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On the collapse of certain Eilenberg–Moore spectral sequences

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Abstract

Let Z be a path connected H-space with $H^*(Z;\mathbb{Z}_p)$ concentrated in even degrees. Then the Eilenberg–Moore spectral sequences associated to the path loop fibrations

$$\Omega Z \to PZ \to Z,$$
$$\Omega^2 Z \to P\Omega Z \to \Omega Z$$

collapse at the E_2 term.

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1. Introduction

In this note we prove the following theorem:

Theorem 1. *Let p be a prime. Let Z be a path connected H-space with $H^*(Z;\mathbb{Z}_p)$ concentrated in even degrees. Consider the path loop fibrations*

$$\Omega Z \to PZ \to Z, \tag{1}$$
$$\Omega^2 Z \to P\Omega Z \to \Omega Z. \tag{2}$$

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The Eilenberg–Moore spectral sequences associated to the path loop fibrations (1) *and* (2) *collapse at the E_2 term. Hence*

$$\mathrm{Tor}_{H^*(Z;\mathbb{Z}_p)}(\mathbb{Z}_p,\mathbb{Z}_p)\cong \mathcal{G}_r H^*(\Omega Z;\mathbb{Z}_p)$$

and

$$\mathrm{Tor}_{H^*(\Omega Z;\mathbb{Z}_p)}(\mathbb{Z}_p,\mathbb{Z}_p)\cong \mathcal{G}_r H^*\big(\Omega^2 Z;\mathbb{Z}_p\big).$$

We note by [7, Proposition 2.8], *there are coalgebra isomorphisms*

$$\mathrm{Tor}_{H^*(Z;\mathbb{Z}_p)}\cong H^*(\Omega Z;\mathbb{Z}_p)$$

and

$$\mathrm{Tor}_{H^*(\Omega Z;\mathbb{Z}_p)}\cong H^*\big(\Omega^2 Z;\mathbb{Z}_p\big).$$

If X is a simply connected finite H-space, then ΩX is a path connected H-space with $H^*(\Omega X;\mathbb{Z}_p)$ concentrated in even degrees [8,11,12]. *Hence letting $Z=\Omega X$, Theorem* 1 *resolves the following conjecture of Choi and Yoon* [4].

Corollary 1. *The Eilenberg–Moore spectral sequences for the path loop fibrations converging to the mod p homology of the double and triple loop spaces of any simply connected finite H-space collapse at the E_2 term.*

The homology of the triple loop spaces of finite H-spaces has been studied by many authors [1,2,4,5]. For compact simply connected Lie groups G, there is an inclusion of the space of based gauge equivalence classes M_k to a space that is homotopy equivalent to $\Omega^3_k G$ [1]. As k increases more elements of the homology of $\Omega^3_k G$ are contained in the homology of M_k. Collapse of the Eilenberg–Moore spectral sequences facilitate the computation of $H_*(\Omega^3_k G;\mathbb{Z}_p)$.

There are also many other path connected H-spaces with mod p cohomology concentrated in even degrees. We note if Y is a simply connected space with $H^*(Y;\mathbb{Z}_p)$ an exterior algebra on odd degree generators, then $\mathrm{Tor}_{H^*(Y;\mathbb{Z}_p)}(\mathbb{Z}_p,\mathbb{Z}_p)$ is even dimensional. Therefore since differentials alter the total degree by 1, the Eilenberg–Moore spectral sequence associated to the path loop fibration

$$\Omega Y\to PY\to Y$$

collapses. Hence $H^*(\Omega Y;\mathbb{Z}_p)$ is concentrated in even degrees. For example, the homology of $\Omega^2(SU(n+1)/SU(m+1))$ is studied in [16]. One notes $H^*(\Omega SU(n+1)/SU(m+1);\mathbb{Z}_p)$ is concentrated in even degrees because $H^*(SU(n+1)/SU(m+1);\mathbb{Z}_p)$ is exterior. By Theorem 1, the Eilenberg–Moore spectral sequence associated to the path loop fibration

$$\begin{aligned}\Omega^2\big(SU(n+1)/SU(m+1)\big)&\to P\Omega SU(n+1)/SU(m+1)\\&\to \Omega SU(n+1)/SU(m+1)\end{aligned}$$

collapses.

In this paper we make the following assumptions. All spaces are connected and endowed with a basepoint. All homologies and cohomologies will be of finite type and will have coefficients in $\mathbb{Z}_p$, the integers mod p for some prime p. Given a connected Hopf algebra A, $P(A)$ and $Q(A)$ will denote the module of primitives and the module of indecomposables, respectively. The Eilenberg–Moore spectral sequence will be abbreviated EMSS.

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2. Some known results

For any space Z, the evaluation map $\varepsilon : \Sigma\Omega Z \to Z$ defined by $\varepsilon(s, \lambda) = \lambda(s)$ induces the suspension map

$$\sigma^* : H^*(Z) \to H^*(\Sigma\Omega Z) \cong H^{*-1}(\Omega Z).$$

σ^* annihilates decomposables and its image lies in the primitives. Hence we often write

$$\sigma^* : QH^*(Z) \to PH^{*-1}(\Omega Z). \tag{3}$$

Given the path-loop fibration

$$\Omega Z \to PZ \to Z$$

there is an associated second quadrant Eilenberg–Moore spectral sequence (EMSS) with

$$E_2 = \mathrm{Tor}_{H^*(Z;\mathbb{Z}_p)}(\mathbb{Z}_p, \mathbb{Z}_p) \text{ and } E_\infty = \mathcal{G}_r H^*(\Omega Z; \mathbb{Z}_p). \tag{4}$$

The suspension map σ^* can be described via the EMSS [5]. We have

$$QH^*(Z; \mathbb{Z}_p) \cong \mathrm{Tor}^{-1,*}_{H^*(Z;\mathbb{Z}_p)}(\mathbb{Z}_p, \mathbb{Z}_p) = E_2^{-1,*} \to E_\infty^{-1,*} \subseteq H^*(\Omega Z; \mathbb{Z}_p). \tag{5}$$

If Z is an H-space the EMSS is a spectral sequence of differential Hopf algebras.

Lemma 1. *Let Z be a path connected H-space. The EMSS collapses if and only if*

$$\sigma^* : QH^*(Z; \mathbb{Z}_p) \to PH^{*-1}(\Omega Z; \mathbb{Z}_p)$$

is a monomorphism.

Proof. By [6] the differentials in the EMSS send algebra generators to $\mathrm{Tor}^{-1,*}_{H^*(Z;\mathbb{Z}_p)}(\mathbb{Z}_p; \mathbb{Z}_p)$. By (5) the EMSS collapses if and only if $\sigma^* : QH^*(Z; \mathbb{Z}_p) \to PH^*(\Omega Z; \mathbb{Z}_p)$ is a monomorphism. □

Theorem 2. *Let Z be a path connected H-space. Then*

(a) $\sigma^* : QH^k(Z; \mathbb{Z}_p) \to PH^{k-1}(\Omega Z; \mathbb{Z}_p)$ *is monic if* $k \not\equiv 2 \bmod 2p$. σ^* *is epic if* $k - 1 \not\equiv -2 \bmod 2p$.

(b) *If Z is simply connected, then ker $\sigma^* \subseteqq \sum_{i=1}^{\infty} \operatorname{im}\beta_i$ where β_i is the ith Bockstein. If $H^*(Z;\mathbb{Z}_p)$ is concentrated in even degrees, then $\operatorname{Tor}_{H^*(Z;\mathbb{Z}_p)}(\mathbb{Z}_p,\mathbb{Z}_p) \cong \mathcal{G}_r H^*(\Omega Z;\mathbb{Z}_p)$ and the EMSS converging to $H^*(\Omega Z;\mathbb{Z}_p)$ collapses.*

(c) *Let A be a bicommutative Hopf algebra. There is an exact sequence*

$$0 \to P(\xi A) \to P(A) \to Q(A) \to Q(\lambda A) \to 0.$$

Hence if $k \not\equiv 0 \bmod p$, $P^k(A) \cong Q^k(A)$.

Proof. (a) is proved in [3]. (b) is proved in [9, Theorem B] and the EMSS collapses by Lemma 1. (c) is shown in [13]. □

Corollary 2. *Let X be a finite simply connected H-space. The EMSS associated to the path loop fibration*

$$\Omega^2 X \to P\Omega X \to \Omega X$$

collapses.

Proof. By [11,12], $H^*(\Omega X:\mathbb{Z}_p)$ is concentrated in even degrees. By Theorem 2(b) the result follows. □

3. The path loop fibration $\Omega^2 Z \to P\Omega Z \to \Omega Z$

Let Z be a path connected H-space with $H^*(Z;\mathbb{Z}_p)$ concentrated in even degrees. The Bockstein sequence implies $H^*(Z;\mathbb{Z}_{(p)})$ is torsion free, because β_i increases degree by 1. Hence

$$H^1(\Omega Z;\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)} \oplus \cdots \oplus \mathbb{Z}_{(p)}.$$

There exist maps

$$S^1 \times \cdots \times S^1 \xrightarrow{f} \Omega Z \xrightarrow{g} S^1 \times \cdots \times S^1$$

such that gf is a mod p homotopy equivalence. Hence if $\widetilde{Z}$ is the 2-connective cover of Z, we have

$$\begin{aligned} &\Omega Z \simeq_p S^1 \times \cdots \times S^1 \times \Omega\widetilde{Z} \quad \text{and} \\ &QH^{2np+2}(\Omega Z;\mathbb{Z}_p) \cong QH^{2np+2}\big(\Omega\widetilde{Z};\mathbb{Z}_p\big). \end{aligned} \tag{6}$$

Now consider the EMSS associated to the path loop fibration

$$\Omega^2 Z \to P\Omega Z \to \Omega Z.$$

By (6) and Theorem 2(a), $\sigma^*: QH^1(\Omega Z;\mathbb{Z}_p) \to H^0(\Omega^2 Z;\mathbb{Z}_p)$ is monic. Hence if there is a nontrivial differential in the EMSS for ΩZ, there will also be a nontrivial differential in the EMSS for $\Omega\widetilde{Z}$. By Lemma 1 and Theorem 2(a) there will be a nontrivial element of $QH^{2np+2}(\Omega\widetilde{Z};\mathbb{Z}_p) \cap \ker\sigma^*$.

Theorem 3. *Let p be an odd prime. Let Z be a path connected H-space with $H^*(Z;\mathbb{Z}_p)$ concentrated in even degrees. Then the EMSS associated to the path loop fibration $\Omega^2 Z \to P\Omega Z \to \Omega Z$ collapses.*

Proof. By Lemma 1 and Theorem 2(a), it suffices to prove

$$\sigma^* : QH^{2np+2}(\Omega\widetilde{Z};\mathbb{Z}_p) \to PH^{2np+1}(\Omega^2\widetilde{Z};\mathbb{Z}_p)$$

is monic. We have

$$\begin{aligned} QH^{2np+2}\big(\Omega\widetilde{Z};\mathbb{Z}_p\big) &\cong QH^{2np+2}(\Omega Z;\mathbb{Z}_p) && \text{by (6)} \\ &\cong PH^{2np+2}(\Omega Z;\mathbb{Z}_p) && \text{by Theorem 2(c)} \\ &\cong \sigma^* QH^{2np+3}(Z;\mathbb{Z}_p) && \text{by Theorem 2(a)} \\ &= 0 \text{ because } H^*(Z;\mathbb{Z}_p) && \text{is even dimensional.} \end{aligned}$$

Hence σ^* is monic. □

The argument for the prime two is slightly different.

Proposition 1. *Let $p = 2$. Let $\bar{x} \in QH^{4\ell+2}(\Omega\widetilde{Z};\mathbb{Z}_2) \cap \ker\sigma^*$.*
We may choose a representative $x \in H^{4\ell+2}(\Omega\widetilde{Z};\mathbb{Z}_2)$ for $\bar{x}$ that is primitive.

Proof. By Theorem 2(c) if there is no primitive representative, then $\bar{x}$ is dual to the square of an odd generator. Hence $\overline{\Delta}x$ has the form

$$\overline{\Delta}x = y \otimes y + \operatorname{im}\big(1 + T^*\big)$$

where $T : \Omega\widetilde{Z} \times \Omega\widetilde{Z} \to \Omega\widetilde{Z} \times \Omega\widetilde{Z}$ is the twist map and y is an odd generator. Theorem 2(a) implies $\sigma^*(y) \neq 0$. Hence by [17]

$$c\big(\sigma^*(x)\big) = \sigma^*(y) \otimes \sigma^*(y) + \operatorname{im}\big(1 + T^*\big) \neq 0.$$

This implies $\sigma^*(x) \neq 0$. Hence if $\bar{x} \in QH^{4\ell+2}(X;\mathbb{Z}_2) \cap \ker\sigma^*$, $\bar{x}$ must have a primitive representative. □

Proposition 2. *Let Z be an H-space with $H^*(Z;\mathbb{Z}_2)$ concentrated in even degrees. Then*

(a) $PH^{8m+2}(\Omega Z;\mathbb{Z}_2) = 0$*;*
(b) *If $\bar{x} \in QH^{8m-2}(\Omega Z;\mathbb{Z}_2) \cap \ker\sigma^*$, then there is a primitive representative x with $x = \varphi_{2^k}(y)$.*

Proof. By [8, Section 29–5] $PH^*(\Omega Z;\mathbb{Z}_2)/\operatorname{im}\sigma^*$ is spanned by transpotence elements. Hence if $x \in PH^*(\Omega Z;\mathbb{Z}_2)$ projects nontrivially to $PH^*(\Omega Z;\mathbb{Z}_2)/\operatorname{im}\sigma^*$, there is an algebra generator $y \in H^t(Z;\mathbb{Z}_2)$ truncated at height 2^k and x is represented by $\varphi_{2^k}(y)$. We have degree $\varphi_{2^k}(y) = 2^k t - 2$, for $k \geqslant 2$. Since $H^*(Z;\mathbb{Z}_2)$ is concentrated in even degrees, t is even. Hence there are no transpotence elements in degrees congruent to 2 mod 8.

Therefore

$$\begin{aligned} PH^{8m+2}(\Omega Z;\mathbb{Z}_2) &= \sigma^* QH^{8m+3}(Z;\mathbb{Z}_2) \\ &= 0 \text{ since } H^*(Z;\mathbb{Z}_2) \quad \text{is even dimensional.} \end{aligned}$$

Now if $\bar{x} \in QH^{8m-2}(\Omega Z;\mathbb{Z}_2) \cap \ker \sigma^*$, by Proposition 1 there is a primitive representative x for $\bar{x}$. The element x cannot be a suspension since $QH^{8m-1}(Z;\mathbb{Z}_2) = 0$. Hence $x = \varphi_{2^k}(y)$ for some $y \in H^t(Z;\mathbb{Z}_2)$ □

Theorem 4. *Let Z be a path connected H-space with $H^*(Z;\mathbb{Z}_2)$ concentrated in even degrees. Then the* mod 2 *EMSS associated to the path loop fibration*

$$\Omega^2 Z \to P\Omega Z \to \Omega Z$$

collapses.

Proof. By Theorem 2(a) and Lemma 1, it suffices to prove that

$$\sigma^* : QH^{4\ell+2}(\Omega Z;\mathbb{Z}_2) \to PH^{4\ell+1}\big(\Omega^2 Z;\mathbb{Z}_2\big)$$

is monic. Let $\bar{x} \in QH^{4\ell+2}(\Omega Z;\mathbb{Z}_2) \cap \ker \sigma^*$. By Proposition 1 and Proposition 2, $\bar{x}$ has a primitive representative $x = \varphi_{2^k}(y)$ for $k \geqslant 2$ and deg $x = 8m-2$. A theorem of Kraines [9, Theorem B] shows elements of $\ker \sigma^*$ in degrees $8m-2$ must have one of the following forms

(i) $\beta_2 Sq^{4m-2} Sq^{2m-1} u_{2m}$.
(ii) $\beta_1 w$ where w has the form $\psi_r(v)$ [3, Section 3].

Suppose $x = \beta_2 Sq^{4m-2} Sq^{2m-1} u_{2m}$. By Theorem 2(c)

$$Sq^{2m-1} u_{2m} = y + d \quad \text{where } y \in PH^{4m-1}(\Omega Z;\mathbb{Z}_2),$$

d is decomposable and $y = \sigma^*(z)$ by Theorem 2(a). Hence

$$x = \beta_2 Sq^{4m-2} \sigma^*(z) = \beta_2 \sigma^*\big(Sq^{4m-2} z\big).$$

But $Sq^{4m-2} z$ is an integral class since $H^*(Z;\mathbb{Z}_2)$ is even dimensional. Hence $x = 0$. Now suppose $x = \beta_1 w$. Then since x is indecomposable so is w. We may assume w is primitive by Theorem 2(c).

By Theorem 2(a) $w = \sigma^*(u)$. Hence $x = \sigma^*(\beta_1 u) = \varphi_{2^k}(y)$. If $x = \sigma^*(\beta_1 u)$, then x is an A_∞ class [14,15]. But if $x = \varphi_{2^k}(y)$ then $A_{2^k-1}(x) \neq 0$ by [10, Theorem 4.7]. This is a contradiction. Hence $\sigma^* : QH^{4\ell+2}(\Omega Z;\mathbb{Z}_2) \to PH^{4\ell+1}(\Omega^2 Z;\mathbb{Z}_2)$ is monic and the mod 2 EMSS collapses. □

Corollary 3. *Let X be a simply connected finite H-space. Then the* mod 2 *EMSS associated to the path loop fibration $\Omega^3 X \to P\Omega^2 X \to \Omega^2 X$ collapses.*

Proof. By [8,11,12] for any prime p, $H^*(\Omega X;\mathbb{Z}_p)$ is concentrated in even degrees. □

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