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An improvement of a recent closed graph theorem

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A B S T R A C T

We obtain a new closed graph theorem which is a substantial improvement of a recent result.

There is a series of generalizations and improvements of the classical closed graph theorem, such as [1–4]. Especially, [4] has given a closed graph theorem which is available for all linear operators and many more nonlinear mappings.

Let X be a vector space and Y a topological vector space. A mapping $f: X \to Y$ is said to be quasi-linear if f satisfies the following (1)–(3):

(1) if $f(x_n) \to 0$ and $f(u_n) \to 0$, then $f(x_n + u_n) \to 0$;
(2) if $f(x_n - x) \to 0$ and $t_n \to t$ in the scalar field $\mathbb{K}$, then $f(t_n x_n - tx) \to 0$;
(3) $f(x_n) \to f(u)$ if and only if $f(x_n - u) \to 0$.

Note that, when Y is Hausdorff, taking $x_n = u = 0$ in (3) yields

(4) $f(0) = 0$.

As was shown in [4], the family of quasi-linear mappings is a large extension of the family of linear operators. Just taking the conditions (1)–(3), Shuhui Zhong and Ronglu Li [4] established a closed graph theorem as follows.

Theorem. *(See [4, Theorem 2.1].) Let X, Y be Fréchet spaces. If $f: X \to Y$ is quasi-linear and its graph $G = \{(x, f(x)):\ x \in X\}$ is closed in $X \times Y$, then f is continuous.*

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In this paper we would like to give a further improvement of this recent result. For topological vector spaces X and Y, a mapping $f : X \to Y$ is said to be weakly quasi-linear if f satisfies (1), (2) and

(3′) if $x_n - u \to 0$ in X, then $f(x_n) \to f(u)$ if and only if $f(x_n - u) \to 0$.

Note that, when Y is Hausdorff, taking $x_n = u = 0$ in (3′) also yields (4).

Evidently, if $f : X \to Y$ satisfies condition (3), then (3′) must hold for f. So every quasi-linear mapping $f : X \to Y$ is weakly quasi-linear. In the following, we first improve the closed graph theorem in [4] by using weakly quasi-linear mappings instead of quasi-linear mappings, and then show that the family of weakly quasi-linear mappings is a large extension of the family of quasi-linear mappings.

1. The closed graph theorem for weakly quasi-linear mappings

A Fréchet space is a complete metrizable linear space. However, a Fréchet space is also a separated complete paranormed space [5, p. 56].

Theorem 1.1. *Let X, Y be Fréchet spaces. If $f : X \to Y$ is weakly quasi-linear and its graph $G = \{(x, f(x)) \colon x \in X\}$ is closed in $X \times Y$, then f is continuous.*

Proof. Let $X = (X, \|\cdot\|_1)$ and $Y = (Y, \|\cdot\|_2)$, where $\|\cdot\|_1$ and $\|\cdot\|_2$ are paranorms [5, p. 15]. Define $d : X^2 \to \mathbb{R}$ by $d(x,u) = \|x-u\|_1 + \|f(x) - f(u)\|_2$, $\forall x, u \in X$. It is easy to see that d is a metric on X.

Suppose $\{x_n\}$ is Cauchy in (X, d), i.e., $d(x_n, x_m) = \|x_n - x_m\|_1 + \|f(x_n) - f(x_m)\|_2 \to 0$ as $n, m \to +\infty$ so $\{x_n\}$ and $\{f(x_n)\}$ are Cauchy in $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ respectively. Since X, Y are complete, there exist $x \in X$ and $y \in Y$ such that $\|x_n - x\|_1 \to 0$, $\|f(x_n) - y\|_2 \to 0$. Then $y = f(x)$ for f has closed graph in $X \times Y$. Now $d(x_n, x) = \|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 = \|x_n - x\|_1 + \|f(x_n) - y\|_2 \to 0$. Therefore, (X, d) is complete.

If $x_n \to x$ and $u_n \to u$ in (X, d), then $d(x_n, x) = \|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 \to 0$ and $d(u_n, u) = \|u_n - u\|_1 + \|f(u_n) - f(u)\|_2 \to 0$. By (3′), $\|f(x_n - x)\|_2 \to 0$, $\|f(u_n - u)\|_2 \to 0$ and $\|f(x_n + u_n - x - u)\|_2 \to 0$ by (1). Since $\|x_n + u_n - (x+u)\|_1 \leqslant \|x_n - x\|_1 + \|u_n - u\|_1 \to 0$, by (3′) again, $\|f(x_n + u_n) - f(x+u)\|_2 \to 0$. Thus, $d(x_n + u_n, x + u) \to 0$, that is, the additive operation is continuous in (X, d).

Suppose that $t_n \to t$ in the scalar field $\mathbb{K}$ and $x_n \to x$ in (X, d). Then $d(x_n, x) = \|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 \to 0$, i.e., $\|x_n - x\|_1 \to 0$, $\|f(x_n) - f(x)\|_2 \to 0$. So $\|f(x_n - x)\|_2 \to 0$ by (3′) and $\|f(t_n x_n - tx)\|_2 \to 0$ by (2), and since $\|t_n x_n - tx\|_1 \to 0$, $\|f(t_n x_n) - f(tx)\|_2 \to 0$ by (3′) again. Then $d(t_n x_n, tx) \to 0$, and therefore the scalar multiplication is also continuous in (X, d).

Thus, (X, d) is a complete metrizable linear space, that is, (X, d) is a Fréchet space and, letting $I(x) = x$ for $x \in X$, $I : (X, d) \to (X, \|\cdot\|_1)$ is continuous, one-to-one and onto. By the open mapping theorem [5, p. 58], the converse $I^{-1} : (X, \|\cdot\|_1) \to (X, d)$ is also continuous.

Now let $x_n \to x$ in $(X, \|\cdot\|_1)$. Then $x_n = I^{-1}(x_n) \to I^{-1}(x) = x$ in (X, d), that is, $\|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 \to 0$ so $\|f(x_n) - f(x)\|_2 \to 0$, i.e., $f(x_n) \to f(x)$ in $(Y, \|\cdot\|_2)$. This shows that $f : (X, \|\cdot\|_1) \to (Y, \|\cdot\|_2)$ is continuous. □

Clearly, the recent result in [4] is a special case of this new closed graph theorem. Moreover, the continuity version Theorem 2.2 of [4] is also a special case of the following Corollary 1.1.

Corollary 1.1. *Let $(X, \mathfrak{T})$, $(Y, \mathfrak{I})$ be Fréchet spaces and τ a Hausdorff topology for Y which is coarser than $\mathfrak{I}$. If $f : X \to (Y, \mathfrak{I})$ is weakly quasi-linear and $\mathfrak{T} - \tau$ continuous, then f is $\mathfrak{T} - \mathfrak{I}$ continuous.*

Proof. Hausdorff τ and continuous f ensure that the graph G is closed in $(X, \mathfrak{T}) \times (Y, \tau)$, and thus also in $(X, \mathfrak{T}) \times (X, \mathfrak{I})$, since the latter has a finer topology. The conclusion now follows from Theorem 1.1. □

2. Weakly quasi-linear mappings

Let $wql(X, Y)$ denote the family of all weakly quasi-linear mappings from the topological vector space X to the topological vector space Y.

Remark. It is obvious that $f \in wql(X, Y)$ is sequentially continuous at every point if it is sequentially continuous at one point.

Proposition 2.1. *If X, Y are Hausdorff and X is finite-dimensional, then every $f \in \mathrm{wql}(X, Y)$ is continuous.*

Proof. Let $e_1, \ldots, e_k$ be a Hamel basis for X. Let $z_n \to 0$ in X, where $z_n = \sum_{i=1}^{k} t_{ni} e_i$. For each $i \leqslant k$ we have $\lim_n t_{ni} = 0$, so that (4) and (2), with $t = 0$ and each $x_n = x = e_i$, imply that

$$\lim_n f(t_{ni} e_i) = \lim_n f(t_{ni} e_i - t e_i) = 0.$$

An inductive version of (1) gives us

$$\lim_n f(z_n) = \lim_n f\left(\sum_{i=1}^{k} t_{ni} e_i\right) = 0 = f(0).$$

Therefore f is sequentially continuous at 0, and hence is sequentially continuous everywhere. Since $\dim X < +\infty$, f is continuous everywhere. □

We have a simple fact which is helpful to our knowledge of weakly quasi-linear mappings.

Proposition 2.2. *A nonzero $\varphi : \mathbb{R} \to \mathbb{R}$ is weakly quasi-linear if and only if*

(I) $\varphi(0) = 0$, *and*
(II) φ *is continuous,* $\varphi(x) \neq 0$ *for all* $x \neq 0$ *and* $\varphi(x_n) \nrightarrow 0$ *whenever* $x_n \to \infty$.

Proof. If $\varphi \in wql(\mathbb{R}, \mathbb{R})$ and $\varphi \neq 0$, then $\varphi(0) = 0$ by (4), and φ is continuous by Proposition 2.1.

Suppose $\varphi(x_0) = 0$ for some $x_0 \neq 0$. Let $t \in \mathbb{R}$ and for every $n \in \mathbb{N}$ let $t_n = t$ and $x_n = x_0$. Then $t_n \to t$ and $\varphi(x_n - 0) = \varphi(x_0) = 0$ so $\varphi(t x_0) = \varphi(t_n x_n - t \cdot 0) \to 0$ by (2). Thus, $\varphi(t x_0) = 0$ for all $t \in \mathbb{R}$ so $\varphi(x) = 0$, $\forall x \in \mathbb{R}$, i.e., $\varphi = 0$, a contradiction. Hence, $\varphi(x) \neq 0$ for all $x \neq 0$.

Suppose that $x_n \to \infty$ and $\varphi(x_n) \to 0$. Then $\frac{1}{x_n} \to 0$ and $\varphi(x_n - 0) = \varphi(x_n) \to 0$. By (2), $\varphi(1) = \varphi(\frac{1}{x_n} \cdot x_n - 0 \cdot 0) \to 0$, i.e., $\varphi(1) = 0$. As was stated above, $\varphi = 0$. This is a contradiction so $\varphi(x_n) \nrightarrow 0$ for every $x_n \to \infty$.

Conversely, suppose that both (I) and (II) hold for φ. Since $\varphi(0) = 0$ and φ is continuous, $\varphi(x_n) \to 0$ when $x_n \to 0$. Suppose that $x_n \nrightarrow 0$ but $\varphi(x_n) \to 0$. By passing to a subsequence if necessary, we assume that $|x_n| \geqslant \varepsilon > 0$ for all n. If $\{x_n\}$ is bounded, then there is a subsequence $x_{n_k} \to x$ with $|x| \geqslant \varepsilon$ and $\varphi(x) = \lim_k \varphi(x_{n_k}) = 0$, a contradiction. If $\{x_n\}$ is not bounded then there is a subsequence $x_{n_k} \to \infty$ and $\lim_k \varphi(x_{n_k}) = \lim_n \varphi(x_n) = 0$. This is also a contradiction and so $\varphi(x_n) \to 0$ implies $x_n \to 0$. Thus, $\varphi(x_n) \to 0$ if and only if $x_n \to 0$.

If $\varphi(x_n) \to 0$ and $\varphi(u_n) \to 0$, then $x_n \to 0$ and $u_n \to 0$ so $x_n + u_n \to 0$ and $\varphi(x_n + u_n) \to 0$. Thus, (1) holds for φ. If $\varphi(x_n - x) \to 0$ and $t_n \to t$ in $\mathbb{K}$, then $x_n - x \to 0$ so $t_n x_n - t x \to 0$ and $\varphi(t_n x_n - t x) \to 0$. Thus, (2) holds for φ. Let $x_n - u \to 0$ in $\mathbb{R}$. Then $x_n \to u$. Since φ is continuous, both $\varphi(x_n - u) \to \varphi(0) = 0$ and $\varphi(x_n) \to \varphi(u)$ hold. This shows that if $x_n - u \to 0$ then $\varphi(x_n) \to \varphi(u)$ if and only if $\varphi(x_n - u) \to 0$, i.e., (3′) holds for φ. □

Obviously, condition (II) of Proposition 2.2 is much weaker than (II) of Proposition 1.4 in [4]. So condition (3′) is much looser than (3) and it becomes very easy to find weakly quasi-linear mappings which are not quasi-linear.

Example 2.1. Let

$$\varphi(x) = \begin{cases} 2 + \sin(x - 2), & x > 2, \\ |x|, & x \leqslant 2. \end{cases}$$

By Proposition 2.2, $\varphi \in wql(\mathbb{R}, \mathbb{R})$. But φ is not monotonic, so φ is not quasi-linear [4, Proposition 1.4].

We would like to say that the family of weakly quasi-linear mappings is an important object in analysis because if $(X, \|\cdot\|)$ is a normed space and $\|\cdot\| \neq 0$ then the norm $\|\cdot\| : (X, \|\cdot\|) \to \mathbb{R}$ is not quasi-linear and so not linear but it must be weakly quasi-linear.

Proposition 2.3. *Let $(X, \|\cdot\|)$ be a nontrivial paranormed space* [5, p. 15]. *Define $f : X \to \mathbb{R}$ by $f(x) = \|x\|$, $\forall x \in X$. Then f is weakly quasi-linear but f is not quasi-linear when $\|\cdot\| \neq 0$.*

Proof. Since $f(x_n) \to 0$ means that $x_n \to 0$ in $(X, \|\cdot\|)$, (1) and (2) hold for f.

If $x_n - u \to 0$ and $f(x_n - u) \to 0$, then $|\|x_n\| - \|u\|| \leqslant \|x_n - u\| \to 0$ so $f(x_n) - f(u) = \|x_n\| - \|u\| \to 0$. If $x_n - u \to 0$ and $f(x_n) - f(u) \to 0$, then $f(x_n - u) = \|x_n - u\| \to 0$. Thus, (3′) holds for f.

If $\|\cdot\| \neq 0$ then (3) fails to hold for f. To see this, pick an $x \in X$ for which $\|x\| > 0$. Then $\|\frac{x}{2}\| \geqslant \frac{\|x\|}{2} > 0$. Letting $x_n = -\frac{x}{2}$ for $n \in \mathbb{N}$, $f(x_n) = \|-\frac{x}{2}\| = \|\frac{x}{2}\| \to \|\frac{x}{2}\| = f(\frac{x}{2})$ but $f(x_n - \frac{x}{2}) = f(-\frac{x}{2} - \frac{x}{2}) = f(-x) = \|-x\| = \|x\| > 0$ so $f(x_n - \frac{x}{2}) \nrightarrow 0$. □

Many Banach spaces contain a copy of $(c_0, \|\cdot\|_\infty)$ or $(\ell^1, \|\cdot\|_1)$ or a reflexive Banach space. Hence, linearly homeomorphic embedding $T : X \to Y$ happens frequently. Especially, for every complex Banach space X and every continuous linear operator $S : X \to X$, $\lambda I - S : X \to X$ is a linear homeomorphism for each $\lambda \in \mathbb{C} \setminus \sigma(S) \supset \{\lambda \in \mathbb{C}: |\lambda| > \|S\|\}$.

Proposition 2.4. *Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a continuous function such that*

$$0 < \mu = \inf_{t \geqslant 0} \varphi(t) \leqslant \sup_{t \geqslant 0} \varphi(t) = M < +\infty.$$

Let X, Y be normed spaces and $T : X \to Y$ a linearly homeomorphic embedding. If $f : X \to Y$ is defined by $f(x) = \varphi(\|x\|)T(x)$, $\forall x \in X$, then f is weakly quasi-linear.

Proof. Since $0 < \mu \leqslant \varphi(t) \leqslant M < +\infty$ for all $t \geqslant 0$ and T is a linear homeomorphism of X onto $T(X)$, $f(x_n) \to 0$ if and only if $T(x_n) \to 0$ and if and only if $x_n \to 0$. Thus, (1) and (2) hold for f.

If $x_n - u \to 0$ and $f(x_n - u) \to 0$, then $\big| \|x_n\| - \|u\| \big| \leqslant \|x_n - u\| \to 0$ i.e., $\|x_n\| \to \|u\|$. By the continuity of φ and T, $f(x_n) - f(u) = \varphi(\|x_n\|)T(x_n) - \varphi(\|u\|)T(u) \to 0$. If $x_n - u \to 0$ and $f(x_n) - f(u) \to 0$, then $f(x_n - u) = \varphi(\|x_n - u\|)T(x_n - u) \to \varphi(0)T(0) = 0$ since both φ and T are continuous. Thus, $(3')$ holds for f. □

Obviously, although the condition of Proposition 2.4 is much weaker than that of Proposition 3.1 in [4], the proof becomes much simpler. So condition $(3')$ is much looser than (3).

Note that if X is an infinite-dimensional Fréchet space then for every nontrivial Fréchet space Y there exist many linear operators from X to Y which are not continuous and, of course, many more weakly quasi-linear mappings from X to Y which are not continuous. The new closed graph theorem just shows that a weakly quasi-linear $f : X \to Y$ is continuous if and only if f has closed graph.

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