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Representation rings of quantum groups

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Abstract

Generators and relations are given for the subalgebra of cocommutative elements in the quantized coordinate rings $\mathcal{O}(G_q)$ of the classical groups, where q is transcendental. This is a ring theoretic formulation of the well-known fact that the representation theory of G_q is completely analogous to its classical counterpart. The subalgebras of cocommutative elements in the corresponding FRT-bialgebras (defined by Faddeev, Reshetikhin, and Takhtadzhyan) are explicitly determined, using a bialgebra embedding of the FRT-bialgebra into the tensor product of the quantized coordinate ring and the one-variable polynomial ring. A parallel analysis of the subalgebras of adjoint coinvariants is carried out as well, yielding similar results with similar proofs. The basic adjoint coinvariants are interpreted as quantum traces of representations of the corresponding quantized universal enveloping algebra.

Keywords: Quantized function algebra; Classical group; Adjoint coaction; Cocommutative element; Quantum trace; FRT-bialgebra

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1. Introduction

A good deal of classical invariant theory concerns the so-called classical groups, their action on vectors and covectors, and their adjoint representation. It is therefore tempting to look for counterparts of this topic in the context of quantum groups, as is shown by various approaches in the literature. Our starting point here is [5], where two quantum versions of the invariant theory of the conjugation action of the general linear group have been studied. Both the (right) *adjoint coaction* $\beta : f \mapsto \sum f_2 \otimes S(f_1) f_3$ (given in Sweedler's notation) and the (right) coaction $\alpha : f \mapsto \sum f_2 \otimes f_3 S(f_1)$ of $\mathcal{O}(GL_q(N))$, the coordinate ring of the quantum general linear group, on the coordinate ring of $N \times N$ quantum matrices, can be considered as quantum deformations of the classical conjugation action. In [5], explicit generators of the subalgebra of coinvariants were determined both for α and β (under the assumption that q is not a root of unity). Both algebras are N-variable commutative polynomial algebras. Note also that an element is an α-coinvariant if and only if it is cocommutative.

Some fragments of this picture had appeared in prior work already, in greater generality. Motivated by the theory of integrable Hamiltonian systems, pairwise commuting q-analogues of the functions $\mathrm{tr}(L^n)$ $(n = 1, 2, \ldots)$ were constructed in [18] for algebras $\mathcal{A}(R)$ generated by N^2 elements u^i_j, subject to the relations $R\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_2\mathbf{u}_1 R$ (see Section 3 for explanation of this notation), where R is an $N^2 \times N^2$ matrix satisfying the Yang–Baxter equation. One can check that the elements constructed by Maillet are cocommutative in the bialgebra $\mathcal{A}(R)$ (though this is not touched in [18]).

Another set of elements of $\mathcal{A}(R)$ was constructed in [2], see also [19, Corollary 10.3.9]. They arise as quantum traces of powers of u with respect to the so-called covariantized (or transmuted) product in $\mathcal{A}(R)$. These elements are adjoint coinvariants, and pairwise commute, so they are also appropriate quantum analogues of the classical functions $\mathrm{tr}(L^n)$.

The adjoint coaction is not multiplicative (neither is the version α). Majid developed a theory for coquasitriangular matrix bialgebras $\mathcal{A}(R)$ which remedies this defect. Namely, a new covariantized product can be introduced on $\mathcal{A}(R)$ in a canonical way. The adjoint coinvariants become central in this new *braided matrix algebra*, known also as a *reflection equation algebra*. This process (called *transmutation* in [19]) provides a bridge between the results of [5], and certain results on the reflection equation algebra. There is a number of papers dealing with the adjoint action (or coaction) on reflection equation algebras. For example, [11] and [7] (see also the references therein) make use of adjoint invariants (central elements) of the reflection equation algebra to study quantizations of coadjoint orbits of $SL(N)$. (Staying in the framework of quantum matrices, related results were obtained in [4].) See also [17] and [16] for discussion of other versions of the reflection equation algebra. There are various versions of the Cayley–Hamilton theorem for quantum matrix algebras or the reflection equation algebra, see [12,14,27]. These imply relations among the above mentioned adjoint coinvariants (respectively cocommutative elements).

Now let us briefly describe the subject of the present paper, where the point of view of invariant theory is adopted, and we look for generators and relations for subalgebras of coinvariants. Our focus is on the matrix bialgebras $\mathcal{A}(G_q)$, associated with the classical group G and the parameter $q \in \mathbb{C}^\times$ by Faddeev, Reshetikhin, and Takhtadzhyan in [22]. These algebras (called *FRT-bialgebras*) are defined in terms of generators and relations. They have a natural bialgebra structure, where the comultiplication reflects the rule for matrix multiplication. Following [22], by the coordinate ring $\mathcal{O}(G_q)$ of the quantum group G_q we mean the quotient of $\mathcal{A}(G_q)$ by an explicitly given ideal. The algebra $\mathcal{A}(G_q)$ is endowed with the adjoint coaction of $\mathcal{O}(G_q)$. Our main result, Theorem 3.3 presents explicit generators and relations for the subalgebra $\mathcal{A}(G_q)^{\mathrm{coc}}$ of cocommutative elements in $\mathcal{A}(G_q)$ under the assumption that q is transcendental (the method of proof probably works when q is not a root of unity). We indicate also how the same thing can be done for the subalgebra $\mathcal{A}(G_q)^\beta$ of adjoint coinvariants in $\mathcal{A}(G_q)$. This recovers the results of [5] as the special case of $GL_q(N)$, $SL_q(N)$. For the other classical groups these results seem to be new. The description of $\mathcal{A}(G_q)^{\mathrm{coc}}$ and $\mathcal{A}(G_q)^\beta$ is obtained from the description of the corresponding subalgebras in $\mathcal{O}(G_q)$ (see Theorem 2.4), where the assertion is essentially a consequence of the Peter–Weyl decomposition, due to Hayashi [13]. Let us note however that from our point of view, the algebra $\mathcal{A}(G_q)$ is closer to the flavour of classical invariant theory (dealing with commutative polynomial algebras), than $\mathcal{O}(G_q)$: it is a graded (Noetherian) algebra, having the same Hilbert series as its classical counterpart. The finite generation property of the subalgebra of coinvariants follows from a general Hilbert type argument, see [6].

After a first draft of this paper was written, we learnt from Stephen Donkin that independently from us, strongly related results were obtained by him on the conjugation action of quantum groups on their coordinate algebra in [9], with no restriction on the deformation parameter q and on the base field (in particular, the case when q is a root of unity is covered as well). Moreover, his work involves the study of the structure of the coordinate ring of the quantum group as a module over the subalgebra of coinvariants.

2. Cocommutative elements in $\mathcal{O}(G_q)$

We work over the base field $\mathbb{C}$ of complex numbers. Let $\mathcal{O}(G_q)$ be any of the coordinate algebras of the quantum groups $GL_q(N)$, $SL_q(N)$, $O_q(N)$, $SO_q(N)$, $Sp_q(N)$, defined in [15, Sections 9.2, 9.3], following [22]. Assume that the complex parameter q is not a root of unity when G_q is $GL_q(N)$ or $SL_q(N)$, and assume that q is transcendental in all other cases. We allow also the case $q = 1$, when we get the commutative coordinate algebra $\mathcal{O}(G)$ of the classical group G corresponding to G_q. The assumption on q guarantees that $\mathcal{O}(G_q)$ is cosemisimple, and its corepresentation theory is completely analogous to its classical counterpart. The results presented in this paper depend crucially on the work of Hayashi [13], concerning the Peter–Weyl decomposition of $\mathcal{O}(G_q)$.

Recall that an element $f \in \mathcal{O}(G_q)$ is *cocommutative* if $\tau \circ \Delta(f) = \Delta(f)$, where $\Delta : \mathcal{O}(G_q) \to \mathcal{O}(G_q) \otimes \mathcal{O}(G_q)$ is the comultiplication, and τ is the flip $\tau(f \otimes g) = g \otimes f$. The cocommutative elements form a subalgebra $\mathcal{O}(G_q)^{\mathrm{coc}}$. We would like to point out that as an immediate corollary of the representation theory of G_q, generators and the structure of $\mathcal{O}(G_q)^{\mathrm{coc}}$ can be described explicitly. This is based on the following well-known statement, which is a reformulation of Schur's lemma.

Lemma 2.1. *The cocommutative elements in a simple coalgebra form a one-dimensional subspace.*

Proof. Since our base field is $\mathbb{C}$, any simple coalgebra C is isomorphic to the dual of the matrix algebra $M(N,\mathbb{C})$ for some N. The trace function on $M(N,\mathbb{C})$ fixes a vector space isomorphism $a \mapsto \mathrm{Tr}(a \cdot \_)$ between $M(N,\mathbb{C})$ and C. Under this isomorphism the center of $M(N,\mathbb{C})$ is mapped onto the space of cocommutative elements in C. $\square$

Given a finite dimensional corepresentation $\varphi : V \to V \otimes \mathcal{O}(G_q)$, write $\mathrm{tr}(\varphi)$ for the sum of the diagonal matrix coefficients of φ (see, for example, [15, 1.3.2] for the notion of matrix coefficients of a corepresentation). If φ is irreducible, then $\mathrm{tr}(\varphi)$ spans the space of cocommutative elements in the coefficient coalgebra of φ by Lemma 2.1. Clearly, $\mathrm{tr}(\varphi \oplus \psi) = \mathrm{tr}(\varphi) + \mathrm{tr}(\psi)$ and $\mathrm{tr}(\varphi \otimes \psi) = \mathrm{tr}(\varphi) \cdot \mathrm{tr}(\psi)$.

The isomorphism classes of irreducible corepresentations of $\mathcal{O}(G_q)$ are parameterized by a set $P(G_q) = P(G)$. This set is independent of q, so it is the same as in the classical case $q = 1$, when it is clearly in a natural bijection with the set of isomorphism classes of irreducible rational representations of the affine algebraic group G. It is a convenient tradition to represent $P(G)$ as a set of certain sequences of integers, see [13, formulae (4.17), (6.2), and Theorem 6.4], or [15, Section 11.2.3] for details. When $G = Sp(N)$, $SL(N)$, or $GL(N)$, then it is natural to identify $P(G)$ with the semigroup of dominant integral weights for the corresponding reductive Lie algebra $\mathfrak{g}$, whereas when $G = SO(N)$, then $P(G)$ consists of those dominant integral weights for so_N, which appear as a highest weight in some tensor power of the vector representation of so_N. When $G = O(N)$, then following [26], $P(G)$ is usually identified with the set of partitions, such that the sum of the length of the first two columns of their Young diagram is at most N.

For $\mathbf{n} \in P(G_q)$, write $\varphi_{\mathbf{n}}$ for the corresponding irreducible corepresentation of $\mathcal{O}(G_q)$.

Proposition 2.2. *The set* $\{\mathrm{tr}(\varphi_{\mathbf{n}}) \mid \mathbf{n} \in P(G_q)\}$ *is a* $\mathbb{C}$*-vector space basis of* $\mathcal{O}(G_q)^{\mathrm{coc}}$. *The structure constants of the algebra* $\mathcal{O}(G_q)^{\mathrm{coc}}$ *with respect to this basis are independent of* q*: they are the same as in the classical case* $q = 1$.

Proof. Start with the Peter–Weyl decomposition of $\mathcal{O}(G_q)$ due to [13] (respectively [20] for $GL_q(N)$); see also [15, 11.2.3, Theorem 22 and 11.5.4, Theorem 51]. We have $\mathcal{O}(G_q) = \bigoplus_{\mathbf{n} \in P(G_q)} C(\varphi_{\mathbf{n}})$, where $C(\varphi_{\mathbf{n}})$ is the coefficient coalgebra of $\varphi_{\mathbf{n}}$. It follows that $\mathcal{O}(G_q)^{\mathrm{coc}} = \bigoplus_{\mathbf{n} \in P(G_q)} C(\varphi_{\mathbf{n}})^{\mathrm{coc}}$. By Lemma 2.1, $C(\varphi_{\mathbf{n}})^{\mathrm{coc}} = \mathbb{C}\,\mathrm{tr}(\varphi_{\mathbf{n}})$, showing the first assertion. For the second assertion, decompose the tensor product $\varphi_{\mathbf{n}} \otimes \varphi_{\mathbf{m}} \cong \bigoplus_{\mathbf{p}} m_{\mathbf{p}}^{\mathbf{n},\mathbf{m}} \varphi_{\mathbf{p}}$. The multiplicities $m_{\mathbf{p}}^{\mathbf{n},\mathbf{m}}$ here are the same as in the classical case $q = 1$, since this holds for the decompositions of tensor products of the corresponding representations of quantized universal enveloping algebras (see, for example, [15, 7.2] or [3, Proposition 10.1.16]; for the case of $O_q(N)$, see Appendix A, Proposition 6.3, and the remark afterwards). On the other hand, they are the structure constants of $\mathcal{O}(G_q)^{\mathrm{coc}}$: we have $\mathrm{tr}(\varphi_{\mathbf{n}}) \cdot \mathrm{tr}(\varphi_{\mathbf{m}}) = \sum_{\mathbf{p}} m_{\mathbf{p}}^{\mathbf{n},\mathbf{m}}\, \mathrm{tr}(\varphi_{\mathbf{p}})$. $\square$

The following immediate corollary is a ring theoretic formulation of the well-known fact that the representation theory of G_q is essentially the same as the representation theory of G.

Proposition 2.3. *The algebra $\mathcal{O}(G_q)^{\mathrm{coc}}$ is isomorphic to its classical counterpart $\mathcal{O}(G)^{\mathrm{coc}}$, via an isomorphism mapping $\mathrm{tr}(\varphi_{\mathbf{n}}) \in \mathcal{O}(G_q)$ to $\mathrm{tr}(\varphi_{\mathbf{n}}) \in \mathcal{O}(G)$ for all $\mathbf{n} \in P(G)$.*

There is a natural right coaction of $\mathcal{O}(G_q)$ on the *quantum exterior algebra* $\bigwedge(G_q)$, [15, see Sections 9.2 and 9.3]. The quantum exterior algebra is graded. Its degree d homogeneous component is a subcomodule of dimension $\binom{N}{d}$, write ω_d for the corepresentation of $\mathcal{O}(G_q)$ on this space, for $d = 1, \dots, N$, and set $\sigma_d = \mathrm{tr}(\omega_d)$. In the classical case $q = 1$ the representation corresponding to ω_d is the dth exterior power of the defining representation of G. When q is transcendental, the multiplicities of the irreducible summands of ω_d are the same as in the classical case $q = 1$, since $\bigwedge(G_q)$ has the same kind of weight space decomposition as in the classical case. In particular, for $SO_q(2l)$ we have $\omega_l = \omega_{l,0} \oplus \omega_{l,1}$ is the direct sum of two non-isomorphic irreducibles; in this case set $\sigma_{l,0} = \mathrm{tr}(\omega_{l,0})$ and $\sigma_{l,1} = \mathrm{tr}(\omega_{l,1})$, so $\sigma_{l,0} + \sigma_{l,1} = \sigma_l$. Generators and relations for the commutative algebra $\mathcal{O}(G_q)^{\mathrm{coc}}$ are the following.

Theorem 2.4.

(i) (*cf.* [6]) $\mathcal{O}(SL_q(l+1))^{\mathrm{coc}}$ *is an l-variable commutative polynomial algebra generated by $\sigma_1, \dots, \sigma_l$.*

(ii) $\mathcal{O}(Sp_q(2l))^{\mathrm{coc}}$ *is an l-variable commutative polynomial algebra generated by $\sigma_1, \dots, \sigma_l$.*

(iii) $\mathcal{O}(O_q(2l+1))^{\mathrm{coc}}$ *is generated by $\sigma_1, \dots, \sigma_l, \sigma_{2l+1}$, subject to the relation $\sigma_{2l+1}^2 = 1$. So it is a rank two free module generated by 1 and σ_{2l+1} over the l-variable commutative polynomial algebra $\mathbb{C}[\sigma_1, \dots, \sigma_l]$.*

(iv) $\mathcal{O}(SO_q(2l+1))^{\mathrm{coc}}$ *is the l-variable commutative polynomial algebra generated by $\sigma_1, \dots, \sigma_l$.*

(v) $\mathcal{O}(O_q(2l))^{\mathrm{coc}}$ *is generated by $\sigma_1, \dots, \sigma_l, \sigma_{2l}$, subject to the relations $\sigma_{2l}^2 = 1$, $\sigma_l \sigma_{2l} = \sigma_l$. So it is the vector space direct sum $\mathbb{C}[\sigma_1, \dots, \sigma_l] \oplus \sigma_{2l}\mathbb{C}[\sigma_1, \dots, \sigma_{l-1}]$ of the l-variable commutative polynomial algebra $\mathbb{C}[\sigma_1, \dots, \sigma_l]$, and the rank one free module generated by σ_{2l} over the $(l-1)$-variable commutative polynomial algebra $\mathbb{C}[\sigma_1, \dots, \sigma_{l-1}]$.*

(vi) $\mathcal{O}(SO_q(2l))^{\mathrm{coc}}$ *is generated by $\sigma_1, \dots, \sigma_{l-1}, \sigma_{l,0}, \sigma_{l,1}$, subject to the relation*

$$(\sigma_{l,0} - \sigma_{l,1})^2 = \left(\sigma_l + 2\sum_{i=0}^{l-1} \sigma_i\right)\left(\sigma_l + 2\sum_{i=0}^{l-1} (-1)^{l-i} \sigma_i\right),$$

where $\sigma_l = \sigma_{l,0} + \sigma_{l,1}$. So $\mathcal{O}(SO_q(2l))^{\mathrm{coc}}$ is a rank two free module generated by 1 and $\sigma_{l,0} - \sigma_{l,1}$ over the l-variable polynomial algebra $\mathbb{C}[\sigma_1, \dots, \sigma_l]$.

(vii) (*cf.* [5]) $\mathcal{O}(GL_q(N))^{\mathrm{coc}}$ *is the commutative Laurent polynomial ring generated by $\sigma_1, \dots, \sigma_N, \sigma_N^{-1}$ (note that σ_N is the quantum determinant).*

Proof. By Proposition 2.3 the result follows from its special case $q = 1$. In the classical case the structure of $\mathcal{O}(G)^{\mathrm{coc}}$ is well-known: it can be derived from the representation theory of G. For sake of completeness we give some references and hints in Appendix B. $\square$

The quantum exterior algebra $\bigwedge(G_q)$ has a basis consisting of formally the same set of monomials as in the classical case, and a general monomial can be easily rewritten in terms of this basis, using the defining relations; see [15, 9.2.1 Proposition 6, 9.3.2 Proposition 15, 9.3.4 Proposition 17]. So in principle one can express the σ_i for each concrete case of Theorem 2.4 as a polynomial in the generators of $\mathcal{O}(G_q)$; an example will be given in Section 3. (The cases (i) and (vii) were handled by different methods in [5,6]; then the σ_i are sums of principal minors of the generic quantum matrix.) However, we do not know how to get such an expression for $\sigma_{l,0}$ (or $\sigma_{l,1}$) in (vi).

3. Cocommutative elements in the FRT-bialgebra

Throughout this section G_q is one of $SL_q(N)$, $O_q(N)$, $Sp_q(N)$, and we retain the assumptions on q made in Section 2, so that the results of [13] on the Peter–Weyl decomposition can be applied.

By definition, $\mathcal{O}(G_q)$ is the quotient of the so-called *FRT-bialgebra* $\mathcal{A}(G_q)$ modulo the ideal generated by $\mathcal{D}_q - 1$, where $\mathcal{D}_q$ is a central group-like element, having degree N in the case of $SL_q(N)$, and having degree 2 in the cases of $O_q(N)$, $Sp_q(N)$. The algebra $\mathcal{A}(G_q)$ was defined in [22] as the associative $\mathbb{C}$-algebra with N^2 generators u^i_j $(i, j = 1, \ldots, N)$, subject to the relations

$$R\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_2\mathbf{u}_1 R. \tag{1}$$

Here R is an $N^2 \times N^2$ matrix, the R-matrix of the vector representation of the Drinfeld–Jimbo algebra $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is the simple Lie algebra corresponding to G_q, and $\mathbf{u}_1 = \mathbf{u} \otimes I$, $\mathbf{u}_2 = I \otimes \mathbf{u}$ are Kronecker products of the $N \times N$ matrices $\mathbf{u} = (u^i_j)$ and the identity matrix in the two possible orders. The relations (1) are homogeneous of degree 2 in the generators u^i_j, therefore $\mathcal{A}(G_q)$ is a graded algebra, with the generators u^i_j having degree 1. Moreover, $\mathcal{A}(G_q)$ is a bialgebra with comultiplication $\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j$ and counit $\varepsilon(u^i_j) = \delta_{i,j}$. Let V be an N-dimensional $\mathbb{C}$-vector space with basis $e_1, \ldots, e_N$. Write $\omega : V \to V \otimes \mathcal{A}(G_q)$ for the $\mathcal{A}(G_q)$-corepresentation given by $\omega(e_i) = \sum_j e_j \otimes u^j_i$, and call ω the *fundamental corepresentation* of $\mathcal{A}(G_q)$. Note that the generators u^i_j are nothing but the matrix coefficients of ω (with respect to the basis $e_1, \ldots, e_N$). It is clear then that the degree r homogeneous component of $\mathcal{A}(G_q)$ is the coefficient space of the rth tensor power $\omega^{\otimes r}$ of the fundamental corepresentation.

Write $\pi : \mathcal{A}(G_q) \to \mathcal{O}(G_q)$ for the natural surjection. A corepresentation φ of $\mathcal{A}(G_q)$ induces the corepresentation $\varphi_{\mathcal{O}(G_q)} = (\mathrm{id} \otimes \pi) \circ \varphi$ of $\mathcal{O}(G_q)$. For $r \in \mathbb{N}_0$ denote by $P_r(G_q)$ the subset of $P(G_q)$ consisting of the $\mathbf{n}$ such that $(\omega^{\otimes r})_{\mathcal{O}(G_q)}$, the rth tensor power of the fundamental corepresentation considered as a corepresentation of $\mathcal{O}(G_q)$, contains a subcorepresentation isomorphic to $\varphi_{\mathbf{n}}$; the explicit form of $P_r(G_q)$ can be found in [13, (4.17)]. Up to isomorphism, there is a unique $\mathcal{A}(G_q)$-subcorepresentation $\varphi_{\mathbf{n},r}$ in $\omega^{\otimes r}$ with $(\varphi_{\mathbf{n},r})_{\mathcal{O}(G_q)} \cong \varphi_{\mathbf{n}}$. The coefficient space $C(\varphi_{\mathbf{n},r})$ is a simple subcoalgebra of the degree r homogeneous component of $\mathcal{A}(G_q)$, and by [15, 11.2.3 Theorems 21 and 22] we have the decomposition

$$\mathcal{A}(G_q) = \bigoplus_{r=0}^{\infty} \bigoplus_{\mathbf{n} \in P_r(G_q)} C(\varphi_{\mathbf{n},r}). \tag{2}$$

The polynomial ring $\mathbb{C}[z]$ is a sub-bialgebra of the coordinate ring $\mathbb{C}[z, z^{-1}]$ of the multiplicative group of $\mathbb{C}$. The map $u^i_j \mapsto \delta_{i,j} z$ extends to a bialgebra homomorphism $\kappa : \mathcal{A}(G_q) \to \mathbb{C}[z]$. This follows from the defining relations (1): specializing $\mathbf{u}$ to any scalar matrix, $\mathbf{u}_1\mathbf{u}_2$ and $\mathbf{u}_2\mathbf{u}_1$ specialize to the same scalar matrix, hence (1) is fulfilled. Therefore there exists an algebra homomorphism κ with the prescribed images of the generators. It is easy to check on the generators that this is a coalgebra homomorphism as well, moreover, that κ has the following *centrality* property:

$$(\mathrm{id} \otimes \kappa) \circ \Delta_{\mathcal{A}(G_q)} = \tau \circ (\kappa \otimes \mathrm{id}) \circ \Delta_{\mathcal{A}(G_q)}, \tag{3}$$

where τ is the flip map $\tau(a \otimes b) = b \otimes a$.

Proposition 3.1. *The map $\iota = (\pi \otimes \kappa) \circ \Delta_{\mathcal{A}(G_q)}$ is a bialgebra injection of $\mathcal{A}(G_q)$ into the tensor product bialgebra $\mathcal{O}(G_q) \otimes \mathbb{C}[z]$. The subcoalgebra $C(\varphi_{\mathbf{n},r})$ is mapped onto $C(\varphi_{\mathbf{n}}) \otimes z^r$ for all $r \in \mathbb{N}_0$ and $\mathbf{n} \in P_r(G_q)$.*

Proof. The map ι is defined as a composition of algebra homomorphisms, hence it is an algebra homomorphism. Property (3) can be used to verify that it is a coalgebra homomorphism as well. The only thing left to show is that ι is injective. The algebra $\mathcal{A}(G_q)$ is graded, the generators u^i_j have degree 1. Similarly, the usual grading on the polynomial ring $\mathbb{C}[z]$ induces a grading on $\mathcal{O}(G_q) \otimes \mathbb{C}[z]$, and the map ι is obviously homogeneous. Therefore, the kernel of ι is spanned by homogeneous elements. Take a homogeneous element f from $\ker(\iota)$, say of degree r. Then $\iota(f) = \pi(f) \otimes z^r$, hence $\pi(f) = 0$. It follows that f is a multiple of $\mathcal{D}_q - 1$. The element $\mathcal{D}_q$ is not a zero-divisor in $\mathcal{A}(G_q)$ by [13, Theorem 5.7(1)]; see also 11.2.3 Lemma 25, and the beginning of the proof of Theorem 22 on p. 414 in [15]. (Note that $\mathcal{A}(G_q)$ is not always a domain, as we shall see later.) Clearly 1 is not a zero-divisor. Therefore no non-zero multiple of $\mathcal{D}_q - 1$ is homogeneous. Thus we have $f = 0$. □

Write $\mathcal{A}(G)$ for the classical counterpart of the FRT-bialgebra. For $SL_q(N)$, this is just the N^2-variable commutative polynomial algebra, that we obtain when we specialize q to 1 in the defining relations (1). It is crucial to note however that in the cases of $O_q(N)$ and $Sp_q(N)$, the algebra $\mathcal{A}(G)$ is different from the N^2-variable commutative polynomial algebra (although specializing q to 1 in relations (1), we end up with the N^2-variable commutative polynomial algebra in these cases as well); see also [8] for this point. To get the right definition of $\mathcal{A}(G)$ for $G = O(N)$ or $G = Sp(N)$, recall that the symmetric matrix $\widehat{R}(q) = \tau \circ R$ has a spectral decomposition

$$\widehat{R}(q) = qP_+(q) - q^{-1}P_-(q) + \epsilon q^{\epsilon - N} P_0(q),$$

where $\epsilon = 1$ for $O_q(N)$ and $\epsilon = -1$ for $Sp_q(N)$; see [15, Section 9.3]. For q transcendental, the eigenvalues $q, -q^{-1}, \epsilon q^{\epsilon-N}$ are pairwise different, therefore (1) is a short expression of the equivalent set of relations

$$P_+(q)\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2 P_+(q), \qquad P_-(q)\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2 P_-(q),$$
$$P_0(q)\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2 P_0(q). \tag{4}$$

When we specialize q to 1, the eigenvalues $\epsilon q^{\epsilon-N}$ and q (respectively $-q^{-1}$) become equal in the orthogonal case (respectively in the symplectic case), and that is why the relations obtained from (1) are not strong enough. Instead, we can write down a third set of relations equivalent to (1) or (4):

$$\widehat{R}(q)\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2\widehat{R}(q) \quad \text{and} \quad \mathrm{K}(q)\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2\mathrm{K}(q), \tag{5}$$

where $\mathrm{K}(q) = (1 + \epsilon(q - q^{-1})^{-1}(q^{N-\epsilon} - q^{\epsilon-N}))P_0(q)$. It is clear that (5) is equivalent to (1), though (5) is trivially redundant for $q \neq 1$. The advantage of (5) compared to (4) is that $\mathrm{K}(q)$ has a rather simple form. Write $\mathrm{C}(q)$ for the matrix of the metric defined in [15, p. 317]. Its non-zero entries all lie on the anti-diagonal, and up to sign, they are q-powers. Note that $\mathrm{C}(1)$ is the matrix of the symmetric (respectively skew-symmetric) bilinear form that appears in the usual definition of the orthogonal (respectively symplectic) group. Now the entries of the $N^2 \times N^2$ matrix $\mathrm{K}(q)$ are given by $\mathrm{K}(q)^{ji}_{mn} = \epsilon\mathrm{C}(q)^j_i\mathrm{C}(q)^m_n$, see [15, p. 318]. So the non-zero entries of $\mathrm{K}(q)$ are all q-powers up to sign. In particular, $K(1)$ makes sense. After these preparations it is natural to define $\mathcal{A}(G)$ as the algebra with generators u^i_j, $i, j = 1, \dots, N$, subject to the relations

$$\widehat{R}(1)\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2\widehat{R}(1) \quad \text{and} \quad \mathrm{K}(1)\mathbf{u}_1\mathbf{u}_2 = \mathbf{u}_1\mathbf{u}_2\mathrm{K}(1). \tag{6}$$

It is a bialgebra with comultiplication and counit given by the same formulae as for $\mathcal{A}(G_q)$. Specializing q to 1 in $\mathcal{D}_q$ we get a group-like element $\mathcal{D}$ of $\mathcal{A}(G)$. As we shall point out below, the quotient of $\mathcal{A}(G)$ modulo the ideal generated by $\mathcal{D} - 1$ can be identified with $\mathcal{O}(G)$, such that the images of the generators u^i_j become the coordinate functions on G, with its usual embedding into the space $M(N, \mathbb{C})$ of $N \times N$ matrices.

A close inspection of the proofs of the statements cited in this section from [15] about the coalgebra structure of $\mathcal{A}(G_q)$ shows that they remain valid for $\mathcal{A}(G)$. Indeed, the key point in the proof of (2) is [15, 11.2.3 Proposition 20], which is a consequence of the quantum Brauer–Schur–Weyl duality, that is, that the commutant algebra of $\widetilde{U}_q(\mathfrak{g})$ acting on a tensor power of the vector representation is generated by the 'shifts' of $\widehat{R}(q)$, see [15, 8.6.3 Theorem 38] for a precise statement. Now in the classical Brauer–Schur–Weyl duality, the corresponding commutant algebra is generated by the shifts of $\widehat{R}(1) = \tau$ and

K(1), therefore the proof of [15, 11.2.3 Proposition 20] works for the algebra $\mathcal{A}(G)$ defined in terms of $\widehat{R}(1)$ and K(1). This yields a version of [15, 11.2.3 Theorem 21] for $\mathcal{A}(G)$, and in turn the decomposition (2) for $\mathcal{A}(G)$:

$$\mathcal{A}(G) = \bigoplus_{r=0}^{\infty} \bigoplus_{\mathbf{n} \in P_r(G)} C(\varphi_{\mathbf{n},r}),$$

where $P_r(G) = P_r(G_q)$, since the multiplicities of the irreducible summands of the rth tensor power of the fundamental corepresentation of $\mathcal{O}(G_q)$ are the same as for $\mathcal{O}(G)$ (cf. [15, 8.6.2 Corollary 37(i)]). Similarly, Proposition 3.1 holds in the case $q = 1$ as well.

So we have defined $\mathcal{A}(G)$ as an algebra given in terms of generators and relations. The path we have followed expresses explicitly that $\mathcal{A}(G)$ is obtained as the special case $q = 1$ of $\mathcal{A}(G_q)$. Moreover, we will need to compare the coalgebra structures of $\mathcal{A}(G)$ and $\mathcal{A}(G_q)$, and this definition makes possible a uniform approach: one can get the above mentioned statements about $\mathcal{A}(G)$ and the corresponding statements on $\mathcal{A}(G_q)$ with q transcendental simultaneously. However, $\mathcal{A}(G)$ has a description in simple geometric terms as well. Namely, $\mathcal{A}(G)$ is the coordinate ring of the Zariski closure of the cone $\mathbb{C}G$ of G, where by this cone we mean the image of the map $\mu : G \times \mathbb{C} \to M(N, \mathbb{C})$, $(g, t) \mapsto tg$. Indeed, the first set of the relations (6) says that the u^i_j pairwise commute (note that $\widehat{R}(1) = \tau$). By the proof of [15, 9.3.1 Lemma 12], the second set of the above relations says that

$$\mathbf{u}\mathrm{C}(1)^{-1}\mathbf{u}^T\mathrm{C}(1) = \mathrm{C}(1)^{-1}\mathbf{u}^T\mathrm{C}(1)\mathbf{u} = \text{a scalar multiple of the identity},$$

where the scalar above is the quadratic group-like element $\mathcal{D}$. Theorems (5.2C) and (6.3B) of [26] describe the generators of the vanishing ideal in the coordinate ring of $M(N, \mathbb{C})$ of the full orthogonal group and the symplectic group. This result can be paraphrased by saying that the quotient of $\mathcal{A}(G)$ modulo the ideal generated by $\mathcal{D} - 1$ is indeed $\mathcal{O}(G)$, as we claimed before. Furthermore, we obtained that the locus of solutions of the equations (6) in $M(N, \mathbb{C})$ is the $N \times N$ matrix semigroup $\mathcal{M}$ consisting of the matrices A such that $A\mathrm{C}(1)^{-1}A^T\mathrm{C}(1)$ and $\mathrm{C}(1)^{-1}A^T\mathrm{C}(1)A$ are equal scalar matrices (we allow the scalar zero). Clearly the subset of invertible elements in $\mathcal{M}$ is $\mathbb{C}^\times G$. Therefore $\mathcal{M} \supseteq \overline{\mathbb{C}G}$, there exist natural surjections $\pi_1 : \mathcal{A}(G) \to \mathcal{O}(\overline{\mathbb{C}G})$ and $\pi_2 : \mathcal{O}(\overline{\mathbb{C}G}) \to \mathcal{O}(G)$, and their composition is the natural surjection $\pi = \pi_2 \circ \pi_1 : \mathcal{A}(G) \to \mathcal{O}(G)$. So, as we noted already, Proposition 3.1 makes sense and is valid for $\mathcal{A}(G)$. It is easy to see that in this case the map ι is the composition $\mu^* \circ \pi_1$ of the comorphism of μ and π_1. Consequently, the injectivity of ι implies that π_1 is an isomorphism, hence $\mathcal{M} = \overline{\mathbb{C}G}$, and $\mathcal{A}(G)$ is the coordinate ring of $\mathcal{M}$. (Alternatively, instead of using Proposition 3.1, it is possible to derive directly from the results of [26] cited above that the vanishing ideal of the Zariski closure of $\mathbb{C}G$ in $M(N, \mathbb{C})$ is generated by the polynomials coming from the second set of relations in (6). For sake of completeness we present this elementary argument in Appendix D.) Note that, being the coordinate ring of a linear algebraic semigroup, $\mathcal{A}(G)$ is naturally a bialgebra; the comultiplication and counit structures coming from this geometric interpretation of $\mathcal{A}(G)$ agree with the one specified before.

Proposition 3.2. *The subalgebra $\mathcal{A}(G_q)^{\mathrm{coc}}$ of cocommutative elements in the FRT-bialgebra is isomorphic to its classical counterpart via an isomorphism mapping* $\mathrm{tr}(\varphi_{\mathbf{n},r}) \in \mathcal{A}(G_q)$ *to* $\mathrm{tr}(\varphi_{\mathbf{n},r}) \in \mathcal{A}(G)$ *for all* $r \in \mathbb{N}_0$, $\mathbf{n} \in P_r(G) = P_r(G_q)$.

Proof. By Lemma 2.1 and (2) we know that $\mathcal{A}(G_q)^{\mathrm{coc}}$ has $\mathrm{tr}(\varphi_{\mathbf{n},r})$, $r \in \mathbb{N}_0$, $\mathbf{n} \in P_r(G)$ as a vector space basis. Identify $\mathcal{A}(G_q)$ with its image under ι from Proposition 3.1. Then $\mathcal{A}(G_q)^{\mathrm{coc}}$ is identified with the subspace of $\mathcal{O}(G_q)^{\mathrm{coc}} \otimes \mathbb{C}[z]$ spanned by the $\mathrm{tr}(\varphi_{\mathbf{n}}) \otimes z^r$ with $\mathbf{n} \in P_r(G_q) = P_r(G)$. The assertion now immediately follows from Proposition 2.3. $\square$

The corepresentation ω_d of $\mathcal{O}(G_q)$ from Section 2 is defined as $(\Omega_d)_{\mathcal{O}(G_q)}$, where Ω_d is a natural right coaction of $\mathcal{A}(G_q)$ on the degree d homogeneous component of the quantum exterior algebra $\bigwedge(G_q)$, for $d = 1, \dots, N$. Set $\rho_d = \mathrm{tr}(\Omega_d)$. Then ρ_d is a cocommutative element in $\mathcal{A}(G_q)$, and $\pi(\rho_d) = \sigma_d$. Another cocommutative element is $\mathcal{D}_q$. Under the bialgebra injection ι, the element $\mathcal{D}_q$ is mapped to $1 \otimes z^2$ (to $1 \otimes z^N$ in the case of $SL_q(N)$), and ρ_d is mapped to $\sigma_d \otimes z^d$. The elements ρ_d can be expressed as polynomials of the generators u^i_j in each concrete case, using the well-known basis and the defining relations of $\bigwedge(G_q)$. The expression for $\mathcal{D}_q$ can be found in [15, 9.3.1 Lemma 12].

Example. The quantum exterior algebra $\bigwedge(O_q(3))$ (we need to use the version on [15, p. 322], and not the one given in [22]) has three generators y_1, y_2, y_3, subject to the relations

$$y_1^2 = y_3^2 = 0, \qquad y_2^2 = \left(q^{1/2} - q^{-1/2}\right) y_1 y_3,$$
$$y_1 y_2 = -q^{-1} y_2 y_1, \qquad y_2 y_3 = -q^{-1} y_3 y_2, \qquad y_1 y_3 = -y_3 y_1.$$

For $1 \leqslant i < j \leqslant 3$ we have $\Omega_2(y_i y_j) = \sum_{s,t=1}^3 y_s y_t \otimes u^s_i u^t_j$. The degree two homogeneous component of $\bigwedge(O_q(3))$ has the basis $y_1 y_2$, $y_2 y_3$, $y_1 y_3$, and using the above relations it is easy to rewrite any monomial $y_s y_t$ as a linear combination of the basis elements. Thus one can easily get that

$$\rho_2 = \mathrm{tr}(\Omega_2) = u^1_1 u^2_2 - q u^2_1 u^1_2 + u^2_2 u^3_3 - q u^3_2 u^2_3 + u^1_1 u^3_3 - u^3_1 u^1_3 + \left(q^{1/2} - q^{-1/2}\right) u^2_1 u^2_3.$$

An expression for the element $\mathcal{D}_q$ is

$$\mathcal{D}_q = u^1_1 u^3_3 + q^{1/2} u^2_1 u^2_3 + q u^3_1 u^1_3.$$

The explicit generators and relations for $\mathcal{A}(G_q)^{\mathrm{coc}}$ are the following:

Theorem 3.3.

(i) (*cf.* [5]) *The algebra $\mathcal{A}(SL_q(N))^{\mathrm{coc}}$ is the N-variable commutative polynomial algebra generated by $\rho_1, \rho_2, \dots, \rho_N = \mathcal{D}_q$. In particular, its Hilbert series is $\prod_{i=1}^N (1 - t^i)^{-1}$.*

(ii) *For $Sp_q(N)$, $N = 2l$, $l \in \mathbb{N}$, the cocommutative elements $\mathcal{A}(Sp_q(N))^{\mathrm{coc}}$ form an $(l+1)$-variable commutative polynomial algebra generated by $\mathcal{D}_q, \rho_1, \rho_2, \dots, \rho_l$. In particular, the Hilbert series of $\mathcal{A}(Sp_q(N))^{\mathrm{coc}}$ is $(1-t^2)^{-1}\prod_{i=1}^{l}(1-t^i)^{-1}$.*

(iii) *For $O_q(N)$, $N = 2l$ or $2l+1$, $l \in \mathbb{N}$, $N \geqslant 3$, we have that $\mathcal{A}(O_q(N))^{\mathrm{coc}}$ is the commutative algebra generated by $\mathcal{D}_q, \rho_1, \rho_2, \dots, \rho_N$, subject to the relations*

$$\rho_{N-i}\rho_{N-j} = \rho_i\rho_j\mathcal{D}_q^{N-i-j} \quad (0 \leqslant i \leqslant j \leqslant l),$$
$$\rho_i\rho_{N-j}\mathcal{D}_q^{j-i} = \rho_j\rho_{N-i} \quad (0 \leqslant i < j \leqslant l),$$

where we set $\rho_0 = 1$ for notational convenience. A $\mathbb{C}$-vector space basis of $\mathcal{A}(O_q(N))^{\mathrm{coc}}$ is $B(N)$, where

$$B(2l) = \bigl\{\rho_1^{i_1}\cdots\rho_l^{i_l}\mathcal{D}_q^{j},\ \rho_N\rho_1^{j_1}\cdots\rho_{l-1}^{j_{l-1}}\mathcal{D}_q^{k},\ \rho_{N-a}\mathcal{D}_q^{b}\rho_1^{k_1}\cdots\rho_{a-b-1}^{k_{a-b-1}}\rho_a^{k_a}\cdots\rho_{l-1}^{k_{l-1}} \ \big|\ j,k,i_s,j_s,k_s \in \mathbb{N}_0,\ 0 \leqslant b < a \leqslant l-1\bigr\}$$

and

$$B(2l+1) = \bigl\{\rho_1^{i_1}\cdots\rho_l^{i_l}\mathcal{D}_q^{j},\ \rho_N\rho_1^{j_1}\cdots\rho_l^{j_l}\mathcal{D}_q^{k},\ \rho_{N-a}\mathcal{D}_q^{b}\rho_1^{k_1}\cdots\rho_{a-b-1}^{k_{a-b-1}}\rho_a^{k_a}\cdots\rho_l^{k_l} \ \big|\ j,k,i_s,j_s,k_s \in \mathbb{N}_0,\ 0 \leqslant b < a \leqslant l\bigr\}.$$

In particular, the Hilbert series of $\mathcal{A}(O_q(N))^{\mathrm{coc}}$ is

$$\frac{1 + t^N(1-t^l) + (1-t^2)(1-t^l)\sum_{0\leqslant b<a\leqslant l-1} t^{N-a+2b}\prod_{k=a-b}^{a-1}(1-t^k)}{(1-t^2)\prod_{i=1}^{l}(1-t^i)},$$

when $N = 2l$, and

$$\frac{1 + t^N + (1-t^2)\sum_{0\leqslant b<a\leqslant l} t^{N-a+2b}\prod_{k=a-b}^{a-1}(1-t^k)}{(1-t^2)\prod_{i=1}^{l}(1-t^i)},$$

when $N = 2l+1$.

Proof. By Proposition 3.2, it is sufficient to prove the result in the classical case. Generators of $\mathcal{A}(G)^{\mathrm{coc}}$ can be obtained from an old result of [23]. The relations among the generators can be determined using the classical case of Proposition 3.1 and Theorem 2.4. A sketch of the details is given in Appendix C. □

The relation $\rho_N^2 = \mathcal{D}_q^N$ in $\mathcal{A}(O_q(N))$ (the special case $i = j = 0$ of the first type relations in Theorem 3.3(iii)) has already been obtained in [13] and [25]. It shows that $\mathcal{A}(O_q(2l))$ is not a domain.

4. Dually paired Hopf algebras and quantum traces

In this preparatory section we collect some standard generalities on Hopf algebras in a form that we shall need later.

Let $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{O} \to \mathbb{C}$ be a dual pairing of Hopf algebras $\mathcal{U}$ and $\mathcal{O}$; see, for example, [15, 1.2.5] for the notion of a dual pairing. Assume that $\langle u, f \rangle = 0$ for all u implies $f = 0$. Then the map $f \mapsto \langle \cdot, f \rangle$ is an injection of $\mathcal{O}$ into the dual space $\mathcal{U}^*$ of $\mathcal{U}$. This injection identifies $\mathcal{O}$ with a Hopf subalgebra of the finite dual $\mathcal{U}^\circ$ of $\mathcal{U}$; in the sequel we shall freely make this identification.

Let $\varphi : V \to V \otimes \mathcal{O}$, $v \mapsto \sum v_0 \otimes v_1$ be a corepresentation of $\mathcal{O}$ on V. (We say then that V is a right $\mathcal{O}$-comodule.) Denote by $L(V)$ the algebra of linear transformations on V. Then $\hat{\varphi} : \mathcal{U} \to L(V)$ defined by the formula $\hat{\varphi}(u)v := \sum \langle u, v_1 \rangle v_0$, $u \in \mathcal{U}$, $v \in V$, is an algebra homomorphism. Thus the corepresentation φ on V induces a representation $\hat{\varphi}$ of $\mathcal{U}$ on V. In other words, a right $\mathcal{O}$-comodule V automatically becomes a left $\mathcal{U}$-module, and the following basic properties hold.

Proposition 4.1. *Let $\varphi : V \to V \otimes \mathcal{O}$ be a corepresentation of $\mathcal{O}$, and let $\hat{\varphi}$ be the corresponding representation of $\mathcal{U}$.*

(i) *A subspace W of V is an $\mathcal{O}$-subcomodule if and only if W is an $\mathcal{U}$-submodule.*
(ii) *An element $v \in V$ is an $\mathcal{O}$-coinvariant if and only if v is a $\mathcal{U}$-invariant.*
(iii) *The coefficient space $C(\varphi)$ of φ coincides with the space of matrix elements $M(\hat{\varphi})$ of $\hat{\varphi}$, provided that V is finite dimensional.*

Recall that $C(\varphi)$ is the smallest subspace C in $\mathcal{O}$ such that $\varphi(V) \subseteq V \otimes C$; it is a subcoalgebra of $\mathcal{O}$. For a finite dimensional representation T of $\mathcal{U}$ the space of matrix elements is

$$M(T) := \mathrm{Span}_{\mathbb{C}} \left\{ c_v^\xi \mid \xi \in V^*,\ v \in V \right\} \subset \mathcal{U}^*,$$

where V^* is the dual space of V, and for $\xi \in V^*$, $v \in V$ the linear function c_v^ξ on $\mathcal{U}$ maps $x \in \mathcal{U}$ to $\xi(T(x)v)$.

In the sequel we write S for the antipode, and Δ for the comultiplication in the Hopf algebras considered. The *right adjoint coaction* $\beta : \mathcal{O} \to \mathcal{O} \otimes \mathcal{O}$ is given in Sweedler's notation by

$$\beta(f) = \sum f_2 \otimes S(f_1) f_3,$$

and the *right adjoint action* ad of $\mathcal{U}$ on itself is given by

$$\mathrm{ad}(a)b = \sum S(a_1) b a_2, \quad a, b \in \mathcal{U},$$

see, for example, [15, 1.3.4] for these definitions. The connection between ad and β can be explained in terms of the left action ad° of $\mathcal{U}$ on its dual space $\mathcal{U}^*$, defined by the formula

$$\big(\mathrm{ad}^\circ(a)\xi\big)(b) := \xi\big(\mathrm{ad}(a)b\big), \quad a, b \in \mathcal{U},\ \xi \in \mathcal{U}^*.$$

Proposition 4.2. *The representation* $\hat{\beta}$ *coincides with the subrepresentation of* ad° *on the* $\mathcal{U}$*-invariant subspace* $\mathcal{O}$ *of* $\mathcal{U}^*$.

Proof. For $a, b \in \mathcal{U}$ and $f \in \mathcal{O}$ we have

$$\begin{aligned}\big\langle b, \hat{\beta}(a)f\big\rangle &= \Big\langle b, \sum \big\langle a, S(f_1)f_3\big\rangle f_2\Big\rangle = \sum \big\langle a, S(f_1)f_3\big\rangle \langle b, f_2\rangle \\ &= \sum \big\langle a_1, S(f_1)\big\rangle \langle a_2, f_3\rangle \langle b, f_2\rangle = \sum \big\langle S(a_1), f_1\big\rangle \langle b, f_2\rangle \langle a_2, f_3\rangle \\ &= \sum \big\langle S(a_1)b, f_1\big\rangle \langle a_2, f_2\rangle = \sum \big\langle S(a_1)ba_2, f\big\rangle = \big\langle \mathrm{ad}(a)b, f\big\rangle.\end{aligned}$$

This implies $\hat{\beta}(a)f = \mathrm{ad}^\circ(a)f$. $\square$

Suppose that there exists an invertible element $\mathcal{K}$ in $\mathcal{U}$ such that

$$S^2(a) = \mathcal{K}a\mathcal{K}^{-1} \quad \text{for all } a \in \mathcal{U}. \tag{7}$$

Then for an arbitrary finite dimensional representation $T : \mathcal{U} \to L(V)$ we define the *quantum trace* of T by

$$\mathrm{tr}_q\, T(a) := \mathrm{Tr}\big(T\big(\mathcal{K}^{-1}a\big)\big), \quad a \in \mathcal{U}, \tag{8}$$

where Tr is the ordinary trace function. So $\mathrm{tr}_q\, T$ is an element of $M(T)$, which is determined by the isomorphism class of T. Obviously this quantum trace depends on the choice of $\mathcal{K}$. It follows from (7) and usual properties of Tr that $\mathrm{ad}^\circ(a)\,\mathrm{tr}_q\, T = \varepsilon(a)\,\mathrm{tr}_q\, T$, or, in other words, that $\mathrm{tr}_q\, T$ is invariant with respect to the action ad°.

Proposition 4.3. *If* $T : \mathcal{U} \to L(V)$ *is a finite dimensional irreducible representation of* $\mathcal{U}$*, such that* $T \otimes T^*$ *and* $T^* \otimes T$ *are isomorphic representations of* $\mathcal{U}$*, then up to scalar multiple,* $\mathrm{tr}_q\, T$ *is the only* ad°*-invariant element in* $M(T)$.

Proof. We use a sequence of natural isomorphisms of $\mathcal{U}$-modules

$$L(V) \cong V \otimes V^* \cong V^* \otimes V \cong M(T). \tag{9}$$

The first isomorphism associates with $v \otimes \xi \in V \otimes V^*$ the linear transformation $x \mapsto \xi(x)v$. This is an isomorphism of the $\mathcal{U}$-representations $T \otimes T^*$ and ad^T, where

$$\mathrm{ad}^T(a)\phi := \sum T(a_1)\phi T\big(S(a_2)\big).$$

By assumption, there exists a linear isomorphism $R_{TT^*}: V \otimes V^* \to V^* \otimes V$ intertwining between $T \otimes T^*$ and $T^* \otimes T$; this is the second isomorphism in (9). The third map $c: V^* \otimes V \to M(T)$ is the linear map sending $\xi \otimes v$ to c_v^ξ. It is surjective by the definition of $M(T)$. This map intertwines between the representations $T^* \otimes T$ and ad°, as one can easily check. (In particular, this shows that $M(T)$ is an ad°-invariant subspace of $\mathcal{U}^*$.) Since our base field $\mathbb{C}$ is algebraically closed, the irreducibility of T implies that $T(\mathcal{U}) = L(V)$, hence the dimension of $M(T)$ is $\dim(V)^2$. Therefore, the surjective linear map c goes between vector spaces of the same dimension. Thus c must be an isomorphism.

Since $\mathcal{K}$ is invertible, $T(\mathcal{K}^{-1})$ is non-zero, and so there exists a $\phi \in L(V)$ such that $\mathrm{Tr}(T(\mathcal{K}^{-1})\phi)$ is non-zero. Choose $a \in \mathcal{U}$ with $T(a) = \phi$. Then $\mathrm{tr}_q\, T(a)$ is non-zero, showing that $\mathrm{tr}_q\, T$ is a non-zero element of $M(T)$. Therefore by the $\mathcal{U}$-module isomorphisms of (9), it is sufficient to show that the subspace of ad^T-invariants in $L(V)$ is one-dimensional. The latter statement is the assertion of Schur's lemma, because $\mathrm{ad}^T(a)\phi = \varepsilon(a)\phi$ for all $a \in \mathcal{U}$ if and only if $T(a)\phi = \phi T(a)$ for all $a \in \mathcal{U}$ (this equivalence can be proved by a straightforward modification of the well known proof of the statement that the center of $\mathcal{U}$ coincides with the subspace of ad-invariant elements). $\square$

A nice example to apply the above considerations is the case when $\mathcal{U}$ is *almost cocommutative*. This means that there exists an invertible element $\mathcal{R}$ in $\mathcal{U} \otimes \mathcal{U}$ such that

$$\tau \circ \Delta(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \quad \text{for all } a \in \mathcal{U},$$

where τ is the flip map. Set $\mathcal{K} := \mu(\mathrm{id} \otimes S)(\mathcal{R}^{-1})$, where μ is the multiplication map in $\mathcal{U}$. Then $\mathcal{K}$ is an invertible element of $\mathcal{U}$, with inverse $\mu(\mathrm{id} \otimes S)(\mathcal{R})$. Formula (7) holds by [3, Proposition 4.2.3], and the remarks afterwards. Thus, using this $\mathcal{K}$, formula (8) gives an ad°-invariant quantum trace. Moreover, for an arbitrary representation T of $\mathcal{U}$ the representations $T \otimes T^*$ and $T^* \otimes T$ are isomorphic; an isomorphism between them is $\tau \circ (T \otimes T^*)\mathcal{R}$, where $\tau(v \otimes \xi) = \xi \otimes v$, see, for example, [3, 4.2, p. 119]. Therefore, we may apply Proposition 4.3 to conclude that if T is irreducible, then up to scalar multiple, $\mathrm{tr}_q\, T$ is the only ad°-invariant element in $M(T)$. We note that in this case $\mathrm{tr}_q\, T$ is the image of $\mathrm{id}_V \in L(V)$ under the composition of the isomorphisms (9), with the isomorphism $\tau \circ (T \otimes T^*)\mathcal{R}$ being used in the middle.

5. Adjoint coinvariants in $\mathcal{O}(G_q)$

For an arbitrary Hopf algebra $\mathcal{O}$, the space $\mathcal{O}^{\mathrm{coc}}$ coincides with the space $\mathcal{O}^\alpha = \{f \in \mathcal{O} \mid \alpha(f) = f \otimes 1\}$ of α-coinvariants, where α is the right coaction of $\mathcal{O}$ on itself given in Sweedler's notation by the formula $\alpha: f \mapsto \sum f_2 \otimes f_3 S(f_1)$, see [5]. So in Section 2 we were dealing with $\mathcal{O}(G_q)^\alpha$; a parallel analysis of the space $\mathcal{O}(G_q)^\beta$ of β-coinvariants is carried out in this section, where β is the adjoint coaction $\beta: f \mapsto \sum f_2 \otimes S(f_1) f_3$. The results (and the proofs) are essentially the same as those of Section 2, but the natural interpretation of them involves the quantized enveloping algebra $\mathcal{U}(G_q)$ associated to G_q, fitting into the general framework formalized in Section 4.

For $G_q = SL_q(N), Sp_q(N), SO_q(2l), SO_q(2l+1)$, the Hopf algebra $\mathcal{U}(G_q)$ is the Drinfeld–Jimbo algebra $U_q(\mathrm{sl}_N)$, $U_q(\mathrm{sp}_N)$, $U_q(\mathrm{so}_{2l})$, $U_{q^{1/2}}(\mathrm{so}_{2l+1})$, respectively. The algebra $\mathcal{U}(O_q(N))$ is $\widetilde{U}_q(\mathrm{so}_N)$, defined in [15, 8.6.1], following [13] (see Appendix A of the present paper). The algebra $\mathcal{U}(GL_q(N))$ is $U_q(\mathrm{gl}_N)$, defined in [15, p. 163]. There is a dual pairing $\langle\cdot,\cdot\rangle : \mathcal{U}(G_q) \times \mathcal{O}(G_q) \to \mathbb{C}$, given in [15, 9.4]. We still assume that q is transcendental (or q is not a root of unity for $GL_q(N)$, $SL_q(N)$). Then this dual pairing is non-degenerate by [13] (see also [15, pp. 410 and 440]). In particular, the map $f \mapsto \langle\cdot, f\rangle$ injects $\mathcal{O}(G_q)$ into the finite dual $\mathcal{U}(G_q)^\circ$ of $\mathcal{U}(G_q)$. In the sequel we shall often consider $\mathcal{O}(G_q)$ as a Hopf-subalgebra of $\mathcal{U}(G_q)^\circ$ in this way.

The representation $\hat{\omega}$ induced by the fundamental corepresentation ω is the so-called *vector representation* of $\mathcal{U}(G_q)$. More generally, set $T_{\mathbf{n}} = \hat{\varphi}_{\mathbf{n}}$ for $\mathbf{n} \in P(G_q)$. When $G_q = Sp_q(N)$, $SL_q(N)$, or $GL_q(N)$, then $\{T_{\mathbf{n}} \mid \mathbf{n} \in P(G_q)\}$ is a complete list of the isomorphism classes of the so-called *type 1 finite dimensional irreducible representations of* $\mathcal{U}(G_q)$. When $G_q = O_q(N)$ or $SO_q(N)$, then $\{T_{\mathbf{n}} \mid \mathbf{n} \in P(G_q)\}$ is a complete list of the isomorphism classes of those (type 1) irreducible representations, which appear as a direct summand in some tensor power of the vector representation.

Let us introduce the following ad hoc terminology. By the *basic representations of* $\mathcal{U}(G_q)$ we mean $\hat{\omega}_1, \ldots, \hat{\omega}_l$ for $G_q = SL_q(l+1), Sp_q(2l), SO_q(2l+1)$, the representations $\hat{\omega}_1, \ldots, \hat{\omega}_N$ for $O_q(N)$, $N = 2l, 2l+1$, the representations $\hat{\omega}_1, \ldots, \hat{\omega}_{l-1}, \hat{\omega}_{l,0}, \hat{\omega}_{l,1}$ for $SO_q(2l)$, and the representations $\hat{\omega}_1, \ldots, \hat{\omega}_N, \hat{\omega}_N^*$ for $GL_q(N)$.

We set $\mathcal{K} = K_{2\rho} \in \mathcal{U}(G_q)$, where $K_{2\rho}$ is defined in [15, p. 164]. So $\rho = \sum_{i=1}^{l} n_i\alpha_i$ is the half-sum of positive roots, α_i are the simple roots of $\mathfrak{g}$, and $K_{2\rho} = K_1^{n_1} \cdots K_l^{n_l}$, where K_i are usual generators of the Drinfeld–Jimbo algebra $U_q(\mathfrak{g})$. For $GL_q(N)$, we set $\mathcal{K} = K_1^{N-1}K_2^{N-3}K_3^{N-5} \cdots K_N^{-N+1}$, where $K_1, \ldots, K_N$ denote the same generators of $\mathcal{U}(GL_q(N))$ as in [15, 6.1.2, p. 163]. Using [15, 6.1.2 Proposition 6] it is easy to check that formula (7) holds for $\mathcal{K}$. Therefore formula (8) defines an ad°-invariant quantum trace $\mathrm{tr}_q\, T$ for an arbitrary finite dimensional representation T of $\mathcal{U}(G_q)$. It is well known that for arbitrary finite dimensional representations T_1, T_2 of $\mathcal{U}(G_q)$ we have $T_1 \otimes T_2 \cong T_2 \otimes T_1$. Therefore by Proposition 4.3, we obtain that for any irreducible finite dimensional representation of $\mathcal{U}(G_q)$, the quantum trace $\mathrm{tr}_q\, T$ spans the subspace of ad°-invariants in $M(T)$.

Obviously, for finite dimensional representations T_1, T_2 we have

$$\mathrm{tr}_q(T_1 \oplus T_2) = \mathrm{tr}_q\, T_1 + \mathrm{tr}_q\, T_2. \tag{10}$$

Since $\mathcal{K}$ is group-like, by [15, 7.1.6] we have

$$\mathrm{tr}_q(T_1 \otimes T_2) = (\mathrm{tr}_q\, T_1) \star (\mathrm{tr}_q\, T_2), \tag{11}$$

where $\star$ is the convolution multiplication in the dual of $\mathcal{U}(G_q)$; so when the irreducible summands of T_1, T_2 are contained in $\{T_{\mathbf{n}} \mid \mathbf{n} \in P(G_q)\}$, then the right-hand side of (11) is the product of $\mathrm{tr}_q\, T_1$ and $\mathrm{tr}_q\, T_2$ in $\mathcal{O}(G_q)$.

Theorem 5.1. *The quantum traces* $\{\mathrm{tr}_q\, T_{\mathbf{n}} \mid \mathbf{n} \in P(G_q)\}$ *form a* $\mathbb{C}$*-vector space basis of the space of* β*-coinvariants in* $\mathcal{O}(G_q)$*. The linear map* $\mathcal{O}(G_q)^\beta \to \mathcal{O}(G)^\beta$, $\mathrm{tr}_q\, T_{\mathbf{n}} \mapsto \mathrm{tr}\,\varphi_{\mathbf{n}}$, $\mathbf{n} \in P(G)$*, is an algebra isomorphism between* $\mathcal{O}(G_q)^\beta$ *and its classical counterpart*

$\mathcal{O}(G)^\beta = \mathcal{O}(G)^{\mathrm{coc}}$. *As a* $\mathbb{C}$*-algebra,* $\mathcal{O}(G_q)^\beta$ *is generated by the quantum traces of the basic representations of* $\mathcal{U}(G_q)$*, subject to the same relations as the corresponding cocommutative elements in Theorem* 2.4.

Proof. Identifying $\mathcal{O}(G_q)$ with a subspace of the dual of $\mathcal{U}(G_q)$, the Peter–Weyl decomposition is written as $\mathcal{O}(G_q) = \bigoplus_{\mathbf{n}\in P(G_q)} M(T_\mathbf{n})$. It is clearly a decomposition as a direct sum of β-subcomodules. Therefore we have $\mathcal{O}(G_q)^\beta = \bigoplus_{\mathbf{n}\in P(G_q)} M(T_\mathbf{n})^\beta$, hence the elements $\mathrm{tr}_q\, T_\mathbf{n}$ form a basis in $\mathcal{O}(G_q)^\beta$ by Proposition 4.3. The structure constants of the algebra $\mathcal{O}(G_q)^\beta$ with respect to this basis are the multiplicities appearing in the tensor product decompositions $T_\mathbf{n} \otimes T_\mathbf{m} \cong \bigoplus_\mathbf{p} m_\mathbf{p}^{\mathbf{n},\mathbf{m}} T_\mathbf{p}$ by (10) and (11). Since the multiplicities $m_\mathbf{p}^{\mathbf{n},\mathbf{m}}$ are the same as in the classical case $q = 1$ (see [15, 7.2] or [3, Proposition 10.1.16], and Appendix A for the case of $O_q(N)$), we obtain the statement about the algebra isomorphism $\mathcal{O}(G_q)^\beta \cong \mathcal{O}(G)^{\mathrm{coc}}$. Then the statement about the generators and relations follows from the known classical case (see Appendix B). □

The definition of the adjoint coaction of $\mathcal{O}(G_q)$ on itself can be modified to make it a coaction β of $\mathcal{O}(G_q)$ on the FRT-bialgebra $\mathcal{A}(G_q)$ as follows: $\beta(f) = \sum f_2 \otimes S(\pi(f_1))\pi(f_3)$. The results of Theorem 5.1 imply a description of $\mathcal{A}(G_q)^\beta$ both as a vector space and as an algebra with explicit generators and relations. This can be derived from the bialgebra embedding ι in Proposition 3.1 in the same way as the results on $\mathcal{A}(G_q)^{\mathrm{coc}}$. The algebra $\mathcal{A}(G_q)^\beta$ turns out to be isomorphic to $\mathcal{A}(G_q)^{\mathrm{coc}} \cong \mathcal{A}(G)^{\mathrm{coc}} = \mathcal{A}(G)^\beta$ as graded algebras (but $\mathcal{A}(G_q)^\beta$ and $\mathcal{A}(G_q)^{\mathrm{coc}}$ are two different subsets of $\mathcal{A}(G_q)$ when $q \neq 1$). We omit the obvious details.

Example. Let us compute $\mathrm{tr}_q\, \hat{\omega}_m$ in the case of $GL_q(N)$. For subsets $I, J \subseteq \{1, \ldots, N\}$ with $|I| = |J| = m$, write $[I|J]$ for the corresponding quantum minor of (u^i_j). So $[I|J]$ is the quantum determinant of the $m \times m$ quantum matrix $(u^i_j)^{i\in I}_{j\in J}$. Fix $J_0 = \{1, \ldots, m\}$, and write $e_I = [J_0|I]$ for the quantum minors belonging to the first m rows. Since $\Delta(e_I) = \sum_{|J|=m} e_J \otimes [J|I]$, the subspace in $\mathcal{O}(GL_q(N))$ spanned by $\{e_J \mid m = |J|\}$ is a subcomodule with respect to the right coaction Δ; the corresponding corepresentation is ω_m, see [15, 11.5.3]. The coefficient space $C(\omega_m)$ of ω_m is the subspace of $\mathcal{O}(GL_q(N))$ spanned by all the $m \times m$ quantum minors. By definition of $\hat{\omega}_m$, for $x \in \mathcal{U}(GL_q(N))$ we have

$$\hat{\omega}_m(x)e_I = \sum_J \langle x, [J|I]\rangle e_J.$$

It follows from the explicit formulae giving the dual pairing in [15, 9.4, p. 328] that

$$\hat{\omega}_m(K_i)e_J = \begin{cases} q^{-1}e_J, & \text{if } i \in J; \\ e_J, & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\hat{\omega}_m\big(\mathcal{K}^{-1}\big)e_J = \hat{\omega}_m\left(\prod_{i=1}^{N} K_i^{-N-1+2i}\right)e_J = \big(q^{-1}\big)^{\sum_{i\in J}(-N-1+2i)}e_J$$
$$= q^{m(N+1)}q^{-2(\sum_{i\in J} i)}e_J;$$

that is, the matrix of $\hat{\omega}_m(\mathcal{K}^{-1})$ with respect to the basis $\{e_J \mid m = |J|\}$ is diagonal. Thus

$$\mathrm{tr}_q\,\hat{\omega}_m(x) = \mathrm{Tr}\big(\hat{\omega}_m\big(\mathcal{K}^{-1}\big)\hat{\omega}_m(x)\big) = \sum_{|J|=m} q^{(m(N+1)-2\sum_{i\in J} i)}\big\langle x, [J|J]\big\rangle.$$

This means that for $m = 1, \ldots, N$, we have

$$\mathrm{tr}_q\,\hat{\omega}_m = \sum_{|J|=m} q^{(m(N+1)-2\sum_{i\in J} i)}[J|J],$$

where the summation ranges over the m-element subsets J of $\{1, \ldots, N\}$. Note that a scalar multiple of this element appears as the basic coinvariant τ_m introduced in [5]. Since it is convenient to perform computations in $\mathcal{O}(GL_q(N))$, the results of this section can be viewed as an explicit determination of the quantum traces of finite dimensional representations of type 1 of $\mathcal{U}(GL_q(N))$, as elements of $\mathcal{O}(GL_q(N))$.

Appendix A

This appendix deals with the algebra $\mathcal{U}(O_q(2l))$ associated with $O_q(2l)$ by Hayashi [13]. We prove the assertion on multiplicities of irreducibles in tensor product decompositions, used in the proof of Proposition 2.2.

Throughout this appendix $q \in \mathbb{C}^\times$ is not a root of unity, or $q = 1$. Write E_i, F_i, K_i, K_i^{-1} $(i = 1, \ldots, l)$ for the usual generators of the Drinfeld–Jimbo algebra $U_q = U_q(\mathrm{so}_{2l})$, defined, for example, in [15, 6.1.2]; when $q = 1$, the algebra U_1 can be defined using an integral form of the Drinfeld–Jimbo algebra, see the proof of Proposition 6.2. The universal enveloping algebra $U = U(\mathrm{so}_{2l})$ is the homomorphic image of U_1, with kernel generated by $K_i - 1$, $i = 1, \ldots, l$. The Dynkin diagram D_l of so_{2l} has an involutive automorphism interchanging the nodes $l-1$ and l. Denote by χ the corresponding involutive automorphism of U_q, so $E_i^\chi = E_{\chi(i)}$, $F_i^\chi = F_{\chi(i)}$, $K_i^\chi = K_{\chi(i)}$, where $\chi(i) = i$ for $i = 1, \ldots, l-2$, $\chi(l-1) = \chi(l)$, and $\chi(l) = \chi(l-1)$. This extends to a Hopf algebra automorphism of U_q by [15, 6.1.6 Theorem 16]. In the case $q = 1$, the automorphism χ of U_1 (see the proof of Proposition 6.2 for the definition of χ on U_1) induces an automorphism (denoted by χ as well) of the quotient U. (The algebra U is generated by the images of E_i, F_i, and χ permutes them by the same rule as above.) Write $\mathbb{C}[\chi]$ for the group algebra of the two-element group generated by χ, and set $\widetilde{U}_q = \mathbb{C}[\chi] \rtimes U_q$, the right crossed product algebra with commutation rule $\chi a \chi = a^\chi$, $a \in U_q$. Similarly, we set $\widetilde{U} = \mathbb{C}[\chi] \rtimes U$.

Let P denote the weight lattice of so_{2l}, and P_+ the subset of dominant integral weights. For $\lambda \in P_+$ denote by T_λ the finite dimensional irreducible type 1 representation of U_q with highest weight λ. The type 1 finite dimensional irreducible representations of $\widetilde{U}_q$

can be determined from the corresponding list for U_q by standard arguments, see [15, 8.6.1 Proposition 34]. Namely, the automorphism χ induces an involutory action on the set of isomorphism classes of irreducible finite dimensional representations of U_q. Given a representation T of U_q define the representation T^χ by $T^\chi(a) = T(a^\chi)$, $a \in U_q$. A dominant integral weight λ can be represented by a sequence of non-negative integers $\lambda = (\lambda_1, \dots, \lambda_l)$, where $\lambda_i = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i)$, and $\alpha_1, \dots, \alpha_l$ are the simple roots. Then $T_\lambda^\chi \cong T_{\lambda^\chi}$, where $\lambda^\chi = (\lambda_1, \dots, \lambda_{l-2}, \lambda_l, \lambda_{l-1})$. If $\lambda^\chi = \lambda$, then there are exactly two non-equivalent extensions of T_λ to a representation of $\widetilde{U}_q$ on the same underlying space, denote them by $\widetilde{T}_\lambda$ and $\widetilde{T}_\lambda^\circ$. They are distinguished by $\widetilde{T}_\lambda(\chi)v = v$ and $\widetilde{T}_\lambda^\circ(\chi)v = -v$ for a highest weight vector v of T_λ. If $\lambda^\chi \neq \lambda$, then the U_q-representation $T_\lambda \oplus T_{\lambda^\chi}$ extends to an irreducible representation $\widetilde{T}_\lambda$ of $\widetilde{U}_q$; the transformation $\widetilde{T}_\lambda(\chi)$ interchanges the underlying subspaces of T_λ and T_{λ^χ}. Now

$$\mathcal{T} = \left\{\widetilde{T}_\lambda, \widetilde{T}_\lambda^\circ, \widetilde{T}_\mu \;\middle|\; \lambda, \mu \in P_+,\ \lambda^\chi = \lambda,\ \mu^\chi \neq \mu\right\}$$

is a complete list of isomorphism classes of type 1 finite dimensional irreducible representations of $\widetilde{U}_q$ (note that q is assumed to be not a root of unity, or $q = 1$).

The Hopf algebra $\mathcal{U}(O_q(2l))$ was defined to be $\widetilde{U}_q$. This can be justified as follows. The element χ may be identified with a suitable reflection in the full orthogonal group $O(2l)$, such that the tangent map of the conjugation by $\chi \in O(2l)$ on the special orthogonal group $SO(2l)$, which is a Lie algebra automorphism of so_{2l}, induces the automorphism χ of the universal enveloping algebra U, defined above. A representation T of $O(2l)$ induces naturally a representation $\widetilde{T}$ of $\widetilde{U}$: on U it is the tangent representation of T, whereas $\widetilde{T}(\chi) = T(\chi)$. Obviously, $\widetilde{T}$ determines T. Writing P'_+ for the subset of P_+ consisting of those λ for which T_λ is the tangent representation of a representation of the group $SO(2l)$ (note that with the notation of Section 2, P'_+ may be naturally identified with $P(SO(2l))$), consider the set $\mathcal{T}' = \{\widetilde{T}_\lambda, \widetilde{T}_\lambda^\circ, \widetilde{T}_\mu \mid \lambda, \mu \in P'_+, \lambda^\chi = \lambda, \mu^\chi \neq \mu\}$ of $\widetilde{U}_q$-representations. In the case $q = 1$, $\mathcal{T}'$ is a set of $\widetilde{U}$-representations (to be more precise, $\widetilde{U}_1$ representations factoring through $\widetilde{U}$), and it coincides with the set of isomorphism classes of $\widetilde{T}$, as T ranges over the set of isomorphism classes of irreducible representations of $O(2l)$. So in the classical case $q = 1$ we may think of the elements of $\mathcal{T}'$ as representations of the full orthogonal group $O(2l)$.

Denote by $\mathcal{H}$ the subalgebra of U_q generated by $K_1^{\pm1}, \dots, K_l^{\pm1}$, and by $\mathcal{H}\langle\tau\rangle$ the subalgebra of $\widetilde{U}_q$ generated by χ over $\mathcal{H}$. Let V be a type 1 finite dimensional $\widetilde{U}_q$-module. (When $q = 1$, the elements K_i act trivially on a type 1 module, so V is actually a module over U.) It has a weight space decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$, and its $\mathcal{H}$-module structure is described by the weight multiplicities $(d_\lambda \mid \lambda \in P)$, $d_\lambda = \dim_{\mathbb{C}} V_\lambda$. Multiplication by χ interchanges the weight spaces V_λ and V_{λ^χ}, hence $d_\lambda = d_{\lambda^\chi}$. For $\lambda = \lambda^\chi$, the subspace V_λ is preserved by χ, and χ acts as an involutory linear automorphism of V_λ; denote by d_λ^+ and d_λ^- the multiplicity of 1 and -1 as an eigenvalue of χ on V_λ, so $d_\lambda^+ + d_\lambda^- = d_\lambda$. Clearly, the $\mathcal{H}\langle\tau\rangle$-module structure of V is determined by the collection of non-negative integers $(d_\lambda^+, d_\lambda^-, d_\mu \mid \lambda, \mu \in P, \lambda = \lambda^\chi, \mu \neq \mu^\chi)$, that we shall call the *formal character* $\mathrm{char}_{\mathcal{H}\langle\tau\rangle} V$ of the $\widetilde{U}_q$-module V. (We keep this notation for the formal character also when $q = 1$, although in this case the weight multiplicities carry more information than just the $\mathcal{H}$-module structure.)

Proposition 6.1. *The structure of a type* 1 *finite dimensional* $\widetilde{U}_q$*-module* V *is determined by its formal character* $\mathrm{char}_{\mathcal{H}\langle\tau\rangle} V$.

Proof. By our assumption on q, we know that the representation T of $\widetilde{U}_q$ on V decomposes as a direct sum of irreducibles from $\mathcal{T}$. One can determine this decomposition by the following process. The formal character determines the weight multiplicities, hence we know how V decomposes over the Drinfeld–Jimbo algebra U_q. Take a maximal weight $\lambda \in P_+$ such that T_λ occurs with multiplicity $m > 0$ in the decomposition over U_q.

Case 1. If $\lambda^\chi \neq \lambda$, then T_{λ^χ} also occurs with multiplicity m in the decomposition over U_q, and T must contain $\widetilde{T}_\lambda$ as a summand with multiplicity m. Subtract m times the formal character of $\widetilde{T}_\lambda$ from the formal character of V, and continue the same process.

Case 2. If $\lambda^\chi = \lambda$, then $d_\lambda^+ + d_\lambda^- = m$, and $\widetilde{T}_\lambda$ must occur with multiplicity d_λ^+ in T, whereas $\widetilde{T}_\lambda^\circ$ must occur with multiplicity d_λ^- in T. Subtract the formal character of these summands from $\mathrm{char}_{\mathcal{H}\langle\tau\rangle} V$, and continue the same process. □

For notational simplicity, set $\widetilde{T}_\lambda^\circ = \widetilde{T}_\lambda$, when $\lambda^\chi \neq \lambda$.

Proposition 6.2. *The formal character of each of the* $\widetilde{U}_q$*-modules* $\widetilde{T}_\lambda$ *and* $\widetilde{T}_\lambda^\circ$ *is independent of* q*, and is the same as in the classical case of* $\widetilde{U}$.

Proof. Recall that the weight multiplicities for $T_\lambda : U_q \to \mathrm{End}_{\mathbb{C}} V(\lambda)$ are independent of q and are the same as in the classical case of U, see [3, Corollary 10.1.15]. If $\lambda^\chi \neq \lambda$, then $\widetilde{T}_\lambda|_{U_q} \cong T_\lambda \oplus T_{\lambda^\chi}$, and the action of χ interchanges the weight subspaces for T_λ and T_{λ^χ}, so the assertion is obvious. Fix now $\lambda = \lambda^\chi \in P_+$, and consider $\widetilde{T}_\lambda$ (the case of $\widetilde{T}_\lambda^\circ$ is similar). A weight subspace $V(\lambda)_\mu$ for $\mu \neq \mu^\chi$ is interchanged by $\widetilde{T}_\lambda(\chi)$ with $V(\lambda)_{\mu^\chi}$, hence the assertion is clear for the contribution of this part in the formal character. Assume from now on that $\mu = \mu^\chi \in P$, and denote by $d^+(q)$, $d^-(q)$ the multiplicity of $+1$, -1 as an eigenvalue of $\widetilde{T}_\lambda(\chi)$ restricted to the weight subspace $V(\lambda)_\mu$. What is left to show is that $d^+(q)$ and $d^-(q)$ do not depend on q.

To this end we need to recall an integral form of the Drinfeld–Jimbo algebra. Let t be an indeterminate, and consider the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$. Denote by $U_t = U_t^{\mathbb{Q}(t)}(\mathrm{so}_{2l})$, the Drinfeld–Jimbo algebra over the field $\mathbb{Q}(t)$. Let U^{res} be the $\mathbb{Z}[t^{\pm 1}]$-subalgebra of U_t defined in [3, 9.3A]. Write V for the irreducible U_t-module with highest weight λ, say $v \in V$ is a fixed highest weight vector. Consider the U^{res}-module $V^{\mathrm{res}} = U^{\mathrm{res}} v$, and recall some of its properties from [3, Proposition 10.1.4]. The module V^{res} has a weight subspace decomposition $V^{\mathrm{res}} = \bigoplus_{\mu \in P} V_\mu^{\mathrm{res}}$, where V_μ^{res} is the intersection of V^{res} and V_μ. Moreover, each V_μ^{res} is a free $\mathbb{Z}[t^{\pm 1}]$-module. For $q \in \mathbb{C} \setminus \{0\}$ define $U_{t\mapsto q}^{\mathrm{res}} = U^{\mathrm{res}} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{C}$ and $V_{t\mapsto q}^{\mathrm{res}} = V^{\mathrm{res}} \otimes_{\mathbb{Z}[t^{\pm 1}]} \mathbb{C}$, where $\mathbb{C}$ is made into a $\mathbb{Z}[t^{\pm 1}]$-module via the homomorphism $\mathbb{Z}[t^{\pm 1}] \to \mathbb{C}$, $t \mapsto q$. Then $U_{t\mapsto q}^{\mathrm{res}}$ is the complex Drinfeld–Jimbo algebra U_q (when $q = 1$, this can be taken as the definition of U_1), and $V_{t\mapsto q}^{\mathrm{res}}$ is a U_q-module with highest weight λ (for q not a root of unity or $q = 1$, this is the irreducible module associated with λ). Moreover, a free $\mathbb{Z}[t^{\pm 1}]$-module basis of V_μ^{res} is mapped onto a $\mathbb{C}$-basis of the weight space $(V_{t\mapsto q}^{\mathrm{res}})_\mu$.

We define the automorphism χ of U_t in the same way as for the complex Drinfeld–Jimbo algebra. Then χ permutes the $\mathbb{Z}[t^{\pm 1}]$-algebra generators of U^{res}, hence χ preserves U^{res}. The automorphism χ of U^{res} induces an automorphism of $U^{\mathrm{res}}_{t\mapsto q}$ in an obvious manner, and the resulting automorphism clearly coincides with the automorphism of U_q called χ already (when $q=1$, this can be taken as the definition of χ). The $\mathbb{Q}(t)$-linear endomorphism $\widetilde{T}_\lambda(\chi)$ of the U_t-module V preserves V^{res}. Indeed, take an element $a\cdot v \in V^{\mathrm{res}}$, $a \in U^{\mathrm{res}}$. Then

$$\widetilde{T}_\lambda(\chi)(a\cdot v) = (\chi\cdot a)\cdot v = \left(a^\chi\cdot\chi\right)\cdot v = a^\chi\cdot\left(\widetilde{T}_\lambda(\chi)(v)\right) = \pm a^\chi\cdot v \in U^{\mathrm{res}}v,$$

since $a^\chi \in U^{\mathrm{res}}$. So $\widetilde{T}_\lambda(\chi)$ restricts to an automorphism of the free $\mathbb{Z}[t^{\pm 1}]$-module V^{res}_μ (recall that $\mu^\chi = \mu$). This free $\mathbb{Z}[t^{\pm 1}]$-module automorphism is represented by a square matrix B with entries from $\mathbb{Z}[t^{\pm 1}]$, such that $B^2 = I$, the identity matrix. Denote by $B_{t\mapsto 1}$ the integer matrix obtained from B by specializing t to 1. Then $B_{t\mapsto 1}$ is a matrix which represents χ, acting on $(V^{\mathrm{res}}_{t\mapsto 1})_\mu$ via the classical irreducible representation $\widetilde{T}_\lambda$ of $\widetilde{U}$. Clearly we have $\operatorname{rank}_{\mathbb{Q}(t)}(B-I) \geqslant \operatorname{rank}_{\mathbb{C}}(B_{t\mapsto 1}-I)$ and $\operatorname{rank}_{\mathbb{Q}(t)}(B+I) \geqslant \operatorname{rank}_{\mathbb{C}}(B_{t\mapsto 1}+I)$, implying $d^+(t) \leqslant d^+(1)$ and $d^-(t) \leqslant d^-(1)$. On the other hand, $d^+(t)+d^-(t) = \dim_{\mathbb{Q}(t)} V_\mu = \dim_{\mathbb{C}}(V^{\mathrm{res}}_{t\mapsto 1})_\mu = d^+(1)+d^-(1)$, so we have equality in both of the above inequalities. Similarly, for $q\in\mathbb{C}\setminus\{0\}$ write $B_{t\mapsto q}$ for the complex matrix obtained from B by specializing t to q. Then $B_{t\mapsto q}$ represents χ, acting on $(V^{\mathrm{res}}_{t\mapsto q})_\mu$ via the representation $\widetilde{T}_\lambda$ of $\widetilde{U}_q$. The obvious inequalities $\operatorname{rank}_{\mathbb{Q}(t)}(B-I) \geqslant \operatorname{rank}_{\mathbb{C}}(B_{t\mapsto q}-I)$ and $\operatorname{rank}_{\mathbb{Q}(t)}(B+I) \geqslant \operatorname{rank}_{\mathbb{C}}(B_{t\mapsto q}+I)$ imply $d^+(t) \leqslant d^+(q)$ and $d^-(t) \leqslant d^-(q)$. On the other hand, $d^+(t)+d^-(t) = \dim_{\mathbb{Q}(t)} V_\mu = \dim_{\mathbb{C}}(V^{\mathrm{res}}_{t\mapsto q})_\mu = d^+(q)+d^-(q)$. Hence we get $d^+(q) = d^+(t) = d^+(1)$ and $d^-(q) = d^-(t) = d^-(1)$. □

Proposition 6.3. *Assume that $q\in\mathbb{C}\setminus\{0\}$ is not a root of unity or $q=1$. Then the tensor product of any pair of representations $T_1, T_2 \in \mathcal{T}$ decomposes as*

$$T_1\otimes T_2 \cong \bigoplus_{T\in\mathcal{T}} m_T T,$$

and the multiplicities m_T here are independent of q (they are the same as in the classical case $q=1$).

Proof. $T_1\otimes T_2$ is a finite dimensional $\widetilde{U}_q$-module of type 1, hence is the direct sum of modules from $\mathcal{T}$. The formal characters of T_1 and T_2 are the same as in the classical case $q=1$ by Proposition 6.2. They determine the formal character of $T_1\otimes T_2$, so it is again the same as in the case $q=1$. So the assertion on the multiplicities follows by Proposition 6.1. □

In the special case when T_2 is the vector representation of $\widetilde{U}_q$ (the irreducible representation with highest weight $(1,0,\dots,0)$, the above result is proved in [13, Proposition 4.2(1)] (see also [15, 8.6.2 Proposition 36]) by different methods.

Finally, note that in the odd dimensional case, the full orthogonal group $O(2l+1)$ is generated over $SO(2l+1)$ by the central element $-I$, which acts as a scalar $+1$ or -1 in any irreducible representation of $O(2l+1)$. Therefore the algebra $\mathcal{U}(O(2l+1))$ is defined as the tensor product $\mathbb{C}[\chi] \otimes U_{q^{1/2}}(\mathrm{so}_{2l+1})$, where χ here is just an abstract generator of the two-element group, and $\mathbb{C}[\chi]$ is the corresponding group algebra. Then an irreducible $U_{q^{1/2}}(\mathrm{so}_{2l+1})$-representation has always two extensions to an $\mathcal{U}(O(2l+1))$-representation on the same underlying space: the element χ acts as a scalar $+1$ or -1. The analogue of Proposition 6.3 holds obviously in this case.

Appendix B

Here we sketch a proof of Theorem 2.4 in the classical case $q = 1$.

When $G = SO(2l+1)$ or when G is simple and simply connected, $\mathcal{O}(G)^{\mathrm{coc}}$ is a polynomial algebra generated by the characters of the fundamental representations, see, for example, [24]. For $SL(l+1)$ or $SO(2l+1)$ the fundamental representations are the first l exterior powers of the defining representation, hence we have (i) and (iv). The rth exterior power of the defining representation of $Sp(2l)$ for $r = 1, \dots, l$ is the direct sum of the rth fundamental representation and some copies of the fundamental representations with strictly lower index, see [10, Section 5.1.3]. Therefore $\sigma_1, \dots, \sigma_l$ is another generating system of $\mathcal{O}(Sp(2l))^{\mathrm{coc}}$, and we get (ii). Since $O(2l+1) \cong SO(2l+1) \times \mathbb{Z}_2$, and ω_{2l+1} is trivial on $SO(2l+1)$ whereas it gives the non-trivial irreducible representation on $\mathbb{Z}_2$, the statement (iii) immediately follows from (iv).

(v) Note that G acts on itself by conjugation, and $\mathcal{O}(G)^{\mathrm{coc}}$ is the corresponding algebra of polynomial invariants. Identify $O(2l)$ with the subset of the space $M(2l, \mathbb{C})$ of $(2l \times 2l)$ matrices consisting of matrices A with $AA^T = I$. The group $O(2l)$ acts on $M(2l, \mathbb{C})$ by conjugation, and the corresponding algebra of polynomial invariants is generated by the functions $A \mapsto \mathrm{Tr}(f(A, A^T))$ as f ranges over the possible monomials in A and A^T, see [23] or [21]. Using $A^T = A^{-1}$ for $A \in O(2l)$, we get that the algebra $\mathcal{O}(G)^{\mathrm{coc}}$ is generated by the functions $A \mapsto \mathrm{Tr}(A^d)$, $d = 1, \dots, 2l$ (the upper bound on d comes from the Cayley–Hamilton identity). Note that $\sigma_i(A)$ is the ith characteristic coefficient of the matrix A, hence $\sigma_1, \dots, \sigma_{2l}$ also generate $\mathcal{O}(G)^{\mathrm{coc}}$. For $r = 1, \dots, l$ we have the well known isomorphisms $\bigwedge^r \mathbb{C}^{2l} \otimes \bigwedge^{2l} \mathbb{C}^{2l} \cong \bigwedge^{2l-r} \mathbb{C}^{2l}$ of $O(2l)$-representations (see, for example, [10, Exercise 6 in Section 5.1.8]). This implies $\sigma_{2l-r} = \sigma_r \sigma_{2l}$, $r = 1, \dots, l$, hence $\mathcal{O}(G)^{\mathrm{coc}}$ is generated by $\sigma_1, \dots, \sigma_l, \sigma_{2l}$, and the relations $\sigma_{2l}^2 = 1$ and $\sigma_l \sigma_{2l} = \sigma_l$ hold. We need to show that there are no further relations among these generators. Realize now $O(2l)$ as the group of invertible matrices $\{A \mid JA = (A^T)^{-1}J\}$, where $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus \cdots \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and restrict the functions in $\mathcal{O}(G)$ to the subset $Y \sqcup Z$, where Y consists of the diagonal matrices $\mathrm{diag}(z_1, z_1^{-1}, \dots, z_l, z_l^{-1})$ and Z consists of the matrices $\left(\begin{smallmatrix} 0 & z_1 \\ z_1^{-1} & 0 \end{smallmatrix}\right) \oplus \mathrm{diag}(z_2, z_2^{-1}, \dots, z_l, z_l^{-1})$ with $z_i \in \mathbb{C}^\times$. Now suppose that $f(\sigma_1, \dots, \sigma_l) + \sigma_{2l} g(\sigma_1, \dots, \sigma_{l-1}) = 0$ holds in $\mathcal{O}(G)$. Clearly the restrictions of $\sigma_1, \dots, \sigma_l$ to Y are algebraically independent. So restricting the above relation to Y we get that $f(t_1, \dots, t_l) = -g(t_1, \dots, t_{l-1})$ as polynomials in $t_1, \dots, t_l$, so the above relation is $(1 - \sigma_{2l}) g(\sigma_1, \dots, \sigma_{l-1}) = 0$. Now restricting this relation to Z one sees that $g(t_1, \dots, t_{l-1})$ is the zero polynomial.

(vi) The highest weights of the representations corresponding to $\sigma_1, \ldots, \sigma_{l-1}, \sigma_{l,0}, \sigma_{l,1}$ generate the semigroup of the highest weights of all irreducible representations of $SO(2l)$, see, for example, [10, p. 102 and 234]. Using the usual partial ordering on the weight semigroup, an inductive argument shows that the trace of an arbitrary irreducible $SO(2l)$-representation can be expressed as a polynomial of $\sigma_1, \ldots, \sigma_{l-1}, \sigma_{l,0}, \sigma_{l,1}$. The elements $\sigma_1, \ldots, \sigma_l$ are algebraically independent by the same argument as in (v). The full orthogonal group $O(2l)$ acts on $SO(2l)$ by conjugation, and this induces an action on $\mathcal{O}(SO(2l))$. For $\chi \in O(2l) \setminus SO(2l)$ we have $\chi(\sigma_{l,0} - \sigma_{l,1}) = -(\sigma_{l,0} - \sigma_{l,1})$, because the automorphism of $SO(2l)$ induced by χ interchanges the representations $\omega_{l,0}$ and $\omega_{l,1}$. Consequently, $(\sigma_{l,0} - \sigma_{l,1})^2$ is an $O(2l)$-invariant in $\mathcal{O}(SO(2l))$, hence it is a polynomial of $\sigma_1, \ldots, \sigma_l$ by (v). The $\mathbb{C}[\sigma_1, \ldots, \sigma_l]$-module generated by 1 and $\sigma_{l,0} - \sigma_{l,1}$ is free (of rank two), because $\mathcal{O}(SO(2l))$ is a domain, and the elements of $(\sigma_{l,0} - \sigma_{l,1})\mathbb{C}[\sigma_1, \ldots, \sigma_l]$ are not invariant with respect to the action of the full orthogonal group, whereas $\sigma_1, \ldots, \sigma_l$ are $O(2l)$-invariants.

The $SO(2l)$-invariant $\sigma_{l,0} - \sigma_{l,1}$ and the relation $(\sigma_{l,0} - \sigma_{l,1})^2 = h(\sigma_1, \ldots, \sigma_l)$ can be seen more explicitly as follows. Think of $SO(2l)$ as the set of determinant 1 matrices A with $JA = (A^T)^{-1}J$. It is not difficult to check that up to sign, $\sigma_{l,0} - \sigma_{l,1}$ is the function mapping $A \in SO(2l)$ to the Pfaffian $\mathrm{Pf}(JA - A^T J)$ of the skew symmetric $(2l \times 2l)$ matrix $JA - A^T J$. (For the definition and basic properties of the Pfaffian see the appendix of [10]; the $SO(2l)$-invariant $A \mapsto \mathrm{Pf}(JA - A^T J)$ appears in [1].) Indeed, both $\sigma_{l,0} - \sigma_{l,1}$ and $A \mapsto \mathrm{Pf}(JA - A^T J)$ span a 1-dimensional $O(2l)$-invariant subspace in $\mathcal{O}(SO(2l))$ on which $O(2l)$ acts by the determinant representation. Both of them are contained in the space of matrix elements of the lth tensor power of the defining representation of $SO(2l)$, and in this $O(2l)$-invariant subspace of $\mathcal{O}(SO(2l))$ the determinant representation of $O(2l)$ occurs with multiplicity one. So $\sigma_{l,0} - \sigma_{l,1}$ and $A \mapsto \mathrm{Pf}(JA - A^T J)$ are non-zero scalar multiples of each other. Restricting them to the maximal torus of $SO(2l)$ one can check that in fact they coincide (up to sign). For $A \in SO(2l)$ we have

$$\mathrm{Pf}^2\big(JA - A^T J\big) = \det\big(JA - JA^{-1}\big) = (-1)^l \det(A + I)\det(A - I).$$

Therefore the relation

$$(\sigma_{l,0} - \sigma_{l,1})^2 = (-1)^l \left(\sigma_l + 2\sum_{i=0}^{l-1} \sigma_i\right)\left((-1)^l \sigma_l + 2\sum_{i=0}^{l-1} (-1)^i \sigma_i\right)$$

holds.

Appendix C

Here we deduce the assertion of Theorem 3.3 in the classical case $q = 1$, for $G = O(N)$ or $Sp(N)$. Recall that $\mathcal{A}(G)$ is the coordinate ring $\mathcal{O}(\mathcal{M})$ of the Zariski closure $\mathcal{M}$ of the cone $\mathbb{C}G$. The group G acts on $M(N, \mathbb{C})$ by conjugation, and $\mathcal{M}$ is a G-stable subvariety in $M(N, \mathbb{C})$. We claim that $\mathcal{A}(G)^{\mathrm{coc}}$ coincides with the algebra $\mathcal{O}(\mathcal{M})^G$ of G-invariants. This follows from the well-known fact that for any affine algebraic group H, the algebra

$\mathcal{O}(H)^{\mathrm{coc}}$ coincides with the algebra of adjoint invariants $\mathcal{O}(H)^H$. Applying this for $H = \mathbb{C}^\times G$, and observing that the conjugation action of $\mathbb{C}^\times$ on $M(N,\mathbb{C})$ is trivial, we obtain that

$$\begin{aligned}\mathcal{A}(G)^{\mathrm{coc}} &= \mathcal{A}(G) \cap \mathcal{O}\big(\mathbb{C}^\times G\big)^{\mathrm{coc}} = \mathcal{O}(\mathcal{M}) \cap \mathcal{O}\big(\mathbb{C}^\times G\big)^{\mathbb{C}^\times G} = \mathcal{O}(\mathcal{M}) \cap \mathcal{O}\big(\mathbb{C}^\times G\big)^G \\ &= \mathcal{O}(\mathcal{M})^G.\end{aligned}$$

Consider the natural surjection $\mathcal{O}(M(N,\mathbb{C}))^G \to \mathcal{O}(\mathcal{M})^G$. Generators of $\mathcal{O}(M(N,\mathbb{C}))^G$ are known from [21,23], these are the functions

$$A \mapsto \mathrm{Tr}\big(A^{i_1}\big(A^*\big)^{j_1} \cdots A^{i_s}\big(A^*\big)^{j_s}\big),$$

where A^* denotes the adjoint of $A \in M(N,\mathbb{C})$ with respect to the invariant bilinear form determining G. By definition of $\mathcal{M}$, if $A \in \mathcal{M}$, then $AA^* = A^*A$ equals the scalar matrix $\mathcal{D}(A)I$. Note that $\mathrm{Tr}(A^*) = \mathrm{Tr}(A)$. Therefore, for $A \in \mathcal{M}$ we have

$$\mathrm{Tr}\big(A^{i_1}\big(A^*\big)^{j_1} \cdots A^{i_s}\big(A^*\big)^{j_s}\big) = \mathcal{D}(A)^k \,\mathrm{Tr}\big(A^d\big),$$

where $k = \min\{i_1 + \cdots + i_s, j_1 + \cdots + j_s\}$, and $d = |i_1 + \cdots + i_s - j_1 - \cdots - j_s|$. Taking into account the Cayley–Hamilton theorem, we get that $\mathcal{O}(\mathcal{M})^G$ is generated by the functions $A \mapsto \mathcal{D}(A)$, $A \mapsto \mathrm{Tr}(A^j)$, $j = 1,\ldots,N$. By the Newton formulae the elements $\mathcal{D}$, $\rho_1,\ldots,\rho_N$ generate the same algebra.

Next we determine the relations among the above generators. Identify $\mathcal{A}(G)$ with its image under the map $\iota : \mathcal{A}(G) \to \mathcal{O}(G) \otimes \mathbb{C}[z]$ from Proposition 3.1. Thus $\rho_j = \sigma_j z^j$ and $\mathcal{D} = z^2$ (we suppress the $\otimes$ sign from the notation). Since $\sigma_1,\ldots,\sigma_l$ are algebraically independent in $\mathcal{O}(G)$ (see Theorem 2.4), the elements $\sigma_1 z,\ldots,\sigma_l z^l, z^2$ are algebraically independent in $\mathcal{A}(G)$.

When $G = Sp(N)$ $(N = 2l)$, we have $\sigma_N = 1$ and $\sigma_{N-i} = \sigma_i$ for $i = 1,\ldots,l$ (this follows from the well known G-module isomorphism $\bigwedge^i \mathbb{C}^N \otimes \bigwedge^N \mathbb{C}^N \cong \bigwedge^{N-i} \mathbb{C}^N$, and the fact that the Nth exterior power of $\mathbb{C}^N$ is the trivial $Sp(N)$-module). Thus

$$\rho_{N-i} = \sigma_{N-i} z^{N-i} = \sigma_i z^i \big(z^2\big)^{l-i} = \rho_i \mathcal{D}^{l-i}$$

for $i = 0,\ldots,l-1$. So $\mathcal{A}(Sp(N))^{\mathrm{coc}}$ is generated by $\mathcal{D}, \rho_1,\ldots,\rho_l$.

Finally, assume $G = O(N)$. In $\mathcal{O}(O(N))$ the relations $\sigma_N^2 = 1$ and $\sigma_i\sigma_N = \sigma_{N-i}$ for $i = 1,\ldots,l$ hold, see Theorem 2.4. Therefore in $\mathcal{A}(G)$ we have

$$\rho_{N-i}\rho_{N-j} = \sigma_{N-i} z^{N-i} \sigma_{N-j} z^{N-j} = (\sigma_N)^2 \sigma_i z^i \sigma_j z^j z^{2(N-i-j)} = \rho_i \rho_j \mathcal{D}^{N-i-j}$$

for $0 \leqslant i \leqslant j \leqslant l$, and

$$\rho_i \rho_{N-j} \mathcal{D}^{j-i} = \sigma_i z^i \sigma_{N-j} z^{N-j} z^{2(j-i)} = \sigma_i \sigma_j \sigma_N z^{N-i+j} = \sigma_{N-i} z^{N-i} \sigma_j z^j = \rho_{N-i}\rho_j$$

for $0 \leqslant i < j \leqslant l$. It is an elementary exercise to show that modulo these relations an arbitrary monomial of $\mathcal{D}, \rho_1,\ldots,\rho_N$ can be rewritten into a monomial contained in $B(N)$:

using the relations of the first type we can get rid of those products of the generators which contain at least two factors from $\{\rho_{l+1},\dots,\rho_N\}$. In the case $N=2l$, by the relation $\rho_{N-i}\rho_l=\rho_i\rho_l\mathcal{D}^{l-i}$ (the special case $j=l$ of the second type relations) we eliminate the products which contain ρ_l and a factor from $\{\rho_{l+1},\dots,\rho_N\}$. By the relations $\rho_{N-j}\mathcal{D}^j=\rho_j\rho_N$ (the special case $i=0$ of the second type relations) we get rid of the products which contain the subword $\rho_{N-j}\mathcal{D}^j$ for $j=1,\dots,l$ if $N=2l+1$ and for $j=1,\dots,l-1$ if $N=2l$. Take a product of the generators which is not ruled out by the above reductions, and which is not contained in $B(N)$. Then it must contain a subword $\rho_{N-j}\rho_i\mathcal{D}^{j-i}$ with $1\leqslant i<j\leqslant l$ (respectively $1\leqslant i<j\leqslant l-1$) when $N=2l+1$ (respectively $N=2l$). Replace this subword by $\rho_j\rho_{N-i}$, using the second type relations. In this way we increase the index of the unique factor of this monomial from the set $\{\rho_{l+1},\dots,\rho_{N-1}\}$. After finitely many such steps we end up in $B(N)$ or with a monomial eliminated already. So we have proved that the elements in $B(N)$ span $\mathcal{A}(O(N))^{\mathrm{coc}}$. We know from Theorem 2.4(iii) and (v) that $\sigma_1^{i_1}\cdots\sigma_l^{i_l}$, $\sigma_{2l+1}\sigma_1^{j_1}\cdots\sigma_l^{j_l}$ $(i_s,j_s\in\mathbb{N}_0)$ are linearly independent in $\mathcal{O}(O(2l+1))$, and $\sigma_1^{i_1}\cdots\sigma_l^{i_l}$, $\sigma_{2l}\sigma_1^{j_1}\cdots\sigma_{l-1}^{j_{l-1}}$ $(i_s,j_s\in\mathbb{N}_0)$ are linearly independent in $\mathcal{O}(O(2l))$. Using again the embedding ι we easily get that the elements of $B(N)$ are linearly independent in $\mathcal{A}(O(N))$. Finally, the fact that $B(N)$ is a basis of $\mathcal{A}(O(N))^{\mathrm{coc}}$ implies that the set of relations used to rewrite arbitrary products of the generators as linear combinations of elements of $B(N)$ is complete: namely, the ideal of relations among the generators $\mathcal{D},\rho_1,\dots,\rho_N$ is generated by the relations given in the statement of Theorem 3.3.

Appendix D

Here we give a direct proof of the fact that for $G=O(N)$ or $Sp(N)$, the algebra $\mathcal{A}(G)$ defined in terms of generators and relations (6) in Section 3, coincides with the coordinate ring of the Zariski closure $\mathcal{M}$ of the cone $\mathbb{C}G$, where G is embedded into $M(N,\mathbb{C})$ in the usual way; that is, $G=\{A\in M(N,\mathbb{C})\mid A\mathrm{C}^{-1}A^T\mathrm{C}=I\}$, where C is the matrix of a symmetric (respectively skew-symmetric) non-degenerate bilinear form (the matrix $\mathrm{C}=\mathrm{C}(1)$ is specified in Section 3). In other words, we claim that the vanishing ideal $I(\mathcal{M})$ of $\mathcal{M}$ in $\mathcal{O}(M(N,\mathbb{C}))=\mathbb{C}[u^i_j\mid i,j=1,\dots,N]$ is generated by the entries of $\mathrm{K}(1)\mathbf{u}_1\mathbf{u}_2-\mathbf{u}_1\mathbf{u}_2\mathrm{K}(1)$ (notation explained in Section 3). Write B for the set of entries of this matrix, and write $\langle B\rangle$ for the ideal generated by these homogeneous quadratic elements. One sees directly from the definition of K(1) that $\langle B\rangle\subseteq I(\mathcal{M})$, see, for example, the proof of [15, 9.3.1 Lemma 12]. Since $\mathbb{C}\mathcal{M}=\mathcal{M}$, the ideal $I(\mathcal{M})$ is homogeneous. Take an arbitrary $f\in I(\mathcal{M})$. Our aim is to show that f is contained in $\langle B\rangle$. We may assume that f is homogeneous of degree d. Clearly $f\in I(G)$, since $G\subset\mathcal{M}$. Now Theorems (5.2 C) and (6.3 B) of [26] assert that B and $\mathcal{D}-1$ generate $I(G)$ in a nice way; that is, there are elements $f_b,h\in\mathbb{C}[u^i_j]$ $(b\in B)$, such that

$$f=(\mathcal{D}-1)h+\sum_{b\in B}bf_b, \tag{12}$$

moreover, $\deg(f_b) \leqslant d-2$ and $\deg(h) \leqslant d-2$. We may assume that h has the minimal possible number of non-zero homogeneous components. Suppose that $h \neq 0$. Write $h = \hat{h} + \tilde{h}$, where $\tilde{h}$ is the minimum degree homogeneous component of h. Then

$$(\mathcal{D}-1)h = -\tilde{h} + \text{higher degree terms}.$$

Since $\deg(\tilde{h}) < d = \deg(f)$, it follows from (12) that $-\tilde{h}$ is killed by the appropriate homogeneous component of $\sum_{b\in B} bf_b$, hence $\tilde{h} = \sum_{b\in B} bh_b$ for some h_b, with $\deg(h_b) \leqslant d-4$. Thus we have

$$f = (\mathcal{D}-1)\hat{h} + \sum_{b\in B} b\big(f_b + h_b(\mathcal{D}-1)\big). \tag{13}$$

Note that in (13) we have $\deg(f_b + h_b(\mathcal{D}-1)) \leqslant d-2$, and $\hat{h}$ has fewer non-zero homogeneous components than h in (12). This contradiction implies that $h = 0$ in (12), so $f = \sum_{b\in B} bf_b$ is contained in $\langle B\rangle$.

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