ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Journal of Number Theory 121 (2006) 224–233
JOURNAL OF Number Theory
www.elsevier.com/locate/jnt

A congruence involving products of q-binomial coefficients

Hao Pan [a,∗], Hui-Qin Cao [b]

[a] *Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China*
[b] *Department of Applied Mathematics, Nanjing Audit University, Nanjing 210029, People's Republic of China*

Received 30 August 2005
Available online 22 March 2006
Communicated by David Goss

Abstract

In this paper we establish a q-analogue of a congruence of Sun concerning the products of binomial coefficients modulo the square of a prime.

MSC: primary 11B65; secondary 05A10, 05A30, 11A07

1. Introduction

In [G], Granville proved the following interesting congruence:

$$(-1)^{(p-1)(m-1)/2}\prod_{k=1}^{m-1}\binom{p-1}{\lfloor kp/m\rfloor}\equiv m^p-m+1 \pmod{p^2} \tag{1.1}$$

for any prime $p\geqslant 5$ and $m\geqslant 2$, where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x. Later Sun [S] extended Granville's result and showed that

$$(-1)^{\frac{p-1}{2}\lfloor\frac{m}{2}\rfloor}\prod_{1\leqslant k\leqslant\lfloor m/2\rfloor}\binom{p-1}{\lfloor pk/m\rfloor}$$

∗ Corresponding author.
E-mail addresses: haopan79@yahoo.com.cn (H. Pan), caohq@nau.edu.cn (H.-Q. Cao).

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2006.02.004

$$\equiv \begin{cases} \left(\frac{m}{p}\right) + \mathrm{eq}_p(m)mp \pmod{p^2} & \text{if } 2 \nmid m, \\ \left(\frac{2m}{p}\right) + \left(\frac{2}{p}\right)\mathrm{eq}_p(m)mp + 2\left(\frac{m}{p}\right)\mathrm{eq}_p(2)p \pmod{p^2} & \text{if } 2 \mid m, \end{cases} \tag{1.2}$$

where $(\frac{\cdot}{p})$ is the Legendre symbol and

$$\mathrm{eq}_p(m) = \frac{m^{(p-1)/2} - \left(\frac{m}{p}\right)}{p}$$

is the Euler quotient.

For an integer m prime to p, define the Fermat quotient $\mathrm{q}_p(m)$ by

$$\mathrm{q}_p(m) = \frac{m^{p-1} - 1}{p}.$$

Observe that

$$\mathrm{q}_p(2) = \frac{2^{p-1} - 1}{p} = \frac{\left(2^{(p-1)/2} - \left(\frac{2}{p}\right)\right)\left(2^{(p-1)/2} + \left(\frac{2}{p}\right)\right)}{p} \equiv 2\left(\frac{2}{p}\right)\mathrm{eq}_p(2) \pmod{p}.$$

Then (1.2) can be rewritten as

$$(-1)^{\frac{p-1}{2}\lfloor\frac{m}{2}\rfloor}\left(\frac{m}{p}\right)\left(\frac{2}{p}\right)^{m-1} \prod_{1\leqslant k\leqslant\lfloor m/2\rfloor} \binom{p-1}{\lfloor pk/m\rfloor}$$
$$\equiv 1 + \left(\frac{m}{p}\right)\mathrm{eq}_p(m)mp + \left(2\lfloor m/2\rfloor + 1 - m\right)\mathrm{q}_p(2)p \pmod{p^2}. \tag{1.3}$$

For a non-negative integer n, let

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$$

and

$$(x;q)_n = \begin{cases} (1-x)(1-xq)\cdots(1-xq^{n-1}) & \text{if } n \geqslant 1, \\ 1 & \text{if } n = 0. \end{cases}$$

And the q-binomial coefficients are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q)}$$

for any $k, n \in \mathbb{N}$. The arithmetic properties of q-binomial coefficients have been investigated by several authors (e.g., see [A,C,F]). Recently Pan [P] established a q-analogue of Granville's congruence (1.1). If $p \geqslant 5$ is a prime and $m \geqslant 2$ is an integer with $p \nmid m$, then we have

$$(-1)^{(p-1)(m-1)/2} q^{m\sum_{k=1}^{m-1}\binom{\lfloor kp/m\rfloor+1}{2}} \prod_{k=1}^{m-1} \begin{bmatrix} p-1 \\ \lfloor kp/m\rfloor \end{bmatrix}_{q^m}$$
$$\equiv \frac{m(q^m;q^m)_{p-1}}{(q;q)_{p-1}} - m + 1 \pmod{[p]_q^2}. \tag{1.4}$$

In this paper we will give a q-analogue of Sun's congruence (1.3). Suppose that p is an odd prime and $m \geqslant 2$ is an integer prime to p. It is not difficult to prove that

$$\frac{(q^m;q^m)_{p-1}}{(q;q)_{p-1}} = \prod_{j=1}^{p-1} \frac{1-q^{jm}}{1-q^j} \equiv 1 \pmod{[p]_q}.$$

So we can define the q-Fermat quotient by

$$\mathrm{Q}_p(m,q) = \frac{(q^m;q^m)_{p-1}/(q;q)_{p-1} - 1}{[p]_q}.$$

For any integer x, we denote by $\langle x\rangle_p$ the least non-negative residue of x modulo p. Let

$$R_p(m) = \{1 \leqslant j < p/2\colon\ \langle jm\rangle_p > p/2\}.$$

Then well-known Gauss's lemma asserts that

$$\left(\frac{m}{p}\right) = (-1)^{|R_p(m)|}.$$

Soon we will show that

$$q^{\sum_{j\in R_p(m)}(p-\langle jm\rangle_p)} \frac{(q^m;q^m)_{\frac{p-1}{2}}}{(q;q)_{\frac{p-1}{2}}} \equiv \left(\frac{m}{p}\right) \pmod{[p]_q}.$$

Define the q-Euler quotient by

$$\mathrm{EQ}_p(m,q) = \frac{q^{\sum_{j\in R_p(m)}(p-\langle jm\rangle_p)}(q^m;q^m)_{\frac{p-1}{2}}/(q;q)_{\frac{p-1}{2}} - \left(\frac{m}{p}\right)}{[p]_q}.$$

Theorem 1.1. *Let $p \geqslant 5$ be a prime and $m \geqslant 2$ be an integer with $p \nmid m$. Then*

$$(-1)^{\frac{p-1}{2}\lfloor\frac{m}{2}\rfloor}\left(\frac{m}{p}\right)\left(\frac{2}{p}\right)^{m-1} q^{2m\sum_{k=1}^{\lfloor m/2\rfloor}\binom{\lfloor kp/m\rfloor+1}{2}} \prod_{k=1}^{\lfloor m/2\rfloor} \begin{bmatrix} p-1 \\ \lfloor kp/m\rfloor \end{bmatrix}_{q^{2m}}$$
$$\equiv 1 + m[p]_q \mathrm{EQ}_p^*(m,q) + \big(2\lfloor m/2\rfloor + 1\big)[p]_{q^m}\mathrm{Q}_p\big(2,q^m\big) - m[p]_q\mathrm{Q}_p(2,q)$$
$$+ m\left(\big|R_p(m)\big| + 2\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\right)\big(1-q^p\big) \pmod{[p]_q^2}, \tag{1.5}$$

where

$$\begin{aligned}\mathrm{EQ}_p^*(m,q) &= \left(\frac{m}{p}\right)\frac{(1+q^p)\mathrm{EQ}_p(m,q^2)}{1+q}\\ &= \frac{(\frac{m}{p})q^{2\sum_{j\in R_p(m)}(p-\langle jm\rangle_p)}(q^{2m};q^{2m})_{\frac{p-1}{2}}/(q^2;q^2)_{\frac{p-1}{2}}-1}{[p]_q}.\end{aligned}$$

The proof of Theorem 1.1 will be given in the next sections.

2. Some lemmas

Below we assume that $p \geqslant 5$ is a prime and m is a positive integer prime to p.

Lemma 2.1.

$$\sum_{j=1}^{(p-1)/2}\frac{1}{[j]_{q^2}} \equiv -(1+q)\mathrm{Q}_p(2,q) \pmod{[p]_q}.$$

Proof. This is an immediate consequence of Theorem 1.1 in [P] by observing that

$$\sum_{j=1}^{(p-1)/2}\frac{1}{[j]_{q^2}} = (1+q)\sum_{j=1}^{(p-1)/2}\frac{1}{[2j]_q}. \qquad \square$$

Lemma 2.2. *Let m' be an integer such that*

$$m'm \equiv 1 \pmod p.$$

Then

$$2\sum_{j\in R_p(m)}\frac{1}{[2j]_q} \equiv \left|R_p(m)\right|(1-q) + \frac{\mathrm{Q}_p(2,q^{m'})}{[m']_q} - \mathrm{Q}_p(2,q) \pmod{[p]_q}.$$

Proof. Clearly

$$\begin{aligned}R_p(-m) &= \left\{1 \leqslant j < p/2\colon\ \langle -jm\rangle_p > p/2\right\}\\ &= \left\{1 \leqslant j < p/2\colon\ \langle jm\rangle_p < p/2\right\}\\ &= \left\{1,2,\ldots,(p-1)/2\right\} \setminus R_p(m).\end{aligned} \tag{2.1}$$

Then applying Lemma 2.1,

$$\sum_{j\in R_p(m)}\frac{1}{[j]_{q^2}} + \sum_{j\in R_p(-m)}\frac{1}{[j]_{q^2}} = \sum_{j=1}^{(p-1)/2}\frac{1}{[j]_{q^2}} \equiv -(1+q)\mathrm{Q}_p(2,q) \pmod{[p]_q}. \tag{2.2}$$

On the other hand, observe that $\langle jm\rangle_p > p/2$ if and only if there exists a $1 \leqslant k < p/2$ such that

$$jm \equiv -k \pmod p,$$

or equivalently,

$$j \equiv -km' \pmod p.$$

It follows that

$$R_p(m) = \bigl\{\langle -km'\rangle_p\colon\ k \in R_p(m')\bigr\},$$

whence

$$\sum_{j\in R_p(m)} \frac{1}{[j]_{q^2}} \equiv \sum_{k\in R_p(m')} \frac{1}{[-km']_{q^2}} \quad \bigl(\operatorname{mod} [p]_{q^2}\bigr).$$

Thus

$$\begin{aligned}
&\sum_{j\in R_p(m)} \frac{1}{[j]_{q^2}} - \sum_{j\in R_p(-m)} \frac{1}{[j]_{q^2}} \\
&\quad\equiv \sum_{k\in R_p(m')} \frac{1}{[-km']_{q^2}} - \sum_{k\in R_p(-m')} \frac{1}{[km']_{q^2}} \\
&\quad= -\sum_{k\in R_p(m')} \frac{q^{2km'}}{[km']_{q^2}} - \sum_{k\in R_p(-m')} \frac{1}{[km']_{q^2}} \\
&\quad= \bigl(1-q^2\bigr)\bigl|R_p(m')\bigr| - \sum_{k\in R_p(m')} \frac{1}{[m']_{q^2}[k]_{q^{2m'}}} - \sum_{k\in R_p(-m')} \frac{1}{[m']_{q^2}[k]_{q^{2m'}}} \\
&\quad\equiv \bigl(1-q^2\bigr)\bigl|R_p(m)\bigr| + \frac{(1+q^{m'})\mathrm{Q}_p(2,q^{m'})}{[m']_{q^2}} \quad \bigl(\operatorname{mod} [p]_q\bigr).
\end{aligned} \tag{2.3}$$

Adding (2.2) and (2.3), we obtain that

$$\sum_{j\in R_p(m)} \frac{1}{[j]_{q^2}} \equiv \frac{|R_p(m)|}{2}\bigl(1-q^2\bigr) + \frac{(1+q^{m'})\mathrm{Q}_p(2,q^{m'})}{2[m']_{q^2}} - \frac{1+q}{2}\mathrm{Q}_p(2,q) \quad \bigl(\operatorname{mod} [p]_q\bigr). \tag{2.4}$$

Finally,

$$\begin{aligned}
2\sum_{j\in R_p(m)} \frac{1}{[2j]_q} &= \frac{2}{1+q} \sum_{j\in R_p(m)} \frac{1}{[j]_{q^2}} \\
&\equiv \bigl|R_p(m)\bigr|(1-q) + \frac{\mathrm{Q}_p(2,q^{m'})}{[m']_q} - \mathrm{Q}_p(2,q) \quad \bigl(\operatorname{mod} [p]_q\bigr). \qquad \square
\end{aligned}$$

Lemma 2.3.

$$\frac{\left(\frac{m}{p}\right)q^{2\sum_{j\in R_p(m)}(p-\langle jm\rangle_p)}(q^{2m};q^{2m})_{\frac{p-1}{2}}/(q^2;q^2)_{\frac{p-1}{2}}-1}{[p]_q}$$
$$\equiv 2\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\frac{q^{2jm}}{[2jm]_q}+\mathrm{Q}_p(2,q)-\frac{\mathrm{Q}_p(2,q^m)}{[m]_q}-\left|R_p(m)\right|(1-q)\pmod{[p]_q}.$$

Proof. Let $r_j=\langle jm\rangle_p$ for each $j\in\mathbb{Z}$. From (2.1), we have

$$\frac{(q^{2m};q^{2m})_{(p-1)/2}}{(q^2;q^2)_{(p-1)/2}}=\prod_{j=1}^{(p-1)/2}\frac{[jm]_{q^2}}{[j]_{q^2}}=\prod_{j\in R_p(m)}\frac{[jm]_{q^2}}{[j]_{q^2}}\prod_{j\in R_p(-m)}\frac{[jm]_{q^2}}{[j]_{q^2}}.$$

Now

$$\prod_{j\in R_p(-m)}\frac{[jm]_{q^2}}{[j]_{q^2}}=\prod_{j\in R_p(-m)}\frac{1-q^{2r_j}}{1-q^{2j}}\left(1+\frac{q^{2r_j}(1-q^{2\lfloor jm/p\rfloor p})}{1-q^{2r_j}}\right),$$

and

$$\prod_{j\in R_p(m)}\frac{[jm]_{q^2}}{[j]_{q^2}}=\prod_{j\in R_p(m)}\frac{1-q^{2(r_j-p)}}{1-q^{2j}}\left(1+\frac{q^{2(r_j-p)}(1-q^{2(\lfloor jm/p\rfloor+1)p})}{1-q^{2(r_j-p)}}\right).$$

It is easy to check that

$$\left\{\langle -jm\rangle_p:\ j\in R_p(m)\right\}\cup\left\{\langle jm\rangle_p:\ j\in R_p(-m)\right\}=\left\{1,2,\dots,(p-1)/2\right\},$$

whence

$$\prod_{j\in R_p(m)}\frac{1-q^{2(r_j-p)}}{1-q^{2j}}\prod_{j\in R_p(-m)}\frac{1-q^{2r_j}}{1-q^{2j}}$$
$$=(-1)^{|R_p(m)|}q^{-2\sum_{j\in R_p(m)}(p-r_j)}\prod_{j\in R_p(m)}\frac{1-q^{2(p-r_j)}}{1-q^{2j}}\prod_{j\in R_p(-m)}\frac{1-q^{2r_j}}{1-q^{2j}}$$
$$=\left(\frac{m}{p}\right)q^{-2\sum_{j\in R_p(m)}(p-r_j)}.$$

Hence

$$\left(\frac{m}{p}\right)q^{2\sum_{j\in R_p(m)}(p-r_j)}\prod_{j=1}^{(p-1)/2}\frac{[jm]_{q^2}}{[j]_{q^2}}$$
$$=\prod_{j\in R_p(m)}\left(1+\frac{q^{2(r_j-p)}(1-q^{2(\lfloor jm/p\rfloor+1)p})}{1-q^{2(r_j-p)}}\right)\prod_{j\in R_p(-m)}\left(1+\frac{q^{2r_j}(1-q^{2\lfloor jm/p\rfloor p})}{1-q^{2r_j}}\right)$$

$$\equiv \prod_{j\in R_p(m)}\left(1+\frac{q^{2jm}(1-q^{2(\lfloor jm/p\rfloor+1)p})}{1-q^{2jm}}\right)\prod_{j\in R_p(-m)}\left(1+\frac{q^{2jm}(1-q^{2\lfloor jm/p\rfloor p})}{1-q^{2jm}}\right)$$

$$\equiv 1+[p]_{q^2}\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\frac{q^{2jm}}{[jm]_{q^2}}+[p]_{q^2}\sum_{j\in R_p(m)}\frac{q^{2jm}}{[jm]_{q^2}} \quad \bigl(\mathrm{mod}\ [p]_{q^2}^2\bigr),$$

where in the last step we use the congruence

$$\frac{1-q^{jp}}{1-q^p}=1+q^p+\cdots+q^{(j-1)p}\equiv j \quad \bigl(\mathrm{mod}\ [p]_q\bigr).$$

Applying (2.4), we have

$$\begin{aligned}
&\sum_{j\in R_p(m)}\frac{q^{2jm}}{[jm]_{q^2}}\\
&=\sum_{j\in R_p(m)}\frac{1}{[m]_{q^2}[j]_{q^{2m}}}-\bigl(1-q^2\bigr)\bigl|R_p(m)\bigr|\\
&\equiv\frac{(1+q^{m'm})\mathrm{Q}_p(2,q^{m'm})}{2[m]_{q^2}[m']_{q^{2m}}}-\frac{1+q^m}{2[m]_{q^2}}\mathrm{Q}_p\bigl(2,q^m\bigr)+\left(\frac{1-q^{2m}}{2[m]_{q^2}}-\bigl(1-q^2\bigr)\right)\bigl|R_p(m)\bigr|\\
&\equiv\frac{1+q}{2}\mathrm{Q}_p(2,q)-\frac{1+q^m}{2[m]_{q^2}}\mathrm{Q}_p\bigl(2,q^m\bigr)-\frac{|R_p(m)|}{2}\bigl(1-q^2\bigr) \quad \bigl(\mathrm{mod}\ [p]_q\bigr).
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{\left(\frac{m}{p}\right)q^{2\sum_{j\in R_p(m)}(p-r_j)}(q^{2m};q^{2m})_{\frac{p-1}{2}}/(q^2;q^2)_{\frac{p-1}{2}}-1}{[p]_q}\\
&=\frac{1+q^p}{1+q}\cdot\frac{\left(\frac{m}{p}\right)q^{2\sum_{j\in R_p(m)}(p-r_j)}(q^{2m};q^{2m})_{\frac{p-1}{2}}/(q^2;q^2)_{\frac{p-1}{2}}-1}{[p]_{q^2}}\\
&\equiv\frac{2}{1+q}\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\frac{q^{2jm}}{[jm]_{q^2}}+\frac{2}{1+q}\sum_{j\in R_p(m)}\frac{q^{2jm}}{[jm]_{q^2}}\\
&\equiv 2\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\frac{q^{2jm}}{[2jm]_q}+\mathrm{Q}_p(2,q)-\frac{\mathrm{Q}_p(2,q^m)}{[m]_q}-\bigl|R_p(m)\bigr|(1-q) \quad \bigl(\mathrm{mod}\ [p]_q\bigr). \qquad \square
\end{aligned}$$

3. Proof of Theorem 1.1

We write

$$\begin{aligned}\begin{bmatrix} p-1 \\ \lfloor kp/m \rfloor \end{bmatrix}_{q^{2m}} &= \prod_{j=1}^{\lfloor kp/m \rfloor} \frac{[p-j]_{q^{2m}}}{[j]_{q^{2m}}} \\ &= \prod_{j=1}^{\lfloor kp/m \rfloor} \frac{[p]_{q^{2m}} - [j]_{q^{2m}}}{q^{2jm}[j]_{q^{2m}}} \\ &= (-1)^{\lfloor kp/m \rfloor} q^{-2m\binom{\lfloor kp/m \rfloor+1}{2}} \prod_{j=1}^{\lfloor kp/m \rfloor} \left(1 - \frac{[p]_{q^{2m}}}{[j]_{q^{2m}}}\right).\end{aligned}$$

Then

$$\begin{aligned}&(-1)^{\sum_{k=1}^{\lfloor m/2 \rfloor} \lfloor kp/m \rfloor} q^{2m \sum_{k=1}^{\lfloor m/2 \rfloor} \binom{\lfloor kp/m \rfloor+1}{2}} \prod_{k=1}^{\lfloor m/2 \rfloor} \begin{bmatrix} p-1 \\ \lfloor kp/m \rfloor \end{bmatrix}_{q^{2m}} \\ &\equiv 1 - [p]_{q^{2m}} \sum_{k=1}^{\lfloor m/2 \rfloor} \sum_{j=1}^{\lfloor kp/m \rfloor} \frac{1}{[j]_{q^{2m}}} \\ &= 1 - [p]_{q^{2m}} \sum_{j=1}^{(p-1)/2} \sum_{k=\lceil jm/p \rceil}^{\lfloor m/2 \rfloor} \frac{1}{[j]_{q^{2m}}} \\ &= 1 - [p]_{q^{2m}} \sum_{j=1}^{(p-1)/2} \frac{\lfloor m/2 \rfloor - \lfloor jm/p \rfloor}{[j]_{q^{2m}}} \pmod{[p]_{q^{2m}}^2}.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}[p]_{q^{2m}} \sum_{j=1}^{(p-1)/2} \frac{\lfloor m/2 \rfloor}{[j]_{q^{2m}}} &\equiv -\left\lfloor \frac{m}{2} \right\rfloor [p]_{q^{2m}} \big(1+q^m\big) \mathrm{Q}_p\big(2, q^m\big) \\ &= -\left\lfloor \frac{m}{2} \right\rfloor \frac{1-q^{2mp}}{1-q^m} \mathrm{Q}_p\big(2, q^m\big) \\ &\equiv -2\lfloor m/2 \rfloor [p]_{q^m} \mathrm{Q}_p\big(2, q^m\big) \pmod{[p]_{q^m}^2}\end{aligned}$$

and

$$\begin{aligned}[p]_{q^{2m}} \sum_{j=1}^{(p-1)/2} \frac{\lfloor jm/p \rfloor}{[j]_{q^{2m}}} &= [mp]_{q^2} \sum_{j=1}^{(p-1)/2} \frac{\lfloor jm/p \rfloor}{[jm]_{q^2}} \\ &= [mp]_{q^2} \sum_{j=1}^{(p-1)/2} \left\lfloor \frac{jm}{p} \right\rfloor \frac{q^{2jm}}{[jm]_{q^2}} + \big(1-q^{2mp}\big) \sum_{j=1}^{(p-1)/2} \left\lfloor \frac{jm}{p} \right\rfloor.\end{aligned}$$

From Lemma 2.3, we deduce that

$$
\begin{aligned}
&[mp]_{q^2}\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\frac{q^{2jm}}{[jm]_{q^2}}\\
&\quad\equiv 2m[p]_q\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\frac{q^{2jm}}{[2jm]_q}\\
&\quad\equiv m[p]_q\left(\mathrm{EQ}_p^*(m,q)-\mathrm{Q}_p(2,q)+\frac{\mathrm{Q}_p(2,q^m)}{[m]_q}+\left|R_p(m)\right|(1-q)\right)\pmod{[p]_q^2}.
\end{aligned}
$$

Since $p\nmid 2m$, we have both $[p]_{q^m}$ and $[p]_{q^{2m}}$ are divisible by $[p]_q$. Also note that

$$[p]_{q^m}=[mp]_q/[m]_q\equiv m[p]_q/[m]_q\pmod{[p]_q^2}$$

and

$$1-q^{2mp}\equiv 2m(1-q)[p]_q\pmod{[p]_q^2}.$$

Therefore we obtain that

$$
\begin{aligned}
&(-1)^{\sum_{k=1}^{\lfloor m/2\rfloor}\lfloor kp/m\rfloor}q^{2m\sum_{k=1}^{\lfloor m/2\rfloor}\binom{\lfloor kp/m\rfloor+1}{2}}\prod_{k=1}^{\lfloor m/2\rfloor}\begin{bmatrix}p-1\\ \lfloor kp/m\rfloor\end{bmatrix}_{q^{2m}}\\
&\quad\equiv 1+m[p]_q\mathrm{EQ}_p^*(m,q)+\bigl(2\lfloor m/2\rfloor+1\bigr)[p]_{q^m}\mathrm{Q}_p\bigl(2,q^m\bigr)-m[p]_q\mathrm{Q}_p(2,q)\\
&\qquad+m\left(\left|R_p(m)\right|+2\sum_{j=1}^{(p-1)/2}\left\lfloor\frac{jm}{p}\right\rfloor\right)(1-q)[p]_q\pmod{[p]_q^2}.
\end{aligned}
$$

Finally, by Lemma 3.1 in [S],

$$\sum_{k=1}^{\lfloor m/2\rfloor}\left\lfloor\frac{kp}{m}\right\rfloor\equiv\frac{p-1}{2}\left\lfloor\frac{m}{2}\right\rfloor+\frac{(p^2-1)(m-1)}{8}-\left|R_p(m)\right|\pmod 2,$$

which implies that

$$(-1)^{\sum_{k=1}^{\lfloor m/2\rfloor}\lfloor kp/m\rfloor}=(-1)^{\frac{p-1}{2}\lfloor\frac{m}{2}\rfloor}\left(\frac{m}{p}\right)\left(\frac{2}{p}\right)^{m-1}.$$

All are done.

Acknowledgment

We thank our advisor, Professor Zhi-Wei Sun, for his help on this paper.

References

[A] G.E. Andrews, q-Analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher, Discrete Math. 204 (1999) 15–25.

[C] W.E. Clark, q-Analogue of a binomial coefficient congruence, Int. J. Math. Math. Sci. 18 (1995) 197–200.

[F] R.D. Fray, Congruence properties of ordinary and q-binomial coefficients, Duke Math. J. 34 (1967) 467–480.

[G] A. Granville, Arithmetic properties of binomial coefficients I: Binomial coefficients modulo prime powers, in: Organic Mathematics, Burnady, BC, 1995, in: CMS Conf. Proc., vol. 20, Amer. Math. Soc., Providence, RI, 1997, pp. 253–276.

[P] H. Pan, A q-analogue of Lehmer's congruence, preprint, arXiv: math.NT/0507511.

[S] Z.-W. Sun, Products of binomial coefficients modulo p^2, Acta Arith. 97 (2001) 87–98.