The Mathematical Work of Joel Lee Brenner*

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INTRODUCTION

Joel Lee Brenner was born on 2 August 1912, in Boston, Massachusetts, the older of the two sons of Anna (Aronie) and Samuel Brenner. His mother was a trained teacher and educated her sons at home for a time. Joel then entered public schools in Boston and graduated from high school high in his class at the age of 14. He recalls a "work problem" which was presented by his seventh-grade teacher: "A can mow a lawn in three days and B can mow the same lawn in four days. How long will it take if the two work together?" The teacher allowed herself to be convinced that her answer, $3\frac{1}{2}$ days, should be corrected to $\frac{12}{7}$ days. While intermediate algebra was the highest level of math taught at his high school, he studied more advanced texts which were furnished him by friends. Following his graduation from high school, he was admitted to Harvard with advanced credit in several subjects including "Advanced Algebra." In 1930, at the age of 17, Joel graduated with honors with a bachelors degree in chemistry. At this time he switched to mathematics, as M. H. Stone and J. von Neumann had done before him, and came under the influence of H. W. Brinkmann, G. Birkhoff, and M. H. Stone, all of whom furnished the sympathy and understanding an immature young man needed. Joel received his masters degree in 1931, but because of financial difficulties, he left school to teach at New York University as a full-time instructor in 1931–32 and also in 1935. His students in 1931 were startled to

*With Contributions by R. P. Boas, Northwestern University, Evanston, Illinois; John dePillis, University of California, Riverside, California; Roger Lyndon, University of Michigan, Ann Arbor, Michigan; and James Wiegold, University College, Cardiff, Wales.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/87/$3.50

note that he was younger than any of them. However, his teaching at NYU was successful, and he returned to Harvard to study in 1932. In spite of the interruptions, he was granted his Ph.D. in February 1936. Garrett Birkhoff suggested his thesis topic and offered advice as needed, but M. H. Stone signed as his supervisor, since Birkhoff was not yet on the faculty. Only Stone himself had received a Ph.D. in mathematics from Harvard at a younger age.

For a wonderful glimpse of life at Harvard in the 1930s, see [81].

Since receiving his Ph.D., Joel has taught at approximately 12 universities and colleges, and also has worked as a mathematician at private firms and for governmental agencies. He spent 12 years (1956–1968) as Senior Mathematician at Stanford Research Institute.

Joel and his wife Frances had 3 children. Mary, the oldest, is a TV producer in New York City. The other two, Elizabeth and Charles, are more mathematical. Elizabeth is a systems engineer in the San Francisco Bay area, and Charles is a mathematician specializing in computer design. Joel's wife, Frances, was on the Palo Alto city council until her death in 1981. Joel lives at 10 Phillips Road, Palo Alto, California 94303; his telephone number is (415) 326-2970. He welcomes visits and telephone calls from mathematicians ("and former mathematicians"). While mathematics involves most of his time, he also enjoys music. He plays violin and viola with many of his visitors.

TEACHING AND SERVICE

Joel was officially employed in teaching for some twenty-five years. While he has never produced a Ph.D. student, he has had a great deal of influence on many young mathematicians. John dePillis writes:

> One thing I remember about my encounter with Joel Brenner some fifteen years ago. ...It changed my direction of research.
>
> We first met about 1970 when we discussed some topics in linear algebra which led to an investigation of eigenvalues of partitioned matrices. I became interested in matrices over finite fields and modules after Joel presented some ideas along these lines. Joint work followed which included [56] and [58].
>
> In [56], generalizations were developed for positive semidefinite matrices which are partitioned into blocks. If the blocks are one by one, then the Hadamard and Fisher inequalities result.
>
> In [58], there is a nice idea that Joel introduced, in proving "Fermat's last theorem" for two by two matrices over the module of integers. This result uses the so called Dieudonné determinant in a pretty way.
>
> Joel has always had a stream of ideas and knowledge that is broad and deep. For example, Joel would almost certainly link my own interests of the moment with something unknown to me. This ability of his was made apparent to me very early on.

It is no wonder, then, that working with Joel would lead to a change in my own direction of research. His ideas were always provocative and imaginative. It is my great good fortune that we met.

The experiences of the author are quite similar. I first met Joel when I was a graduate student and he was visiting the University of British Columbia in 1966–67. Shortly after arriving at UBC, Joel circulated a memo to all the graduate students, informing them that he had several open problems in various areas of mathematics and would share them with willing students. Hoping to get a problem in group theory that I might work into a thesis, I went to his office and inquired about the problems. He presented me the Van der Waerden conjecture, which he informed me would be quite difficult, and after defining the permanent for me sent me off with several problems concerning the permanent function. His encouragement and enthusiasm persevered through several "proofs" of the Van der Waerden conjecture, and soon some of the less well-known problems had been solved. He would always tell me how a proposed attack would work and leave me to fight out the details. Those exchanges led to the publication of my first paper, and I became his thirteenth coauthor. By the time Joel had left UBC in the spring of 1967, I was firmly entrenched in matrix theory.

In a letter to the author Joel relates the following incident which is typical of his inventiveness and enthusiasm:

A student I had after the war was Myers—I think John Myers. He was taking an extra credit course in solid geometry, but he wasn't doing well. In fact, he got every problem wrong on each of the first three tests—a straight zero. I tried a special approach on him. We were coming soon to the volume of a pyramid; this is hard to teach because the three pyramids that make up a triangular prism are not easily seen to be congruent, even though they have the same volume; and they are nested in a special way. I asked this student to make me a wooden model of the three pyramids and the way they fit. This he did, even from redwood; this is a very hard wood to plane and cut, because it splinters so easily. His model was most useful, and I kept it for many years.

After that, Myers never got anything wrong. If he had to multiply 3.1416 by 22 by 144, his answer was correct to the last decimal place, the first time he tried it. His grade went from straight zero to straight 100. This is the largest improvement any student of mine (or anyone else's) has ever shown.

Joel's service to the mathematical community has not consisted solely of teaching. He was the editor of the elementary problems section of the M.A.A. *Monthly* from 1978 to 1983. The editor of the *Monthly* during most of that time was R. P. Boas, who writes:

My editor for elementary problems resigned, and I called Joel to ask for suggestions. He volunteered to take on the job himself. He should have been put in charge of the whole problems section, but I couldn't do this; however,... Joel did most of the work; also Lyndon, who was the editor for advanced problems, was out of the country a good deal, and Joel took over that section by default.

He was extremely helpful and had high standards. He must have spent enormous amounts of time on the problems section, and I thought he brought it into a better state than anyone before or since.

Joel has recently been appointed, at the suggestion of R. P. Boas, an associate editor of the *Journal of Mathematical Analysis and Applications.*

RESEARCH

As a glance at the attached list of publications will indicate, Joel has produced a lot of mathematics and worked with a lot of mathematicians, approximately 35 coauthors. A closer look at Joel's publication list will also reveal the varied nature of his publications—from the study of satellites in near-earth orbit, to generators of groups. Most of his published work falls into three general areas: matrix theory, group theory, and combinatorics. However, as R. C. Lyndon notes, "The majority of Joel's papers deal with Matrix Theory, pure and simple."

Contributions to Matrix Theory

Joel's first mathematical paper [2] was a proof that if the orders of the elements of an abelian group are bounded then the group is decomposable into the direct product of cyclic groups. He obtained this result in 1935; however, it did not appear until after the publication of his thesis. It is now known that this result has as a consequence the possibility of transforming any matrix (over an algebraically closed field) into the Jordan canonical form. Joel's proof is in two paragraphs. The first paragraph states the induction hypothesis, and the second, also short, gives the argument.

Joel's next work, which contained the results of his thesis [1] and subsequent work [4, 25], deals with the group of matrices of nonzero determinant with entries from the ring of integers modulo p^r, where p is a rational prime. This ring has turned out to be a suggestive one, and many papers have since been written on matrices over general rings, usually rings that have one or another of the special properties enjoyed by the ring Z/p^rZ.

During the Second World War, Joel's research waned, but after the war, largely because of the encouragement of D. H. Lehmer, he returned to research, publishing several articles on bounds for determinants and eigenvalue location. In fact, Joel wrote the first article in which the matrix being considered was thought of as a block partitioned matrix [15]. In 1954, while at the Ballistics Research Laboratories in Maryland as a visiting scientist, Joel discovered a theorem that appeared in 1959, entitled "Relations among the minors of a matrix with dominant diagonal" [24]. In *Permanents* [Vol. 6 of the Encyclopedia of Mathematics and its Applications (Addison-Wesley, 1978)], Henryk Minc credits this article as one of the three that caused a reawakening of interest in permanents of matrices, a subject which had been dormant from the mid 1920s. The theorem is:

Let A be a matrix of p rows and n (> p) columns. Suppose the principal elements are nonzero and dominate:

$$a_{ii} \neq 0; \qquad \sigma_i |a_{ii}| = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad 0 \leqslant \sigma_i \leqslant 1, \quad i = 1, \ldots, p.$$

then if $(p =)\ r_0 \leqslant r_1 \leqslant \cdots \leqslant r_{p-1}\ (= n)$ *are any fixed integers, the following inequality is valid (empty sums are* 0, *empty products are* 1):

$$\sigma_1 \sigma_2 \cdots \sigma_p \det A[1,2,\ldots,p|1,2,\ldots,p]$$

$$\geqslant \sum_{s=1}^{p} \left\{ \sigma_1 \sigma_2 \cdots \sigma_{p-s} \sum_{t \in O_s} \det A[1,2,\ldots,p|1,2,\ldots,p-s,t_1,\ldots,t_s] \right\},$$

where $\det A[\alpha_1,\ldots,\alpha_q|\beta_1,\ldots,\beta_q]$ *is the absolute value of the determinant of the submatrix of A lying in rows* $\alpha_1,\ldots,\alpha_q$ *and columns* $\beta_1,\ldots,\beta_q$; *and* O_s *is the (possibly empty) set of s-tuples of strictly increasing sequences,* $r_{s-1} < t_1 < t_2 < \cdots < t_s \leqslant r_s$.

Since the permanent can be expanded by minors, the same theorem holds on replacing "det" by "per." From this theorem it was shown that if A is row stochastic (or substochastic)—that is, the sum of the absolute values of all entries on any row is 1 (at most 1)—then $\operatorname{per}(I - A) \geqslant 0$; and that the permanental roots, i.e. roots of $\operatorname{per}(zI - A)$, lie within the Geršgorin disks [41, 45]: facts that had been conjectured by M. Marcus and M. Newman in "Inequalities for the permanent function," *Ann. Math* 75:47–62 (1962). In a series of articles [42, 43, 51, 53, 54] appearing in the 1960s, Joel developed several generalizations of Geršgorin's theorem, most using the results of [24].

In 1951, H. Flanders ["Elementary divisors of AB and BA," *Proc. Amer. Math. Soc.* 2:871–874 (1951)] derived the relation between the elementary divisors of AB and BA, a deep question in case A and B are both singular. He showed that the elementary divisors that correspond to nonzero eigenvalues are the same; but the elementary divisors corresponding to 0 can be different, though they can differ in degree by at most one. Twenty-one years later, in [66], Joel raised the question concerning the elementary divisors of ABC, BCA, and CAB. Since any pair of ABC, BCA, and CAB can be written in the form XY, YX, Flander's theorem applies; but it does not tell the whole story. Joel showed what other necessary condition must be enjoyed by the degrees of the respective elementary divisors of this set of three matrices.

Contributions to Group Theory[1]

Joel Brenner's work in group theory goes right back to the late thirties, covers more than twenty-five items, and is still going strong. The common feature of all the papers, or nearly all, is that they are "elementary" in that they deal with complex calculations with elements (be they permutations or matrices), rather than structural. As Roger Lyndon has remarked to me, by now Joel probably knows more than anyone living (or dead?) on the behavior of actual real live permutations, and it is to be hoped that he will commit this knowledge to writing for posterity.

After some early work on linear homogeneous groups, Joel proved some nice theorems in 1955 on free groups of matrices [16], a theme he returned to in 1975 [70]. The seventies saw a great upsurge of group-theoretical work, when he began to be interested in covering theorems for nonabelian simple groups [63, 64, 67, 69, 72, 74, 76, 77, 92]. These papers give detailed information on the following sort of problem. What is the least n such that $C^n = G$, where C is a class of conjugate elements in a finite simple group G? This work was carried out with several collaborators (another common feature of Joel's writings) and has been taken up in a big way in Israel. My own very happy collaboration [71, 73, 82, 95] with Joel sprang up as a result of a query he directed to me concerning covering the alternating group with powers of classes of cyclic elements. We got interested in the concept of *spread*. A two-generator group has spread r if every set of r nontrivial elements $x_1, x_2, \ldots, x_r$ of G have a common mate y, that is, G is generated by y and x_i, for each $i = 1, 2, \ldots, r$. By now it is known what the exact spreads of the alternating groups of even degree are, though it is still a conjecture (due, as I recall, to Joel) that the spread of A_{2n+1} tends to ∞ with n. As well as the spread of the projective special linear group, where his knowledge of matrices was crucial, we have considered the Mathieu groups from this angle (with Guralnick in Reference [97]). The concept has been

[1]Contributed by James Wiegold, University College, Wales.

used by Dr. M. J. Evans (Ph.D. thesis, University of Wales) to prove that the Suzuki groups have just one T_3-system each. In addition to this, and many of the papers that I have not mentioned, Joel's rich collaboration with Lyndon on the properties of permutations has a strong group-theoretical as well as combinatorial flavor.

Contribution to Combinatorics and the Study of Permutations[2]

Joel has worked a lot with the problem of generating sets of familiar groups, especially finite simple groups. It seems to be established that most (probably all) finite simple groups are generable by an element a of order 2 together with an element b of prime order. (M.D.E. Conder has recently strengthened results of G. Higman in showing that, for every $k \geqslant 7$, all but finitely many finite alternating and symmetric groups are generated by a and b of respective orders 2,3 satisfying the additional condition that ab is of order k.) Joel has worked extensively on the question of when a permutation group is covered by a class of elements of specified cycle structure, that is, by the product of two conjugate elements of (the same) cycle structure. He has shown, for example:

(i) *If $n \geqslant 9$ and if $1 \leqslant s_1 \leqslant s_2 < n$, $s_1 + s_2 = n$, then every permutation P in A_n is expressible as the product $P = QR$ of two (conjugate) permutations Q, R, both of type $s_1^1 s_2^1$, that is, with cycles of respective lengths s_1, s_2, and no others, in their canonical decomposition.*

(ii) (with J. Riddell) *Every permutation P in A_{4s} can be written as the product $P = QR$ of two permutations Q, R in S_{4s}, where Q, R are conjugate and both have just 4-cycles in their canonical decompositions.*

(iii) *If $n > 3$, $q > n+1$, the group $G = PSL(n, q)$ has a class C such that $CC = G$. This recovers and strengthens some results of R. C. Thompson, who showed that every element in such G is a commutator.*

(iv) (with R. J. Evans and D. M. Silberger) *If $n > 4$ and $n > \frac{5}{2}\log m$, m = largest square-free divisor of rs, then every element P in A_n can be written $P = x^r y^s$, for x, y in A_n.* This improves similar results of M. Droste.

(v) (unpublished) *If $n \geqslant k+2$, and if $4 \leqslant k \leqslant 20$, k even, then every permutation in A_n can be written as the product $P = QR$ of two conjugate permutations Q, R of period k (in A_n), and hence also in the form $Q^k R^k QR$.*

(vi) ([111]; with L. B. Beasley, P. Erdős, M. Szalay, A. Williamson) *The proportion of classes in A_n that contain a pair of generators is $1 - \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$.*

[2]Contributed by Roger C. Lyndon, University of Michigan.

Joel's work with me can be viewed as arising from a variation on the problem above. A problem of B. H. Neumann (1933), arising from foundations of geometry and posed in terms of the modular group, can be interpreted as that of classifying all pairs of permuations a and b of a countably infinite set Ω such at $a^2 = 1$, $b^3 = 1$, and ab is transitive on Ω. We solved this by a classification of all coset graphs of the groups generated by a and b in the modular group. (The same argument was used independently by W. W. Stothers). We applied similar pictorial methods to other similar problems; for example: if $b \in \operatorname{sym}\Omega$ satisfies certain obvious conditions (notably, if b contains some infinite cycle), then there exists a such that $a^2 = 1$ and ab is transitive.

A theorem of G. A. Miller (1900) states that if α, β, and γ, are integers greater than 1, then there exist permutations a and b of some finite set Ω such that a, b, and ab have orders α, β, and γ. (This is the hard part of Fenchel's theorem that every Fuchsian group contains a torsion free subgroup of finite index.) By methods similar to those used above, we showed that ab can always be chosen with exactly one or two (according to parities) nontrivial cycles. Miller (1902) showed that such a, b can be chosen, with the group they generate transitive on Ω, for finite sets Ω of unbounded cardinality d. There is an obvious lower bound for $d = d(\alpha, \beta, \gamma)$ in terms of α, β, γ. We found an explicit (infinite) set of cases, when one of α, β, γ is 2, where this value cannot be attained, together with extensive evidence for the conjecture that it can be attained in all other cases.

Our methods led us to consider graphs with a doubly Eulerian trail: a reduced path that traverses each edge exactly once in each direction. We showed, among other things, that (the 1-skeleton of) every regular tessellation of the Euclidean or hyperbolic plane admits such a trail. Joel's playful spirit and athletic prowess is exemplified by the discovery that the infinite n-dimensional jungle gym (cubic lattice) also admits a doubly Eulerian trail.

Current Activities and Future Plans

Joel writes:

Mathematically, I am still working on inequalities and permutations. Perhaps a monograph on permutations will appear within the next five years.

The canard that "significant mathematics is the purview of men and women under 35" will be contradicted by examples of important mathematics by 100 individual mature mathematicians over the past 150 years. Please send me your contribution to overthrowing this popular misconception—your nomination.

Socially and musically, I plan to enjoy the creations of my colleagues, and to connect my life with theirs. Eventually, I may take up clarinet, saxophone, flute, guitar, piano, and perhaps harmonica.

His friends and associates wish him success in his endeavors and a happy, healthy, and long life.

I would like to thank all those individuals who contributed to the writing of this article, in particular James Wiegold, Roger Lyndon, R. P. Boas, John dePillis, and especially Joel himself.

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BOOKS TRANSLATED AND EDITED BY JOEL LEE BRENNER

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Received 9 May 1986