An improvement of a recent closed graph theorem

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ABSTRACT

We obtain a new closed graph theorem which is a substantial improvement of a recent result.

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There is a series of generalizations and improvements of the classical closed graph theorem, such as [1–4]. Especially, [4] has given a closed graph theorem which is available for all linear operators and many more nonlinear mappings.

Let X be a vector space and Y a topological vector space. A mapping f : X → Y is said to be quasi-linear if f satisfies the following

(1) if f(x_n) → 0 and f(u_n) → 0, then f(x_n + u_n) → 0;
(2) if f(x_n − x) → 0 and t_n → t in the scalar field K, then f(t_n x_n − tx) → 0;
(3) f(x_n) → f(u) if and only if f(x_n − u) → 0.

Note that, when Y is Hausdorff, taking x_n = u = 0 in (3) yields

(4) f(0) = 0.

As was shown in [4], the family of quasi-linear mappings is a large extension of the family of linear operators. Just taking the conditions (1)–(3), Shuhui Zhong and Ronglu Li [4] established a closed graph theorem as follows.

Theorem. (See [4, Theorem 2.1].) Let X, Y be Fréchet spaces. If f : X → Y is quasi-linear and its graph \( G = \{(x, f(x)) : x \in X\} \) is closed in \( X \times Y \), then f is continuous.
In this paper we would like to give a further improvement of this recent result. For topological vector spaces $X$ and $Y$, a mapping $f : X \to Y$ is said to be weakly quasi-linear if $f$ satisfies (1), (2) and

$$(3') \text{ if } x_n - u \to 0 \text{ in } X, \text{ then } f(x_n) \to f(u) \text{ if and only if } f(x_n - u) \to 0.$$ 

Note that, when $Y$ is Hausdorff, taking $x_n = u = 0$ in $(3')$ also yields (4).

Evidently, if $f : X \to Y$ satisfies condition (3), then $(3')$ must hold for $f$. So every quasi-linear mapping $f : X \to Y$ is weakly quasi-linear. In the following, we first improve the closed graph theorem in [4] by using weakly quasi-linear mappings instead of quasi-linear mappings, and then show that the family of weakly quasi-linear mappings is a large extension of the family of quasi-linear mappings.

1. The closed graph theorem for weakly quasi-linear mappings

A Fréchet space is a complete metrizable linear space. However, a Fréchet space is also a separated complete paranormed space [5, p. 56].

**Theorem 1.1.** Let $X$, $Y$ be Fréchet spaces. If $f : X \to Y$ is weakly quasi-linear and its graph $G = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$, then $f$ is continuous.

**Proof.** Let $X = (X, \| \cdot \|_1)$ and $Y = (Y, \| \cdot \|_2)$, where $\| \cdot \|_1$ and $\| \cdot \|_2$ are paranorms [5, p. 15]. Define $d : X^2 \to \mathbb{R}$ by $d(x, u) = \|x - u\|_1 + \|f(x) - f(u)\|_2$, $\forall x, u \in X$. It is easy to see that $d$ is a metric on $X$.

Suppose $(x_n)$ is Cauchy in $(X, d)$, i.e., $d(x_n, x_m) = \|x_n - x_m\|_1 + \|f(x_n) - f(x_m)\|_2 \to 0$ as $n, m \to +\infty$ so $(x_n)$ and $(f(x_n))$ are Cauchy in $(X, \| \cdot \|_1)$ and $(Y, \| \cdot \|_2)$ respectively. Since $X$, $Y$ are complete, there exist $x \in X$ and $y \in Y$ such that $\|x_n - x\|_1 \to 0$, $\|f(x_n) - y\|_2 \to 0$. Then $y = f(x)$ for $f$ has closed graph in $X \times Y$. Now $d(x_n, x) = \|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 = \|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 \to 0$. Therefore, $(X, d)$ is complete.

If $x_n \to x$ and $u_n \to u$ in $(X, d)$, then $d(x_n, x) = \|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 \to 0$ and $d(u_n, u) = \|u_n - u\|_1 + \|f(u_n) - f(u)\|_2 \to 0$. By $(3')$, $\|f(x_n - x)\|_2 \to 0$ and $\|f(u_n - u)\|_2 \to 0$ and $\|f(x_n + u_n - x - u)\|_2 \to 0$ by (1). Since $\|x_n + u_n - (x + u)\|_1 \leq \|x_n - x\|_1 + \|u_n - u\|_1 \to 0$, by $(3')$ again, $\|f(x_n + u_n) - f(x + u)\|_2 \to 0$. Thus, $d(x_n + u_n, x + u) \to 0$, that is, the additive operation is continuous in $(X, d)$.

Suppose that $t_n \to t$ in the scalar field $\mathbb{K}$ and $x_n \to x$ in $(X, d)$. Then $d(x_n, x) = \|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 \to 0$, i.e., $\|x_n - x\|_1 \to 0$, $\|f(x_n) - f(x)\|_2 \to 0$. So $\|f(x_n - x)\|_2 \to 0$ by $(3')$ and $\|f(t_nx_n - tx)\|_2 \to 0$ by (2), and since $\|t_nx_n - tx\|_1 \to 0$, $\|f(t_nx_n) - f(tx)\|_2 \to 0$ by $(3')$ again. Thus $d(t_nx_n, tx) \to 0$, and therefore the scalar multiplication is also continuous in $(X, d)$.

Thus, $(X, d)$ is a complete metrizable linear space, that is, $(X, d)$ is a Fréchet space and, letting $l(x) = x$ for $x \in X$, $l : (X, d) \to (X, \| \cdot \|_1)$ is continuous, one-to-one and onto. By the open mapping theorem [5, p. 58], the converse $l^{-1} : (X, \| \cdot \|_1) \to (X, d)$ is also continuous.

Now let $x_n \to x$ in $(X, \| \cdot \|_1)$. Then $x_n = l^{-1}(x_n) \to l^{-1}(x) = x$ in $(X, d)$, that is, $\|x_n - x\|_1 + \|f(x_n) - f(x)\|_2 \to 0$ so $\|f(x_n) - f(x)\|_2 \to 0$, i.e., $f(x_n) \to f(x)$ in $(Y, \| \cdot \|_2)$. This shows that $f : (X, \| \cdot \|_1) \to (Y, \| \cdot \|_2)$ is continuous. □

Clearly, the recent result in [4] is a special case of this new closed graph theorem. Moreover, the continuity version Theorem 2.2 of [4] is also a special case of the following Corollary 1.1.

**Corollary 1.1.** Let $(X, \Sigma)$, $(Y, \Im)$ be Fréchet spaces and $\tau$ a Hausdorff topology for $Y$ which is coarser than $\Im$. If $f : X \to (Y, \Im)$ is weakly quasi-linear and $\Sigma - \tau$ continuous, then $f$ is $\Sigma - \Im$ continuous.

**Proof.** Hausdorff $\tau$ and continuous $f$ ensure that the graph $G$ is closed in $(X, \Sigma) \times (Y, \tau)$, and thus also in $(X, \Sigma) \times (X, \Im)$, since the latter has a finer topology. The conclusion now follows from Theorem 1.1. □

2. Weakly quasi-linear mappings

Let $wql(X, Y)$ denote the family of all weakly quasi-linear mappings from the topological vector space $X$ to the topological vector space $Y$.

**Remark.** It is obvious that if $f \in wql(X, Y)$ is sequentially continuous at every point if it is sequentially continuous at one point.

**Proposition 2.1.** If $X$, $Y$ are Hausdorff and $X$ is finite-dimensional, then every $f \in wql(X, Y)$ is continuous.
**Proposition 2.2.** A nonzero \( \varphi : \mathbb{R} \to \mathbb{R} \) is weakly quasi-linear if and only if

1. \( \varphi(0) = 0 \), and
2. \( \varphi \) is continuous, \( \varphi(x) \neq 0 \) for all \( x \neq 0 \) and \( \varphi(x_n) \to 0 \) whenever \( x_n \to \infty \).

**Proof.** If \( \varphi \in \text{wql}(\mathbb{R}, \mathbb{R}) \) and \( \varphi \neq 0 \), then \( \varphi(0) = 0 \) by (4), and \( \varphi \) is continuous by Proposition 2.1.

Suppose \( \varphi \in \text{wql}(\mathbb{R}, \mathbb{R}) \) and \( \varphi \neq 0 \), then \( \varphi(0) = 0 \) by (4), and \( \varphi \) is continuous by Proposition 2.1.

We have a simple fact which is helpful to our knowledge of weakly quasi-linear mappings.

**Example 2.1.** Let

\[
\varphi(x) = \begin{cases} 
2 + \sin(x - 2), & x > 2, \\
|\lambda|, & x \leq 2.
\end{cases}
\]

By Proposition 2.2, \( \varphi \in \text{wql}(\mathbb{R}, \mathbb{R}) \). But \( \varphi \) is not monotonic, so \( \varphi \) is not quasi-linear [4, Proposition 1.4].

We would like to say that the family of weakly quasi-linear mappings is an important object in analysis because if \( (X, \| \cdot \|) \) is a normed space and \( \| \cdot \| \neq 0 \) then the norm \( \| \cdot \| : (X, \| \cdot \|) \to \mathbb{R} \) is not quasi-linear and so not linear but it must be weakly quasi-linear.

**Proposition 2.3.** Let \( (X, \| \cdot \|) \) be a nontrivial paranormed space [5, p. 15]. Define \( f : X \to \mathbb{R} \) by \( f(x) = \| x \| \), \( \forall x \in X \). Then \( f \) is weakly quasi-linear but \( f \) is not quasi-linear when \( \| \cdot \| \neq 0 \).

**Proof.** Since \( f(x_0) \to 0 \) means that \( x_0 \to 0 \) in \( (X, \| \cdot \|) \), (1) and (2) hold for \( f \).

If \( x_n \to u \) and \( f(x_n - u) \to 0 \), then \( \| x_n - u \| \leq \| x_n - u \| \to 0 \) so \( f(x_n) - f(u) = \| x_n \| - \| u \| \to 0 \). If \( x_n \to u \) and \( f(x_n - f(u)) \to 0 \), then \( f(x_n - u) = \| x_n - u \| \to 0 \). Thus, (3) holds for \( f \).

If \( \| \cdot \| \neq 0 \) then (3) fails to hold for \( f \). To see this, pick an \( x \in X \) for which \( \| x \| > 0 \). Then \( \| x \| \geq \frac{1}{\| x \|} > 0 \). Letting \( x_n = -\frac{1}{2} \) for \( n \in \mathbb{N} \), \( f(x_n) = \| -\frac{1}{2} \| = \frac{1}{2} \to 0 \) but \( f(x_n - x) = f(-\frac{1}{2}) = f(x_n - x) = f(x_n) = \| x_n \| \to \| x \| \geq 0 \) so \( f(x_n - x) \to 0 \).

Many Banach spaces contain a copy of \( (c_0, \| \cdot \|_\infty) \) or \( (c^1, \| \cdot \|_1) \) or a reflexive Banach space. Hence, linearly homeomorphic embedding \( T : X \to Y \) happens frequently. Especially, for every complex Banach space \( X \) and every continuous linear operator \( S : X \to X \), \( \lambda I - S : X \to X \) is a linear homeomorphism for each \( \lambda \in \mathbb{C} \setminus \sigma(S) \cup \{ \lambda \in \mathbb{C} : |\lambda| > \| S \| \} \).
Proposition 2.4. Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a continuous function such that

$$0 < \mu = \inf_{t \geq 0} \varphi(t) \leq \sup_{t \geq 0} \varphi(t) = M < +\infty.$$ 

Let $X, Y$ be normed spaces and $T : X \to Y$ a linearly homeomorphic embedding. If $f : X \to Y$ is defined by $f(x) = \varphi(\|x\|)T(x)$, $\forall x \in X$, then $f$ is weakly quasi-linear.

**Proof.** Since $0 < \mu \leq \varphi(t) \leq M < +\infty$ for all $t \geq 0$ and $T$ is a linear homeomorphism of $X$ onto $T(X)$, $f(x_n) \to 0$ if and only if $T(x_n) \to 0$ and if and only if $x_n \to 0$. Thus, (1) and (2) hold for $f$.

If $x_n - u \to 0$ and $f(x_n - u) \to 0$, then $\|x_n\| - \|u\| \leq \|x_n - u\| \to 0$ i.e., $\|x_n\| \to \|u\|$. By the continuity of $\varphi$ and $T$, $f(x_n) - f(u) = \varphi(\|x_n\|)T(x_n) - \varphi(\|u\|)T(u) \to 0$. If $x_n - u \to 0$ and $f(x_n) - f(u) \to 0$, then $f(x_n - u) = \varphi(\|x_n - u\|)T(x_n - u) \to \varphi(0)T(0) = 0$ since both $\varphi$ and $T$ are continuous. Thus, $(3')$ holds for $f$. $\Box$

Obviously, although the condition of Proposition 2.4 is much weaker than that of Proposition 3.1 in [4], the proof becomes much simpler. So condition $(3')$ is much looser than $(3)$.

Note that if $X$ is an infinite-dimensional Fréchet space then for every nontrivial Fréchet space $Y$ there exist many linear operators from $X$ to $Y$ which are not continuous and, of course, many more weakly quasi-linear mappings from $X$ to $Y$ which are not continuous. The new closed graph theorem just shows that a weakly quasi-linear $f : X \to Y$ is continuous if and only if $f$ has closed graph.

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**References**