The Stochastic Korteweg–de Vries Equation in $L^2(\mathbb{R})$

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We present here two results of global existence and uniqueness of the Cauchy problem concerning the Korteweg–de Vries equation forced by a random term of white noise type in the functional framework of weighted spaces on $L^2(\mathbb{R})$. This work is motivated by both physical experiments where a forced generation of weakly nonlinear waves by localized disturbances was observed [1, 15] and a result of uniqueness in weighted space in the homogeneous case [8, 14].

1. INTRODUCTION

It is well known that the Korteweg–de Vries equation models the evolution in time of long, unidirectional weakly non linear waves at the surface of a fluid. When the pressure above is not constant or the bottom is not flat, a forcing term is added which is either the pressure gradient or the gradient of the function whose graph defines the bottom (See [1, 15]).

This paper focuses on the case where the forcing term is random, which is a natural approach if we assume that the outer pressure is generated by a turbulent velocity field. We furthermore assume that this random term is of white noise type, which leads us to study the following stochastic partial differential equation

$$
\frac{\partial u}{\partial t} + \frac{3}{2} \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial x} = \Phi \frac{\partial^3 B}{\partial t \partial x},
$$

(1.1)

where $u = u(x, t)$ is a real valued random process defined on $\mathbb{R} \times \mathbb{R}_+$, $\Phi$ is a linear operator and $B$ is a brownian sheet defined on $\mathbb{R} \times \mathbb{R}_+$. We emphasize the fact that here $\Phi$ does not depend on the unknown $u$, i.e. the noise is assumed to be additive.

We recall that $\{B(t, x)\}_{(t, x) \in \mathbb{R}_+ \times \mathbb{R}}$ is a zero mean gaussian process such that

$$
\mathbb{E} B(t, x) B(s, y) = (t \land s)(x \land y),
$$

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for $t, s \geq 0, x, y \in \mathbb{R}$. We rewrite the right hand side of the equation as the time-derivative of a cylindrical Wiener process on $L^2(\mathbb{R})$ by setting

$$W(t) = \partial_t B = \sum_{i \in \mathbb{N}} \beta_i(t) e_i,$$

where $\{e_i\}_{i \in \mathbb{N}}$ denotes a Hilbertian basis of $L^2(\mathbb{R})$ and $\{\beta_i\}_{i \in \mathbb{N}}$ is a family of real brownian motions mutually independent in a fixed probability space (see [5, 18]). We shall rewrite (1.1) in the Ito form

$$\frac{du}{dt} + \frac{3u}{x^3} + u \frac{\partial u}{\partial x} \ dt = \Phi \ dW,$$

(1.2)

with the initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}.$$ (1.3)

Our main interest in this article is to understand the Cauchy problem associated to (1.2)-(1.3). A forthcoming article will be devoted to the numerical study of the solution of (1.2) and to the description of the influence of the noise on well known phenomena such as propagation, interaction of solitons, emission of a soliton by a localized forcing term. Considering the deterministic equations, existence and uniqueness for smooth solutions have been considered in [2, 3, 12, 17, 20]. More recently, C.E. Kenig, G.P. Ponce and L. Vega [13] have been able to prove existence and uniqueness in $H^s(\mathbb{R})$ using techniques from harmonic analysis. Also, J. Ginibre and Y. Tsutsumi [8], using the structure of the Airy function and a smoothing effect discovered by T. Kato [11], have proved uniqueness in weighted $L^2(\mathbb{R})$ spaces (see also [14]). Before describing the theory of the stochastic Korteweg–de Vries equation (1.2), we first point out that the linear part of the Korteweg–de Vries equation defines a unitary group, denoted by $\{S(t)\}_{t \in \mathbb{R}}$. Thus, it seems difficult to obtain solutions in $H^s(\mathbb{R})$ for $s \in \mathbb{R}$ if $\Phi$ is not a Hilbert–Schmidt operator from $L^2(\mathbb{R})$ to $H^s(\mathbb{R})$. Indeed, the solution of the linear problem

$$\begin{cases} \frac{du}{dt} + \frac{3u}{x^3} \ dt = \Phi \ dW \\ u(0) = 0 \end{cases}$$

is given by

$$W_L(t) = \int_0^t S(t - \tau) \ \Phi \ dW(\tau)$$ (1.4)
and it can be seen directly that
\[ E \left[ |W_E(t)|^2_{H^1(\mathbb{R})} \right] = t \left| \Phi \right|_{L^2_{2}(\mathbb{R}, H^{2}(\mathbb{R}))}^2, \]
where \( \cdot \) denotes the norm in the space of Hilbert–Schmidt operators from \( L^2(\mathbb{R}) \) to \( H^1(\mathbb{R}) \).

A previous work [6] has generalized the techniques of C.E. Kenig, G.P. Ponce and L. Vega [13] under the assumption that \( \Phi \) is a Hilbert–Schmidt operator from \( L^2(\mathbb{R}) \) into \( H^1(\mathbb{R}) \), proving existence and uniqueness in \( \mathbb{R}([0, T], H^1(\mathbb{R})) \). In this work, we would like to consider a more general covariance operator \( \Phi \). We notice that the physical model of forced Korteweg–de Vries equations has been derived under the assumption of a localized forcing term [1, 15]. Thus we can consider a noise which is, in a sense to be precised, localized. On the mathematical point of view, this remark enables us to work in weighted \( L^2(\mathbb{R}) \) spaces and to use the techniques of J. Ginibre and Y. Tsutsumi [8]. So that, we want to replace regularity assumptions on \( \Phi \) by a localization property. Roughly speaking, we will assume that \( \Phi \) is a Hilbert–Schmidt operator from \( L^2(\mathbb{R}) \) into itself and is small at \(+ \infty\).

To be more precise, let us introduce the following weighted spaces
\[ X^{\alpha, q}([0, T]) = \{ h_u \in L^4([0, T], L^4(\mathbb{R})) \}, \]
and
\[ Y = \{(1 + x^2)^{3/8} u \in L^2(\mathbb{R})\}, \]
where \( \alpha > 0, q \) an integer such that \( \frac{1}{3} < q < \frac{1}{4} \) and \( h_u \) is a \( C^\infty \) and increasing function which is equal to 1 if \( x > 1 \) and to \( e^{|x|} \) if \( x < 0 \).

J. Ginibre and Y. Tsutsumi have proved the following smoothing effect of \( \{ S(t) \}_{t \in \mathbb{R}} \) (see Proposition 2.9),
\[ \left| \int_0^\tau S(t - \tau) \frac{\partial (u v)}{\partial x} \, dt \right|_{X^{\alpha, q}([0, T])} \leq C(T) \sup_{[0, T]} |u|_{X^{\alpha, q}([0, T])} |v|_{L^4([0, T])}, \quad (1.5) \]
which can be used to prove uniqueness of solutions in the above spaces. However, we cannot use this property to construct a solution by a fixed point strategy since it implies contraction in \( X^{\alpha, q}([0, T]) \) for a solution in the second weighted space. This phenomenon comes from the asymmetry of the weighted functions which seems to be imposed by the structure of the Airy function. This difficulty can be overcome considering a sequence of smooth solutions obtained with regularized data if we are able to prove an
We assume $a \text{ priori}$ estimate in $Y$. Then it follows easily that the sequence of approximation is almost surely Cauchy.

Our main assumptions are

\begin{align*}
u_0 &\in Y, \quad \Phi \in L^2_{\text{loc}}(\mathbb{R}, Y),
\end{align*}

(1.6)

(1.7)

The main difficulty is that the $a \text{ priori}$ estimate of the solution in $Y$ requires an estimate on $\partial W_{\nu} / \partial x$, the derivative of the solution of the linear equation, in $L^1(0, T), L^\infty(\mathbb{R}))$. This difficulty is due to the presence of a derivative in the nonlinear term. This particular estimate is obtained thanks to a sharp smoothing property of $[S(t)]_{t \in \mathbb{R}}$ discovered in [13] which enables us to get the bound

\begin{equation}
\sup_{t \in \mathbb{R}} \mathbb{E} \left( \sup_{x \in \mathbb{R}} \left| \frac{\partial}{\partial x} \int_0^t S(t - \tau) \Phi \, dW(\tau) \right|^p \right) \leq C_p \| \Phi \|_{L^2_0(\mathbb{R}, H^s(\mathbb{R})))} (1.8)
\end{equation}

for any $p$. We note that (1.8) is an improvement of a similar estimate in [6]. Unfortunately, we get this result only when we assume the additional property

\begin{equation}
\Phi \in L^2_0([0, T], H^s(\mathbb{R})))
\end{equation}

(1.9)

for $\varepsilon$ an arbitrary positive number. Thus, one of our main results states that when (1.6), (1.7) and (1.9) hold, there exists a unique solution in $\mathcal{H}([0, T], Y) \cap X^s([0, T])$. Moreover, since a contracting argument is used, it is obtained in a constructive way.

When we only assume (1.6) and (1.7), it is still possible to prove existence and uniqueness but we use an abstract argument based on the existence of martingale solutions and a probabilistic argument. Indeed, using T. Kato’s smoothing effect and techniques borrowed from [7], we can derive $a \text{ priori}$ estimates in spaces of the type $L^2(\Omega, X_1)$ where $X_1$ is a metric space compactly embedded in another space $X_2$. As it is usual in the context of stochastic partial differential equations, this and Skorohod’s Theorem are used to construct a martingale solution, i.e. a solution of (1.2)-(1.3) for another Wiener $\tilde{W}$. We prove that this solution belongs to the above mentioned weighted spaces where we know that uniqueness holds. Then, it suffices to use an argument of T. Yamada and S. Watanabe [22], generalized by M. Viot [21], which states that existence of a martingale solution and pathwise uniqueness imply the existence of a unique solution for any given Wiener process.

The paper is organized as follows. In Section 2, after some notations and preliminaries, we shall expose the different notions of existence of solution
that we shall need afterwards. We end this section by presenting the result of pathwise uniqueness in weighted space using the technics developed in [8]. In Section 3, we shall expose in Theorem 3.1 the result of existence of strong solutions in \( L^{\infty}([0, T], Y) \) almost surely, when (1.6), (1.7) and the extra assumption (1.9) hold. We shall first prove the estimates on the linear problem which are necessary for the proof of Theorem 3.1 which is postponed until the end of the section. In Section 4, we shall present in Theorem 4.1 the same result but with only the assumptions (1.6) and (1.7).

We think that both results (Theorems 3.1 and 4.1) are interesting and complementary. Theorem 4.1 gives the same conclusion as Theorem 3.1 under weaker assumptions but the method is not constructive like the method used for Theorem 3.1. We recall that it is important to have constructive arguments when justifying a numerical scheme.

2. NOTATIONS AND PRELIMINARIES

2.1. Deterministic Framework

Let \( X \) be any Banach space and \( I \) any interval in \( \mathbb{R} \). We shall denote by \( L^p(I, X) \), \( 1 \leq p \leq +\infty \), the space of functions which are Bochner-integrable from \( I \) to \( X \). When \( X = \mathbb{R} \), we simply use the notation \( L^p(I) \). We shall also denote by \( \mathcal{C}([0, T], X) \) (resp. \( \mathcal{C}^\beta([0, T], X) \)) the space of continuous (resp. Hölder continuous with exponent \( \beta \)) functions from \([0, T]\) to \( X \) and by \( \mathcal{C}_w([0, T], X) \) the space of weakly continuous functions from \([0, T]\) to \( X \). We shall denote by \( \| \cdot \|_Y \) the norm on a Banach space \( Y \). In the case of \( Y = L^2(\mathbb{R}) \), we will denote \( \| \cdot \| \) (resp. \( (\cdot, \cdot) \)) the \( L^2 \)-norm (resp. the \( L^2 \)-inner product).

Given any number \( \sigma \), the Sobolev space \( H^\sigma(\mathbb{R}) \) is defined as the space of tempered distributions \( u \) such that

\[
\int_{\mathbb{R}} (1 + \xi^2)^{\sigma} |\mathcal{F} u(\xi)|^2 d\xi < +\infty,
\]

where \( \mathcal{F} \) is the Fourier transform. In Section 3, we shall use the following linear operators \( D \) and \( \mathcal{H} \) (the Hilbert transform) which are defined by means of the Fourier transform:

\[
Du = \mathcal{F}^{-1}(\xi \mathcal{F} u)
\]

and

\[
\mathcal{H} u = \mathcal{F}^{-1}\left( \frac{\xi}{|\xi|} \mathcal{F} u \right).
\]
We shall also use the Sobolev space in the time variable \( W^{\alpha, p}([0, T], X) \) with \( \alpha > 0 \) and \( 1 \leq p \leq +\infty \), which is defined as the space of functions \( u \) such that
\[
\left( \int_{[0, T]^2} \frac{|u(t) - u(s)|^p}{|t-s|^{1+\alpha p}} \, dt \, ds \right)^{\frac{1}{p}} < +\infty.
\]

In the frame of the spaces \( H^\sigma(\mathbb{R}) \), it is well known that the linear part of the KdV equation generates a unitary group \( \{S(t)\}_{t \in \mathbb{R}} \). More precisely, the solution of the linear problem
\[
\left\{ \begin{array}{l}
\frac{\partial u}{\partial t} + 3u \frac{\partial u}{\partial x} = 0, \\
(u(x, 0) = u_0(x), x \in \mathbb{R}
\end{array} \right.
\]
with \( u_0 \in H^\sigma(\mathbb{R}) \), \( \sigma > 0 \), is given by
\[
u(t) = S(t) u_0 = \mathcal{F}^{-1}(e^{it\mathcal{F}} u_0).
\]

We mention here a useful estimate concerning the previous linear problem analogous to the Strichartz estimates related to the Schrödinger equation (see [8, Lemma 2.1, p. 1392]).

**Lemma 2.1.** For any \( f \) in \( L^2(\mathbb{R}) \) and for any pair of integers \((p, r)\) such that \( 2/p = \frac{1}{4} (1 - 2/r) \) and \( 2 \leq r \leq +\infty \), there exists a constant \( C > 0 \) such that
\[
|S(\cdot) f|_{L^p([0, T], L^r(\mathbb{R}))} \leq C |f|_{L^2(\mathbb{R})}.
\]

In Section 3, when no confusion is possible, we shall use shorter notations. For example, if \( x \in \mathbb{R} \) and \( t \in [0, T] \), we will use \( L^p([0, T], L^r(\mathbb{R})) \) as the usual space \( L^p([0, T], L^r(\mathbb{R})) \).

In Section 4, we shall use the local spaces \( H^\sigma_0(\mathbb{R}) \), \( \sigma \in \mathbb{R} \). Let \( u \) a distribution on \( \mathbb{R} \), \( u \) is in \( H^\sigma_0(\mathbb{R}) \) for a \( \sigma > 0 \), if and only if \( u \) is in \( H^\sigma([a, b]) \) for any \((a, b) \in \mathbb{R}^2 \). We recall that \( H^\sigma_0(\mathbb{R}) \) is a complete metrizable locally convex space or a Fréchet space, e.g., endowed with the metric
\[
d(u, v) = \sum_{k \geq 0} \frac{1}{2^k} \min(|u - v|_{H^\sigma([-k, k])}, 1).
\]

We denote by \( H^\sigma_{-0}(\mathbb{R}) \) with \( \sigma > 0 \), the set of the distributions \( u \) such that for any \((a, b) \in \mathbb{R}^2 \), \( u \) is in \( H^{-\sigma}([a, b]) \), the topological dual of \( H^\infty_0([a, b]) \) itself the closure of \( \mathcal{D}([a, b]) \) in \( H^\infty([a, b]) \). We recall that \( H^\sigma_{-0}(\mathbb{R}) \) is also a Fréchet space.
In Section 4, we shall also use the following compactness lemma whose proof is based on a classical compact embedding theorem (see [16]), the Ascoli-Arzelà theorem, and on diagonal extraction.

**Lemma 2.2.** Let $T > 0$, $\alpha > 0$, $\beta > 0$. Let $\mathcal{A}$ be a set of distributions $u$ such that

(i) $\mathcal{A}$ is bounded in $L^2([0, T], H^s_{\text{loc}}(\mathbb{R})) \cap W^{\infty}([0, T], H^{-s}_\text{loc}(\mathbb{R}))$;

(ii) $\mathcal{A}$ is bounded in $C^\alpha([0, T], H^{s_1}_\text{loc}(\mathbb{R})) \cap C([0, T], H^{s_2}_\text{loc}(\mathbb{R}))$, for any $s_1 < 1$ and $s_2 > 2$.

Then $\mathcal{A}$ is relatively compact in $L^2([0, T], H^s_{\text{loc}}(\mathbb{R})) \cap C([0, T], H^{-s}_\text{loc}(\mathbb{R}))$.

### Definition of solutions

We call stochastic basis a system $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \in [0, T]}, P, \{W(t)\}_{t \in [0, T]})$ where $(\Omega, \mathcal{G}, P)$ is a probability space, $\{\mathcal{G}_t\}_{t \in [0, T]}$ a filtration and $\{W(t)\}_{t \in [0, T]}$ a cylindrical Wiener process on $L^2(\mathbb{R})$ adapted to this filtration (see [5]). We mean by $L^p(\Omega, X)$, $1 < p < +\infty$ the space of random variables $u$ with integrable $p$th power on $\Omega$, with values in $X$ and we set

$$
\mathbb{E} |u|^p = \int_\Omega |u(\omega)|^p d\mathbb{P}(\omega) < +\infty.
$$

We shall sometimes use shorter notations as for example $L^p(\Omega, X)$ for the space $L^p(\Omega, X)$.

Let $\Phi$ be a linear operator from $L^2(\mathbb{R})$ into a Hilbert space $H$, $\Phi$ is said to be *Hilbert–Schmidt* iff the term

$$
|\Phi|^2_{L^2(\mathbb{R}, H)} \equiv \sum_{i \geq 0} |\Phi e_i|_H^2 < +\infty,
$$

is finite where $\{e_i\}_{i \geq 0}$ is a Hilbertian basis of $H$. When $H = H^s(\mathbb{R})$ for $s > 0$, we shall write $L^2_\sigma(L^2(\mathbb{R}), H^s(\mathbb{R})) = L^2_{\sigma,s}$ and for $s = 0$, we will simply use the notation $L^2_{\sigma}.

The application of the Ito formula in the following sections will not be always rigorously justified due to some lack of regularity (in Proposition 3.2 for instance). However the results are correct and can be easily justified by means of a regularization. In Section 4, we shall construct martingale solutions, or weak solutions. In order for this notion to be more precise, we first introduce the definition of *strong solution*.

**Definition 2.3 [Strong Solution].** Let $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \in [0, T]}, P, \{W(t)\}_{t \in [0, T]})$ be a stochastic basis for some $T > 0$. Let $u_0 \in L^2(\mathbb{R})$ and $\Phi \in L^2_{\sigma}$. We call a strong solution of (1.2)–(1.3), a stochastic process $u$ adapted to this basis such that
(i) \( u \in L^\infty([0, T], L^2(\mathbb{R})) \cap \mathcal{C}([0, T], H^1_{\text{loc}}(\mathbb{R})), \ \mathbb{P} \ \text{a.s., for some} \ \gamma > 0, \)
(ii) \( u(t) - u_0 + \int_0^t \left( t \partial^3 u / \partial x^3 + u(\partial u / \partial x) \right) \, dt = \Phi W(t), \ \mathbb{P} \ \text{a.s.,} \ t \in [0, T] \)
in the distribution sense.

A martingale solution will have a weaker sense. More precisely,

**Definition 2.4 [Martingale Solution].** Let \( T > 0. \) Let \( u_0 \in L^2(\mathbb{R}) \) and \( \Phi \in L^2_\gamma. \) We call a martingale solution or weak solution of (1.2)–(1.3), a pair of random functions \( (\tilde{u}, \tilde{W}) \) defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) adapted to a filtration \( \{ \mathcal{F}_t \}_{t \in [0, T]} \) such that

(i) \( \{ \tilde{W}(t) \}_{t \in [0, T]} \) is a cylindrical Wiener process on \( L^2(\mathbb{R}); \)
(ii) \( \tilde{u} \in L^\infty([0, T], L^2(\mathbb{R})) \cap \mathcal{C}([0, T], H^1_{\text{loc}}(\mathbb{R})), \ \mathbb{P} \ \text{a.s., for} \ \gamma > 0; \)
(iii) \( \tilde{u}(t) - u_0 + \int_0^t \left( t \partial^3 \tilde{u} / \partial x^3 + \tilde{u}(\partial \tilde{u} / \partial x) \right) \, dt = \Phi \tilde{W}(t), \ \mathbb{P} \ \text{a.s.,} \ t \in [0, T] \)
in the distribution sense.

In Section 3, we shall construct a solution \( u \) satisfying the following mild formulation, for any \( t \in [0, T], \)

\[
u(t) = S(t)u_0 - \frac{1}{2} \int_0^t S(t-\tau) \frac{\partial^2 u^2}{\partial x^2} \, d\tau + \int_0^t S(t-\tau) \Phi \, dW(\tau).
\]

Let us note that a strong solution such as Definition 2.1, i.e., a weak one in the partial differential equation sense, such that

\[
\frac{\partial}{\partial x}(u^2) \in L^1([0, T], L^2_{\text{loc}}(\mathbb{R}))
\]
satisfy the mild formulation (see [5]). Indeed, it is the case when \( u \) is sufficiently smooth, for instance, \( u \in L^2([0, T], H^1_{\text{loc}}(\mathbb{R})). \)

In order to conclude to the existence of strong solutions according to the sense of Definition 2.1, the following concept will be used.

**Definition 2.5 [Pathwise Uniqueness].** We say that pathwise uniqueness holds for (1.2) and for a pair of Lusin space \((X, Y), \) if, whenever \( u_1 \) and \( u_2 \) are any two solutions defined on the same probability space \((\Omega, \mathcal{F}, \mathbb{P}), \) adapted to the same filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) and with the same Wiener process \( \{ W_t \}_{t \geq 0} \) itself adapted to the previous filtration such that \( u_1(0) = u_2(0) \) in \( X, \) then \( u_1 = u_2 \) in \( Y \) a.s.

In Section 4, we will use the following theorem due to M. Viot [21], T. Yamada and S. Watanabe [22] which allows us to get strong solutions from weak ones provided pathwise uniqueness holds in some space.
Theorem 2.6. Let $Y$ be a Lusin space such that there exists a pathwise uniqueness for initial data in some space $X$ and such that a martingale solution having values in $Y$ can be constructed.

Then, for any stochastic basis $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \geq 0}, \mathbb{P}, \{W_t\}_{t \geq 0})$, there exists a unique solution with values in $Y$.

2.3. Pathwise Uniqueness

This idea of working with weighted spaces comes from the intention of replacing the assumptions of regularity by assumptions of localization. This is motivated by previous results of uniqueness of the Cauchy problem in the homogeneous case in weighted spaces (see [8, 14]). Before discussing these results, let us introduce the underlying weight functions. Using the same notation as in [8], we denote by $\{h^0_t\}_{t \geq 0}$ positive and increasing smooth functions from $\mathbb{R}$ into itself such that

$$h^0_t(x) = e^{\alpha x} \quad \text{for } x \leq 0$$
$$h^0_t = C t^\theta \quad \text{at } +\infty$$

for some $C > 0$. As regards the existence of such functions, the reader is referred to [8] p. 1397. We shall simply use $h^0_t$ (resp. $h^\theta_t$) instead of $h^0_t$ (resp. $h^\theta_t$). The following property whose proof is straightforward will be used throughout this paper.

Lemma 2.7. There exist positive constants $c_i, i = 1, ..., 3$, such that

$$h^t \leq c_1 (1 + h)$$
$$h^{n2} \leq c_2 (1 + h) h'$$
$$h''' \leq c_3 (1 + h)$$

Remark 2.8. When $h = h^\theta_t$, we shall use implicitly $h^\theta_t(x) \geq \delta > 0$, for any $x \in \mathbb{R}$, thus verifying inequalities (2.1) with $1 + h$ being replaced by $h$.

Another consequence is that

$$\{(h^\theta_t)^{1/2} u \in L^2(\mathbb{R})\} \subset L^2(\mathbb{R})$$

with continuous embedding.

We are now able to introduce the following result from [8, Lemma 2.6, p. 1397]. For any $T > 0$, for any $\alpha > 0$ and for any integer $q$, let us define the spaces $X^{\alpha,q}([0,T])$ and $Y$ as

$$X^{\alpha,q}([0,T]) = \{h^0_t u \in L^q([0,T], L^4(\mathbb{R}))\},$$
$$Y = \{(1 + x_+)^{3/2} u \in L^2(\mathbb{R}))\}.$$
Proposition 2.9. Let $\alpha > 0$, $q$ an integer such that $\frac{1}{4} < 1/q < \frac{1}{2}$ and $T > 0$. There exists $C(T) > 0$ such that for any $u$ in $X^{\alpha,q}([0,T])$ and for any $v$ in $L^\infty([0,T], Y)$, the following smoothing effect holds

$$\left| \int_0^t S(t - \tau) \frac{\partial (uv)}{\partial x} dt \right|_{X^{\alpha,q}([0,T])} \leq C(T) \| u \|_{X^{\alpha,q}([0,T])} \| v \|_{L^\infty([0,T], Y)},$$

where $C(\cdot)$ is uniformly bounded on the compact sets of $\mathbb{R}$.

This proposition leads us to the result of uniqueness in weighted space.

More precisely, we have

Corollary 2.10. Let $T > 0$, $\alpha > 0$ and $q$ as in Proposition 2.2. Then there exists at most one solution $u$ of (1.2)-(1.3) such that

$$u \in L^\infty([0,T], Y), \quad a.s. \quad (2.2)$$

$$u \in X^{\alpha,q}([0,T]), \quad a.s. \quad (2.3)$$

Proof. Let $u_1$ and $u_2$ be two solutions of (1.2)-(1.3) such that (2.2)-(2.3) hold. Let $w = u_1 - u_2$. Then $w$ verifies the equation

$$\frac{\partial w}{\partial t} + \frac{\partial^3 w}{\partial x^3} + \frac{1}{2} \frac{\partial}{\partial x} (w(u_1 + u_2)) = 0$$

with $w(0) = 0$. If now we write the mild form of this equation, we obtain

$$w(t) = \int_0^t S(t - \tau) \frac{\partial}{\partial x} ((w(\tau)(u_1 + u_2)(\tau))) d\tau.$$

Let $t^*$ be such that

$$C(T)(t^*)^{3/8} |u_i|_{L^\infty([0,T], Y)} \leq \delta < \frac{1}{2},$$

then by Proposition 2.9, we get successively

$$|w|_{X^{\alpha,q}([0,T])} \leq C(T) |u_1 + u_2|_{L^\infty([0,T], Y)} |w|_{X^{\alpha,q}([0,T])}$$

$$\leq C(T)(t^*)^{3/8} \sum_{i=1,2} |u_i|_{L^\infty([0,T], Y)} |w|_{X^{\alpha,q}([0,T])}$$

$$\leq 2\delta |w|_{X^{\alpha,q}([0,T])}.$$

Thus, $w = 0$ on $[0, t^*]$. We conclude by iterating this argument. \[\square\]
3. CONSTRUCTIVE METHOD

In this section we are interested in the stochastic KdV equation with an additive noise written in the following Ito form

\[
du + \left( \frac{\partial^3 u}{\partial x^3} + uu_x \right) \, dt = \Phi \, dW, \quad (3.1)
\]

for \( x \in \mathbb{R}, \, t \in [0, T], \, T > 0 \), with the initial condition

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \quad (3.2)
\]

Here, \( \{ W(t) \}_{t \in [0, T]} \) denotes a cylindrical Wiener process on \( L^2(\mathbb{R}) \) adapted to a given filtration \( \{ \mathcal{F}_t \}_{t \in [0, T]} \) on a given probability space \( (\Omega, \mathcal{F}, P) \).

We assume here that \( \Phi \) is a linear operator from \( L^2(\mathbb{R}) \) into itself such that

\[
1 + x + x^3 \Phi \in L^2_x \quad \text{(3.3)}
\]

and, contrary to Section 4, we take the extra assumption

\[
\Phi \in L^{3, \varepsilon}_x \quad \text{(3.4)}
\]

for some \( \varepsilon > 0 \).

We seek a solution \( u \) of (3.1)-(3.2) under the following \textit{mild} form

\[
u(t) = S(t)u_0 - \frac{1}{2} \int_0^t S(t-\tau) \frac{\partial^2 (u^2)}{\partial x^2} \, d\tau + \int_0^t S(t-\tau) \Phi \, dW(\tau). \quad (3.5)
\]

To that purpose, we will prove that a sequence of approximate smooth solutions obtained with smooth data converges almost surely to a solution satisfying (3.5) in \( X^{q, q}([0, T]) \) with \( q > 0 \) and \( q \) an integer such that

\[
\frac{1}{4} < \frac{1}{q} < \frac{1}{2}.
\]

The main result of this section is contained in the following theorem.

\textbf{Theorem 3.1.} \textit{Let} \( T > 0 \). \textit{Let} \( u_0 \) be such that

\[
u_0 \in Y. \quad (3.6)
\]

Under the assumptions (3.3), (3.4), and (3.6), there exists a unique stochastic process \( u \) which is a global strong solution of (3.1)(3.2) such that
u \in X^\alpha \cdot q([0, T]), \quad \text{a.s.,}

u \in L^\infty([0, T], Y), \quad \text{a.s.,}

\frac{\partial u}{\partial x} \in L^2([0, T], L^2_{loc}(\mathbb{R})), \quad \text{a.s.}

Moreover

u \in \mathcal{E}([0, T], Y), \quad \text{a.s.}

The fact that here the noise is additive (i.e., \Phi independent of u) allows us to consider the following linear problem apart

\frac{du}{dt} + \frac{\partial u}{\partial x} dt = \Phi dW, \quad (3.7)

u(0) = 0, \quad (3.8)

whose solution is given by

W^x_L(t) = \int_0^t S(t - \tau) \Phi dW(\tau). \quad (3.9)

The proof of Theorem 3.6 essentially consists in deriving \textit{a priori} estimates on \( W^x_L \). Indeed, in order to obtain pathwise estimates on the solutions of (3.1)–(3.2), we need to prove that \( W^x_L \) belongs to \( X^\alpha \cdot q([0, T]) \cap L^\infty([0, T], Y) \) and that \( \partial W^x_L / \partial x \) belongs to \( L^2([0, T], L^2(\mathbb{R})) \). This last property being the most difficult to prove.

3.1. The Linear Problem

\textbf{Proposition 3.2.} Let \( h = h^\beta \) for some \( \beta > 0 \). Let \( T > 0 \) and \( \varepsilon > 0 \). Let finally \( \alpha > 0 \) and \( q \) be an integer such that \( 2 < q < 4 \). Then there exist various constants \( C_i(T), i = 1, \ldots, 4 \) such that

\begin{align*}
E \| W^x_L \|_{L^\infty([0, T], L^q(\mathbb{R}))}^2 &\leq C_1(T) \| \Phi \|_{L^2}^2, \quad (3.10) \\
E \| h^{1/2} W^x_L \|_{L^\infty([0, T], L^q(\mathbb{R}))}^2 &\leq C_2(T) \| h^{1/2} \Phi \|_{L^2}^2, \quad (3.11) \\
E \| W^x_L \|_{L^\infty([0, T], L^q(\mathbb{R}))}^2 &\leq C_3(T) \| \Phi \|_{L^2}^2, \quad (3.12) \\
E \left\| \frac{\partial W^x_L}{\partial x} \right\|_{L^2([0, T], L^2(\mathbb{R}))}^2 &\leq C_4(T) \| \Phi \|_{L^2}^2. \quad (3.13)
\end{align*}
Remark 3.3.

- When \( \beta = \frac{3}{4} \), (3.11) implies that \( W_L \in L^\infty([0, T], Y) \).
- As previously mentioned, the main difficulty is to prove (3.13). It is on the proof of this estimate, and only there, that we need assumption (3.4).

Proof of (3.10). We apply the Ito formula to \( W_L \) for the functional \( \frac{1}{2} \frac{|h^{1/2}W_L(t)|^2}{2} \) and obtain for any \( t \in [0, T] \)

\[
\| W_L(t) \|^2 = 2 \int_0^t \langle W_L(\tau), \Phi \, dW(\tau) \rangle + t \| \Phi \|^2_{L^2}.
\]

Using a martingale inequality (see [5, Theorem 3.14]), we get the result

\[
\mathbb{E} \sup_{t \in [0, T]} \| W_L(t) \|^2 \leq 6 \mathbb{E} \left( \int_0^T \| W_L(\tau) \|^2 \| \Phi \|^2_{L^2} \, d\tau \right)^{1/2} + T \| \Phi \|^2_{L^2}
\]

\[
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} \| W_L(t) \|^2 + C(T) \| \Phi \|^2_{L^2}.
\]

Proof of (3.11). We apply the Ito formula to \( W_L \) for the functional \( \frac{1}{2} \frac{|h^{1/2}W_L(t)|^2}{2} \). We obtain for any \( t \in [0, T] \) after several integrations by parts

\[
\frac{1}{2} \left| h^{1/2}W_L(t) \right|^2 + \frac{3}{2} \int_0^t \left( \frac{\partial}{\partial x} \left( h^{1/2}W_L(t) \right) \right)^2 \, dt = \frac{1}{2} \int_0^t d\tau \left| h^{1/2}W_L(t) \right|^2 + \int_0^t \left( h^{1/2}W_L(t) \right) \, dW(t)
\]

Owing to Remark 2.2, the following inequality holds for any \( t \in [0, T] \):

\[
\frac{1}{2} \left| h^{1/2}W_L(t) \right|^2 \leq \frac{C_1}{2} \int_0^t \left| h^{1/2}W_L(t) \right|^2 \, dt + \frac{1}{2} \left| h^{1/2}W_L(t) \right|^2.
\]

Then the Gronwall lemma yields

\[
\sup_{t \in [0, T]} \left| h^{1/2}W_L(t) \right|^2 \leq C(T) \left( \left| h^{1/2}W_L(t) \right|^2 + \sup_{t \in [0, T]} \left( \left| h^{1/2}W_L(t) \right|^2 \right) \right).
\]

Using the same techniques as those in the proof of (3.10), we eventually get the result.
Proof of (3.12). We first have, after a Hölder inequality with respect to |ω| and several applications of the Fubini theorem with respect to (t, ω) and to (x, ω),
\[
E \left( \int_0^T dt \left( \int_{\mathbb{R}} dx |W_L(t, x)|^4 \right)^{q/4} \right)^{2/q} \leq \left( \int_0^T dt E \left( \int_{\mathbb{R}} dx |W_L|^4 \right)^{q/4} \right)^{2/q},
\]
since \(q\) is such that \(2 < q < 4\).

As \(W_L(t, x)\) is a gaussian process, we get, from the definition of the stochastic integral, for any \((x, t) \in \mathbb{R} \times [0, T] , \)
\[
E |W_L(t, x)|^4 \leq C(E |W_L(t, x)|^2)^2 
\leq C \left( \sum_{n \geq 0} \left( \int_{0}^t |S(t - \tau) \Phi \phi_n|^2 (x) d\tau \right)^2 \right) 
\leq C(T) \left( \sum_{n \geq 0} \left( \int_{0}^t |S(t - \tau) \Phi \phi_n|^4 (x) d\tau \right)^{1/2} \right)^2,
\]
where a Hölder inequality in \(\mathbb{R}\) has been used in the last term. Then an integration on \(\mathbb{R}\) gives, by means of the generalized Minkowski inequality,
\[
\left( \int_{\mathbb{R}} dx E(|W_L(t, x)|^4) \right)^{q/4} \leq C(T) \left( \sum_{n \geq 0} \left( \int_{0}^t \int_{\mathbb{R}} |S(t - \tau) \Phi \phi_n|^4 (x) d\tau \right)^{1/2} \right)^{q/2} 
\leq C(T) \left( \sum_{n \geq 0} \left( \int_{0}^t \int_{\mathbb{R}} |S(t - \tau) \Phi \phi_n|^4 (x) d\tau \right)^{1/2} \right)^{q/2} 
\leq C(T) \left( \sum_{n \geq 0} \left( \int_{0}^t |S(t - \tau) \Phi \phi_n|^2 (x) d\tau \right)^{1/2} \right)^{q/2}
\]
for any \(t \in [0, T] \). Eventually, we integrate on \([0, T] \) and, using again the generalized Minkowski inequality, we get, thanks to (3.14),
\[
E |W_L|^2_{L^2([0, T], L^q(\mathbb{R}))} \leq C(T) \sum_{n \geq 0} \left( \int_{0}^T dt |S(t - \cdot) \Phi \phi_n|^2_{L^2([0, T], L^q(\mathbb{R}))} \right)^{2/q}.
\]

We finally obtain (3.12) by means of Lemma 2.1 with \((p, r) = (12, 4)\) and the fact that \(h_q(x) \leq 1\), for any \(x \in \mathbb{R}\).
Proof of (3.13). This proof uses a technical lemma of interpolation which can be found in the appendix of [6] and a very sharp property of smoothness of the Airy group due to [13, (Lemma 2.1, p. 329)].

Let us first notice that
\[
\mathbb{E} \int_0^T \left( \frac{\partial W_L(t)}{\partial x} \right)_L^2 \ dt = \int_0^T \mathbb{E} \left( \left( \frac{\partial W_L(t)}{\partial x} \right)_L^2 \right) \ dt.
\]

We shall in fact estimate
\[
\sup_{t \in [0, T]} \left( \mathbb{E} \left( \left( \frac{\partial W_L(t)}{\partial x} \right)_L^p \right) \right)^{2/p}
\]
for some \( p \geq 2 \). More precisely, we shall prove that the following estimate holds
\[
|D^p W_L|_{L^p_t(L^2_x)}^2 \leq C |\Phi|_{L^2_x}^2,
\]
(3.15)

for some \( p \geq 2 \).

First of all, let us note that (3.4), the gaussianity of \( W_L \) and the fact that the Airy group is unitary in \( L^2(\mathbb{R}) \) lead directly to the existence of a constant \( C(p, T) > 0 \) such that
\[
|D^p W_L|_{L^p_t(L^2_x)}^2 \leq C(p, T) |\Phi|_{L^2_x}^2.
\]
(3.16)

Indeed, we have for any \( t \in [0, T] \),
\[
\int_\mathbb{R} dx \mathbb{E}(|D^p W_L(x, t)|^p)^{2/p} \leq C_p \int_\mathbb{R} dx \mathbb{E}(|D^p W_L(x, t)|^2)
\]
\[
= C_p \sum_{n \in \mathbb{N}} \int_0^T dt |S(t - \tau) D^p \Phi e_n|^2
\]
\[
\leq C(p, T) |\Phi|_{L^2_x}^2.
\]

Then we also have
\[
|D^{1 + \epsilon} W_L|_{L^{p/(1+\epsilon)}_t(L^{2/(1+\epsilon)}_x)} \leq C(p) |\Phi|_{L^2_x}^2.
\]
(3.17)

Indeed, for any \((x, t) \in \mathbb{R} \times [0, T] \), thanks to the gaussianity of \( D^{1 + \epsilon} W_L \), one has successively
\[
(\mathbb{E} |D^{1 + \epsilon} W_L(x, t)|^p)^{1/p} \leq C(p) (\mathbb{E} |D^{1 + \epsilon} W_L(x, t)|^2)^{1/2}
\]
\[
\leq C(p) \left( \int_0^T \sum_{n \geq 0} |DS(t - \tau) D^p \Phi e_n|^2 \ dt \right)^{1/2}.
\]
Now, using the result previously quoted (\([13]\)), namely
\[
\int_{\mathbb{R}} dt |DS(t-\tau) D^r \Phi e_i(x)|^2 = C |D^r \Phi e_i|^2_{L^2(\mathbb{R})},
\]
for any \(x \in \mathbb{R}\), we obtain directly (3.17) with \(p \geq 2\).

We quote here the interpolation result from [6].

**Lemma 3.4.** Let \(X\) be a Banach space and \(u\) a function from \(\mathbb{R}\) into \(X\) such that, for at least one \(p \in ]1, +\infty[\) and one \(\sigma > 0\),
\[
u \in L^p(X) \quad \text{and} \quad D^\sigma u \in L^\infty(X).
\]

Then for any \(\alpha \in [0, \sigma]\), \(D^\alpha u \in L^p(X)\) with
\[
\frac{1}{p_\alpha} = \frac{1}{p} \left( 1 - \frac{\alpha}{\sigma} \right)
\]
and there exists \(C > 0\) such that
\[
|D^\alpha u|_{L^p(X)} \leq C |u|_{L^p(X)}^{\frac{1-\alpha}{\sigma}} |D^\sigma u|_{L^\infty(X)}^{\frac{\alpha}{\sigma}}.
\]

We apply the previous lemma with \(u = D^r \Phi e_i(-, t)\) at a fixed time \(t\), \(p = 2\), \(X = L^p(\Omega)\) for \(p \geq 2\). Then (3.16) and (3.17) lead to
\[
|D^{\alpha + \varepsilon} W_i|_{L^p(\Omega)} \leq C |\Phi|_{L^p_i}, \quad (3.18)
\]
for some \(\alpha \in [0, 1]\) such that \(1/p = \frac{1}{2}(1 - \alpha)\).

We need one more estimate in order to conclude. It is not difficult to see that, for the same reasons as those mentioned in the proofs of (3.16) and (3.17), we also have, after the application of Lemma 3.4,
\[
|D^{\alpha + \varepsilon} W_i|_{L^p(\Omega)} \leq C |\Phi|_{L^p_i} \leq C^* |\Phi|_{L^p_i}, \quad (3.19)
\]
for the same \(\alpha\) as in (3.18).

For one fixed \(\varepsilon > 0\), let us now fix the constants \(\alpha\) and \(p\)
\[
\alpha = 1 - \frac{\varepsilon}{2} < 1, \quad (3.20)
\]
i.e.,
\[
p = \frac{4}{\varepsilon}.
\]
Thus since $L^p_r(L^p_u)$ is isomorphic to $L^p_u(L^p_r)$, (3.18), (3.19) and (3.20) give
\[ |DW|_{L^p(L^p_r(W^\omega, (R)))}^2 \leq C(T) |\Phi|_{L^2}^2, \]  
(3.21)

We conclude to (3.15) thanks to (3.20), (3.21) and the Sobolev embedding
\[ \mathcal{W}^{\omega, p}(R) \subset L^\omega(R) \]
because $\omega/2 > 1$. Finally, from (3.15) we get successively
\[ \frac{\partial W}{\partial x} \mid_{L^q(L^2)} \leq C |\mathcal{H}_3|_{L^2} \leq C |\mathcal{H}_3|_{L^2}, \]
and thus the result (3.13) thanks to Hölder inequality in $\omega$ which is possible since $p \geq 2$.

3.2. Pathwise a priori Estimates

We shall see that the estimate (3.12) on the linear problem allows us to solve (3.1)–(3.2) locally in time in $X^q([0, T])$ using Proposition 2.9. This proposition supposes the obtention of an a priori estimate of the solutions in a weighted space based on $L^1(R)$. This is the purpose of the following propositions.

**Proposition 3.5.** Let $u$ be a mild solution of (3.1)–(3.2). Then, the following inequality holds for any $T > 0$, a.s.,
\[ |u|_{X^q(I, T)} \leq C(T, |u|_{L^q(I, L^2)}, |u|_{L^q(I, Y)}, |W|_{X^q(I, T)}), \]  
(3.22)

where $q_1 > 8/3$, $\alpha > 0$ and $C_S$ is a non-decreasing function.

**Proof.** $I = [0, T]$ is divided into $N$ intervals $\{I_i\}_{i=1}^N$ where $I = [s_i, s_{i+1}]$ of the same length $\delta > 0$. Since $1/q \in ]1/4, 1/2[ \subset ]1/4, 5/8[$, we can apply Proposition 2.2 together with a Hölder inequality in time to the nonlinear term of (3.5) on each interval $I_i$,
\[ |u|_{X^q(I, I)} \leq |h_s S(-s) u(s)|_{L^q(I, L^q(R))} + C(\delta) |u|_{X^q(I, I)} |u|_{L^q(I, Y)} |I_i|^\theta \]
\[ + |h_s \int_{s_i}^{s_{i+1}} S(-\tau) \Phi dW(\tau)|_{L^q(I, L^q(R))}, \]  
(3.23)

where $\theta = 3/8 - 1/q_1$. 

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The first term of the right hand side of (3.23) can be estimated using Lemma 2.1 with \((p, q) = (12, 4)\) and a Hölder inequality as follows, since, clearly, \(h \in \mathbb{L}^1\),

\[ |h \cdot \xi \cdot (s_i - s_j) - s_i u(s_i) ||_{\mathbb{L}^2} \leq C |I_i|^{1/4 - 1/12} |u(s_i) ||_{\mathbb{L}^8} \cdot \]

We now choose \(\delta(\omega) > 0\) such that

\[ C(T) |u|_{\mathbb{L}^{\phi}(\mathbb{L}^2, Y)} \delta(\omega)^{\phi} = \frac{1}{2} \]

and after summation of (3.23) with respect to \(i\), we obtain

\[ |u|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)} \leq 2^{q - 1} \left( |u|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)} \delta \right)^{1/12} + 2 |W_L|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)} N \).

Using now that \(N(\omega) = T/\delta(\omega)\), the previous choice of \(\delta\) yields

\[ |u|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)} \leq C(T, \omega) |u|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)} + |u|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)} |W_L|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)}. \]

Remark 3.6. A local estimate in time in \(\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)\) can also be obtained. For that, it is sufficient to write (3.23) on a small interval near the origin. Then we get for some \(T > 0\) and a.s. that

\[ |u|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)} \leq C(T, |u_0|, |u|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)}) + |W_L|_{\mathbb{L}^{q}(\mathbb{L}^2, \mathbb{L}^q)}. \]

Thus, thanks to (3.12), the assertion (2.3) of Corollary 2.10 can be removed providing that we have an \(a \ priori\) estimate in \(\mathbb{L}^2(\mathbb{R})\) and in the weighted space \(Y([0, T])\).

**Proposition 3.7.** Let \(T > 0\) and \(h = h^\beta\) for some \(\beta > 0\) satisfying (2.1). Then for any smooth solution \(u\) of (3.1)(3.2), we get, \(\mathbb{P}\) a.s.

\[ |u|_{\mathbb{L}^{\phi}(\mathbb{L}^2)} \leq C_6 \left( |u_0|_{\mathbb{L}^2}, \frac{\partial W_L}{\partial x} |_{\mathbb{L}^2}, |W_L|_{\mathbb{L}^{\phi}(\mathbb{L}^2)} \right), \quad \text{(3.24)} \]

\[ |h^{1/2} u|_{\mathbb{L}^{\phi}(\mathbb{L}^2)} \leq C_7 \left( T, |u_0|_{\mathbb{L}^2}, |h^{1/2} u_0|_{\mathbb{L}^2}, \frac{\partial W_L}{\partial x} |_{\mathbb{L}^2}, |h^{1/2} W_L|_{\mathbb{L}^{\phi}(\mathbb{L}^2)} \right), \quad \text{(3.25)} \]

\[ \frac{\partial u}{\partial x} \in \mathbb{L}^{2}([0, T], L_{1/2}^{2}(\mathbb{R})), \quad \text{(3.26)} \]

where \(C_i, i = 6, 7\) are non-decreasing functions of their arguments.
Proof of (3.24). Let us set \( v = u - W_L \), then \( v \) verify the following equation

\[
\begin{aligned}
\frac{\partial v}{\partial t} + \frac{\partial^3 v}{\partial x^3} + \frac{1}{2} \frac{\partial}{\partial x} (v + W_L)^2 &= 0, \\
v(0) &= u_0.
\end{aligned}
\]

For \( v \in L^\infty([0, T], H^3(\mathbb{R})) \) for example, we multiply Eq. (3.27) by \( v \) and after an integration by parts, we get

\[
\frac{1}{2} \frac{d}{dt} |v(t)|^2 = - \int_{\mathbb{R}} W_L \frac{\partial W_L}{\partial x} v \, dx - \frac{1}{2} \int_{\mathbb{R}} \frac{\partial W_L}{\partial x} v^2 \, dx.
\]

After classical computations this leads to

\[
\frac{1}{2} \frac{d}{dt} |v(t)|^2 \leq \left[ \frac{\partial W_L}{\partial x} \right]_{L^\infty(\mathbb{R})} |v(t)|^2 + \frac{1}{2} \left[ \frac{\partial W_L}{\partial x} \right]_{L^\infty(\mathbb{R})} |W_L(t)|^2.
\]

Then an application of the Gronwall lemma gives for any \( t \in [0, T] \),

\[
|v(t)|^2 \leq |u_0|^2 \exp \left( 2 \int_0^T \frac{\partial W_L}{\partial x} |v(t)|^2 + \frac{1}{2} \left[ \frac{\partial W_L}{\partial x} \right]_{L^\infty(\mathbb{R})} |W_L(t)|^2 \right),
\]

(3.28)

(3.24) follows immediately from (3.28) and (3.10).

Proof of (3.25). Let us set again \( v = u - W_L \) and assume that \( v \in L^\infty([0, T], H^3(\mathbb{R})) \). We first consider the case \( h \in C_0^\infty(\mathbb{R}, \mathbb{R}_+) \) and verifies (2.1). After multiplication of (3.27) by \( h v \) and several integrations by parts, we obtain at time \( t \)

\[
\frac{1}{2} \frac{d}{dt} |h^{1/2} v|^2 + \frac{3}{2} |h^{1/2} \frac{\partial v}{\partial x}|^2 = \frac{1}{2} \left( (v, h^n v) + \frac{1}{2} \left( h v, \frac{\partial}{\partial x} (v + W_L)^2 \right) \right). \quad (3.29)
\]

The first term of the right hand side of (3.29) can be estimated thanks to (2.1),

\[
\frac{1}{2} |(v, h^n v)| \leq \frac{c_2}{2} |h^{1/2} v|^2.
\]
As regards the other ones
\[
\left| \frac{1}{2} \left( hv, \frac{\partial}{\partial x} (v + W_L)^2 \right) \right| \leq \frac{1}{3} \left| \int_R h v^3 \, dx \right| + |h|^{1/2} v \left| \frac{\partial W_L}{\partial x} \right|_{L^\infty(R)} \nonumber \\
\quad + \frac{3}{2} |h|^{1/2} v \left| \frac{\partial W_L}{\partial x} \right|_{L^\infty(R)} + \frac{1}{2} |W_L|_{L^\infty(R)} |h|^{1/2} v^2, \nonumber 
\]
and we obtain finally
\[
\left| \frac{1}{2} (hv, \frac{\partial}{\partial x} (v + W_L)^2) \right| \leq \frac{1}{3} \left| \int_R h v^3 \, dx \right| + \left| \frac{\partial W_L}{\partial x} \right|_{L^\infty(R)} \left[ \frac{3}{2} |h|^{1/2} v^2 + |h|^{1/2} W_L \right] \nonumber \\
\quad + \frac{c_1}{2} |W_L|_{L^\infty(R)} |h|^{1/2} v^2. \quad (3.30)
\]

The estimate of the cubic term of (3.30) comes from Sobolev embeddings and standard linear interpolation in the Sobolev spaces. More precisely, we get
\[
\left| \int_R h v^3 \, dx \right| \leq \left( \int_R h^2 v^4 \, dx \right)^{1/2} \left( \int_R v^2 \, dx \right)^{1/2}, \quad (3.31)
\]
with
\[
\left| \int_R h^2 v^4 \, dx \right| \leq C |h|^{1/2} v^3 |h|^{1/2} v |h^1(R). \quad (3.32)
\]

Then, thanks to (2.1), the $H^1(\mathbb{R})$-term of (3.32) is estimated by
\[
|h|^{1/2} v^2 |h^1(\mathbb{R}) \leq \left( c_1 + \frac{c_2}{2} \right) |h|^{1/2} v^2 + 2 \left| h^{1/2} \frac{\partial v}{\partial x} \right|^2. \quad (3.33)
\]

We finally estimate the cubic term by substituting (3.31) with (3.32) and (3.33) and obtain for any $\eta > 0$
\[
\left| \int_R h v^3 \, dx \right| \leq \eta |v| \left| h^{1/2} \frac{\partial v}{\partial x} \right|^2 + C_\eta |v| |h|^{1/2} v^2. \quad (3.34)
\]

Now, thanks to (3.28), we know that
\[
\forall u_0 \in L^2(\mathbb{R}), \quad \forall T > 0, \quad \mathbb{P} \text{ a.s., } \exists M(T, |u_0|, \omega) > 0
\]
such that for any $v$ smooth solution, we have
\[
|v|_{L^\infty([0, T], L^2(\mathbb{R}))} \leq M(T, u_0, \omega).
\]
We now deduce from (3.34) that for $\eta$ such that
\[
\eta \frac{3}{M(T, u_0, \alpha)} \leq \frac{1}{2},
\]
one has
\[
\frac{1}{2} \frac{d}{dt} |h^{1/2}v|^2 + \left| h^{1/2} \frac{\partial v}{\partial x} \right|^2 \leq C(T, |u_0|, h) |h^{1/2}v|^2 + C \left| \frac{\partial W_L}{\partial x} \right|_{L^\infty(\mathbb{R})} \left[ |h^{1/2}v|^2 + |h^{1/2}W_L|^2 \right] + C |W_L|_{L^\infty(\mathbb{R})} |h^{1/2}v|^2.
\]

Thanks to the estimates of the linear problem (3.10)–(3.13) and to the Gronwall lemma, we eventually get
\[
|h^{1/2}v|^2 + 2 \int_0^t \left| h^{1/2} \frac{\partial v}{\partial x} \right|^2 \, dt \leq |h^{1/2}u_0|^2 + C(T, |u_0|, \alpha, h) \int_0^t |h^{1/2}v|^2 + C'(\alpha, h) \tag{3.35}
\]

Hence (3.25) holds when $h \in C_0^\infty(\mathbb{R})$. We now consider a sequence \{h_n\}_{n \geq 0} of $C_0^\infty(\mathbb{R}, \mathbb{R}_+)$ functions such that $h_n$ verifies (2.1) uniformly with respect to $n$ and such that $h_n(x) \rightarrow h^R(x)$ for any $x \in \mathbb{R}$. Then we apply (3.35) with $h$ replaced by $h_n$ and obtain an estimate of $|h_n^{1/2}v|^2$ uniformly in $n$ by the Gronwall lemma. Taking the limit $n \rightarrow +\infty$ the Beppo-Levi theorem yields the required estimates for $h^R$.

**Proof of (3.26).** Estimate (3.26) comes from (3.35) and the fact that $h = h^R$ is increasing, thus making $h^R$ greater than a positive constant on any compact set of $\mathbb{R}$.

### 3.3. Proof of Theorem 3.1

First, we construct a sequence of smooth solutions \{u_n\}_{n \geq 0} of (3.1)–(3.2). For instance, for smooth data $(\Phi_n, u_{0n})$ such that
\[
\begin{align*}
\Phi_n &\in L^0_2, \\
u_{0n} &\in H^\infty(\mathbb{R}),
\end{align*}
\]

and using (1 + $x_+$)\$3/8$ $\Phi_n \in L^0_2$, \(1 + x_+\)\$3/8$ $u_{0n} \in L^\infty(\mathbb{R})$,
\[
\tag{3.36}
\]
with

\[
\begin{align*}
\Phi_n \to \Phi & \in L^6_{\frac{1}{2}}, \\
(1+x^+)^{3\Phi} \Phi_n & \to (1+x^+)^{3\Phi} \Phi \in L^6_{\frac{1}{2}}, \quad (3.37) \\
u_{0,n} & \to u_0 \in L^2(\mathbb{R}).
\end{align*}
\]

Standard computations lead to the existence of \(u_n\) solution of (3.1)-(3.2), \((\Phi, u_0)\) being replaced by \((\Phi_n, u_{0,n})\). For that, we can generalize easily to the stochastic case the techniques of construction of weak solutions in \(H^1(\mathbb{R})\) developed in [20] in the determinist frame, which use the invariants of the Korteweg-de Vries equation. We can also generalize the fixed point method, introduced in [10], which construct mild solutions for smooth data.

Both methods yield the global existence of \(u_n\) in \(L^\infty([0, T], H^3(\mathbb{R}))\), a.s., such that

\[
u(t) = S(t)u_{0,n} \mathop{\to}^{\scriptscriptstyle{\text{a.s.}}} \int_0^t S(t-\tau) \frac{\partial (u^2_n)(\tau)}{\partial x} d\tau + W_{L,n}(t), \quad (3.38)
\]

where \(W_{L,n}(t)\) is given by (3.9), \(\Phi\) being replaced by \(\Phi_n\). Now, thanks to (3.37) and (3.12), there exists a subsequence that we shall denote again \([W_{L,n}]_{n \geq 0}\) such that

\[
W_{L,n} \to W_L \in X^{s,q}([0, T]) \quad \text{a.s.} \quad (3.39)
\]

From now on we shall fix an \(\omega\) such that convergence (3.39) holds. We are going to prove that \([u_n]_{n \geq 0}\) is a Cauchy sequence in \(X^{s,q}([0, T(\omega)])\) for some \(T(\omega) > 0\).

**Lemma 3.8.** There exist \(T(\omega) > 0\) and \(C(\omega) > 0\) such that for any \(p, q\) in \(\mathbb{N}\),

\[
|u_p - u_q|_{X^{s,q}([0, T(\omega)])} \leq C(\omega) \left[ |u_{0,p} - u_{0,q}| + |W_{L,p} - W_{L,q}|_{X^{s,q}([0, T(\omega)])} \right]. \quad (3.40)
\]

**Proof.** Let us write \(u_p - u_q\) using the mild formulation (3.38). We get for any \(t > 0\)

\[
\begin{align*}
u_p(t) - u_q(t) &= S(t)(u_{0,p} - u_{0,q}) - \frac{1}{2} \int_0^t S(t-\tau) \frac{\partial}{\partial x} [(u_p - u_q)(u_p + u_q)] d\tau \\
&+ W_{L,p}(t) - W_{L,q}(t).
\end{align*}
\]
We use the same techniques as in Proposition 3.5 and apply the result of Proposition 2.9 to (3.41), and obtain

\[
|u_p - u_q|_{\mathcal{X}^n([0, T])} \leq CT^{1/q - 1/12} |u_{0,p} - u_{0,q}| \\
+ C(T) T^\theta |u_p - u_q|_{\mathcal{X}^n([0, T])}, |u_p + u_q|_{L^p(Y)} \\
+ |W_{L,p} - W_{L,q}|_{\mathcal{X}^n([0, T])}.
\] (3.42)

Thanks to the a priori estimate (3.25) and estimates (3.10)–(3.13) with \( h = h^N \), there exists \( M(T, |u_0|_Y, \omega) > 0 \) such that for any \( T > 0 \), for any integer \( n \)

\[
|u_n|_{L^\infty([0, T], Y)} \leq M(T, |u_0|_Y, \omega).
\]

By choosing \( T(\omega) \) such that

\[
2C(T) T^{\theta} M(T, |u_0|_Y, \omega) \leq \frac{1}{2}
\] (3.43)

and by substituting (3.42) with (3.43), we finally obtain (3.40).

Since \( \{u_n\}_{n \geq 0} \) is a Cauchy sequence in \( \mathcal{X}^n([0, T(\omega)]) \), there exists \( u \) such that

\[
u_n \to u \quad \text{in} \quad \mathcal{X}^n([0, T(\omega)]).
\] (3.43)

Let now \( T_0 \) be any positive real number. (3.25) then yields the existence of a \( \tilde{u} \) such that

\[
\tilde{u} \in L^\infty([0, T_0], Y),
\]

and

\[
u_n \to \tilde{u} \quad \text{in} \quad L^\infty([0, T_0], Y) \text{ weak-star},
\]

which implies that

\[
u_n \to \tilde{u} \quad \text{in} \quad \mathcal{D}'([0, T_0] \times \mathbb{R}).
\]

Also, it is not difficult to see that (3.44) implies

\[
u_n \to u \quad \text{in} \quad \mathcal{D}'([0, T(\omega)] \times \mathbb{R})
\]

Therefore, \( \tilde{u}|_{[0, T(\omega)]} = u \) and thus \( |\nu(T(\omega))|_Y \leq |\tilde{u}|_{L^\infty([0, T_0], Y)} \). The interval in time \( T(\omega) \) of the local existence being estimated in term of the norm of the initial condition in the weighted space \( Y \), we can iterate the process on \([T(\omega), T(\omega) + T_1(\omega)]\), \( T_1(\omega) \) depending on \( |\tilde{u}|_Y \) and so on until \([0, T_0] \) is entirely covered.
Thus we have constructed a global strong solution of (3.1)–(3.2). The uniqueness follows from Corollary 2.10. Now, it remains to prove the continuity of the trajectories $t \mapsto u(t, \omega)$ in the weighted space based on $L^2(\mathbb{R})$ a.s. This result comes from an a.s. energy equality in this space and from the weak continuity in this space.

**Lemma 3.9.** Let $T > 0$. Let $u$ be a solution of (3.1)–(3.2) such that

$$u \in L^\infty([0, T], L^2(\mathbb{R})) \quad \text{a.s.}$$

$$\frac{\partial u}{\partial x} \in L^\infty([0, T], L^2_{loc}(\mathbb{R})) \quad \text{a.s.}$$

Let $h \in C^\infty_c(\mathbb{R}, \mathbb{R})$ such that (2.1) holds. We furthermore assume that

$$h^{1/2}u \in L^\infty([0, T], L^2(\mathbb{R})) \quad \text{a.s.}$$

$$(h')^{1/2} \frac{\partial u}{\partial x} \in L^2([0, T], L^2(\mathbb{R})) \quad \text{a.s.}$$

$$h^{1/2}\Phi \in L^2_{c}.$$ 

Then, the following equality holds a.s., for any $t \in [0, T]$,

$$\frac{1}{2} |h^{1/2}u(t)|^2 + \frac{3}{2} \int_0^t \left| h^{1/2} \frac{\partial u}{\partial x} \right|^2 \, dt$$

$$= \frac{1}{2} |h^{1/2}u_0|^2 + \frac{1}{2} \int_0^t |h^{1/2}u|^2 \, dt + \frac{1}{3} \int_0^t \int dx \, h' u^3$$

$$+ \int_0^t (hu, \Phi dW(\tau)) + \frac{t}{2} |h^{1/2}\Phi|_{L^2_{c}}^2.$$ 

**Proof.** This proof is rather similar to the proof of estimate (3.25) of Proposition 3.7. First, functions $h$ with compact support will be used. So let $\{u_n\}_{n \geq 0}$ such that

$$u_n = \rho_n \ast (\theta u)$$

where $\{\rho_n\}_{n \geq 0}$ is a smoothing kernel that is chosen such that

$$|\rho_n \ast u| \leq |u|, \quad \forall u \in L^2(\mathbb{R})$$

For instance, a truncature in the Fourier space.
and where $\theta$ is a function of $C^0_0(\mathbb{R})$ such that $\theta = 1$ in a neighborhood of $\text{Supp}(h)$. $u_n$ therefore verifies the following stochastic partial differential equation

$$
\frac{du_n}{dt} + \left( \frac{\partial^3 u_n}{\partial x^3} + \frac{1}{2} \rho_n \ast \theta \frac{\partial u_n^2}{\partial x} \right) dt = \rho_n \ast \theta \Phi \, dW + \left( 3 \rho_n \ast \frac{\partial^2 \theta u}{\partial x^2} - 3 \rho_n \ast \frac{\partial \theta u}{\partial x} \right) dt + \rho_n \ast \theta'' u.
$$

Then, applying the Ito formula to $u_n$, it is not difficult to see that, thanks to (3.45), (3.46) and (3.51), we can take the limit in all the terms provided suitable integrations by parts have been done. The terms which contain derivatives of $\theta$ are null at the limit.

Let us take now the limit on the functions $h$. Let $\{h_n\}_{n \geq 0}$ be a sequence of functions $C^3_0(\mathbb{R}, \mathbb{R}^+)$ such that

- $h_n(x) \nearrow h(x)$, $\forall x \in \mathbb{R}$;
- $h_n$ verifies (2.1) uniformly with respect to $n$;
- $|h_n'| \leq c_4 |h'|$.

We take the limit in (3.50) with $h$ replaced by $h_n$ either by the Beppo-Levi theorem, or by the dominated convergence theorem thanks to (3.47) and (3.48). The stochastic integral converges in $L^2(\Omega)$ thanks to (3.47) and (3.49), thus a subsequence converges a.s.

The proof of the continuity of the trajectories will be achieved when the weak-continuity of the solution $u$ in $L^\infty([0, T], Y)$ is proved. First, it is not difficult to see that

$$
u \in C([0, T], H^{-2}(\mathbb{R})).$$

Moreover, since $u \in L^\infty([0, T], Y)$ and since the embedding of the weighted space $Y$ into $H^{-2}(\mathbb{R})$ is dense and continuous, the weak-continuity is deduced from the Strauss Lemma (see e.g., Lemma 1.4, p. 263 in [19]).

Finally, (3.50) with $h = h^{3/4}$, which implies the continuity of the norm, together with the property of weak-continuity of $u$ yields the result. This ends the proof of the Theorem 3.1.

4. ABSTRACT METHOD

The purpose of this section is to obtain a result of existence and uniqueness of (3.1)–(3.2) under assumptions (1.6) and (1.7) only. The method
used here is similar to a compactness method for deterministic PDEs, this being possible thanks to the local smoothing effects of the linear part of the equation (see [11, 4]). First and foremost, let us point out the essential difficulty of the use of such methods in the framework of stochastic partial differential equations. Contrary to the previous section, we can no longer use a pathwise construction of the solution and we will first work in $L^2(\Omega)$. In order to give some idea of the problems encountered, let us consider $V$ and $H$ two reflexive and separable Hilbert spaces. Let $\mathcal{H} = L^2(\Omega, V)$ and $\mathcal{H} = L^2(\Omega, H)$. Then, if $V \subset H$ with compact embedding, this is no more the case for $\mathcal{H}$ and $\mathcal{H}$. But, if $\mathcal{H}$ designates a set of Radon measure in $V$-weak which verifies the Prokhorov criterion of tightness, the same should apply in $H$-strong. Such a method leads via the Skorohod theorem to martingale solutions, i.e., solutions in another probability space and with another Wiener process. The uniqueness result that appears in the preliminaries shows us that for a given Wiener process and a given probability space, there is at most one solution in $L^\infty([0, T], Y)$ for initial data in $Y$. The conclusion follows then from Theorem 2.6.

The main result of this section is contained in the following theorem.

**Theorem 4.1.** Let $T > 0$ and $u_0$ be such that (3.6) holds. Under the assumption of localization of the noise (3.3), there exists a unique solution $u$ of (3.1)-(3.2). Moreover the following estimates hold a.s.:

$$u \in L^\infty([0, T], Y), \quad (4.1)$$

$$u \in C([0, T], H^s_{loc}(\mathbb{R})), \quad \forall s > 2, \quad (4.2)$$

$$\frac{\partial u}{\partial x} \in L^2([0, T], H^s_{loc}(\mathbb{R})), \quad \forall s < 0. \quad (4.3)$$

Let $h = h^{1/4}$. Let $\{u_{0,n}\}_{n \geq 0}$ be a sequence of elements of $H^1(\mathbb{R})$ such that

$$h^{1/2}u_{0,n} \to h^{1/2}u_0 \quad \text{in} \quad L^2(\mathbb{R}). \quad (4.4)$$

Let $\{\Phi_n\}_{n \geq 0}$ be a sequence of elements of $L^0_{-4}$ such that

$$h^{1/2}\Phi_n \to h^{1/2}\Phi \quad \text{in} \quad L^0_2. \quad (4.5)$$

A global solution $u_n$ at (3.1)-(3.2) can be associated to the data $(u_{0,n}, \Phi_n)$ (see the beginning of Subsection 3.3). Moreover, one has a.s.

$$u_n \in L^\infty([0, T], H^3(\mathbb{R})).$$
4.1. A priori Estimates

**Proposition 4.2.** For any $T > 0$,
\[
\{u_n\}_{n \geq 0} \text{ is bounded in } L^2(\Omega, L^\infty([0, T], L^2(\mathbb{R}))).
\]
\[
\left\{ \frac{\partial u_n}{\partial x} \right\}_{n \geq 0} \text{ is bounded in } L^2(\Omega, L^2([0, T], L^2_{loc}(\mathbb{R}))).
\]

The proof of Proposition 4.6 follows from the two following lemmas.

**Lemma 4.3.** For any $T > 0$, there exist positive constants $C_1(T)$ and $C_2(T)$ such that
\[
E \sup_{t \in [0, T]} |u_n(t)|^2 \leq C_1(T) (|u_0|^2 + |\phi|_{L^2_0})
\]
(4.6)
\[
E \sup_{t \in [0, T]} |u_n(t)|^4 \leq C_2(T) (|u_0|^4 + |\phi|_{L^2_0}^4).
\]
(4.7)

**Lemma 4.4.** For any $T > 0$, there exist positive constants $C_3(T)$ and $C_4(T)$ such that
\[
E \sup_{t \in [0, T]} |\frac{\partial u_n}{\partial t}(t)|^2 \leq C_3(T) (1 + |u_0|^4 + |u_0|^2 + |\phi|_{L^2_0}^4 + |\phi|_{L^2_{loc}(\mathbb{R}), Y})
\]
(4.8)
\[
E \left[ \int_0^T |\frac{\partial u_n}{\partial x}|^2 \, dt \right] \leq C_4(T) (1 + |u_0|^4 + |u_0|^2 + |\phi|_{L^2_0}^4 + |\phi|_{L^2_{loc}(\mathbb{R}), Y}).
\]
(4.9)

**Proof of Lemma 4.3.** First, let us note that the estimate (4.6) can be easily computed using the same method than those used in the proof of (3.10) in Proposition 3.2.

Now let us prove (4.7). Since $u_n$ is sufficiently smooth, Itô’s formula can be applied to $u_n$ verifying (3.1) with the functional $\frac{1}{\sqrt{2}} |\phi|^4$. Standard computations lead to the following of the right member.

\[
E \sup_{t \in [0, T]} |u_n(t)|^4 \leq 4E \sup_{t \in [0, T]} \left| \int_0^t |u_n(\tau)|^2 (u_n(\tau), \Phi_n \, dW(\tau)) \right|
\]
\[
+ 6E \int_0^T |u_n(\tau)|^2 |\phi_n|_{L^2_0}^2 \, d\tau.
\]
(4.10)

Next, we apply a classical martingale inequality (see [5], Theorem 3.14) to the second term of the right member.
Using (4.11) in (4.10), we get finally

$$
\mathbb{E} \sup_{t \in [0, T]} |u_n(t)|^4 \leq 2 |u_{0,n}|^4 + C_1(T) |\Phi_n|^4_{L^2}. \quad \text{(4.11)}
$$

Then (4.4) and (4.5) give the result. \( \square \)

Proof of Lemma 4.4. As previously, the Ito formula is applied to \( u_n \) verifying (3.1) with the functional \( \Phi \equiv \frac{1}{2} |h^{1/2} \phi|^2 \). We obtain the following inequality, for any \( t \in [0, T] \), a.s.:

$$
\frac{1}{2} |h^{1/2} u_n(t)|^2 + \int_0^t \left( h u_n \cdot \frac{\partial^3 u_n}{\partial x^3} + u_n \cdot \frac{\partial u_n}{\partial x} \right) dt
$$

$$
= \frac{1}{2} |h^{1/2} u_{0,n}|^2 + \int_0^t \left( h u_n \cdot \Phi_n dW(\tau) \right) + \frac{1}{2} |h^{1/2} \Phi_n|^2_{L^2}. \quad \text{(4.12)}
$$

Let us note that

$$
\left( h u_n, \frac{\partial^3 u_n}{\partial x^3} \right) = \frac{3}{2} \left| h^{1/2} \frac{\partial u_n}{\partial x} \right|^2 - \frac{1}{2} (u_n, h^{2} \cdot u_n),
$$

and

$$
\left( h u_n, \frac{\partial u_n}{\partial x} \right) = - \frac{1}{2} \int_R h |u_n|^2 dx.
$$

Thanks to (2.1), one has immediately for any \( t \in [0, T] \)

$$
\left| \int_0^t (u_n, h^{1/2} u_n) dt \right| \leq c_1 \int_0^t |h^{1/2} u_n|^2 dt. \quad \text{(4.13)}
$$

Let us estimate next the cubic term. The computations that follow are similar to those of the previous section. A Hölder inequality with respect
to the space variable followed by a standard inequality of interpolation in the Hilbert spaces leads to, for any $t \in [0, T]$

$$
\left| \frac{1}{3} \int_0^t \int_\mathbb{R} dx \, h' u_n^2 \right| \leq \frac{1}{3} \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left| h^{1/2} u_n |_{L^3(\mathbb{R})} \right| |u_n| \\
\leq \frac{c_0^2}{3} \int_0^t \int_\mathbb{R} dx \left| h^{1/2} u_n \right|^{3/2} \left| h^{1/2} u_n |_{H^1(\mathbb{R})} \right| |u_n|, \quad (4.14)
$$

where $c_0$ is a positive constant such that

$$
|h^{1/2} u_n |_{L^3(\mathbb{R})} \leq c_0 \left| h^{1/2} u_n \right|^{3/4} \left| h^{1/2} u_n |_{H^1(\mathbb{R})} \right| .
$$

And thanks to the assumptions (2.1), we get for any $t \in [0, T]$ and a.s.

$$
|h^{1/2} u_n |_{H^1(\mathbb{R})} \leq \frac{c_2}{2} |h^{1/2} u_n|^2 + |h^{1/2} u_n|^2 + \left| h^{1/2} \frac{\partial u_n}{\partial x} \right|^2 . \quad (4.15)
$$

By substituting (4.14) with (4.15) and using again the assumptions (2.1) but in the term in $\partial u_n / \partial x$, we obtain the following inequality for any $t \in [0, T]$

$$
\left| \frac{1}{3} \int_0^t \int_\mathbb{R} dx \, h' u_n^2 \right| \leq C \left( \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left| h^{1/2} u_n \right|^{3/2} \left| h^{1/2} u_n |_{H^1(\mathbb{R})} \right| \right) \\
+ \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left| u_n (\tau) \right|^3 + \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left| u_n \right|^{5/2} \left| h^{1/2} \frac{\partial u_n}{\partial x} \right|^{1/2} .
$$

Using (4.15) and (4.16) in (4.13), we obtain for any $t \in [0, T]$

$$
F_2(u_n(t)) + \frac{3}{2} \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left| h^{1/2} \frac{\partial u_n}{\partial x} \right|^2 \\
\leq F_2(u_0, n) + \frac{c_3}{2} \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left| h^{1/2} u_n \right| \left( \int_0^t \int_\mathbb{R} dx \left| h^{1/2} u_n \right|^{3/2} \left| h^{1/2} \frac{\partial u_n}{\partial x} \right|^{1/2} \right) \\
+ C \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left| u_n \right|^3 + \int_0^t \frac{1}{3} \int_\mathbb{R} dx \left( h u_n (\tau) \left( \Phi_n, dW(\tau) \right) \right) + \frac{t}{2} \left| h^{1/2} \Phi_n \right|^{2} .
$$
Several Hölder’s inequalities in time followed by Young’s inequalities lead to
\[
\frac{1}{2} F_2(u_n(t)) + \int_0^t dt \left| \frac{1}{2} \frac{\partial u_n}{\partial x} \right|^2 \\
\leq F_2(u_{0,n}) + C_3 \int_0^t dt F_2(u_n(\tau)) + C(T) \int_0^t dt (1 + |u_n|^4) \\
+ \left| \int_0^t (hu_n(\tau), \Phi_n dW(\tau)) \right| + \frac{1}{2} \left| \frac{1}{2} \frac{\partial u_n}{\partial x} \right|^2.
\]

The Gronwall inequality then leads to the following one
\[
\frac{1}{2} \sup_{t \in [0,T]} F_2(u_n(t)) + \int_0^T \left| \frac{1}{2} \frac{\partial u_n}{\partial x} (\tau) \right|^2 \\
\leq C(T) \left( 1 + \sup_{t \in [0,T]} |u_n(t)|^4 + \left| \frac{1}{2} \frac{\partial \Phi_n}{\partial x} \right|^2 \right) \\
+ F_2(u_{0,n}) + \sup_{t \in [0,T]} \left| \int_0^t (hu_n(\tau), \Phi_n dW(\tau)) \right|.
\]

We get (4.8) and (4.9) by the application of a martingale inequality as in the proof of estimate (3.11) and by Lemma 4.3. Since $h'$ is increasing, $h'$ is lower-bounded by a positive constant on every compact set of $\mathbb{R}$.

We shall use now Proposition 4.6 together with the equation (3.1) in order to get another set of estimates given by the following proposition.

**Proposition 4.5.** For any $T > 0$,

\[ \left\{ u_n \right\}_{n \geq 0} \text{ is bounded in } L^2(\Omega, W^{\gamma,2}([0,T], H^{-2}_w(\mathbb{R}))) \]

\[ \left\{ u_n \right\}_{n \geq 0} \text{ is bounded in } L^2(\Omega, C^\delta([0,T], H^{-2}_w(\mathbb{R}))) \]

for any $(\gamma, \delta) \in [0, \frac{1}{2}]^2$.

First of all, let us mention the following result

**Lemma 4.6.** For any $T > 0$, $0 < \gamma < \frac{1}{2}$ and for any $(n, p) \in \mathbb{N}$, there exists a positive constant $C_5(T, p, \gamma)$ such that

\[
\mathbb{E} \left| \Phi_n W \right|^{2p}_{W^{\gamma,\gamma}([0,T], L^2(\mathbb{R}))} \leq C_5(T, p, \gamma) \left| \Phi_n \right|^{2p}_{L^2}.
\]
Proof of Lemma 4.6. The proof directly follows the characterization of the Sobolev space $W^{2p}([0, T], L^2(\mathbb{R}))$ mentioned in Section 2. Indeed, one has

$$
E |\Phi_n W|^2_{W^{2p}([0, T], L^2(\mathbb{R}))} = E \int_0^T \int_0^T \frac{|\Phi_n W(t) - \Phi_n W(s)|^{2p}}{|t-s|^{1+\gamma p}} \, dt \, ds.
$$

Using the gaussian property of $\{W(t)\}_{t\in[0,T]}$, we obtain, after the application of the Fubini theorem in $(t, s)$,

$$
E |\Phi_n W|^2_{W^{2p}([0, T], L^2(\mathbb{R}))} \leq C_p |\Phi_n|^2_{L^2_0} \int_0^T \int_0^T |t-s|^{-(1+(2\gamma-1)p)} \, dt \, ds.
$$

The right member of the previous inequality being finite since $\gamma < \frac{1}{2}$, then (4.18) is obtained thanks to (4.5).

Proof of Proposition 4.5. We write that $u_n$ is solution of (3.1) a.s., so that

$$
u_n(t) = u_{0,n} + \int_0^t \left( \frac{\partial^2 u_n}{\partial x^2} + u_n \frac{\partial u_n}{\partial x} \right) \, d\tau + \Phi_n W(t). \tag{4.19}
$$

Thanks to a Sobolev inequality and a standard interpolation inequality, we find that for any $k \in \mathbb{N}$, there exists a positive constant $C(k)$ such that

$$
\left| \frac{\partial u_n}{\partial x} \right|_{H^{-\gamma}(]-k,k[)} \leq C(k) |u_n|_{L^2(]-k,k[)}^{1/2} |u_n|_{H^2(]-k,k[)}^{1/2},
$$

that is to say, for any $T > 0$,

$$
E \left| u_n \frac{\partial u_n}{\partial x} \right|_{L^2([0,T],H^{-\gamma}(]-k,k[))}^2 \leq C(k) \left( E \right)^{\gamma/4} \left( \int_0^T |u_n|_{L^2(]-k,k[)}^2 \, d\tau \right)^{1/4} \times \left( E \int_0^T |u_n|_{H^2(]-k,k[)}^2 \, d\tau \right)^{1/4}.
$$

Then Proposition 4.2 gives for any $k \in \mathbb{N}$

$$
E \left| u_n \frac{\partial u_n}{\partial x} \right|_{L^2([0,T],H^{-\gamma}(]-k,k[))}^2 \leq C(k). \tag{4.20}
$$
We also have thanks to (4.9)

\[ \mathbb{E} \left| \frac{\partial^3 u_n}{\partial x^3} \right|^2_{L^2([0, T], H^{-1}([0, T]))} \leq C^\ast(k). \tag{4.21} \]

Consequently, (4.18)-(4.21) (with \( p = 1 \)) and (4.4) lead to

\[ \mathbb{E} |u_n|^2_{W^{\gamma, 1}([0, T], H^{-1}([0, T]))} \leq C_6(k), \tag{4.22} \]

for any \( k \in \mathbb{N} \) and for some \( 0 < \gamma < \frac{1}{2} \). Then choosing \( \gamma > 0 \) and \( p \in \mathbb{N} \) such that \( 0 < \delta < \gamma - 1/2p \) for a certain \( \delta \in ]0, \frac{1}{2}[ \), (4.18)-(4.21) and (4.4) give, owing to the Sobolev embedding of \( W^{\gamma', p}([0, T], L^2(\mathbb{R})) \) and \( W^{1, 2}([0, T], H_{\text{loc}}^{-2}(\mathbb{R})) \) in \( C^\delta([0, T], H_{\text{loc}}^{-2}(\mathbb{R})) \)

\[ \mathbb{E} |u_n|^2_{W^{\delta, 1}([0, T], H^{-1}([0, T]))} \leq C_7(k), \tag{4.23} \]

for any \( k \in \mathbb{N} \). Finally, (4.22) and (4.23) end the proof of Proposition 4.5.

4.2. Proof of Theorem 4.1

We first construct martingale solution and will conclude thanks to the pathwise uniqueness and Theorem 2.6. Let us denote by

\[ X_{\gamma, \delta}(T) = L^2([0, T], H^{1}_{\text{loc}}(\mathbb{R})) \cap W^{\gamma, 2}([0, T], H^{-2}_{\text{loc}}(\mathbb{R})) \]

\[ \cap C^\delta([0, T], H^{-2}_{\text{loc}}(\mathbb{R})) \]

for some \( \gamma \) and \( \delta \) in \( ]0, \frac{1}{2}[ \) and

\[ Y_{s, \gamma}(T) = L^2([0, T], H^{s}_{\text{loc}}(\mathbb{R})) \cap C([0, T], H^{-s}_{\text{loc}}(\mathbb{R})) \]

for some \( s < 1 \) and \( s' > 2 \).

**Proposition 4.7.** For any \( T > 0 \), the family of laws \( \{ \mathcal{L}(u_n) \}_{n \geq 0} \) on \( X_{\gamma, \delta}(T) \) is tight in \( Y_{s, \gamma}(T) \).

**Remark 4.8.** The space \( Y_{s, \gamma}(T) \) and \( X_{\gamma, \delta}(T) \) are Fréchet spaces, thus metrizable, complete and separable. Therefore the measures \( \mathcal{L}(u_n) \) are inner regular.

**Proof of Proposition 4.7.** The family of measure \( \{ \mathcal{L}(u_n) \}_{n \geq 0} \) is a family of probability measures. Therefore by the Prokhorov criterion, it is sufficient to prove that for any \( \varepsilon > 0 \), there exists a compact \( K_\varepsilon \) of \( Y_{s, \gamma}(T) \) such that

\[ \mathcal{L}(u_n)(K_\varepsilon) \leq \varepsilon \tag{4.24} \]

for any \( n \in \mathbb{N} \).
Thus, for any $\epsilon > 0$, let us set $B_\epsilon$ the following subset of $X_{\gamma}(T)$,

$$B_\epsilon = \bigcap_{k \geq 1} \left\{ |u|_{H^k([0, T], H^k([0, T]))}^2 + |u|_{H^k([0, T], H^k([0, T]))}^2 \leq \frac{2^k}{\epsilon} \left( C_\delta(k) + C_\gamma(k) + C_\delta(k) \right) \right\}$$

where $C_\delta(k) > 0$ is provided by Proposition 4.2, i.e. for any $k \in \mathbb{N}$,

$$E|u_n|_{L^2([0, T], H^k([0, T]))}^2 \leq C_\delta(k). \quad (4.25)$$

We set now $K_\epsilon$ the closure of $B_\epsilon$ in $Y_{s,s'}(T)$. Then (4.22), (4.23) and (4.25) together with the Bienaymé-Tchebychev inequality and Lemma 2.2 yield the result.

From Proposition 4.7, we infer that there exists a measure $\mu$ on $Y_{s,s'}(T)$ and a subsequence that we denote again $\{ u_{n'} \}_{n \geq 0}$ such that

$$\mathcal{L}(u_n) \rightarrow \mu.$$

The Skohorod theorem (see [9, Theorem 2.7, p. 9]) can be applied to the previous weak-convergence and thus there exists a sequence of random functions $\{ \tilde{u}_n \}_{n \geq 0}$ with value in $Y_{s,s'}(T)$, a stochastic process $\tilde{u}$ with value in the same space, both defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\tilde{u}_n \rightarrow \tilde{u} \quad \text{in} \quad Y_{s,s'}(T), \quad \tilde{\mathbb{P}} \text{ a.e.,} \quad (4.26)$$

$$\mathcal{L}(\tilde{u}_n) = \mathcal{L}(u_n), \quad \forall k \in \mathbb{N}. \quad (4.27)$$

Moreover by (4.8), $\tilde{u} \in L^2(\Omega; L^\infty([0, T], Y))$, we deduce easily that $\tilde{u} \in C_{w'}([0, T], Y)$.

We now have to prove that $\{ \tilde{u}_n \}_{n \geq 0}$ verifies an equation similar to (3.1) but in another probability space, and then to take the limit.

We shall need the following result whose proof is left to the reader.

**Lemma 4.9.** Let $T > 0$, $s \in \frac{1}{2} \mathbb{N}$ and $s' > 2$. Let $w \in H^3(\mathbb{R})$ compactly supported. Then the application

$$\begin{cases}
F_w : Y_{s,s'}(T) \rightarrow \mathcal{C}([0, T]) \\
F_w : v \mapsto \left( v - \int_0^T \left( \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial x} \right) d\tau, w \right)
\end{cases}$$

is continuous.
Next, we set
\[
M^n(t) = u_n(t) - u_{0,n} - \int_0^t \left[ \frac{\partial^3 u_n}{\partial x^3} + \frac{\partial u_n}{\partial x} \right] d\tau, \tag{4.28}
\]
\[
\tilde{M}^n(t) = \tilde{u}_n(t) - u_{0,n} - \int_0^t \left[ \frac{\partial^3 \tilde{u}_n}{\partial x^3} + \frac{\partial \tilde{u}_n}{\partial x} \right] d\tau, \tag{4.29}
\]
\[
\tilde{N}(t) = \tilde{u}(t) - u_0 - \int_0^t \left[ \frac{\partial^3 \tilde{u}}{\partial x^3} + \frac{\partial \tilde{u}}{\partial x} \right] d\tau. \tag{4.30}
\]

By definition, for \( \{e_i\}_{i=0}^\infty \) a Hilbertian basis of \( L^2(\mathbb{R}) \) that we assume to be sufficiently smooth, we have
\[
(M^n(t), e_i) = |\Phi^* e_i| \beta_i(t),
\]
where \( \{\beta_i(t)\}_{t \in [0, T]} \) denotes a family of real brownian motions mutually independent on \( (\Omega, \mathcal{F}, P) \). We then set for any \((i, n) \in \mathbb{N}^2\) and for any \( t \in [0, T] \)
\[
\tilde{\beta}_i^*(t) = \frac{1}{|\Phi^* e_i|} (\tilde{M}^n(t), e_i). \tag{4.31}
\]

Let \( \phi \) be a continuous bounded function from \( \mathbb{R}([0, T]) \) to \( \mathbb{R} \). By definition,
\[
E\phi((M^n(\cdot), e_i)) = E\phi(\beta_i) |\Phi^* e_i|.
\]
But one also has
\[
E\phi((M^n(\cdot), e_i)) = E(\phi + F_n)(u^n).
\]

Then, we deduce from the result of Lemma 4.9 and from (4.27) that for any \((i, n) \in \mathbb{N}^2\)
\[
E\phi(\beta_i) = \mathbb{E}\phi(\tilde{\beta}_i^*),
\]
and so the family of random functions defined by (4.31) is those of real brownian motions mutually independent.

We set then
\[
\tilde{W}^n(t) = \sum_{j \geq 0} \tilde{\beta}_j^*(t) e_j, \tag{4.32}
\]
and we have with (4.31) and (4.32),
\[ u^n(t) = u_{0,n} + \int_0^t \left[ \frac{\partial^n u}{\partial x^n} + \hat{u}_n \frac{\partial u}{\partial x} \right] dt + \Phi_n \tilde{W}(t), \quad \mathbb{P} \text{ a.e.} \quad (4.33) \]

In order to take the limit in (4.33), let us set
\[ \tilde{\beta}_i(t) = \frac{1}{|\theta^* e_i|} (\tilde{M}(t), e_i). \quad (4.34) \]

Thanks to the assumption of smoothness on \( e_i \), Lemma 4.9 and (4.4)–(4.5), we get for any \( i \in \mathbb{N} \)
\[ \tilde{\beta}_n^i \rightarrow \tilde{\beta}_i \quad \text{in} \quad \mathcal{C}([0,T]), \quad \mathbb{P} \text{ a.e.} \]

Since for a gaussian random variable, convergence almost surely implies convergence in \( L^p(\mathcal{Q}) \) for any \( p \geq 1 \), we conclude that \( \{ \tilde{\beta}_i(t) \}_{i \in [0,T]} \) is a family of real brownian motions mutually independent. We set finally
\[ \tilde{W}(t) = \sum_{j \neq 0} \tilde{\beta}_j(t) e_j, \]

and we have
\[ \tilde{u}(t) = u_0 + \int_0^t \left[ \frac{\partial^n \hat{u}}{\partial x^n} + \hat{u} \frac{\partial \hat{u}}{\partial x} \right] dt + \Phi \tilde{W}(t), \quad \mathbb{P} \text{ a.e.} \quad (4.35) \]

The pair \( (\tilde{u}, \tilde{W}) \) is a martingale solution of (3.1)–(3.2) according to the sense of Definition 2.4.

Thus, taking into account Remark 3.6 and the fact that \( \tilde{u} \in \mathcal{C}_w([0,T], Y) \) which is a Lusin space, we can use Theorem 2.6 and obtain the strong existence and uniqueness of solution of (3.1)–(3.2) in the spaces mentioned in Theorem 4.1. This ends the proof of Theorem 4.1.

REFERENCES