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Global analysis of solutions of $x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{A + Bx_n + Cx_{n-1}}$

E. Camouzis

Department of Mathematics, The American College of Greece, Deree College, 6 Gravias Street, Aghia Paraskevi, 15342, Athens, Greece

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Abstract

We study the global character of solutions of the third order rational difference equation

$$x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots,$$

where the parameters β, δ, A are nonnegative, $\beta + \delta > 0$, $B, C > 0$, the initial conditions x_{-2}, x_{-1}, x_0 are nonnegative real numbers and the denominator is always positive.

Keywords: Boundedness; Difference equation; Unbounded solutions

1. Introduction

We study the global character of solutions of the third order rational difference equation

$$x_{n+1} = \frac{\beta x_n + \delta x_{n-2}}{A + Bx_n + Cx_{n-1}}, \tag{1}$$

where the parameters β, δ, A are nonnegative, $\beta + \delta > 0$, $B, C > 0$, the initial conditions x_{-2}, x_{-1}, x_0 are nonnegative real numbers and the denominator is always positive.

E-mail address: camouzis@acgmail.gr.

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When $\delta = 0$, Eq. (1) was studied in [3]. When $\beta = 0$, the following theorem holds:

Theorem A (see [1]). *Assume that $\beta = 0$, $A \geqslant 0$ and $B, C > 0$. Then the solutions of the equation*

$$x_{n+1} = \frac{x_{n-2}}{A + Bx_n + Cx_{n-1}}, \quad n = 0, 1, \ldots, \tag{2}$$

have the following period-three trichotomy behavior:

(a) *When*

$$A > 1,$$

every solution of Eq. (2) *converges to zero.*

(b) *When*

$$A = 1,$$

every solution of Eq. (2) *converges to a period-three solution of the form*

$$\ldots, 0, 0, \phi, 0, 0, \phi, \ldots,$$

with $\phi \geqslant 0$.

(c) *When*

$$0 \leqslant A < 1,$$

Eq. (2) *has unbounded solutions.*

For the rest of the sequel, we assume that $\beta, \delta > 0$. Using an appropriate change of variables Eq. (1) becomes

$$x_{n+1} = \frac{\beta x_n + x_{n-2}}{A + Bx_n + x_{n-1}}, \tag{3}$$

where $A \geqslant 0$, β, $B > 0$, the initial conditions x_{-2}, x_{-1}, x_0 are nonnegative real numbers and the denominator is always positive.

Equation (3) has one or two equilibrium points. When $\beta + 1 \leqslant A$, Eq. (3) has only the zero equilibrium. When $\beta + 1 > A$, and $A > 0$, Eq. (3) has two equilibrium points, namely the zero equilibrium and the positive equilibrium $\bar{x} = \frac{\beta+1-A}{B+1}$. When $A = 0$, Eq. (3) has only the positive equilibrium $\bar{x}$.

A question of great importance in the study of difference equations is whether or not the solutions are bounded. The following theorem which was established in [2] shows the existence of unbounded solutions of Eq. (3).

Theorem B. *Assume that*

$$\beta < B(1 - A). \tag{4}$$

Let $\{x_n\}$ be a solution of Eq. (3) *such that*

$$0 < x_{-1} < 1 + \epsilon, \qquad 0 < x_0 < \frac{\beta}{B} + \epsilon, \qquad x_{-2} > \max\{K, L\}, \tag{5}$$

where

$$K = \frac{\alpha + \beta(\frac{\beta}{B} + \epsilon) + \delta(1+\epsilon)}{\epsilon}, \qquad L = \frac{\alpha + 1 + \epsilon + \delta(\frac{\beta}{B} + \epsilon)}{B\epsilon}$$

and

$$0 < \epsilon < \frac{1 - A - \frac{\beta}{B}}{B+1}.$$

Then

$$\lim_{n\to\infty} x_{3n+1} = \infty, \qquad \lim_{n\to\infty} x_{3n+2} = \frac{\beta}{B}, \qquad \lim_{n\to\infty} x_{3n+3} = 0.$$

2. Global analysis of solutions of Eq. (3)

The following lemmas will be useful in the sequel.

Lemma 2.1. *Assume that* $A = 1$. *Let* $\{x_n\}$ *be a solution of Eq.* (3) *for which there exists* $N > 0$ *such that*

$$0 < x_{N-2}, x_{N-1}, x_N < \frac{\beta}{B}. \tag{6}$$

Then it holds

$$x_{N+1} < \frac{\beta}{B}.$$

Proof. In view of Eq. (3), we get

$$x_{N+1} = \frac{\beta x_N + x_{N-2}}{1 + Bx_N + x_{N-1}} < \frac{\frac{\beta}{B}(\beta+1)}{1+\beta} = \frac{\beta}{B}.$$

The proof is complete. □

Lemma 2.2. *Assume that* $A = 1$. *Let* $\{x_n\}$ *be a solution of Eq.* (3) *for which there exists* $N > 0$ *such that*

$$x_{N+1} \geqslant \frac{\beta}{B}. \tag{7}$$

Then it holds that

$$x_{N-1} < \frac{\beta}{B}. \tag{8}$$

Proof. Suppose for the sake of contradiction that

$$x_{N-1} \geqslant \frac{\beta}{B}.$$

Then in view of Eq. (3), we get

$$x_{N-2} \geqslant \frac{\beta}{B}\left(1 + \frac{\beta}{B}\right) \quad \text{and} \quad x_{N-4} \geqslant \frac{\beta}{B},$$

which implies that

$$x_{N-5} \geqslant \frac{\beta}{B}\left(1+\frac{\beta}{B}\right)^2 \quad \text{and} \quad x_{N-7} \geqslant \frac{\beta}{B}.$$

Inductively, we get

$$x_{N-3k-2} > \frac{\beta}{B}\left(1+\frac{\beta}{B}\right)^{k+1}, \quad k = 0, 1, \ldots,$$

which is a contradiction and the proof is complete. □

Theorem 2.1. *Assume that* $A \geqslant 1$. *Let* $\{x_n\}$ *be a solution of Eq.* (3). *There exists* $N > 0$ *such that*

$$x_n < \frac{\beta}{B}, \quad n \geqslant N. \tag{9}$$

Proof. We will consider two cases. First assume that

$$A = 1.$$

In view of Lemma 2.1 it suffices to show that there exists $N > 0$ such that (6) holds. For the sake of contradiction and in view of (7), (8) assume that there exists $N > 0$ such that for all $n \geqslant 0$,

$$0 < x_{3n+N}, x_{3n+N+1} < \frac{\beta}{B} < x_{3n+N-1}.$$

From Eq. (3), we get

$$x_{3n+N+2} = \frac{\beta x_{3n+N+1} + x_{3n+N-1}}{1 + Bx_{3n+N+1} + x_{3n+N}} < x_{3(n-1)+N+2}.$$

Let

$$S = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} x_{3n+N+5} = \lim_{n\to\infty} x_{3n+N+2} = \lim_{n\to\infty} x_{3n+N-1} \geqslant \frac{\beta}{B}.$$

In addition there exist subsequences, namely $\{x_{3n_i+N+4}\}$, $\{x_{3n_i+N+3}\}$, $\{x_{3n_i+N+1}\}$, $\{x_{3n_i+N}\}$, such that

$$l_4 = \lim_{i\to\infty} x_{3n_i+N+4}, \qquad l_3 = \lim_{i\to\infty} x_{3n_i+N+3},$$

$$l_1 = \lim_{i\to\infty} x_{3n_i+N+1}, \qquad l_0 = \lim_{i\to\infty} x_{3n_i+N} \leqslant \frac{\beta}{B}.$$

From Eq. (3), we get

$$S = \frac{\beta l_1 + S}{1 + Bl_1 + l_0}.$$

If $l_0 > 0$, we get

$$\frac{\beta}{B} \leqslant S = \frac{\beta l_1}{Bl_1 + l_0} < \frac{\beta}{B}$$

which is a contradiction.

On the other hand, if $l_0 = 0$, it follows that $l_4, l_3 > 0$ and so

$$S = \frac{\beta l_4}{Bl_4 + l_3} < \frac{\beta}{B}$$

which is also a contradiction and the proof of (9) is complete when $A = 1$.

Assume that

$$A > 1.$$

For the sake of contradiction assume that there exists N such that

$$x_{N+1} = \frac{\beta x_N + x_{N-2}}{A + Bx_N + x_{N-1}} > \frac{\beta}{B}. \tag{10}$$

Eq. (10) implies that

$$x_{N-2} = \frac{\beta x_{N-3} + x_{N-5}}{A + Bx_{N-3} + x_{N-4}} > \frac{\beta}{B}A$$

which in addition implies that

$$x_{N-5} > A^2 \frac{\beta}{B}.$$

Inductively we have that

$$x_{N-3k-2} > A^{k+1} \frac{\beta}{B}, \quad k = 0, 1, \ldots,$$

which is a contradiction and so the proof of (9) is complete. □

Lemma 2.3. *Assume that* $1 \leqslant A < \beta + 1$. *Let* $\{x_n\}$ *be a positive solution of Eq.* (3). *Then it holds that*

$$S = \limsup_{n \to \infty} x_n > 0. \tag{11}$$

Proof. Assume for the sake of contradiction that $S = 0$. There exists $\epsilon > 0$, $m > 0$, where

$$0 < m = \frac{A + (B+1)\epsilon}{\beta + 1} < 1. \tag{12}$$

Without loss of generality assume that

$$0 < x_n < \epsilon, \quad n \geqslant -2.$$

Choose N large enough. From Eq. (3), we get

$$x_{N+1} = \frac{\beta x_N + x_{N-2}}{A + Bx_N + x_{N-1}} < \epsilon$$

which implies

$$\min\{x_{N-2}, x_N\} < \epsilon m$$

from which it follows that

$$\min\{x_{N-5}, x_{N-3}, x_{N-1}\} < \epsilon m^2.$$

Sufficient repetition of that argument leads to a contraction and so the proof of (12) is complete. □

Theorem 2.2. *Assume that*

$$1 \leqslant B < +\infty, \qquad 1 \leqslant A < \beta + 1. \tag{13}$$

Then every positive solution $\{x_n\}$ *of Eq.* (3) *converges to the positive equilibrium of Eq.* (3).

Proof. In view of Theorem 2.1, the solution $\{x_n\}$ of Eq. (3) is bounded from above by the positive constant $\frac{\beta}{B}$. Let

$$S = \limsup_{n\to\infty} x_n < +\infty, \qquad I = \liminf_{n\to\infty} x_n \geqslant 0.$$

Then in view of Eq. (3), we get

$$S \leqslant \frac{(\beta+1)S}{A+BS}$$

and so in view of (12)

$$S \leqslant \frac{\beta+1-A}{B}.$$

Assume that $S = \frac{\beta+1-A}{B}$. There exist subsequences, namely $\{x_{n_i+1}\}$, $\{x_{n_i}\}$, $\{x_{n_i-1}\}$, $\{x_{n_i-2}\}$, $\{x_{n_i-3}\}$ such that

$$S = \lim_{i\to\infty} x_{n_i+1}, \qquad l_i = \lim_{i\to\infty} x_{n_i-t}, \quad t = 0, 1, 2, 3.$$

Then

$$S = \frac{\beta l_0 + l_{-2}}{A + B l_0 + l_{-1}},$$

which implies $l_0 = l_{-2} = S$ and $l_{-1} = 0$. From Eq. (3), we get

$$l_0 = S = \frac{l_{-3}}{A+S} \leqslant \frac{S}{A+S}$$

which implies that $S \leqslant 1 - A$, a contradiction, and so

$$S < \frac{\beta+1-A}{B}. \tag{14}$$

There exist $\epsilon > 0$, $m > 0$ and $N > 0$

$$0 < m < \min\{x_{N-2}, x_{N-1}, x_N\}$$

and

$$S + \epsilon < \frac{\beta+1-A}{B} < \beta + 1 - A - Bm, \quad \text{when } B > 1$$

or

$$S + \epsilon < \beta + 1 - A - m, \quad \text{when } B = 1$$

such that

$$x_n < S + \epsilon < \beta + 1 - A - Bm, \quad n \geqslant N - 2.$$

In addition

$$x_{N+1} = \frac{\beta x_N + x_{N-1}}{A + Bx_N + x_{N-1}} > \frac{(\beta + 1)m}{A + Bm + \beta + 1 - A - Bm} = m.$$

Inductively we have that

$$m < x_n < \beta + 1 - A - Bm, \quad n \geqslant N - 2.$$

Hence $I > 0$. Then, in view of Eq. (3), we get

$$S \leqslant \frac{(\beta + 1)S}{A + BS + I}$$

which implies that

$$BS + I \leqslant \beta + 1 - A$$

and also

$$I \geqslant \frac{(\beta + 1)I}{A + BI + S},$$

which implies that

$$BI + S \geqslant \beta + 1 - A.$$

Combining the inequalities, we get that

$$(B - 1)(S - I) \leqslant 0$$

and so $I = S$ when $B > 1$.

On the other hand, when $B = 1$, we get

$$S = \frac{\beta l_0 + l_{-2}}{A + l_0 + l_{-1}} \leqslant \frac{(\beta + 1)S}{A + S + I},$$

which implies that

$$S \leqslant \beta + 1 - A - I.$$

Assume that $S = \beta + 1 - A - I$ and $S > I$. Then

$$l_0 = l_{-2} = S, \qquad l_{-1} = I.$$

In addition

$$l_0 = S = \frac{\beta l_{-1} + l_{-3}}{A + l_{-1} + l_{-2}} < \frac{(\beta + 1)S}{A + S + I},$$

which implies that

$$S < \beta + 1 - A - I,$$

a contradiction, and so either $S < \beta + 1 - A - I$ or $S = I$. Assume that $S < \beta + 1 - A - I$. There exist subsequences, namely $\{x_{n_j+1}\}$, $\{x_{n_j}\}$, $\{x_{n_j-1}\}$, $\{x_{n_j-2}\}$, $\{x_{n_j-3}\}$ such that

$$I = \lim_{j\to\infty} x_{n_j+1}, \qquad m_t = \lim_{j\to\infty} x_{n_j-t}, \quad t = 0, 1, 2, 3.$$

In view of Eq. (3), we get

$$I = \frac{\beta m_0 + m_{-2}}{A + m_0 + m_{-1}} \geqslant \frac{(\beta+1)I}{A + I + S}$$

and so $S \geqslant \beta + 1 - A - I$, a contradiction. Hence $S = I$. The proof is complete. □

Conjecture 2.1. *Assume that*

$$1 \leqslant A < \beta + 1, \qquad 0 < B < 1.$$

Show that every positive solution of Eq. (3) *converges to the positive equilibrium of Eq.* (3).

Theorem 2.3. *Assume that*

$$\beta > B(1 - A), \qquad 1 \geqslant A \geqslant 0, \qquad B \geqslant 1. \tag{15}$$

Then $(1 - A, \frac{\beta}{B})$ *is an invariant interval for all positive solutions of Eq.* (3).

Proof. Assume that $\{x_n\}$ is a solution of Eq. (3), with initial conditions x_{-2}, x_{-1}, x_0 such that

$$1 - A < x_{-2}, x_1, x_0 < \frac{\beta}{B}.$$

Then

$$x_1 - \frac{\beta}{B} = \frac{Bx_{-2} - A\beta - \beta x_{-1}}{B(A + Bx_0 + x_{-1})} < \frac{\beta(1 - A - x_{-1})}{B(A + Bx_0 + x_{-1})} < 0.$$

In addition

$$\begin{aligned} x_1 - (1 - A) &= \frac{\beta x_0 + x_{-2}}{A + Bx_0 + x_{-1}} - (1 - A) \\ &= \frac{[\beta - (1 - A)B]x_0 + x_{-2} - (1 - A)(A + x_{-1})}{A + Bx_0 + x_{-1}} \\ &> \frac{(1 - A)[\beta - (1 - A)B + 1 - A - \frac{\beta}{B}]}{A + Bx_0 + x_{-1}} \\ &= \frac{(1 - A)(B - 1)[\beta - (1 - A)B]}{A + Bx_0 + x_{-1}} > 0. \end{aligned}$$

Inductively the result follows. □

Theorem 2.4. *Assume that*

$$\beta > B(1 - A), \qquad 1 \geqslant A \geqslant 0, \qquad B \geqslant 1. \tag{16}$$

Then every solution of Eq. (3) *with initial conditions in the invariant interval* $(1 - A, \frac{\beta}{B})$ *converges to the positive equilibrium of Eq.* (3).

Proof. We will consider two cases. First assume that $B = 1$. Let

$$S = \limsup_{n\to\infty} x_n, \qquad I = \liminf_{n\to\infty} x_n.$$

Then in view of Eq. (3), we get

$$S \leqslant \frac{(\beta+1)S}{A+S+I},$$

which implies that

$$S \leqslant \beta + 1 - A - I.$$

Assume that $S = \beta + 1 - A - I$ and $S > I$. There exist subsequences, namely $\{x_{n_i+1}\}$, $\{x_{n_i}\}$, $\{x_{n_i-1}\}$, $\{x_{n_i-2}\}$, $\{x_{n_i-3}\}$ such that

$$S = \lim_{i\to\infty} x_{n_i+1}, \qquad l_t = \lim_{i\to\infty} x_{n_i-t}, \quad t = 0, 1, 2, 3.$$

In view of Eq. (3), we have that $l_0 = l_{-2} = S$ and $l_{-1} = I$. In addition

$$l_0 = S = \frac{\beta I + l_{-3}}{A+I+S} < \frac{(\beta+1)S}{1+S+I}$$

which implies that

$$S < \beta + 1 - A - I,$$

a contradiction, and so either $S < \beta + 1 - A$ or $S = I$. Assume that $S < \beta + 1 - A$. There exist subsequences, namely $\{x_{n_j+1}\}$, $\{x_{n_j}\}$, $\{x_{n_j-1}\}$, $\{x_{n_j-2}\}$, $\{x_{n_j-3}\}$ such that

$$I = \lim_{j\to\infty} x_{n_j+1}, \qquad m_t = \lim_{j\to\infty} x_{n_j-t}, \quad t = 0, 1, 2, 3.$$

In view of Eq. (3), we get

$$I = \frac{\beta m_0 + m_{-2}}{A+m_0+m_{-1}} \geqslant \frac{(\beta+1)I}{A+I+S}$$

and so $S \geqslant \beta + 1 - A - I$, a contradiction. Hence $S = I$. The proof is complete in the case $B = 1$.

When $B > 1$, the convergence of $\{x_n\}$ is a consequence of the global stability result, Theorem A.0.5 in Ref. [3, p. 205] applied in the invariant interval $[1 - A, \frac{\beta}{B}]$. The only hypothesis of Theorem A.0.5 remaining to be checked is whether the system

$$\begin{cases} M = \frac{(\beta+1)M}{A+BM+m}, \\ m = \frac{(\beta+1)m}{A+Bm+M} \end{cases}$$

has a unique solution. This is clear because $0 < B < 1$. The proof is complete. $\square$

Open Problem 2.1. *Assume that* (16) *holds. Prove that every positive solution of Eq.* (3) *converges to the positive equilibrium of Eq.* (3)*.*

When $\beta + 1 > A$, the positive equilibrium $\bar{x}$ of Eq. (3) is locally asymptotically stable if and only if

$$A \geqslant 1 \tag{17}$$

or

$$A < 1 \quad \text{and} \quad \beta > \frac{-1 - A - 2B + \sqrt{(1+A)^2 + 4B(2 + (2-A)B)}}{2B} = \beta^*. \tag{18}$$

In rational difference equations it often occurs, when we have two equilibrium points one of which is zero and the other is positive, that the local stability of the positive equilibrium implies that the positive equilibrium is a global attractor of all positive solutions of the equation. On the other hand, when $\beta < B(1-A)$, Theorem B predicts that Eq. (3) has unbounded solutions. When

$$1 > A \geqslant 0, \qquad 1 > B \geqslant \frac{\sqrt{5}-1}{2} \quad \text{or} \quad \frac{1}{2} \leqslant B \leqslant \frac{\sqrt{5}-1}{2} \quad \text{and}$$
$$1 > A > \frac{B^2 + B - 1}{B^2 - B} \tag{19}$$

it holds that

$$\beta^* < \beta < B(1-A)$$

and so the positive equilibrium of Eq. (3) is locally asymptotically stable and also there exist solutions of Eq. (3) which are unbounded. Therefore in this particular equation local stability does not imply global attraction.

Open Problem 2.2. *Assume that* (19) *holds.*

(a) *Find the set of initial conditions x_{-2}, x_{-1}, x_0 for which every solution of Eq.* (1) *is unbounded.*
(b) *Find the set of initial conditions x_{-2}, x_{-1}, x_0 for which every solution of Eq.* (1) *converges to the positive equilibrium of Eq.* (3).

When

$$\frac{1}{2} < B < \frac{\sqrt{5}-1}{2} \quad \text{and} \quad A < \frac{B^2 + B - 1}{B^2 - B} \quad \text{or}$$
$$0 < B < \frac{1}{2} \quad \text{and} \quad 0 \leqslant A < 1 \tag{20}$$

it holds that

$$\beta^* > \beta > B(1-A).$$

In this case numerical investigations indicate chaotic behavior of solutions of Eq. (3).

Open Problem 2.3. *Investigate the behavior of solutions of Eq.* (3) *when* (20) *holds.*

Conjecture 2.2. *Assume that*

$$\beta > B(1-A).$$

Prove that all solutions of Eq. (3) *are bounded.*

Theorem 2.5. *Assume that*

$$A \geqslant \beta + 1. \tag{21}$$

Then every solution $\{x_n\}$ *of Eq.* (3) *converges to zero.*

Proof. Let $\{x_n\}$ be a solution of Eq. (3). In view of Theorem 2.1, the solution $\{x_n\}$ is bounded from above. Let

$$S = \limsup_{n\to\infty} x_n.$$

Then from Eq. (3), we get

$$S \leqslant \frac{(\beta+1)S}{A+BS}$$

and so

$$S \leqslant \frac{\beta+1-A}{B} \leqslant 0.$$

The proof is complete. □

3. Period-three solutions of Eq. (1)

In this section we prove the existence of a unique prime period-three solution of Eq. (1) when $B = 1$. Using an appropriate change of variables Eq. (1) becomes

$$x_{n+1} = \frac{x_n + \delta x_{n-2}}{A + x_n + x_{n-1}}, \quad n = 0, 1, \ldots \tag{22}$$

where $\delta > 0$, $A \geqslant 0$, the initial conditions x_{-2}, x_{-1}, x_0 are nonnegative and the denominator is always positive.

When $\delta > A + 1$, there are values of δ, A such that the positive equilibrium of Eq. (22) is locally asymptotically stable. On the other hand condition (4) of Theorem B, with β replaced by $\frac{\beta}{\delta}$ and A replaced by $\frac{A}{\delta}$, and $B = \beta = 1$ becomes, $\delta > A + 1$ and so in this case, Theorem B predicts that Eq. (22) has unbounded solutions. The next theorem predicts that when $\delta > A + 1$, Eq. (22) has periodic solutions of period three.

Theorem 3.1. *Eq.* (22) *possesses a unique prime period-three solution of the form*

$$\ldots, p, q, r, p, q, r, \ldots$$

if and only if

$$\delta > A + 1.$$

Furthermore p, q, r *are the three positive solutions of the cubic equation*

$$-Lx^3 + 2(L^2 + L + 1)x^2 - (L^3 + 3L^2 + 3L + 2)x + L(L^2 + L + 1) = 0, \tag{23}$$

where $L = \delta - A - 1$. *In fact if* p *is one of the solutions of* (23) *the other two solutions are*

$$q = \delta - A - \frac{1}{1 + p + A - \delta}, \qquad r = \delta - A - \frac{p + A - \delta + 1}{p + A - \delta}. \tag{24}$$

Proof. Let

$$x_{-2} = p, \qquad x_{-1} = q, \qquad x_0 = r,$$

where p, q, r are not all equal. Then the triple p, q, r is a prime period-three solution of Eq. (22) if and only if

$$\begin{aligned} r + \delta p &= Ap + rp + qp, \\ p + \delta q &= Aq + qp + qr, \\ q + \delta r &= Ar + qr + rp \end{aligned} \tag{25}$$

where $A + p + q$, $A + p + r > 0$, $A + q + r > 0$. Using the change of variables

$$P = p - \delta + A, \qquad Q = q - \delta + A, \qquad R = r - \delta + A,$$

we get that

$$\begin{aligned} R - P &= R(P - Q), \\ R - Q &= Q(P - R), \\ P - Q &= P(Q - R). \end{aligned} \tag{26}$$

In view of (26), we have that

$$Q = -\frac{1}{1 + P}, \qquad R = -\frac{P + 1}{P}.$$

Substituting $P = p - \delta + A$, $Q = q - \delta + A$ and $R = r - \delta + A$, we have

$$q = \delta - A - \frac{1}{1 + p + A - \delta}, \qquad r = \delta - A - \frac{p + A - \delta + 1}{p + A - \delta}.$$

Substituting q and r in (25), we get

$$\frac{f(p)}{(p + A - \delta)(p + A - \delta + 1)} = \frac{-Lp^3 + 2(L^2 + L + 1)p^2 - (L^3 + 3L^2 + 3L + 2)p + L(L^2 + L + 1)}{(p + A - \delta)(p + A - \delta + 1)} = 0.$$

It holds that $(p + A - \delta)(p + A - \delta + 1) = 0$ if and only if $p = q = r$. Therefore $f(p) = 0$. Similarly we can show that $f(q) = f(r) = 0$. It can be easily shown that Eq. (23) has three distinct positive solutions if and only if

$$\delta > A + 1.$$

The proof is complete. □

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