Induced vacuum currents in anti-de Sitter space with toral dimensions

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A B S T R A C T

We investigate the Hadamard function and the vacuum expectation value of the current density for a charged massive scalar field on a slice of anti-de Sitter (AdS) space described in Poincaré coordinates with toroidally compact dimensions. Along compact dimensions periodicity conditions are imposed on the field with general phases. Moreover, the presence of a constant gauge field is assumed. The latter gives rise to Aharonov–Bohm-like effects on the vacuum currents. The current density along compact dimensions is a periodic function of the gauge field flux with the period equal to the flux quantum. It vanishes on the AdS boundary and, near the horizon, to the leading order, it is conformally related to the corresponding quantity in Minkowski bulk for a massless field. For large values of the length of the compact dimension compared with the AdS curvature radius, the vacuum current decays as power-law for both massless and massive fields. This behavior is essentially different from the corresponding one in Minkowski background, where the currents for a massive field are suppressed exponentially.

1. Introduction

In quantum field theory on curved backgrounds, the anti-de Sitter (AdS) spacetime is interesting from several points of view. The corresponding metric tensor is a maximally symmetric solution of the Einstein equations in the presence of a negative cosmological constant and, because of the high degree of symmetry, numerous physical problems are exactly solvable on its background. These investigations may help to gain deeper understanding of the influence of gravity on quantum matter. The early interest in the dynamics of quantum fields in AdS spacetime was motivated by principal questions of the quantization on curved backgrounds. The presence of both regular and irregular modes and the possibility of getting an interesting causal structure, lead to a number of interesting features that have no analogs in Minkowski spacetime field theory. Being a constant negative curvature manifold, AdS space provides a convenient infrared regulator in interacting quantum field theories [1], cutting infrared divergences without reducing the symmetries. The importance of AdS spacetime as a gravitational background increased after the discovery that it generically arises as a ground state in extended supergravity and in string theories,

and also as the near-horizon geometry of the extremal black holes and branes.

More recent interest in this subject was generated in connection with two types of models where AdS geometry plays a crucial role. The first one, called AdS/CFT correspondence [2] (for a review see [3]), represents a realization of the holographic principle and provides a mapping between string theories or supergravity in the AdS bulk and a conformal field theory living on its boundary. In this mapping the solutions in AdS space play the role of classical sources for the correlators in the boundary field theory. The AdS/CFT correspondence has many interesting consequences and provides an opportunity to investigate non-perturbative field-theoretical effects at strong couplings. Among the recent developments of such a holography is the application to strong-coupling problems in condensed matter physics (holographic superconductors, quantum phase transitions, topological insulators). The second type of model with AdS spacetime as a background geometry is a realization of a braneworld scenario with large extra dimensions, and provides a solution to the hierarchy problem between the gravitational and electroweak mass scales (for reviews see [4]). Braneworlds naturally appear in string/M-theory context and give a novel setting for discussing phenomenological and cosmological issues related to extra dimensions, including the problem of the cosmological constant.

The presence of extra compact dimensions is an inherent feature of all these models. In quantum field theory, the boundary

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conditions imposed on the field operator along compactified dimensions lead to a number of interesting physical effects that include topological mass generation, instabilities in interacting field theories and symmetry breaking. These boundary conditions modify the spectrum of the zero-point fluctuations, as a result the vacuum energy density and the stresses are changed. This is the well-known topological Casimir effect. This effect has been investigated for large number of geometries and has important implications on all scales, from mesoscopic physics to cosmology (for reviews see [5]). The vacuum energy depends on the size of extra dimensions and this provides a stabilization mechanism for moduli fields in Kaluza–Klein-type models and in braneworld scenario. In particular, motivated by the problem of radion stabilization in Randall–Sundrum-type braneworlds, the investigations of the Casimir energy on AdS bulk have attracted a great deal of attention.[1] The Casimir effect in AdS spacetime with compact internal spaces has been considered in [7]. The vacuum energy generated by the compactification of extra dimensions can also serve as a model of dark energy needed for the explanation of the present accelerated expansion of the universe.

An important characteristic associated with charged fields is the vacuum expectation value (VEV) of the current density. Although the corresponding operator is local, because of the global nature of the quantum vacuum, this expectation value carries information about both the geometry and topology of the background space. Moreover, this VEV acts as the source in the semiclassical Maxwell equations and therefore plays an important role in modeling a self-consistent dynamics involving the electromagnetic field. The VEV of the current density for a fermionic field in flat spaces with toral dimensions has been investigated in [8]. Applications were given to the electrons of a graphene sheet rolled into cylindrical and toroidal shapes and described in terms of an effective Dirac theory in a two-dimensional space. Combined effects of the compactification and boundaries are discussed in [9]. The finite temperature effects on the current densities for scalar and fermionic fields in topologically nontrivial spaces have been studied in [10]. The VEV of the current density for charged scalar and Dirac spinor fields in de Sitter spacetime with toroidally compact spatial dimensions are considered in [11]. The effects of nontrivial topology induced by the compactification of a cosmic string along its axis have been discussed in [12].

In the present paper we investigate the VEV of the current density for a charged scalar field in a slice of AdS spacetime covered by Poincaré coordinates assuming that a part of spatial coordinates are compactified to a torus. In addition to the background gravitational field, we also assume the presence of a constant gauge field interacting with the field. Though the corresponding field strength vanishes, the nontrivial spatial topology gives rise Aharonov–Bohm-like effect on the current density along compact dimensions. This current is a source of magnetic fields in the uncompactified subspace, in particular, on the branes in braneworld scenario. The problem under consideration is also of separate interest as an example with gravitational and topological polarizations of the vacuum for charged fields, where one-loop calculations can be performed in closed form.

The outline of the paper is as follows. In the next section we describe the geometry of the problem and evaluate the Hadamard function for a charged massive scalar field obeying general quasiperiodicity conditions along compact dimensions. By using this function, in Section 3, the VEV of the current density is investigated. The behavior of this VEV in various asymptotic regions of the parameters is discussed. The main results of the paper
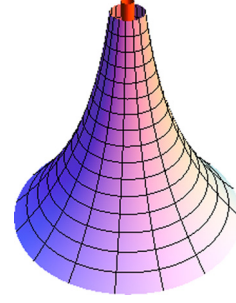
are summarized in Section 4. The main steps for the transformation of the Hadamard function to its final expression are described in Appendix A.

2. Hadamard function

As a background geometry we consider $(D + 1)$-dimensional AdS spacetime. In Poincaré coordinates the corresponding line element is expressed as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = e^{-2y/a}\eta_{ik}dx^i dx^k - dy^2, \qquad (1)$$

where $\eta_{ik} = \mathrm{diag}(1, -1, \ldots, -1)$, $i, k = 0, 1, \ldots, D - 1$, is the metric tensor for D-dimensional Minkowski spacetime and $-\infty < y < +\infty$. The parameter a is the AdS curvature radius and is related with the cosmological constant. Note that the Poincaré coordinates cover a part of the AdS manifold and there is a horizon corresponding to the hypersurface $y = +\infty$. In what follows we assume that the coordinates x^l, with $l = p + 1, \ldots, D - 1$, are compactified to circles with the lengths L_l, so $0 \leqslant x^l \leqslant L_l$. For the coordinates x^l, with $l = 1, 2, \ldots, p$, one has $-\infty < x^l < +\infty$. Hence, the subspace perpendicular to the y-axis has a topology $R^p \times (S^1)^q$, where $q + p = D - 1$. The coordinates in uncompactified and compactified subspaces will be denoted by $\mathbf{x}_p = (x^1, \ldots, x^p)$ and $\mathbf{x}_q = (x^{p+1}, \ldots, x^{D-1})$, respectively.

Introducing a new coordinate

$$z = ae^{y/a}, \quad 0 \leqslant z < \infty, \qquad (2)$$

the line element is presented in a conformally-flat form

$$ds^2 = (a/z)^2\left(\eta_{ik}dx^i dx^k - dz^2\right). \qquad (3)$$

The hypersurfaces identified by $z = 0$ and $z = \infty$ correspond to the AdS boundary and horizon, respectively. In Fig. 1 we have displayed the spatial geometry corresponding to (1) for $D = 2$ embedded into the 3-dimensional Euclidean space. The compact dimension corresponds to the circles. We have also displayed the flux of the gauge field strength which threads the compact dimension (see below). Note that for a given z, the proper length of the lth compact dimension is given by $L_{(p)l} = aL_l/z$ and it decreases with increasing z.

In this paper we are interested in the evaluation of the VEV of the current density

$$j_\mu(x) = ie\left[\varphi^+(x)D_\mu\varphi(x) - \left(D_\mu\varphi^+(x)\right)\varphi(x)\right], \qquad (4)$$

associated with a massive charged scalar field, $\varphi(x)$, in the presence of an external classical gauge field, A_μ. In (4), $D_\mu = \nabla_\mu + ieA_\mu$, with ∇_μ being the standard covariant derivative. The corresponding equation of motion is given by

$$\left(g^{\mu\nu}D_\mu D_\nu + m^2 + \xi R\right)\varphi(x) = 0, \qquad (5)$$

Fig. 1. The spatial section of the geometry under consideration for $D = 2$ embedded into a 3-dimensional Euclidean space.

where ξ is the curvature coupling parameter. The most important special cases correspond to minimally and conformally coupled fields with the parameters $\xi = 0$ and $\xi = (D-1)/(4D)$, respectively. In the geometry under consideration for the scalar curvature one has $R = -D(D+1)/a^2$. In models with nontrivial topology one need also to specify the periodicity conditions obeyed by the field operator along compact dimensions. Here we consider quasiperiodicity conditions

$$\varphi(t, \mathbf{x}_p, \mathbf{x}_q + L_l\mathbf{e}_l) = e^{i\alpha_l}\varphi(t, \mathbf{x}_p, \mathbf{x}_q), \qquad (6)$$

with constant phases α_l and with $\mathbf{e}_l$ being the unit vector along the lth compact dimension, $l = p+1, \ldots, D-1$. The special cases of untwisted and twisted fields correspond to $\alpha_l = 0$ and $\alpha_l = \pi$, respectively.

In what follows we assume that the gauge field is constant, $A_\mu = \text{const}$. Though the corresponding field strength vanishes, the nontrivial topology of the background space gives rise to Aharaonov–Bohm-like effects on the VEVs of physical observables. By a gauge transformation, the problem with a constant gauge field may be reduced to the problem in the absence of the gauge field with the shifted phases in the periodicity conditions (6). Indeed, let us consider two sets of the fields (φ, A_μ) and (φ', A'_μ) related by the gauge transformation $\varphi = \varphi' e^{-ie\chi}$, $A_\mu = A'_\mu + \partial_\mu\chi$. Choosing $\chi = A_\mu x^\mu$, we see that in the new gauge the vector potential vanishes, whereas the field operator obeys the periodicity condition

$$\varphi'(t, \mathbf{x}_p, \mathbf{x}_q + L_l\mathbf{e}_l) = e^{i\tilde{\alpha}_l}\varphi'(t, \mathbf{x}_p, \mathbf{x}_q), \qquad (7)$$

with the new phases

$$\tilde{\alpha}_l = \alpha_l + eA_lL_l. \qquad (8)$$

In this expression, $A_lL_l = \oint dx^l A_l$ is the flux of the field strength which threads the lth compact dimension. Note that in models with compact extra dimensions the gauge field fluxes generate a potential for the moduli fields providing a stabilization mechanism for them (for a review see [13]). In the evaluation of the VEVs we can use both the sets (φ, A_μ) and (φ', A'_μ). In the following, the evaluation procedure for the VEV of the current density is based on the set (φ', A'_μ). The field obeys Eq. (5) and the current density is given by (4) with $D_\mu = \nabla_\mu$ in both the formulas. For simplicity of the notations we shall omit the prime in the discussion below.

The VEV of the current density can be expressed in terms of the Hadamard function

$$G^{(1)}(x, x') = \langle 0|\varphi(x)\varphi^+(x') + \varphi^+(x')\varphi(x)|0\rangle, \qquad (9)$$

being $|0\rangle$ the vacuum state. Here we consider the vacuum corresponding to the Poincaré coordinate system, the so called Poincaré vacuum. For the VEV of the current density one gets

$$\langle 0|j_\mu(x)|0\rangle = \langle j_\mu\rangle = \frac{i}{2}e \lim_{x'\to x}(\partial_\mu - \partial'_\mu)G^{(1)}(x, x'). \qquad (10)$$

Let us denote by $\{\varphi_\sigma^{(+)}(x), \varphi_\sigma^{(-)}(x)\}$ the complete set of normalized positive- and negative-energy solutions to the field equation obeying the periodicity condition (7). The set of quantum numbers σ that label the mode functions will be specified below. Expanding the field operator in terms of the complete set we obtain the following expression for the Hadamard function

$$G^{(1)}(x, x') = \sum_\sigma \sum_{s=\pm} \varphi_\sigma^{(s)}(x)\varphi_\sigma^{(s)*}(x'). \qquad (11)$$

For the problem under consideration, we can choose the mode function in the factorized form

$$\varphi_\sigma^{(\pm)}(x) = e^{i\mathbf{k}_p\mathbf{x}_p + i\mathbf{k}_q\mathbf{x}_q \mp i\omega t}f(z), \qquad (12)$$

where $\mathbf{k}_p = (k_1, \ldots, k_p)$ and $\mathbf{k}_q = (k_{p+1}, \ldots, k_{D-1})$. For the components of the wave vector along uncompactified dimensions one has $-\infty < k_l < +\infty$, $l = 1, \ldots, p$, and for the components along compact dimensions from the condition (7) we find

$$k_l = (2\pi n_l + \tilde{\alpha}_l)/L_l, \quad l = p+1, \ldots, D-1, \qquad (13)$$

with $n_l = 0, \pm 1, \pm 2, \ldots$. Substituting (12) into the field equation we obtain the equation for the function $f(z)$. The solution of the latter is presented in the form

$$f(z) = Cz^{D/2}Z_\nu(\lambda z), \qquad (14)$$

where $Z_\nu(x)$ is a cylinder function of the order

$$\nu = \nu(D) = \sqrt{D^2/4 - D(D+1)\xi + m^2a^2}, \qquad (15)$$

and

$$\omega = \sqrt{\lambda^2 + k^2}, \quad k^2 = \mathbf{k}_p^2 + \mathbf{k}_q^2. \qquad (16)$$

Now, the set of quantum numbers is specified as $\sigma = (\mathbf{k}_p, \mathbf{n}_q, \lambda)$, with $\mathbf{n}_q = (n_{p+1}, \ldots, n_{D-1})$. For imaginary values of ν AdS spacetime is unstable [14] and in what follows we assume that the parameter ν is real.

The factor C in (14) is determined from the normalization condition

$$\int d^Dx\sqrt{|g|}g^{00}\varphi_\sigma^{(j)}(x)\varphi_{\sigma'}^{(j')*}(x) = \frac{1}{2\omega}\delta_{jj'}\delta_{\sigma\sigma'}, \qquad (17)$$

where $\delta_{\sigma\sigma'}$ is understood as Kronecker delta for discrete components of σ and as the Dirac delta function for the continuous ones. For $\nu \geqslant 1$, from the normalizability of the mode functions it follows that one should take $Z_\nu(\lambda z) = J_\nu(\lambda z)$ being $J_\nu(x)$ the Bessel function of the first kind. For the solution with the Neumann function $Y_\nu(x)$, the normalization integral in (17) diverges on the AdS boundary $z = 0$. Note that in AdS/CFT correspondence normalizable and non-normalizable modes are dual to states and sources, respectively. For $0 \leqslant \nu < 1$, in order to uniquely define the mode functions for the quantization procedure, it is necessary to specify boundary conditions at the AdS boundary [14,15]. The Dirichlet and Neumann boundary conditions are the most frequently used ones. The general class of allowed boundary conditions of the Robin type, has been specified in [16] on the base of general analysis in [17]. In what follows, for the modes with $0 \leqslant \nu < 1$, we shall choose $Z_\nu(\lambda z) = J_\nu(\lambda z)$ that corresponds to Dirichlet condition on the AdS boundary. In this case, from (17) for the coefficient in (14) one gets

$$|C|^2 = \frac{a^{1-D}\lambda}{2(2\pi)^p\omega V_q}, \qquad (18)$$

where $V_q = L_{p+1}\cdots L_{D-1}$ is the volume of the compact subspace.

Substituting the mode functions into (11), for the Hadamard function we obtain

$$G^{(1)}(x, x') = \frac{a^{1-D}(zz')^{D/2}}{(2\pi)^p V_q}\sum_{\mathbf{n}_q\in\mathbf{Z}^q}e^{i\mathbf{k}_q\cdot\Delta\mathbf{x}_q}\int d\mathbf{k}_p e^{i\mathbf{k}_p\cdot\Delta\mathbf{x}_p}$$

$$\times \int_0^\infty d\lambda\frac{\lambda}{\omega}J_\nu(\lambda z)J_\nu(\lambda z')\cos(\omega\Delta t), \qquad (19)$$

where $\Delta\mathbf{x}_q = \mathbf{x}_q - \mathbf{x}'_q$, $\Delta\mathbf{x}_p = \mathbf{x}_p - \mathbf{x}'_p$, $\Delta t = t - t'$. By using the transformations described in Appendix A, this function is presented in the final form

$$G^{(1)}(x, x') = \frac{2a^{1-D}}{(2\pi)^{(D+1)/2}} \sum_{\mathbf{n}_q \in \mathbf{Z}^q} e^{i\tilde{\alpha} \cdot \mathbf{n}_q} q_{\nu-1/2}^{(D-1)/2}(v_{\mathbf{n}_q}), \qquad (20)$$

where we have defined

$$v_{\mathbf{n}_q} = 1 + \frac{(\Delta \mathbf{x}_p)^2 + \sum_{i=p+1}^{D-1} (\Delta x^i - L_i n_i)^2 + (\Delta z)^2 - (\Delta t)^2}{2zz'}, \qquad (21)$$

with $\Delta z = z - z'$. In (20), we have introduced the function

$$q_\alpha^\mu(x) = \frac{e^{-i\pi\mu} Q_\alpha^\mu(x)}{(x^2 - 1)^{\mu/2}}$$

$$= \frac{\sqrt{\pi}\, \Gamma(\alpha + \mu + 1)}{2^{\alpha+1} \Gamma(\alpha + 3/2) x^{\alpha+\mu+1}}$$

$$\times F\left(\frac{\alpha+\mu}{2} + 1, \frac{\alpha+\mu+1}{2}; \alpha + \frac{3}{2}; \frac{1}{x^2}\right), \qquad (22)$$

being $Q_\alpha^\mu(x)$ the associated Legendre function of the second kind and $F(a, b; c; u)$ is the hypergeometric function. By using the recurrence relations for the associated Legendre functions we can obtain the following relation

$$\partial_x q_\alpha^\mu(x) = -q_\alpha^{\mu+1}(x). \qquad (23)$$

Note that in (20) the term with $\mathbf{n}_q = 0$ corresponds to the Hadamard function in AdS spacetime with trivial topology of the subspace perpendicular to the y-axis. The remaining terms are induced by the compactification. The quantity $v_{\mathbf{n}_q}$ with $\mathbf{n}_q = 0$ is related to the invariant distance $d(x, x')$ by $v_0 = \cosh(d(x, x'))$.

For a conformally coupled massless field one has $\nu = 1/2$ and the hypergeometric function in (20) is expressed as

$$F\left(\frac{\nu+D/2}{2}, \frac{\nu+D/2+1}{2}; \nu+1; 1/v_{\mathbf{n}_q}^2\right)$$

$$= \frac{v_{\mathbf{n}_q}}{1-D} \sum_{j=-1,1} j(1 + j/v_{\mathbf{n}_q})^{(1-D)/2}. \qquad (24)$$

For the Hadamard function this gives

$$G^{(1)}(x, x') = (zz'/a^2)^{(D-1)/2} G_M^{(1)}(x, x'), \qquad (25)$$

where

$$G_M^{(1)}(x, x') = \frac{\Gamma((D-1)/2)}{2\pi^{(D+1)/2}}$$

$$\times \sum_{\mathbf{n}_q \in \mathbf{Z}^q} e^{i\tilde{\alpha} \cdot \mathbf{n}_q} \left[\left(\Delta x_{\mathbf{n}_q}^{(-)}\right)^{1-D} - \left(\Delta x_{\mathbf{n}_q}^{(+)}\right)^{1-D} \right], \qquad (26)$$

with the notation

$$\left(\Delta x_{\mathbf{n}_q}^{(\pm)}\right)^2 = (\Delta \mathbf{x}_p)^2 + \sum_{i=p+1}^{D-1} \left(\Delta x^i - L_i n_i\right)^2$$

$$+ \left(z \pm z'\right)^2 - (\Delta t)^2. \qquad (27)$$

It can be shown that the function (26) coincides with the Hadamard function for a massless scalar field in $(D + 1)$-dimensional Minkowski spacetme with uncompactified dimensions $(x^1, \ldots, x^p, z)$ and compactified dimensions $(x^{p+1}, \ldots, x^{D-1})$, in the presence of a planar boundary at $z = 0$ on which the field obeys Dirichlet boundary condition. The formula (25) is the standard conformal relation between the conformally connected problems [18]. For a conformally coupled massless field the problem at hand is conformally related to the corresponding problem in Minkowski spacetime with a planar boundary. The presence of the latter is

a consequence of the boundary condition we have imposed on the AdS boundary. Note that the part in the function $G_M^{(1)}(x, x')$ with the first term in the square brackets of (26), is the Hadamard function in boundary-free Minkowski spacetime with toroidally compactified dimensions. The remaining part is induced by the boundary.

3. Vacuum current density

With the Hadamard function from (20), the VEV of the current density is evaluated by making use of formula (10). The charge density and the components along uncompactified dimensions vanish. For the component along the lth compact dimension, by using the relation (23), one gets

$$\langle j^l \rangle = \frac{4ea^{1-D}L_l}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l)$$

$$\times \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1}) q_{\nu-1/2}^{(D+1)/2}\left(1 + g_{\mathbf{n}_q}^2/(2z^2)\right), \qquad (28)$$

where $\mathbf{n}_{q-1} = (n_{p+1}, \ldots, n_{l-1}, n_{l+1}, \ldots, n_{D-1})$,

$$\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1} = \sum_{i=1, \neq l}^{D-1} \tilde{\alpha}_i n_i, \quad g_{\mathbf{n}_q} = \left(\sum_{i=p+1}^{D-1} n_i^2 L_i^2\right)^{1/2}, \qquad (29)$$

and the summation goes over all values of $-\infty < n_i < +\infty$, $i \neq l$. Note that in (28) instead of $\cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1})$ we can also write $\prod_{i=1, \neq l}^{D-1} \cos(\tilde{\alpha}_i n_i)$. As is seen, the current density along the lth compact dimension is an odd periodic function of the phase $\tilde{\alpha}_l$ and an even periodic function of the phases $\tilde{\alpha}_i$, $i \neq l$. In both cases the period is equal to 2π. In particular, the current density is a periodic function of the magnetic fluxes with the period equal to the flux quantum $2\pi/|e|$. In the absence of the gauge field, the current density along the lth compact dimension vanishes for untwisted and twisted fields along that direction. Of course, the latter is a direct consequence of the problem symmetry under the reflection $x^l \to -x^l$ in these special cases.

The charge flux through the $(D - 1)$-dimensional spatial hypersurface $x^l = \text{const}$ is determined by the quantity $n_l \langle j^l \rangle$, where $n_l = a/z$ is the corresponding normal. From (28) we see that this quantity depends on the coordinate lengths of the compact dimensions L_i in the form of the ratio L_i/z. For a given z, the latter is the proper length of the compact dimension, $L_{(p)i} = aL_i/z$, measured in units of the AdS curvature radius a. This property is a consequence of the maximal symmetry of AdS spacetime. Because $\langle j^z \rangle = 0$ and the VEV (28) depends on the coordinate z only, the covariant conservation equation $\nabla_l \langle j^l \rangle = 0$ is satisfied trivially.

The function $q_\alpha^\mu(x)$ in (28) is defined by (22). By using the relation (23), it can also be expressed in the form

$$q_{\nu-1/2}^{(D+1)/2}(x) = (-1)^n \sqrt{\frac{\pi}{2}} \partial_x^n \frac{(x + \sqrt{x^2-1})^{-\nu}}{\sqrt{x^2-1}}, \qquad (30)$$

for $D = 2n$, $n = 1, 2, \ldots$, and in the form

$$q_{\nu-1/2}^{(D+1)/2}(x) = (-1)^n \partial_x^n Q_{\nu-1/2}(x), \qquad (31)$$

for $D = 2n - 1$. In what follows we shall need the asymptotic expressions in different limiting cases. For $x \gg 1$ one has

$$q_{\nu-1/2}^{(D+1)/2}(x) \approx \frac{\sqrt{\pi}\, \Gamma(\nu + D/2 + 1)}{2^{\nu+1/2} \Gamma(\nu+1) x^{D/2+\nu+1}}. \qquad (32)$$

If x is close to 1, $x - 1 \ll 1$, by using the asymptotic formula for the hypergeometric function, we get

$$q_{\nu-1/2}^{(D+1)/2}(x) \approx \frac{\Gamma((D+1)/2)}{2(x-1)^{(D+1)/2}}. \tag{33}$$

Finally, for large values of the order, $\nu \gg 1$, one has

$$q_{\nu-1/2}^{(D+1)/2}(x) \approx \sqrt{\frac{\pi}{2}} \frac{\nu^{D/2}(x+\sqrt{x^2-1})^{-\nu}}{(x^2-1)^{(D+2)/4}}. \tag{34}$$

An alternative expression for the VEV of the current density is obtained by using the integral representation (57) given in Appendix A for the Hadamard function:

$$\langle j^l \rangle = \frac{2eL_l}{(2\pi)^{D/2}a^{D+1}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1})$$
$$\times \int_0^{\infty} dx \frac{x^{D/2}I_\nu(x)}{e^{x[1+g_{\mathbf{n}_q}^2/(2z^2)]}}. \tag{35}$$

This representation is useful in the discussion of limiting cases.

For a conformally coupled massless field one has $\nu = 1/2$ and for the current density we get

$$\langle j^l \rangle = (z/a)^{D+1}\langle j^l \rangle_M^{(b)}, \tag{36}$$

where

$$\langle j^l \rangle_M^{(b)} = 2eL_l \frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1})$$
$$\times \left[\frac{1}{g_{\mathbf{n}_q}^{D+1}} - \frac{1}{(g_{\mathbf{n}_q}^2 + 4z^2)^{(D+1)/2}} \right], \tag{37}$$

is the current density for a massless scalar field in Minkowski spacetime with toroidally compactified dimensions, in the presence of Dirichlet boundary at $z = 0$. The part with the first term in figure braces corresponds to the VEV in the boundary-free Minkowski spacetime. It is obtained from the general result of [10] in the zero mass limit.

Now we turn to the investigation of the current density in various asymptotic regions of the parameters. First let us consider the Minkowskian limit corresponding to $a \to \infty$ for a fixed value of the coordinate y. In this limit $\nu \approx ma \gg 1$ and for the coordinate z one has $z \approx a + y$. Introducing in (35) a new integration variable $x = \nu u$ we use the uniform asymptotic expansion for the function $I_\nu(\nu u)$ for large values of the order [21]. The dominant contribution to the integral comes from large values of u and, after the integration, to the leading order we get

$$\langle j^l \rangle \approx \frac{4eL_l m^{(D+1)/2}}{(2\pi)^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1})$$
$$\times \frac{K_{(D+1)/2}(mg_{\mathbf{n}_q})}{g_{\mathbf{n}_q}^{(D+1)/2}}, \tag{38}$$

being $K_{(D+1)/2}(x)$ the MacDonald function. The expression in the right-hand side coincides with the corresponding result for a scalar field in Minkowski spacetime with toroidally compactified dimensions obtained in [10].

If the proper length of the one of the compact dimensions, say x^i, $i \neq l$, is large compared with the AdS curvature radius, $L_i/z \gg 1$, the dominant contribution into (28) comes from the $n_i = 0$ term and the contribution of the remaining terms is suppressed by the factor $(z/L_i)^{D+2\nu+2}$. To the leading order, we obtain the current density for the topology $R^{p+1} \times (S^1)^{q-1}$ with the uncompactified direction x^i. In the opposite limit corresponding to small proper length of the dimension x^i, $L_i/z \ll 1$, the dominant contribution

comes from large values of n_i and for the corresponding series in (35) we use the relation

$$\sum_{n_i=-\infty}^{+\infty} \cos(\tilde{\alpha}_i n_i)e^{-xL_i^2n_i^2/(2z^2)} \approx \frac{2z}{L_i}\sqrt{\frac{\pi}{2x}}e^{-(\sigma_i z/L_i)^2/(2x)}, \tag{39}$$

with $\sigma_i = \min(\tilde{\alpha}_i, 2\pi - \tilde{\alpha}_i)$, $0 \leqslant \tilde{\alpha}_i < 2\pi$. The behavior of the current density depends crucially on whether the phase $\tilde{\alpha}_i$ is zero or not. For $\tilde{\alpha}_i = 0$, to the leading order, for the combination $(aL_i/z)\langle j^l \rangle$ we obtain the expression which coincides with the formula for $\langle j^l \rangle$ in D-dimensional AdS spacetime, obtained from the geometry under consideration by excluding the dimension x^i, with the replacement $\nu(D-1) \to \nu(D)$, where the function $\nu(D)$ is defined by (15). For $\tilde{\alpha}_i \neq 0$, after the substitution (39), the dominant contribution in the integral of (35) comes from the values $x \gtrsim z/L_i$. Replacing the modified Bessel function by its asymptotic expression for large values of the argument, we can see that the dominant contribution comes from the term $n_l = 1$, $n_r = 0$, $r \neq i, l$, and to the leading order one gets

$$\langle j^l \rangle \approx \frac{2eL_l\sigma_i^{(D-1)/2}\sin(\tilde{\alpha}_l)}{(2\pi)^{(D-1)/2}a^{D+1}} \frac{z^{D+1}e^{-L_l\sigma_i/L_i}}{(L_iL_l)^{(D+1)/2}}. \tag{40}$$

In this case the current density is exponentially small.

Now let us consider the behavior of the current $\langle j^l \rangle$ for large and small values of L_l. For large values of the proper length compared with the AdS curvature radius, $L_l/z \gg 1$, the argument of the function $q_{\nu-1/2}^{(D+1)/2}(x)$ in (28) is large and by using (32) in the leading order we find

$$\langle j^l \rangle \approx \frac{eL_l(2z^2)^{D/2+\nu+1}}{2^{D/2+\nu-1}a^{D+1}\pi^{D/2}} \frac{\Gamma(\nu+D/2+1)}{\Gamma(\nu+1)} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l)$$
$$\times \sum_{\mathbf{n}_{q-1}} \frac{\cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1})}{g_{\mathbf{n}_q}^{D+2\nu+2}}. \tag{41}$$

The dominant contribution comes from large values of n_i, $i \neq l$, and we replace the summation $\sum_{\mathbf{n}_{q-1}}$ by the integration. For $\tilde{\alpha}_i = 0$, $i = p+1, \ldots, D-1$, $i \neq l$, to the leading order one finds

$$\langle j^l \rangle \approx \frac{4e\Gamma(p/2+\nu+2)}{\pi^{p/2+1}\Gamma(\nu+1)a^{D+1}V_q} \frac{z^{D+2\nu+2}}{L_l^{p+2\nu+2}} \sum_{n_l=1}^{\infty} \frac{\sin(\tilde{\alpha}_l n_l)}{n_l^{p+2\nu+3}}, \tag{42}$$

with the power-law decay as a function of L_l for both massless and massive fields. In this sense, the situation for the AdS bulk is essentially different from that in the corresponding problem for Minkowski background. For the latter, in the massless case and for large values of L_l the current density decays as $1/L_l^p$, whereas for a massive field the current is suppressed exponentially, by the factor e^{-mL_l}. This shows that the influence of the background gravitational field on the VEV is crucial. If $L_l/z \gg 1$ and at least one of the phases $\tilde{\alpha}_i$, $i \neq l$, is not equal to zero, the dominant contribution comes from the $n_l = 1$ term and one gets

$$\langle j^l \rangle \approx \frac{2ea^{-1-D}}{\pi^{(p+1)/2}} \frac{\sin(\tilde{\alpha}_l)z^{D+2\nu+2}}{\Gamma(\nu+1)V_qe^{\beta_{q-1}L_l}} \frac{\beta_{q-1}^{(p+3)/2+\nu}}{(2L_l)^{(p+1)/2+\nu}}, \tag{43}$$

where $\beta_{q-1} = (\sum_{i=p+1,\neq l}^{D-1} \tilde{\alpha}_i^2/L_i^2)^{1/2}$. In this case the current density, as a function of L_l, decays exponentially.

For small values of L_l, $L_l/z \ll 1$, the dominant contribution to the current density comes from the term with $\mathbf{n}_{q-1} = 0$ and to the leading order we obtain

$$\langle j^l \rangle \approx \frac{2e\Gamma((D+1)/2)}{\pi^{(D+1)/2}(a/z)^{D+1}L_l^D} \sum_{n_l=1}^{\infty} \frac{\sin(\tilde{\alpha}_l n_l)}{n_l^D}. \tag{44}$$
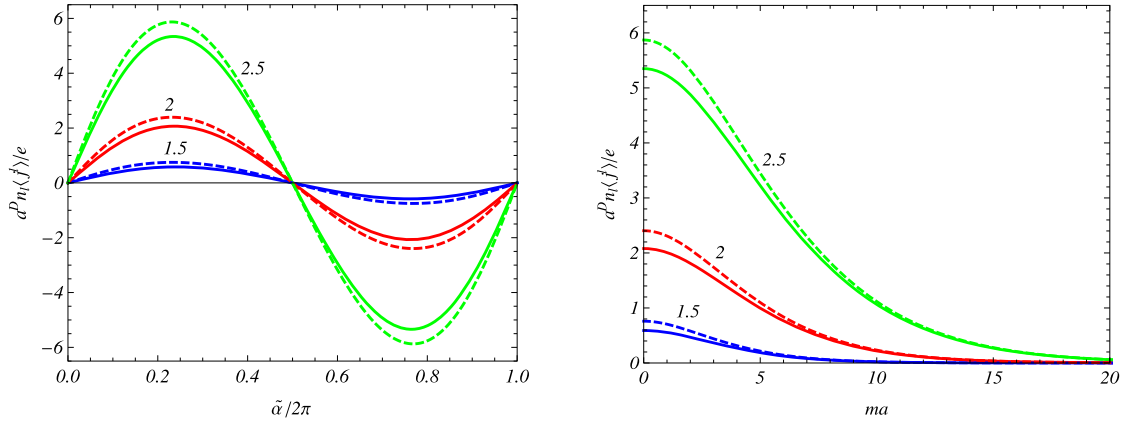
Fig. 2. The quantity $a^D n_l \langle j^l \rangle / e$ as a function of $\tilde{\alpha}$ (left panel) and as a function of ma (right panel) for $D = 4$ AdS space with a single compact dimension x^{D-1}. The numbers near the curves correspond to the values of the ratio z/L and the full/dashed curves are for minimally/conformally coupled fields. For the graphs on the left panel $ma = 0.5$ and for the right panel $\tilde{\alpha} = \pi/2$.

In this limit, the contribution of the terms with $\mathbf{n}_{q-1} \neq 0$ is suppressed by the factor $e^{-\sigma_l |\mathbf{L} \cdot \mathbf{n}_{q-1}|/L_l}$, where $\sigma_l = \min(\tilde{\alpha}_l, 2\pi - \tilde{\alpha}_l)$, $0 < \tilde{\alpha}_l < 2\pi$. The expression in the right-hand side of (44), multiplied by $(a/z)^{D+1}$, coincides with the VEV of the current density for a massless scalar field in $(D+1)$-dimensional Minkowski spacetime compactified along the direction x^l to the circle with the length L_l.

Now let us investigate the behavior of the current density near the AdS boundary and near the horizon for fixed values of the lengths L_i. Near the boundary, $z \to 0$, the argument of the function $q_{\nu-1/2}^{(D+1)/2}(x)$ in (28) is large and to the leading order we find

$$\langle j^l \rangle \approx \frac{4e L_l \Gamma(\nu + D/2 + 1)}{\pi^{D/2} \Gamma(\nu+1) a^{D+1}} z^{D+2\nu+2} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l)$$

$$\times \sum_{\mathbf{n}_{q-1}} \frac{\cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1})}{g_{\mathbf{n}_q}^{D+2\nu+2}}. \tag{45}$$

Hence, the current density vanishes on the boundary. Near the horizon one has $z \to \infty$, the argument of the function $q_{\nu-1/2}^{(D+1)/2}(x)$ is close to 1 and we use the asymptotic expression (33). This gives,

$$\langle j^l \rangle \approx (z/a)^{D+1} \langle j^l \rangle_M, \tag{46}$$

where

$$\langle j^l \rangle_M = 2e L_l \frac{\Gamma((D+1)/2)}{\pi^{(D+1)/2}} \sum_{n_l=1}^{\infty} n_l \sin(\tilde{\alpha}_l n_l) \sum_{\mathbf{n}_{q-1}} \frac{\cos(\tilde{\alpha}_{q-1} \cdot \mathbf{n}_{q-1})}{g_{\mathbf{n}_q}^{D+1}}, \tag{47}$$

is the corresponding current density in Minkowski spacetime for a massless scalar field.

And finally, for large values of the mass, $ma \gg 1$, by using the asymptotic expression (34) we see that the dominant contribution in (28) comes from the term with $n_i = 0$, $i \neq l$, and $n_l = 1$. To the leading order we get

$$\langle j^l \rangle \approx \frac{2e L_l \sin(\tilde{\alpha}_l)}{(2\pi)^{D/2} a^{D+1}} \frac{(ma)^{D/2} u^{-D/2-1}(1+u^2/4)^{-(D+2)/4}}{(1+u^2/2+u\sqrt{1+u^2/4})^{ma}}, \tag{48}$$

with $u = L_l/z$. As we could expect, for large values of the mass we have an exponential suppression. The suppression is stronger near the AdS boundary.

In the special case of a single compact dimension (dimension x^{D-1}) with the length $L_{D-1} = L$ from the general formula one finds
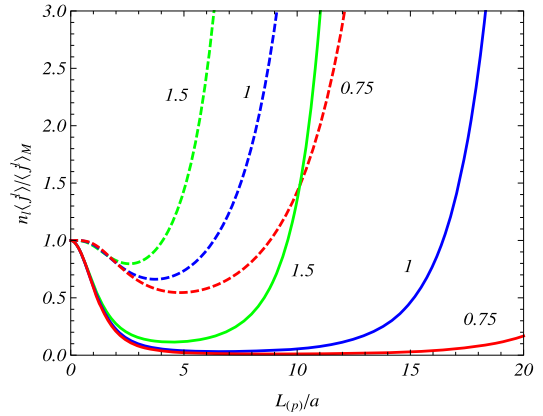

Fig. 3. The ratio of the current densities in AdS and Minkowski backgrounds as a function of the proper length of the compact dimension. The numbers near the curves correspond to the values of ma. The full and dashed curves are for minimally and conformally coupled fields. For the phase in the periodicity condition we have taken $\tilde{\alpha} = \pi/2$.

$$\langle j^{D-1} \rangle = \frac{4e a^{-1-D} L}{(2\pi)^{(D+1)/2}} \sum_{n=1}^{\infty} n \sin(\tilde{\alpha} n) q_{\nu-1/2}^{(D+1)/2}(1 + n^2 L^2/(2z^2)), \tag{49}$$

where $\tilde{\alpha} = \tilde{\alpha}_{D-1}$. For this special case with $D = 4$, in Fig. 2 we have displayed the quantity $a^D n_l \langle j^l \rangle / e$ as a function of the phase in the periodicity condition (left panel) and as a function of the mass (right panel). The numbers near the curves correspond to the values of the ratio z/L and the full/dashed curves are for minimally/conformally coupled fields. On the left panel, the graphs are plotted for $ma = 0.5$ and for the right panel we have taken $\tilde{\alpha} = \pi/2$.

For the same model with $D = 4$, in Fig. 3 we have plotted the ratio of the current densities in AdS and Minkowski bulks for the same proper lengths of the compact dimension, $L_{(p)}$, as a function of the proper length measured in units of the AdS curvature radius. The current density in Minkowski bulk is given by the right-hand side of (38), specified to the special case under consideration. The graphs are plotted for $\tilde{\alpha} = \pi/2$ and the numbers near the curves are the corresponding values of the parameter ma (mass measured in units of the AdS energy scale). As before the full and dashed curves correspond to minimally and conformally coupled fields. Note that in the case of the AdS bulk one has $L_{(p)} = aL/z$, with L being the coordinate length. In the Minkowski spacetime with compact dimension the proper and coordinate lengths coincide.

From Fig. 3 we see the feature already described before on the base of the asymptotic analysis: for a massive field and for large values of the proper length the decay of the current density in the Minkowski bulk is stronger than that for AdS background.

In braneworld models of the Randall–Sundrum type, the background geometry contains one or two parallel branes. The presence of the brane imposes a boundary condition on the field operator at its location (see, for instance, [19,20]). For an untwisted scalar field, the boundary condition is of the Robin type with the coefficient depending on the curvature coupling parameter ξ and on the brane mass term. For a twisted scalar, the field operator obeys the Dirichlet boundary condition. In the presence of the branes, the VEV of the current density is decomposed as $\langle j^l \rangle = \langle j^l \rangle_0 + \langle j^l \rangle_b$, where $\langle j^l \rangle_0$ is the VEV in the geometry without the branes, investigated in the present paper, and $\langle j^l \rangle_b$ is the contribution induced by the branes. The investigation of the latter in the models with single and two branes will be presented elsewhere. Similar to the VEVs of the field squared and the energy-momentum tensor, discussed in [20] for the model with trivial topology, we expect that, at distances from the brane larger than the AdS curvature radius, the brane-induced contribution in the VEV of the current density is suppressed exponentially. Hence, in braneworld models, at large distances from the brane, the total VEV is given by the expressions given above, up to exponentially small terms.

4. Conclusion

In the present paper we have investigated combined effects of geometry and topology on the VEV of the current density for a charged scalar field. In order to have an exactly solvable problem we have taken highly symmetric background geometry corresponding to a slice of AdS spacetime with a toroidally compactified subspace. Currently, the AdS spacetime is among the most popular backgrounds in gravitational physics and appears in a number of contexts. The information on the vacuum fluctuations is encoded in two-point functions and, as the first step, we have evaluated the Hadamard function for general values of the phases in the periodicity conditions obeyed by the field operator along compact dimensions. Additionally, the presence of a constant gauge field is assumed. The latter may be excluded by a gauge transformation that leads to the shift in the phases of periodicity conditions. The shift is expressed in terms of the ratio of the magnetic flux enclosed by compact dimension to the flux quantum. The Hadamard function is expressed in the form (20), where the part corresponding to AdS geometry without compactification is explicitly separated and is presented by the term with $\mathbf{n}_q = 0$. Because the toroidal compactification does not change the local geometry, in this way the renormalization of the VEVs for physical quantities bilinear in the field is reduced to the one for the uncompactified AdS spacetime.

With a given Hadamard function, the VEV of the current density is evaluated by using the formula (10). The vacuum charge density and the components of the current along uncompactified dimensions vanish and the current density along lth compact dimension is given by the expression (28). The latter is an odd periodic function of the phase $\tilde{\alpha}_l$ and an even periodic function of the phases $\tilde{\alpha}_i$, $i \neq l$. In particular, the current density is a periodic function of the magnetic fluxes with the period equal to the flux quantum. For a conformally coupled massless field the current density is conformally related to the corresponding quantity in $(D+1)$-dimensional Minkowski spacetime with toral dimensions, in the presence of a planar boundary on which the field obeys Dirichlet boundary condition. The appearance of the latter is a consequence of the boundary condition we have imposed on the AdS boundary. As an additional check, we have shown that in the Minkowskian limit,

corresponding to $a \to \infty$ for a fixed value of the coordinate y, the result obtained in [12] is recovered.

In order to clarify the dependence of the current density on the values of the parameters characterizing the geometry and the topology, we have considered various limiting cases. Near the AdS boundary, the VEV of the current density behaves as $z^{D+2\nu+2}$ and, hence, it vanishes on the AdS boundary. For fixed values of the coordinate lengths of compact dimensions, this region corresponds to large values of the proper lengths. Near the horizon, corresponding to $z = \infty$, the VEV of the current density is related to the current density in Minkowski spacetime with toral dimensions by the relation (46) and behaves as z^{D+1}. It is also of interest to consider the behavior for large and small proper lengths of compact dimensions for a fixed value of z. If the proper length of the ith compact dimension is small, then the behavior of the current density along the lth dimension depends crucially on whether the phase $\tilde{\alpha}_i$ is zero or not. For the zero value of this phase, in the leading order, the expression for the quantity $L_{(p)i}\langle j^l \rangle$ coincides with the formula for $\langle j^l \rangle$ in D-dimensional AdS spacetime, obtained from the geometry at hand by excluding the dimension x^i and replacing $\nu(D-1) \to \nu(D)$, where the function $\nu(D)$ is defined by (15). For $\tilde{\alpha}_i \neq 0$ the current density $\langle j^l \rangle$ is suppressed by the factor $e^{-L_i \sigma_i / L_i}$, where σ_i is defined in the paragraph after formula (39). For large values of the proper length of the lth dimension, compared with the AdS curvature radius, and for zero values of the phases along the remaining directions, the decay of the VEV $\langle j^l \rangle$, as a function of L_l is power-law, as $1/L_l^{p+2\nu+2}$ for both massless and massive fields. For massive fields this behavior is crucially different from that for the Minkowski bulk where the decay is exponential. If at least one of the phases $\tilde{\alpha}_i$, $i \neq l$, is not equal to zero, the suppression of the current density is exponential, given by (43). For small values of the proper length of the lth dimension, in the leading order, the VEV $\langle j^l \rangle$ does not depend on the phases along other directions and the current density behaves as $1/L_l^{D+1}$.

Appendix A. Transformation of the Hadamard function

For the transformation of the expression (19) we use the integral representation of the Bessel function (see [21])

$$J_\mu(x) = \frac{(x/2)^\mu}{2\pi i} \int_{-\infty}^{(0+)} du \frac{e^{u - x^2/(4u)}}{u^{\mu+1}}, \tag{50}$$

in which the phase of u increases from $-\pi$ to π as u describes the contour. Taking in this representation $\mu = -1/2$ and changing the integration variable to $s = u e^{i\pi}$, we can obtain the following representation

$$\frac{\cos(\omega \Delta t)}{\omega} = -\frac{1}{2\sqrt{\pi}} \int_C \frac{ds}{s^{1/2}} e^{-\omega^2 s + (\Delta t)^2/(4s)}, \tag{51}$$

with the contour of the integration depicted in Fig. 4. Note that the integral can also be presented in the form $\int_C ds = \int_{c_\rho} ds - 2\int_\rho^\infty ds$, where c_ρ is the anticlockwise oriented circle of the radius ρ, with the center at the origin of the complex s-plane.
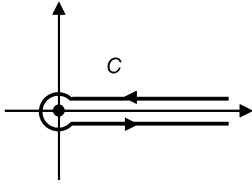
Fig. 4. The contour of the integration in the representation (51).

Substituting into (19) and the changing the order of the integrations, the integral over λ is evaluated by using the formula [22]

$$\int_0^\infty d\lambda\, \lambda\, J_\nu(\lambda z) J_\nu(\lambda z') e^{-\lambda^2 s} = \frac{1}{2s} \frac{I_\nu(zz'/2s)}{e^{(z^2+z'^2)/(4s)}}. \tag{52}$$

For the integral over the momentum along uncompactified dimensions we get

$$\int d\mathbf{k}_p e^{i\mathbf{k}_p \cdot \Delta\mathbf{x}_p - s\mathbf{k}_p^2} = (\pi/s)^{p/2} e^{-|\Delta\mathbf{x}_p|^2/(4s)}. \tag{53}$$

With these formulas, we can see that, under the condition $(\Delta t)^2 < |\Delta\mathbf{x}_p|^2 + z^2 + z'^2$, the part of the s-integral over the circle c_ρ vanishes in the limit $\rho \to 0$ and for the remaining integral one gets $\int_C ds = -2\int_0^\infty ds$. As a result, the following representation for the Hadamard function is obtained:

$$G^{(1)}(x,x') = \frac{a^{1-D}(zz')^{D/2}}{(4\pi)^{(p+1)/2} V_q} \sum_{\mathbf{n}_q \in \mathbf{Z}^q} e^{i\mathbf{k}_q \cdot \Delta\mathbf{x}_q}$$
$$\times \int_0^\infty ds \frac{I_\nu(zz'/2s)}{s^{(p+3)/2}} e^{-s\mathbf{k}_q^2 - [|\Delta\mathbf{x}_p|^2 + z^2 + z'^2 - (\Delta t)^2]/(4s)}. \tag{54}$$

For the further transformation we note that in (54) the series corresponding to the lth compact dimension has the form $\sum_{n_l=-\infty}^{+\infty} e^{ik_l \Delta x^l - s^2 k_l^2}$. From the Poisson resummation formula it directly follows that

$$\sum_{n=-\infty}^{+\infty} f(k_l) = \frac{L_l}{2\pi} \sum_{n_l=-\infty}^{+\infty} e^{in_l\tilde\alpha_l} \tilde f(n_l L_l), \tag{55}$$

where $\tilde f(y) = \int_{-\infty}^{+\infty} dx e^{-iyx} f(x)$. From this formula one gets

$$\sum_{n_l=-\infty}^{+\infty} e^{ik_l \Delta x^l - sk_l^2} = \frac{L_l}{2\sqrt{\pi s}} \sum_{n_l=-\infty}^{+\infty} e^{in_l\tilde\alpha_l} e^{-(\Delta x^l - L_l n_l)^2/(4s)}. \tag{56}$$

With this transformation, the Hadamard function is expressed as

$$G^{(1)}(x,x') = \frac{a^{1-D}}{(2\pi)^{D/2}} \sum_{\mathbf{n}_q \in \mathbf{Z}^q} e^{i\tilde\alpha \cdot \mathbf{n}_q} \int_0^\infty dx\, x^{D/2-1} I_\nu(x) e^{-v_{\mathbf{n}_q} x}, \tag{57}$$

where

$$\tilde\alpha = (\tilde\alpha_{p+1}, \ldots, \tilde\alpha_{D-1}), \tag{58}$$

and $v_{\mathbf{n}_q}$ is defined by (21). The integral in (57) is expressed in terms of the hypergeometric function and we obtain the representation (20).

References

[1] C.G. Callan Jr., F. Wilczek, Nucl. Phys. B 340 (1990) 366.
[2] J.M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231;
    E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253;
    S.S. Gubser, I.R. Klebanov, A.M. Polyakov, Phys. Lett. B 428 (1998) 105.
[3] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Phys. Rep. 323 (2000) 183.
[4] V.A. Rubakov, Phys. Usp. 44 (2001) 871;
    P. Brax, C. Van de Bruck, Class. Quantum Gravity 20 (2003) R201;
    E. Kiritsis, Phys. Rep. 421 (2005) 105;
    R. Maartens, K. Koyama, Living Rev. Relativ. 13 (2010) 5.
[5] G. Plunien, B. Muller, W. Greiner, Phys. Rep. 134 (1986) 87;
    E. Elizalde, S.D. Odintsov, A. Romeo, A.A. Bytsenko, S. Zerbini, Zeta Regularization Techniques with Applications, World Scientific, Singapore, 1994;
    V.M. Mostepanenko, N.N. Trunov, The Casimir Effect and Its Applications, Oxford University Press, Oxford, 1997;
    K.A. Milton, The Casimir Effect: Physical Manifestation of Zero-Point Energy, World Scientific, Singapore, 2002;
    M. Bordag, G.L. Klimchitskaya, U. Mohideen, V.M. Mostepanenko, Advances in the Casimir Effect, Oxford University Press, Oxford, 2009;
    D. Dalvit, P. Milonni, D. Roberts, F. da Rosa (Eds.), Casimir Physics, Lecture Notes in Physics, vol. 834, Springer-Verlag, Berlin, 2011.
[6] E. Elizalde, S.D. Odintsov, A.A. Saharian, Phys. Rev. D 87 (2013) 084003.
[7] A. Flachi, J. Garriga, O. Pujolàs, T. Tanaka, J. High Energy Phys. 08 (2003) 053;
    A. Flachi, O. Pujolàs, Phys. Rev. D 68 (2003) 025023;
    A.A. Saharian, Phys. Rev. D 73 (2006) 044012;
    A.A. Saharian, Phys. Rev. D 73 (2006) 064019;
    A.A. Saharian, Phys. Rev. D 74 (2006) 124009;
    E. Elizalde, M. Minamitsuji, W. Naylor, Phys. Rev. D 75 (2007) 064032;
    R. Linares, H.A. Morales-Técotl, O. Pedraza, Phys. Rev. D 77 (2008) 066012;
    M. Frank, N. Saad, I. Turan, Phys. Rev. D 78 (2008) 055014.
[8] S. Bellucci, A.A. Saharian, Phys. Rev. D 82 (2010) 065011.
[9] S. Bellucci, A.A. Saharian, Phys. Rev. D 87 (2013) 025005.
[10] E.R. Bezerra de Mello, A.A. Saharian, Phys. Rev. D 87 (2013) 045015;
     S. Bellucci, E.R. Bezerra de Mello, A.A. Saharian, Phys. Rev. D 89 (2014) 085002.
[11] S. Bellucci, A.A. Saharian, H.A. Nersisyan, Phys. Rev. D 88 (2013) 024028.
[12] E.R. Bezerra de Mello, A.A. Saharian, Eur. Phys. J. C 73 (2013) 2532;
     S. Bellucci, E.R. Bezerra de Mello, A. de Padua, A.A. Saharian, Eur. Phys. J. C 74 (2014) 2688.
[13] M.R. Douglas, S. Kachru, Rev. Mod. Phys. 79 (2007) 733.
[14] P. Breitenlohner, D.Z. Freedman, Phys. Lett. B 115 (1982) 197;
     P. Breitenlohner, D.Z. Freedman, Ann. Phys. (N.Y.) 144 (1982) 249;
     L. Mezincescu, P.K. Townsend, Ann. Phys. (N.Y.) 160 (1985) 406.
[15] S.J. Avis, C.J. Isham, D. Storey, Phys. Rev. D 18 (1978) 3565.
[16] A. Ishibashi, R.M. Wald, Class. Quantum Gravity 21 (2004) 2981.
[17] A. Ishibashi, R.M. Wald, Class. Quantum Gravity 20 (2003) 3815.
[18] N.D. Birrell, P.C.W. Davis, Quantum Fields in Curved Space, Cambridge University Press, Cambridge, 1982.
[19] A. Flachi, D.J. Toms, Nucl. Phys. B 610 (2001) 144.
[20] A.A. Saharian, Nucl. Phys. B 712 (2005) 196.
[21] G.N. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 1966.
[22] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, Integrals and Series, vol. 2, Gordon and Breach, New York, 1986.