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# Stability of a time-varying fishing model with delay

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**Abstract**

To incorporate ecosystem effects, environmental variability and other factors that affect the population growth, the periodicity of the parameters of the model is assumed. We introduce a delay differential equation model which describes how fish are harvested:

$$\dot{N}(t) = \left[ \frac{a(t)}{1 + \left( \frac{N(\theta(t))}{K(t)} \right)^{\gamma}} - b(t) \right] N(t). \tag{A}$$

In our previous studies the persistence of Eq. (A) and the existence of a periodic solution to this equation were investigated. In the present paper the explicit conditions of global attractivity of the positive periodic solutions to Eq. (A) are obtained. It will also be shown that if the stability conditions are violated, the model exhibits sustained oscillations.


*Keywords:* Fishery; Periodic environment; Delay differential equations; Global and local stability

## 1. Introduction and preliminaries

Virtually all biological systems exist in environments which vary with time, frequently in a periodic way [3]. Ecosystem effects and environmental variability are very important factors and mathematical models cannot ignore, for example, year-to-year changes in weather, habitat destruction and exploitation, the expanding food surplus, and other factors that affect the population growth.

Consider the following differential equation which is widely used in fisheries [1,2]

$$\dot{N} = [\beta(t, N) - M(t, N)]N - F(t)N, \tag{1}$$

where $N = N(t)$ is the population biomass, $\beta(t, N)$ is the per-capita fecundity rate, and $M(t, N)$ is the per-capita mortality rate, and $F(t)$ is the harvesting rate per-capita.

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In Eq. (1) let $\beta(t, N)$ be a Hill's type function [1,2,5]

$$\beta(t, N) = \frac{a}{1 + \left(\frac{N}{K}\right)^{\gamma}}, \tag{2}$$

where $a$ and $K$ are positive constants.

Parameter $\gamma > 0$, that controls how rapidly density dependence sets in, can be regarded as an abruptness parameter. Note that the generalized Ricker curve widely used in fishery

$$\beta(t, N) = \mathrm{e}^{[r(1-(N/K))^{\gamma}]}$$

has a similar inverted sigmoidal shape for $\gamma > 1$ and $r > 0$. It is well-known that the canonical logistic curve has equal periods of slow and fast growth. In contrast, the Hill's curve does not incorporate the symmetry and has a shorter period of fast growth.

We assume that in (2) $a = a(t)$, $K = K(t)$, and $b(t) = M(t, N) - F(t)$ are continuous positive functions.

Generally, fishery models [1,2] recognize that for real organisms it takes time to develop from newborns to reproductively active adults.

Let in Eq. (2) $N = N(\theta(t))$, where $\theta(t)$ is the maturation time delay $0 \le \theta(t) \le t$. If we take into account that delay, then we have the following time-lag model based on Eq. (1)

$$\dot{N}(t) = \left[\frac{a(t)}{1 + \left(\frac{N(\theta(t))}{K(t)}\right)^{\gamma}} - b(t)\right] N(t) \tag{3}$$

for $\gamma > 0$, with the initial function and the initial value

$$N(t) = \varphi(t), \quad t < 0, \; N(0) = N_0 \tag{4}$$

under the following conditions:

(a1) $a(t), b(t), K(t)$ are continuous on $[0, \infty)$, $b(t) \ge b > 0$, $K \ge K(t) \ge k > 0$;
(a2) $\theta(t)$ is a continuous function, where $\theta(t) \le t$, $\limsup_{t\to\infty} \theta(t) = \infty$;
(a3) $\varphi : (-\infty, 0) \to R$ is a continuous bounded function, $\varphi(t) \ge 0$, $N_0 > 0$.

**Definition 1.1.** A function $N : R \to R$ with continuous derivative is called *a (global) solution* to problem (3) and (4), if it satisfies Eq. (3) for all $t \in [0, \infty)$ and equalities (4) for $t \le 0$.

If $t_0$ is the first point, where the solution $N(t)$ to (3) and (4) vanishes, i.e., $N(t_0) = 0$, then we consider only the positive solutions to problems (3) and (4) on the interval $[0, t_0)$.

The objective of this paper is to present a complete qualitative analysis of model (3), including the usual questions such as local stability and global stability of the system. In general, periodic delay differential equations do not possess equilibria, and the study of equilibria and stability is replaced by the more difficult problems of the existence and stability of periodic solutions. We study the combined effects of periodically varying environments and periodic harvesting rates on the fish population. Stability analysis will discover the conditions for existence of delay induced stability (instability). This analysis will show whether delays stabilize or destabilize the system. It will also be shown that if the stability conditions are violated, the model exhibits sustained oscillations.

A basic question in mathematical biology concerns the long-term survival of each component, and many criteria have been used to define the notion of long-term survival. Recently [5] we considered the subject of permanence, i.e., the study of the long-term survival of each species in a set of populations.

**Lemma 1.1.** *Suppose* $a(t) > b(t)$,

$$\sup_{t>0} \int_{\theta(t)}^{t} (a(s) - b(s))\mathrm{d}s < \infty, \qquad \sup_{t>0} \int_{\theta(t)}^{t} b(s)\mathrm{d}s < \infty.$$

*Then there exists the global positive solution to* (3) *and* (4) *and this solution is persistent:*

$$0 < \alpha_N \le N(t) \le \beta_N < \infty.$$

**Lemma 1.2.** *Let $a(t), b(t), K(t), \theta(t)$ be T-periodic functions, $a(t) \geq b(t)$. If at least one of the following conditions hold:*

(b1)

$$\inf_{t\geq 0}\left(\frac{a(t)}{b(t)}-1\right)K^{\gamma}(t) > 1,$$

(b2)

$$\sup_{t\geq 0}\left(\frac{a(t)}{b(t)}-1\right)K^{\gamma}(t) < 1,$$

*then Eq. (3) has at least one periodic positive solution $N_0(t)$.*

In what follows, we use a classical result from the theory of differential equations with delay [4,6].

**Lemma 1.3.** *Suppose that for linear delay differential equation*

$$\dot{x}(t) + r(t)x(h(t)) = 0 \tag{5}$$

*where $0 \leq t - h(t) \leq \sigma$, the following conditions hold:*

$$r(t) \geq r_0 > 0, \tag{6}$$

$$\limsup_{t\to\infty}\int_{h(t)}^{t} r(s)\mathrm{d}s < \frac{3}{2}. \tag{7}$$

*Then for every solution $x$ to Eq. (5) we have $\lim_{t\to\infty} x(t) = 0$.*

## 2. Main results

Let us study global stability of the periodic solutions to Eq. (3).

**Theorem 2.1.** *Let $a(t), b(t), K(t), \theta(t)$ be T-periodic functions, satisfying conditions of Lemma 1.1 and one of conditions (b1) or (b2) of Lemma 1.2. Suppose also that*

$$a(t) \geq a_0 > 0, \qquad \gamma\int_{\theta(t)}^{t} a(s)\mathrm{d}s < 6. \tag{8}$$

*Then there exists the unique positive periodic solution $N_0(t)$ of Eq. (3) and for every positive solution $N(t)$ of system (3)–(4) we have*

$$\lim_{t\to\infty}(N(t) - N_0(t)) = 0,$$

*i.e., the positive periodic solution $N_0(t)$ is a global attractor for all positive solutions to (3).*

**Proof.** Lemma 1.2 implies that there exists a positive periodic solution $N_0(t)$. If that solution is an attractor for all positive solutions then it is the unique positive periodic solution.

We set $N(t) = \exp(x(t))$ and rewrite Eq. (3) in the form

$$\dot{x}(t) = \frac{a(t)}{1+\left(\frac{\mathrm{e}^{x(\theta(t))}}{K(t)}\right)^{\gamma}} - b(t). \tag{9}$$

Suppose $u(t)$ and $v(t)$ are two different solutions to (9). Denote $w(t) = u(t) - v(t)$. To prove Theorem 2.1 it is sufficient to show that $\lim_{t\to\infty} w(t) = 0$.

It follows

$$\dot{w}(t) = a(t)\left[\frac{1}{1+\left(\frac{\mathrm{e}^{u(\theta(t))}}{K(t)}\right)^{\gamma}} - \frac{1}{1+\left(\frac{\mathrm{e}^{v(\theta(t))}}{K(t)}\right)^{\gamma}}\right). \tag{10}$$

Let

$$f(y,t)=\frac{1}{1+\left(\frac{e^y}{K(t)}\right)^\gamma}. \tag{11}$$

Using the mean value theorem, we have for every $t$

$$f(y,t)-f(z,t)=f'(c)(y-z), \tag{12}$$

where $\min\{y,z\}\le c(t)\le\max\{y,z\}$.

Clearly,

$$f'_y(y,t)=-\frac{\gamma\left(\frac{e^y}{K(t)}\right)^\gamma}{\left\{1+\left(\frac{e^y}{K(t)}\right)^\gamma\right\}^2} \tag{13}$$

and $|f'_y(y,t)|<\frac{1}{4}\gamma$.

Equalities (11) and (12) imply that Eq. (10) takes the form

$$\dot{w}(t)=-M(t)w(\theta(t)), \tag{14}$$

where

$$M(t)=\frac{\gamma a(t)\left(\frac{e^{c(t)}}{K(t)}\right)^\gamma}{\left\{1+\left(\frac{e^{c(t)}}{K(t)}\right)^\gamma\right\}^2},$$

and

$$\min\{u(\theta(t)),v(\theta(t))\}\le c(t)\le\max\{u(\theta(t)),v(\theta(t))\}.$$

Now we want to check that for Eq. (14) all conditions of Lemma 1.3 hold.

From (13) we have $M(t)<\frac{1}{4}\gamma a(t)$. Therefore inequality (7) holds. Let us check inequality (6). Set $N_1(t)=e^{u(t)}$, $N_2(t)=e^{v(t)}$, where $N_1(t),N_2(t)$ are two solutions to Eq. (3), corresponding to the solutions $u(t)$ and $v(t)$ to Eq. (9). Lemma 1.1 implies that

$$M(t)\ge\frac{\gamma a_0\left(\frac{\min\{\alpha_{N_1},\alpha_{N_2}\}}{K}\right)^\gamma}{\left(1+\left(\frac{\max\{\beta_{N_1},\beta_{N_2}\}}{k}\right)^\gamma\right)^2}>0,$$

where $\alpha_N$ and $\beta_N$ are defined by Lemma 1.1. Hence inequality (6) holds and therefore Theorem 2.1 is proven. □

Consider now Eq. (3) with proportional coefficients:

$$\dot{N}(t)=\left[\frac{ar(t)}{1+\left(\frac{N(\theta(t))}{K}\right)^\gamma}-br(t)\right)N(t), \tag{15}$$

where $r(t)\ge r_0>0$. Clearly, if $a>b$ then Eq. (15) has the unique positive equilibrium

$$N^*=\left(\frac{a}{b}-1\right)^{\frac{1}{\gamma}}K. \tag{16}$$

**Corollary 1.** *If $a>b$, $\left(\frac{a}{b}-1\right)K^\gamma\ne1$, $r(t)\ge r_0>0$, and*

$$\gamma a\limsup_{t\to\infty}\int_{\theta(t)}^t r(s)\mathrm{d}s<6, \tag{17}$$

*then the equilibrium $N^*$ is a global attractor for all positive solutions to Eq. (15).*

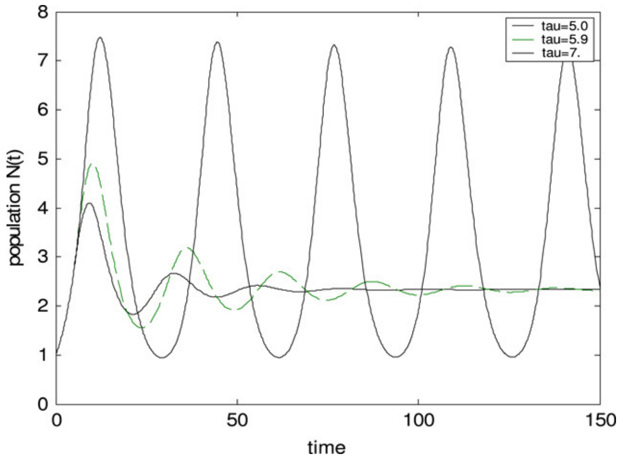


Fig. 1. If $\tau < 6$ then all solutions have a global attractor, otherwise the solution is unstable.

Let us now compare the global attractivity condition (17) with the local stability conditions.

**Theorem 2.2.** *Suppose $a > b, r(t) \geq r_0 > 0$ and*

$$\frac{\gamma(a-b)b}{a} \limsup_{t\to\infty} \int_{\theta(t)}^{t} r(s)\mathrm{d}s < \frac{3}{2}. \tag{18}$$

*Then the equilibrium $N^*$ of Eq. (15) is locally asymptotically stable.*

**Proof.** Set $x = N - N^*$ and from Eq. (15) we have

$$\dot{x}(t) = \left[\frac{ar(t)}{1+\left(\frac{x(\theta(t))+N^*}{K}\right)^{\gamma}} - br(t)\right](x(t)+N^*). \tag{19}$$

Denote

$$F(u,v) = \left[\frac{ar(t)}{1+\left(\frac{u+N^*}{K}\right)^{\gamma}} - br(t)\right](v+N^*).$$

Clearly,

$$F'_u(0,0) = -\frac{\gamma(a-b)b}{a} r(t)$$

and $F'_v(0,0) = 0$. Hence for Eq. (15) the linearized equation has a form

$$\dot{x}(t) = -\frac{\gamma(a-b)b}{a} r(t)x(\theta(t)). \tag{20}$$

Lemma 1.3 and condition (18) imply that Eq. (20) is asymptotically stable, therefore the positive equilibrium $N^*$ of Eq. (15) is locally asymptotically stable.

Compare now Theorems 2.1 and 2.2. We have $\max\{b(a-b)\} = a/4$. Therefore, if

$$a\gamma \limsup_{t\to\infty} \int_{\theta(t)}^{t} r(s)\mathrm{d}s < 6, \tag{21}$$

then Eq. (15) has locally asymptotically stable equilibrium $N^*$. □

The latter condition (21) does not depend on $b$, and is identical to condition (17) that guarantees the existence of a global attractor. Therefore in Theorem 2.2 we obtained the best possible conditions for global attractivity for Eq. (3).

Fig. 1 illustrates the significance of condition (17). In the numerical example we set in Eq. (15) $r(t) = 1$, $K = \gamma = a = 1$, $b = 0.3$ and vary delay $\tau = 5, 5.9$, and 7.

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