p}a\bigl(|\ell|(x-\ell)\bigr)\right|^p dx$$
$$= \int_{\mathbf{R}^n}\sum_{\ell\neq 0}|c_\ell|^p|\ell|^n\bigl|a\bigl(|\ell|(x-\ell)\bigr)\bigr|^p dx = \|a\|_{L^p}^p\sum_{\ell\neq 0}|c_\ell|^p.$$

Next, we estimate $\|f\|_{M_s^{p,q}}$. We note the following facts:

Fact 1. Let $\Psi \in \mathcal{S}(\mathbf{R}^n)$ be such that $\Psi = 1$ on $[-\delta/4, \delta/4]^n$, $\operatorname{supp}\Psi \subset [-3\delta/8, 3\delta/8]^n$, and $|\widehat{\Psi}| \geqslant C > 0$ on $[-2, 2]^n$. Then we have

$$\|f\|_{M_s^{p,q}} \approx \left(\sum_{k\in\mathbf{Z}^n} \big(1+|k|\big)^{sq} \big\| f * (M_k\Psi)\big\|_{L^p}^q \right)^{1/q},$$

where $M_k\Psi(x) = e^{ik\cdot x}\Psi(x)$.

Fact 2. For all $\ell \neq 0$, we have

$$\operatorname{supp} a\big(|\ell|(\cdot - \ell)\big) \subset \ell + \big[-\delta/8|\ell|, \delta/8|\ell|\big]^n \subset \ell + [-\delta/8, \delta/8]^n.$$

Fact 3. For all $x \in m + [-\delta/8, \delta/8]^n$, $m \in \mathbf{Z}^n$, we have

$$\operatorname{supp}\Psi(x - \cdot) \subset x + [-3\delta/8, 3\delta/8]^n \subset m + [-\delta/2, \delta/2]^n.$$

From these facts, we have

$$\begin{aligned}
&\big\|(M_k\Psi) * f\big\|_{L^p}^p \\
&\quad\geqslant \sum_{m\in\mathbf{Z}^n} \int_{\Delta(m,\delta)} \big|(M_k\Psi) * f(x)\big|^p\,dx \\
&\quad= \sum_{m\in\mathbf{Z}^n} \int_{\Delta(m,\delta)} \left| \int_{\mathbf{R}^n} e^{ik\cdot(x-y)}\Psi(x-y) \sum_{\ell\neq 0} c_\ell |\ell|^{\frac{n}{p}} a\big(|\ell|(y-\ell)\big)\,dy \right|^p dx \\
&\quad= \sum_{m\neq 0} \int_{\Delta(m,\delta)} \left| \int_{\mathbf{R}^n} e^{-ik\cdot y}\Psi(x-y) c_m |m|^{\frac{n}{p}} a\big(|m|(y-m)\big)\,dy \right|^p dx,
\end{aligned}$$

where we set $\Delta(m,\delta) = m + [-\delta/8, \delta/8]^n$. If $x \in m + [-\delta/8, \delta/8]^n$ and $y \in \operatorname{supp} a(|m|(\cdot - m))$, then

$$x - y \in \big(m + [-\delta/8, \delta/8]^n\big) - \big(m + [-\delta/8, \delta/8]^n\big) = [-\delta/4, \delta/4]^n,$$

and so $\Psi(x-y) = 1$. Hence,

$$\begin{aligned}
&\big\|(M_k\Psi) * f\big\|_{L^p}^p \\
&\quad\geqslant \sum_{m\neq 0} \int_{\Delta(m,\delta)} \left| \int_{\mathbf{R}^n} e^{-ik\cdot y}\Psi(x-y) c_m |m|^{\frac{n}{p}} a\big(|m|(y-m)\big)\,dy \right|^p dx \\
&\quad= \sum_{m\neq 0} \int_{\Delta(m,\delta)} \left| \int_{\mathbf{R}^n} e^{-ik\cdot y} c_m |m|^{\frac{n}{p}} a\big(|m|(y-m)\big)\,dy \right|^p dx \\
&\quad= \left(\frac{\delta}{4}\right)^n \sum_{m\neq 0} |c_m|^p |m|^{n-pn} \left| \hat{a}\left(\frac{k}{|m|}\right)\right|^p.
\end{aligned}$$

Moreover, using $|\hat{a}(\xi)| \geqslant C > 0$ for all $1/2 \leqslant |\xi| \leqslant 2$, we obtain

$$\begin{aligned}\big\|(M_k\Psi) * f\big\|_{L^p}^p &\geqslant (\delta/4)^n \sum_{m\neq 0} |c_m|^p |m|^{n-pn} \big|\hat{a}\big(k/|m|\big)\big|^p \\ &\geqslant (\delta/4)^n \sum_{|k|/2\leqslant|m|\leqslant 2|k|} |c_m|^p |m|^{n-pn} \big|\hat{a}\big(k/|m|\big)\big|^p \\ &\gtrsim \sum_{|k|/2\leqslant|m|\leqslant 2|k|} |c_m|^p |m|^{n-pn} \gtrsim |k|^{n-pn} \sum_{|k|/2\leqslant|m|\leqslant 2|k|} |c_m|^p\end{aligned}$$

for all $k \neq 0$. Then

$$\begin{aligned}\|f\|_{M_s^{p,q}} &\approx \left(\sum_{k\in\mathbf{Z}^n} \big(1+|k|\big)^{sq} \big\|(M_k\Psi) * f\big\|_{L^p}^q \right)^{1/q} \\ &\gtrsim \left\{ \sum_{k\neq 0} \big(1+|k|\big)^{sq} \left(|k|^{n-pn} \sum_{|k|/2\leqslant|m|\leqslant 2|k|} |c_m|^p \right)^{q/p} \right\}^{1/q} \\ &\gtrsim \left\{ \sum_{k\neq 0} |k|^{(n(1/p-1)+s)q} \left(\sum_{|k|/2\leqslant|m|\leqslant 2|k|} |c_m|^p \right)^{q/p} \right\}^{1/q}.\end{aligned}$$

Therefore, by our assumption $L^p(\mathbf{R}^n) \hookrightarrow M_s^{p,q}(\mathbf{R}^n)$, we have

$$\begin{aligned}&\left\{ \sum_{k\neq 0} |k|^{(n(1/p-1)+s)q} \left(\sum_{|k|/2\leqslant|m|\leqslant 2|k|} |c_m|^p \right)^{q/p} \right\}^{1/q} \\ &\quad\lesssim \|f\|_{M_s^{p,q}} \lesssim \|f\|_{L^p} \lesssim \left(\sum_{\ell\neq 0} |c_\ell|^p \right)^{1/p}. \qquad \square\end{aligned}$$

Proof of Lemma 4.3. Suppose that $s \geqslant -n(1/p + 1/q - 1)$ contrary to our claim. Noting that $q/p < 1$ from the assumption $q < p$, take $\varepsilon > 0$ such that $(1+\varepsilon)q/p < 1$ and define $\{c_k\}_{k\in\mathbf{Z}^n\setminus\{0\}}$ by

$$c_k = \begin{cases} |k|^{-n/p}(\log|k|)^{-(1+\varepsilon)/p} & \text{if } |k| \geqslant N, \\ 0 & \text{if } |k| < N, \end{cases}$$

where N is sufficiently large. Note also that $\{|k|^{-n/r}(\log|k|)^{-\alpha/r}\}_{|k|\geqslant N} \in \ell^r$ if $\alpha > 1$, and $\{|k|^{-n/r}(\log|k|)^{-\alpha/r}\}_{|k|\geqslant N} \notin \ell^r$ if $\alpha \leqslant 1$, where $r < \infty$ (see, for example, [23, Remark 4.3]). Thus

$$\left(\sum_{k\neq 0} |c_k|^p \right)^{1/p} = \left\{ \sum_{|k|\geqslant N} \big(|k|^{-n/p}\big(\log|k|\big)^{-(1+\varepsilon)/p}\big)^p \right\}^{1/p} < \infty.$$

On the other hand, since $n(1/p-1)+s \geqslant -n/q$ and $(1+\varepsilon)q/p < 1$, we see that

$$\left\{\sum_{k\neq 0}|k|^{(n(1/p-1)+s)q}\left(\sum_{|k|/2\leqslant|\ell|\leqslant 2|k|}|c_\ell|^p\right)^{q/p}\right\}^{1/q}$$
$$\geqslant\left\{\sum_{|k|\geqslant 2N}|k|^{(n(1/p-1)+s)q}\left(\sum_{|k|/2\leqslant|\ell|\leqslant 2|k|}\left(|\ell|^{-n/p}\big(\log|\ell|\big)^{-(1+\varepsilon)/p}\right)^p\right)^{q/p}\right\}^{1/q}$$
$$\gtrsim\left\{\sum_{|k|\geqslant 2N}|k|^{(n(1/p-1)+s)q}\big(\log|k|\big)^{-(1+\varepsilon)q/p}\right\}^{1/q}$$
$$\gtrsim\left\{\sum_{|k|\geqslant 2N}\left(|k|^{-n/q}\big(\log|k|\big)^{-\{(1+\varepsilon)q/p\}/q}\right)^q\right\}^{1/q}=\infty.$$

This contradicts Lemma 4.4. □

Proof of Theorem 1.4 ("only if" part). Suppose $M^{p,q}(\mathbf{R}^n)\hookrightarrow L^p_s(\mathbf{R}^n)$. Then we have $s\leqslant n\nu_2(p,q)$ by Corollary 1.2. Particularly in the case $p<q$, we have $s<-n(1/p-1/q)=n\nu_2(p,q)$ for $p\leqslant 2$ by Lemma 4.1, and $s<-n(1/p'+1/q'-1)=n(1/p+1/q-1)=n\nu_2(p,q)$ for $2\leqslant p$ by the dual statement of Lemma 4.3. In the case $p=\infty$, we must have $L^1_{-s}(\mathbf{R}^n)\hookrightarrow M^{1,q'}(\mathbf{R}^n)$, since otherwise the fact $\mathcal{S}(\mathbf{R}^n)$ is dense in both $L^1_{-s}(\mathbf{R}^n)$ and $\mathcal{M}^{1,q'}(\mathbf{R}^n)$ implies that $M^{\infty,q}(\mathbf{R}^n)\not\subset L^\infty_s(\mathbf{R}^n)$, contrary to the assumptions. Hence $L^1(\mathbf{R}^n)\hookrightarrow M^{1,q'}_s(\mathbf{R}^n)$. Then we have $s<-n/q'=n(1/q-1)=n\nu_2(\infty,q)$ for $q\neq 1$ by Lemma 2.4. All of these results yields the necessity of conditions (1)–(4). □

5. Applications

We consider the unimodular Fourier multiplier $e^{i|D|^\alpha}$, $\alpha\geqslant 0$, defined by

$$e^{i|D|^\alpha}f(x)=\int_{\mathbf{R}^n}e^{i|\xi|^\alpha}\hat{f}(\xi)e^{ix\cdot\xi}\,d\xi,\qquad f\in\mathcal{S}\big(\mathbf{R}^n\big).$$

The operator $e^{i|D|^\alpha}$ has an intimate connection with the solution $u(t,x)$ of initial value problem for the dispersive equation

$$\begin{cases} i\partial_t u+|\Delta|^{\alpha/2}u=0,\\ u(0,x)=f(x),\end{cases}$$

$(t,x)\in\mathbf{R}\times\mathbf{R}^n$. The boundedness of $e^{i|D|^\alpha}$ on several function spaces has been studied extensively by many authors. Concerning the L^p-Sobolev spaces L^p_s and the modulation spaces $M^{p,q}_s$, the following theorems are known.

Theorem A. *(See Miyachi [15].) Let $1<p<\infty$, $s\in\mathbf{R}$ and $\alpha>1$. Then $e^{i|D|^\alpha}$ is bounded from $L^p_s(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ if and only if $s\geqslant\alpha n|1/p-1/2|$.*

Theorem B. *(See Bényi, et al. [2].) Let $1\leqslant p,q\leqslant\infty$ and $0\leqslant\alpha\leqslant 2$. Then $e^{i|D|^\alpha}$ is bounded from $M^{p,q}(\mathbf{R}^n)$ to $M^{p,q}(\mathbf{R}^n)$.*

The boundedness of $e^{i|D|^\alpha}$ with $0 \leqslant \alpha \leqslant 2$ on weighted modulation spaces $M_s^{p,q}(\mathbf{R}^n)$ follows from Theorem B and the lifting property of modulation spaces (Lemma 2.3). Indeed, since the operator $T = e^{i|D|^\alpha}$ is translation invariant, it commutes with I_s and we have

$$\|Tf\|_{M_s^{p,q}} \approx \|I_s T f\|_{M^{p,q}} \approx \|T I_s f\|_{M^{p,q}} \lesssim \|I_s f\|_{M^{p,q}} \approx \|f\|_{M_s^{p,q}},$$

which means the boundedness of T on $M_s^{p,q}$.

We remark that Theorem B with $\alpha = 2$ is established in a more general form by [26]. We also remark that Bényi and Okoudjou [3] extends Theorem B to the case $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $0 \leqslant \alpha \leqslant 2$ and the case $n/(n+1) \leqslant p < 1$, $0 < q \leqslant \infty$, $\alpha = 1, 2$. For $\alpha > 2$, we have a different type of boundedness:

Theorem C. *(See Miyachi, et al. [16].) Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbf{R}$ and $\alpha > 2$. Then $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $M^{p,q}(\mathbf{R}^n)$ if and only if $s \geqslant (\alpha - 2)n|1/p - 1/2|$.*

Theorem A says that the operator $e^{i|D|^\alpha}$ is not bounded on $L^p(\mathbf{R}^n)$, and we have generally a loss of regularity of the order up to $\alpha n|1/p - 1/2|$. Theorems B and C describe an advantage of modulation spaces because we have no loss in the case $0 \leqslant \alpha \leqslant 2$ or smaller loss in the case $\alpha > 2$ if we consider the operator $e^{i|D|^\alpha}$ on these spaces.

Then what is the exact order of the loss when we consider the operator $e^{i|D|^\alpha}$ between L^p spaces and modulation spaces. We can answer this question by using our main theorem. The case $0 \leqslant \alpha \leqslant 2$ is rather simple, and we have the following results:

Theorem 5.1. *Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbf{R}$ and $0 \leqslant \alpha \leqslant 2$. Then $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ if and only if $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$.*

Proof. Assume that $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$:

$$\left\|e^{i|D|^\alpha} f\right\|_{L^p} \lesssim \|f\|_{M_s^{p,q}}.$$

Note that $e^{-i|D|^\alpha}$ is also bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$. Then by taking $f = e^{-i|D|^\alpha} g$ we have

$$\|g\|_{L^p} \lesssim \left\|e^{-i|D|^\alpha} g\right\|_{M_s^{p,q}} \lesssim \|g\|_{M_s^{p,q}},$$

which means that $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$ by the equation $\overline{e^{i|D|^\alpha} f} = e^{-i|D|^\alpha} \bar{f}$. Conversely assume that $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$. Since $e^{i|D|^\alpha}$ is bounded on $M_s^{p,q}(\mathbf{R}^n)$ by Theorem B, we have

$$\left\|e^{i|D|^\alpha} f\right\|_{L^p} \lesssim \left\|e^{i|D|^\alpha} f\right\|_{M_s^{p,q}} \lesssim \|f\|_{M_s^{p,q}},$$

which means that $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$. □

The following corollary is straightforwardly obtained from Theorem 5.1 and Theorem 1.4. The second part is just the dual statement of the first part:

Corollary 5.2. *Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbf{R}$ and $0 \leqslant \alpha \leqslant 2$. Then $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ if and only if one of the following conditions is satisfied*:

(1) $q \leqslant p < \infty$ *and* $s \geqslant -n\nu_2(p,q)$;
(2) $p < q$ *and* $s > -n\nu_2(p,q)$;
(3) $p = \infty, q = 1$, *and* $s \geqslant -n\nu_2(\infty, 1)$;
(4) $p = \infty, q \neq 1$, *and* $s > -n\nu_2(\infty, q)$,

and from $L_s^p(\mathbf{R}^n)$ to $M^{p,q}(\mathbf{R}^n)$ if and only if one of the following conditions is satisfied:

(5) $q \geqslant p > 1$ *and* $s \geqslant n\nu_1(p,q)$;
(6) $p > q$ *and* $s > n\nu_1(p,q)$;
(7) $p = 1$, $q = \infty$, *and* $s \geqslant n\nu_1(1,\infty)$;
(8) $p = 1$, $q \neq \infty$ *and* $s > n\nu_1(1,q)$.

For $\alpha > 2$, we have the following results:

Theorem 5.3. *Let $1 \leqslant p,q \leqslant \infty$, $s \in \mathbf{R}$ and $\alpha > 2$. Then $e^{i|D|^\alpha}$ is bounded from $M^{p,q}_{s+(\alpha-2)n|1/p-1/2|}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ if $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$.*

Proof. Assume that $M_s^{p,q}(\mathbf{R}^n) \hookrightarrow L^p(\mathbf{R}^n)$. Since $e^{i|D|^\alpha}$ is bounded from $M^{p,q}_{s+(\alpha-2)n|1/p-1/2|}(\mathbf{R}^n)$ to $M_s^{p,q}(\mathbf{R}^n)$ by Theorem C, we have

$$\big\|e^{i|D|^\alpha} f\big\|_{L^p} \lesssim \big\|e^{i|D|^\alpha} f\big\|_{M_s^{p,q}} \lesssim \|f\|_{M^{p,q}_{s+(\alpha-2)n|1/p-1/2|}},$$

which means that $e^{i|D|^\alpha}$ is bounded from $M^{p,q}_{s+(\alpha-2)n|1/p-1/2|}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$. □

The following corollary is obtained from Theorem 5.3, Theorem 1.4 and the duality argument again:

Corollary 5.4. *Let $1 \leqslant p, q \leqslant \infty$, $s \in \mathbf{R}$ and $\alpha > 2$. Then $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ if one of the following conditions is satisfied*:

(1) $q \leqslant p < \infty$ *and* $s \geqslant -n\nu_2(p,q) + (\alpha-2)n|1/p-1/2|$;
(2) $p < q$ *and* $s > -n\nu_2(p,q) + (\alpha-2)n|1/p-1/2|$;
(3) $p = \infty$, $q = 1$, *and* $s \geqslant -n\nu_2(\infty,1) + (\alpha-2)n|1/p-1/2|$;
(4) $p = \infty$, $q \neq 1$, *and* $s > -n\nu_2(\infty,q) + (\alpha-2)n|1/p-1/2|$,

and from $L_s^p(\mathbf{R}^n)$ to $M^{p,q}(\mathbf{R}^n)$ if one of the following conditions is satisfied:

(5) $q \geqslant p > 1$ *and* $s \geqslant n\nu_1(p,q) + (\alpha-2)n|1/p-1/2|$;
(6) $p > q$ *and* $s > n\nu_1(p,q) + (\alpha-2)n|1/p-1/2|$;
(7) $p = 1$, $q = \infty$, *and* $s \geqslant n\nu_1(1,\infty) + (\alpha-2)n|1/p-1/2|$;
(8) $p = 1$, $q \neq \infty$ *and* $s > n\nu_1(1,q) + (\alpha-2)n|1/p-1/2|$.

Remark 5.5. For pseudo-differential operators

$$\sigma^W(X,D)f(x) = \frac{1}{(2\pi)^n} \iint_{\mathbf{R}^{2n}} \sigma\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\cdot\xi} f(y)\,dy\,d\xi$$

with the symbol $\sigma \in M^{\infty,1}(\mathbf{R}^{2n})$, it is well known that it is bounded on each $M^{p,q}(\mathbf{R}^n)$ [8, Theorem 14.5.2] (see also [18]). Furthermore, this class of pseudo-differential operators is a Wiener algebra, meaning that if in addition $\sigma^W(X,D)$ is invertible on $L^2(\mathbf{R}^n) = M^{2,2}(\mathbf{R}^n)$, then its inverse is of the form $\tau^W(X,D)$ with $\tau \in M^{\infty,1}(\mathbf{R}^{2n})$ (see [9,19]). From these observation it follows if $s \in \mathbf{R}$ and $\sigma \in M^{\infty,1}(\mathbf{R}^{2n})$, then the following is true:

(1) If $M^{p,q}(\mathbf{R}^n) \hookrightarrow L^p_s(\mathbf{R}^n)$, then $\sigma^W(X,D)$ is bounded from $M^{p,q}(\mathbf{R}^n)$ to $L^p_s(\mathbf{R}^n)$.
(2) Suppose that $\sigma^W(X,D)$ is invertible on $L^2(\mathbf{R}^n)$. If $\sigma^W(X,D)$ is bounded from $M^{p,q}(\mathbf{R}^n)$ to $L^p_s(\mathbf{R}^n)$, then $\mathcal{M}^{p,q}(\mathbf{R}^n) \hookrightarrow L^p_s(\mathbf{R}^n)$.

We may now combine (1) and (2) with Theorem 1.4 to obtain similar results to Corollaries 5.2 and 5.4. We omit the details.

Although the converse of Theorem 5.3 is not true, we believe that the converse of Corollary 5.4 is still true. In fact we have at least the following result:

Theorem 5.6. *Let $1 \leqslant p,q \leqslant \infty$, $s \in \mathbf{R}$ and $\alpha > 2$.*

(1) *Suppose that $e^{i|D|^\alpha}$ is bounded from $M^{p,q}_s(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$. Then we have $s \geqslant -n\nu_2(p,q) + (\alpha-2)n|1/p-1/2|$.*
(2) *Suppose that $e^{i|D|^\alpha}$ is bounded from $L^p_s(\mathbf{R}^n)$ to $M^{p,q}(\mathbf{R}^n)$ instead. Then we have $s \geqslant n\nu_1(p,q) + (\alpha-2)n|1/p-1/2|$.*

To prove Theorem 5.6, we use the following lemma.

Lemma 5.7. *Let $1 \leqslant p,q \leqslant \infty$, $s \in \mathbf{R}$ and $\alpha \geqslant 0$. Then*

$$\sup_{k\in\mathbf{Z}^n} \langle k\rangle^{-s} \left\|\varphi(D-k)e^{i|D|^\alpha}\right\|_{L^p\to L^p} \lesssim \left\|e^{i|D|^\alpha}\right\|_{M^{p,q}_s\to L^p}, \tag{6}$$

where φ is a function satisfying (1).

Proof. Let N be a positive integer such that

$$\varphi(\cdot-k) = \sum_{|\ell|\leqslant N} \varphi(\cdot-k)\varphi\big(\cdot-(k+\ell)\big)$$

for all $k \in \mathbf{Z}^n$. Then we have

$$\begin{aligned}
&\left\|\varphi(D-k)e^{i|D|^\alpha} f\right\|_{L^p} \\
&\quad\leqslant \left\|e^{i|D|^\alpha}\right\|_{M^{p,q}_s\to L^p} \left\|\varphi(D-k)f\right\|_{M^{p,q}_s}
\end{aligned}$$

$$= \left\| e^{i|D|^\alpha} \right\|_{M_s^{p,q} \to L^p} \left(\sum_{m \in \mathbf{Z}^n} \langle m \rangle^{sq} \left\| \varphi(D-m)\varphi(D-k) f \right\|_{L^p}^q \right)^{1/q}$$

$$= \left\| e^{i|D|^\alpha} \right\|_{M_s^{p,q} \to L^p} \left(\sum_{|\ell| \leqslant N} \langle k+\ell \rangle^{sq} \left\| \varphi\big(D-(k+\ell)\big)\varphi(D-k) f \right\|_{L^p}^q \right)^{1/q}$$

$$\lesssim \langle k \rangle^s \left\| e^{i|D|^\alpha} \right\|_{M_s^{p,q} \to L^p} \|f\|_{L^p}$$

for $f \in \mathcal{S}(\mathbf{R}^n)$ and $q \neq \infty$. We have easily the same conclusion for $q = \infty$. Hence, we obtain the desired result. □

Remark 5.8. We remark that since

$$\left\| e^{i|D|^\alpha} \right\|_{M_s^{p,\tilde{q}} \to M^{p,\tilde{q}}} \approx \sup_{k \in \mathbf{Z}^n} \langle k \rangle^{-s} \left\| \varphi(D-k) e^{i|D|^\alpha} \right\|_{L^p \to L^p}$$

for $1 \leqslant p, \tilde{q} \leqslant \infty$, $s \in \mathbf{R}$ (see [16, Lemma 2.2]), we have

$$\left\| e^{i|D|^\alpha} \right\|_{M_s^{p,\tilde{q}} \to M^{p,\tilde{q}}} \lesssim \left\| e^{i|D|^\alpha} \right\|_{M_s^{p,q} \to L^p}$$

for all $1 \leqslant p, q, \tilde{q} \leqslant \infty$.

Now, we prove Theorem 5.6.

Proof of Theorem 5.6. Since the latter is just the dual statement of the former, we prove only the former. Suppose that $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$:

$$\left\| e^{i|D|^\alpha} f \right\|_{L^p} \lesssim \|f\|_{M_s^{p,q}}, \quad f \in \mathcal{S}\big(\mathbf{R}^n\big). \tag{7}$$

(i) Let $q \leqslant \min(p, p')$. By the necessary condition of Theorem C and Remark 5.8, we have $s \geqslant (\alpha - 2)n|1/p - 1/2|$. Since $\nu_2(p,q) = 0$, we obtain the desired result.

(ii) Let $1 \leqslant p \leqslant 2$ and $p \leqslant q \leqslant p'$. Note that inequality (7) can be written as

$$\left\| e^{i|D|^\alpha} \langle D \rangle^{-s} f \right\|_{L^p} \lesssim \|f\|_{M^{p,q}}, \quad f \in \mathcal{S}\big(\mathbf{R}^n\big) \tag{8}$$

by the lifting property. Here, we denote $\langle D \rangle^{-s} f = (\langle \cdot \rangle^{-s} \hat{f}(\cdot))^\vee$ for $s \in \mathbf{R}$.

Let $g \in \mathcal{S}(\mathbf{R}^n)$ be such that

$$\operatorname{supp} \hat{g} \subset \left\{ \xi \;\middle|\; 2^{-1} < |\xi| < 2 \right\} \quad \text{and} \quad \hat{g}(\xi) = 1 \quad \text{on } \left\{ \xi \;\middle|\; 2^{-1/2} < |\xi| < 2^{1/2} \right\}, \tag{9}$$

and test (8) with a specific $f = U_\lambda g$, $\lambda \geqslant 1$. Since

$$e^{i|D|^\alpha} \langle D \rangle^{-s} U_\lambda g = U_\lambda \big(e^{i|\lambda D|^\alpha} \langle \lambda D \rangle^{-s} g \big),$$

it follows from Theorem 2.2 that

$$\lambda^{-n/p} \left\| e^{i|\lambda D|^\alpha} \langle \lambda D \rangle^{-s} g \right\|_{L^p} \lesssim \lambda^{n\mu_1(p,q)} \|g\|_{M^{p,q}}.$$

On the other hand, by the change of variable $x \mapsto \lambda^\alpha x$ and the method of stationary phase, we obtain

$$\begin{aligned}\left\| e^{i|\lambda D|^\alpha} \langle \lambda D\rangle^{-s} g \right\|_{L^p} &= \left\| \int_{\mathbf{R}^n} e^{ix\cdot\xi + i|\lambda\xi|^\alpha} \langle \lambda\xi\rangle^{-s} \hat{g}(\xi)\, d\xi \right\|_{L^p} \\ &= \lambda^{\alpha n/p} \left\| \int_{\mathbf{R}^n} e^{i\lambda^\alpha (x\cdot\xi + |\xi|^\alpha)} \langle \lambda\xi\rangle^{-s} \hat{g}(\xi)\, d\xi \right\|_{L^p} \\ &\gtrsim \lambda^{\alpha n/p - \alpha n/2 - s}.\end{aligned}$$

Combining these two estimates, we get

$$\lambda^{n\mu_1(p,q) + n/p - \alpha n/p + \alpha n/2 + s} \gtrsim 1$$

for all $\lambda \geqslant 1$. Letting $\lambda \to \infty$ yields the necessary condition

$$\begin{aligned} s &\geqslant -n(1/q - 1) - n/p + \alpha n/p - \alpha n/2 \\ &= (\alpha - 2)n(1/p - 1/2) + n/p - n/q \\ &= (\alpha - 2)n|1/p - 1/2| - n\nu_2(p,q), \end{aligned}$$

since $\mu_1(p,q) = 1/q - 1$ and $\nu_2(p,q) = -1/p + 1/q$.

(iii) Let $2 \leqslant p \leqslant \infty$ and $p' \leqslant q \leqslant p$. By duality, we have

$$\left\| e^{-i|D|^\alpha} f \right\|_{L^{p'}} \lesssim \|f\|_{M^{p',q'}_{-s}}, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

So, we have only to prove the following lemma.

Lemma 5.9. *Let $1 \leqslant p' \leqslant 2$, $p' \leqslant q' \leqslant p$ and $s \in \mathbf{R}$. If $e^{i|D|^\alpha}$ is bounded from $L^{p'}_s(\mathbf{R}^n)$ to $M^{p',q'}(\mathbf{R}^n)$, then $s \geqslant (\alpha - 2)n|1/p - 1/2| - n\nu_2(p,q)$.*

Proof of Lemma 5.9. Set $f = U_\lambda g$, $\lambda \geqslant 1$, where g is a function satisfying (9). Then, by Lemma 2.2, we have

$$\begin{aligned}\left\| e^{i|D|^\alpha} f \right\|_{M^{p',q'}} &= \left\| e^{i|D|^\alpha} U_\lambda g \right\|_{M^{p',q'}} \\ &= \left\| U_\lambda \left(e^{i|\lambda D|^\alpha} g \right) \right\|_{M^{p',q'}} \\ &\gtrsim \lambda^{n\mu_2(p',q')} \left\| e^{i|\lambda D|^\alpha} g \right\|_{M^{p',q'}}.\end{aligned}$$

In the same way as (ii), we obtain, by the change of variable $x \mapsto \lambda^\alpha x$ and the method of stationary phase,

$$\begin{aligned}
\|e^{i|\lambda D|^\alpha} g\|_{M^{p',q'}} &= \Bigl(\sum_{k\in\mathbf{Z}^n} \|\varphi(D-k)e^{i|\lambda D|^\alpha} g\|_{L^{p'}}^{q'}\Bigr)^{1/q'} \\
&\gtrsim \Bigl\| \int_{\mathbf{R}^n} \varphi(\xi) e^{ix\cdot\xi + i|\lambda\xi|^\alpha} \hat{g}(\xi)\, d\xi \Bigr\|_{L^{p'}} \\
&= \lambda^{\alpha n/p'} \Bigl\| \int_{\mathbf{R}^n} e^{i\lambda^\alpha (x\cdot\xi + |\xi|^\alpha)} \varphi(\xi)\hat{g}(\xi)\, d\xi \Bigr\|_{L^{p'}} \\
&\gtrsim \lambda^{\alpha n/p' - \alpha n/2}.
\end{aligned}$$

Hence, we have

$$\|e^{i|D|^\alpha} f\|_{M^{p',q'}} \gtrsim \lambda^{n\mu_2(p',q')} \lambda^{\alpha n(1/p'-1/2)}.$$

On the other hand, we have

$$\|f\|_{L_s^{p'}} = \|U_\lambda g\|_{L_s^{p'}} \approx \lambda^{-n/p'} \lambda^s.$$

Combining these two estimates, we obtain

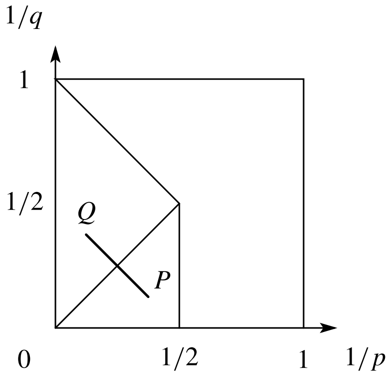
$$\lambda^{s-n/p'-n\mu_2(p',q')-\alpha n(1/p'-1/2)} \gtrsim 1$$

for all $\lambda \geqslant 1$. Letting $\lambda \to \infty$ yields the necessary condition

$$\begin{aligned}
s &\geqslant \alpha n\bigl(1/p'-1/2\bigr) + n/p' + n\bigl(-2/p'+1/q'\bigr) \\
&= (\alpha-2)n\bigl(1/p'-1/2\bigr) + 2n\bigl(1/p'-1/2\bigr) + n/p' + n\bigl(-2/p'+1/q'\bigr) \\
&= (\alpha-2)n\bigl(1/p'-1/2\bigr) + n\bigl(1/p'+1/q'-1\bigr) \\
&= (\alpha-2)n(1/2-1/p) + n(1-1/p-1/q) \\
&= (\alpha-2)n|1/p-1/2| - n\nu_2(p,q),
\end{aligned}$$

since $\mu_2(p',q') = -2/p'+1/q'$ and $\nu_2(p,q) = 1/p+1/q-1$. □

(iv) Let $2 \leqslant p \leqslant \infty$ and $p < q$. Contrary to our claim, suppose that there exists $\varepsilon > 0$ such that $s = (\alpha-2)n|1/p-1/2| - n\nu_2(p,q) - \varepsilon$ implies (7). Then, by interpolation with the estimate for a point $Q(1/p_1, 1/q_1)$ with $2 < p_1 < \infty$, $p_1' < q_1 < p_1$ and $s = (\alpha-2)n|1/p_1-1/2| - n\nu_2(p_1,q_1)$ (which holds by Corollary 5.4), one would obtain an improved estimates of the segment joining $P(1/p, 1/q)$ and $Q(1/p_1, 1/q_1)$, which is not possible. In the same way as above, we can treat the case $1 \leqslant p \leqslant 2$ and $p' < q$, and we have the conclusion.



□

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