Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 244 (2008) 1141–1156
Journal of Differential Equations
www.elsevier.com/locate/jde

Indices of the iterates of $\mathbb{R}^3$-homeomorphisms at Lyapunov stable fixed points ☆

Francisco R. Ruiz del Portal [a,*], José M. Salazar [b]

[a] *Departamento de Geometría y Topología, Facultad de CC.Matemáticas, Universidad Complutense de Madrid, Madrid 28040, Spain*

[b] *Departamento de Matemáticas, Universidad de Alcalá, Alcalá de Henares, Madrid 28871, Spain*

Received 18 April 2007; revised 20 July 2007

Available online 20 September 2007

Abstract

Given any positive sequence $\{c_n\}_{n\in\mathbb{N}}$, we construct orientation preserving homeomorphisms $f:\mathbb{R}^3\to\mathbb{R}^3$ such that $\mathrm{Fix}(f)=\mathrm{Per}(f)=\{0\}$, 0 is Lyapunov stable and $\limsup \frac{|i(f^m,0)|}{c_m}=\infty$. We will use our results to discuss and to point out some strong differences with respect to the computation and behavior of the sequences of the indices of planar homeomorphisms.

MSC: 37C25; 37B30; 54H25

Keywords: Fixed point index; Conley index

1. Introduction

The computation of the sequence of the indices, or the sequence of Lefschetz numbers, of the iterates of a map is an important and non-trivial problem.

When a fixed point is an isolated invariant set of an orientation preserving planar homeomorphism, the problem of the computation of the indices of its iterates was solved by Le Calvez and Yoccoz [13,14] and, by the authors, in the orientation reversing case [18]. Later Le Calvez solved the general problem in the orientation preserving case using the Carathéodory's theory

☆ The authors have been supported by MEC, MTM 2006-0825.
* Corresponding author.
E-mail address: r_portal@mat.ucm.es (F.R. Ruiz del Portal).

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.08.006

of prime ends [15] and the authors, in [19], the general case for orientation reversing planar homeomorphisms.

For orientation preserving planar homeomorphisms there are integers q and $r \geqslant 1$ such that the sequence of indices is as follows:

$$i\left(f^k, p\right) = \begin{cases} 1 - rq & \text{if } k \in r\mathbb{Z}, \\ 1 & \text{if } k \notin r\mathbb{Z}. \end{cases}$$

If the problem in the plane resulted to be hard, the analogous problem in $\mathbb{R}^3$ seems to be strongly non-trivial because of the different dynamical pathologies that can appear. For instance, while Lyapunov stable isolated fixed points of planar homeomorphisms have always index $= 1$, i.e. the Euler characteristic of a disc [6,17], for $\mathbb{R}^n$-vector fields with $n \geqslant 3$, Bonatti and Villadelprat in [3] proved that the index of stable, even in the past and in the future, isolated rest points can be any integer (see also the paper of Erle [8]).

There are not many known results about the behavior of the sequences of fixed point indices of homeomorphisms in dimension 3. For instance it is well known that the sequence must follow Dold's necessary conditions [7]. Shub and Sullivan proved that for C^1-maps (no necessarily injective) the sequence is bounded. Later, Chow, Mallet-Paret and Yorke [5] gave bounds about the form of the sequence of indices in terms of the spectrum of the derivative $Df(p)$. Babenko and Bogatyi [1] proved that these bounds are sharp in dimension 2 and in a recent paper Graff and Nowak-Przygodzki have proved [10] that for C^1-maps the sequence of fixed point indices follows one among exactly seven different periodic patterns.

More recently, the authors, see [20], have solved completely the problem for $\mathbb{R}^3$-homeomorphisms belonging to a special class. A class that is quite natural to study because the corresponding family in $\mathbb{R}^2$ is the set of all planar homeomorphisms such that $\{p\}$ is an isolated invariant set.

Let $U \subset \mathbb{R}^3$ be an open subset and let $\mathcal{B}$ be the set of all homeomorphisms $f : U \subset \mathbb{R}^3 \to f(U) \subset \mathbb{R}^3$ such that there exists a closed 3-dimensional ball, N, with the following properties:

(a) N is an isolating block such that the maximal invariant set contained in N, $\operatorname{Inv}(N, f)$, is $\{p\}$,
(b) ∂N is a locally flat 2-sphere, and
(c) the component of $f(N) \cap N$ containing p is also a closed ball.

In [20] it is shown that for every $f \in \mathcal{B}$ the sequence $\{i(f^k, p)\}_{k \in \mathbb{N}}$ is periodic. Conversely, for any periodic sequence of integers $\{r_k\}_{k \in \mathbb{N}}$ satisfying Dold's necessary congruences, there exists an orientation preserving homeomorphism $f \in \mathcal{B}$ such that $i(f^k, p) = r_k$ for every $k \in \mathbb{N}$.

Shub and Sullivan in [21] also conjectured that for every C^1-map, h, defined in a compact manifold,

$$\limsup \frac{\log(|\operatorname{Per}^m(h)|)}{m} \geqslant \limsup \frac{\log(|\Lambda(h^m)|)}{m},$$

where Λ denotes the Lefschetz number. Obviously every homeomorphism of the n-sphere, S^n, satisfies the above inequality because the sequence of the Lefschetz numbers of its iterates is constant if it preserves orientation.

A more general and slightly different version of the above problem is whether

$$\limsup \frac{\log(|\operatorname{Per}^m(h)|)}{m} \geqslant \limsup \frac{\log(\sum_{p\in\operatorname{Per}^m(h)} |i(h^m, p)|)}{m}.$$

It is well known that there are examples of non-injective continuous maps for which both previous inequalities fail to be true [21].

In this paper we see that for S^3-homeomorphisms the answer to this second problem is negative. We will show that, if for any neighborhood N of the origin, $\operatorname{Inv}(N, f) \cap \partial(N) \neq \emptyset$, then the sequence of indices of the iterates of f may be unbounded. In other words, if in the conjecture of Shub and Sullivan we replace a manifold by a bounded open subset of $\mathbb{R}^3$, a C^1-map by a homeomorphism and the Lefschetz numbers by the fixed point indices, the answer is negative even for stable fixed points. More precisely we shall prove the following theorems which solve, in the negative, Problem 2.3.1 of [22].

Theorem 1. *For each positive sequence $\{c_n\}_{n\in\mathbb{N}}$, there exist orientation preserving $\mathbb{R}^3$-homeomorphisms, f, such that* $\operatorname{Fix}(f) = \operatorname{Per}(f) = \{0\}$ *and* $\limsup \frac{|i(f^m,0)|}{c_m} = \infty$. *Moreover, if $B \subset \mathbb{R}^3$ is any closed ball centered in the origin, $f(B) \cap B$ is a topological ball,* $\operatorname{Inv}(B, f)$ *is the closed 2-disc $B \cap \{z = 0\}$ and f is limit of a sequence of homeomorphisms $\{f_m\}_{m\in\mathbb{N}}$ such that* $\operatorname{Inv}(B, f_m) = \{0\}$ *for every $m \in \mathbb{N}$ and, for every $n \in \mathbb{N}$, there exists m_0 such that $i(f^n, 0) = i((f_m)^n, 0)$ for every $m \geqslant m_0$.*

Theorem 2. *For each positive sequence $\{c_n\}_{n\in\mathbb{N}}$, there exist orientation preserving $\mathbb{R}^3$-homeomorphisms, h, such that* $\operatorname{Fix}(h) = \operatorname{Per}(h) = \{0\}$, 0 *is Lyapunov stable and* $\limsup \frac{|i(h^m,0)|}{c_m} = \infty$. *In particular, there are $\mathbb{R}^3$-homeomorphisms, such that* $\operatorname{Fix}(h) = \operatorname{Per}(h) = \{0\}$, 0 *is Lyapunov stable and* $\limsup \frac{\log(|i(h^m,0)|)}{m} = \infty$.

The techniques used for the computation of the indices are valid for both orientation preserving and orientation reversing homeomorphisms.

If X is a compact ANR (absolute neighborhood retract for metric spaces, see [11]), $i_X(f, p)$ will denote, if it is well defined, the fixed point index of f in a small enough neighborhood of p. When the indices are computed in the Euclidean space we shall write just $i(f, p)$.

The reader is referred to the text of [4,7,16] and the recent book of Jezierski and Marzantowicz [12] for information about the fixed point index theory. The last one is also appropriated to find in a unified way the results of [1,5,21] we mentioned above.

2. Preliminary definitions and some basic examples

Given $A \subset B \subset N$, $\operatorname{cl}(A)$, $\operatorname{cl}_B(A)$, $\operatorname{int}(A)$, $\operatorname{int}_B(A)$, ∂A and $\partial_B A$ will denote the closure of A, the closure of A in B, the interior of A, the interior of A in B, the boundary of A and the boundary of A in B, respectively.

Let $U \subset X$ be an open set. By a (*local*) *semidynamical system* we mean a local homeomorphism $f : U \to X$. The *invariant part* of N, $\operatorname{Inv}(N, f)$, is defined as the set of all $x \in N$ such that there is a full orbit γ with $x \in \gamma \subset N$.

A compact set $S \subset X$ is *invariant* if $f(S) = S$. A compact invariant set S is *isolated with respect to f* if there exists a compact neighborhood N of S such that $\operatorname{Inv}(N, f) = S$. The neighborhood N is called an *isolating neighborhood of S*.

An *isolating block* N is a compactum such that $\mathrm{cl}(\mathrm{int}(N)) = N$ and $f^{-1}(N) \cap N \cap f(N) \subset \mathrm{int}(N)$. Isolating blocks are a special class of isolating neighborhoods.

We consider the *exit set of* N to be defined as

$$N^- = \{x \in N\colon\ f(x) \notin \mathrm{int}(N)\}.$$

Let S be an isolated invariant set and suppose $L \subset N$ is a compact pair contained in the interior of the domain of f. The pair (N, L) is called a *filtration pair* for S (see Franks and Richeson paper [9]) provided N and L are each the closure of their interiors and

(1) $\mathrm{cl}(N \setminus L)$ is an isolating neighborhood of S,
(2) L is a neighborhood of N^- in N, and
(3) $f(L) \cap \mathrm{cl}(N \setminus L) = \emptyset$.

Remark 1. Filtration pairs are easy to construct once we have an isolating block N. In fact, for every small enough closed neighborhood L of N^-, (N, L) is a filtration pair [9].

In [20] we compute the indices of the iterates of $\mathbb{R}^3$-homeomorphisms, f, when there is a block N, that is topological closed ball, such that $\mathrm{Inv}(N, f) = \{0\}$. On the other hand, there are not techniques for the explicit computation of the sequence of the iterates of arbitrary homeomorphisms. Since in this paper we shall deal with homeomorphisms such that for every closed ball, B, centered in 0, $\mathrm{Inv}(B, f) \cap \partial B \neq \emptyset$, we will compute the sequence by approximating adequately our map by a sequence of homeomorphisms $\{f_m\}_{m\in\mathbb{N}}$ such that $\mathrm{Inv}(B, f_m) = \{0\}$ for every B and every $m \in \mathbb{N}$.

In the following examples there will be an isolating block N, which is a solid ball, such that $\mathrm{Inv}(N, f) = \{0\}$. The sequences of indices are easily seen to be periodic. However, they will provide some ingredients we shall need to prove Theorems 1 and 2.

2.1. *Examples where* $\mathrm{Inv}(N, f) = \{0\}$

Example 1. Consider the linear homeomorphism $g : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix

$$\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In this case, $\mathrm{Fix}(g) = \{0\}$, $\{0\}$ is the unique compact g-invariant set and 0 is a hyperbolic fixed point.

The computation of the sequence $\{i(g^k, 0)\}_{k\in\mathbb{N}}$ is a very easy problem using standard methods. However, we are going to calculate the sequence using different ideas.

There exist a filtration pair (N, L) such that N is an isolating block, a closed 3-dimensional ball, and L is a disjoint union of two balls L_1 and L_2. Identifying L_1 and L_2 to two different points q_1 and q_2 we obtain a quotient space, denoted by N_L, and a map induced by g, $\bar{g} : N_L \to N_L$, with $\mathrm{Fix}(\bar{g}) = \mathrm{Per}(\bar{g}) = \{0, q_1, q_2\}$.

Now, the Lefschetz number

$$\Lambda(\bar{g}^k) = 1 = i_{N_L}(\bar{g}^k, 0) + i_{N_L}(\bar{g}^k, q_1) + i_{N_L}(\bar{g}^k, q_2).$$

Since q_1 and q_2 are attractors in N_L, we have that $i_{N_L}(\bar{g}^k, q_1) = i_{N_L}(\bar{g}^k, q_2) = 1$. Then $i(g^k, 0) = i_{N_L}(\bar{g}^k, 0) = 1 - 2 = -1$ for every $k \in \mathbb{N}$.

On the other hand, since g preserves orientation, $i(g^{-k}, 0) = -i(g^k, 0) = 1$ for every $k \in \mathbb{N}$.

Using similar ideas we can compute the sequences of indices of the iterations of g^{-1} in another way. Analogously there exists a filtration pair (N, E) for g^{-1} such that N is again an isolating block, a closed 3-dimensional ball and E is now a solid torus.

Identifying E to a point q we obtain the quotient space N_E which is an ANR having the homotopy type of a 2-sphere.

Now, $2 = \Lambda((\overline{g^{-1}})^k) = i_{N_E}((\overline{g^{-1}})^k, 0) + i_{N_E}((\overline{g^{-1}})^k, q)$.

Then, $i_{\mathbb{R}^3}(g^{-k}, 0) = i_{N_E}((\overline{g^{-1}})^k, 0) = 2 - 1 = 1$ for every $k \in \mathbb{N}$, and again $i_{\mathbb{R}^3}(g^k, 0) = -i_{\mathbb{R}^3}(g^{-k}, 0) = -1$ for every $k \in \mathbb{N}$.

Example 2. Let C be the cube $C = [-1, 1]^3$. Joining 0 with the one skeleton of ∂C we obtain six closed and bounded cones with disjoint interiors $\{c_j\colon\ j = 1, \dots, 6\}$. Let $C_j = \{\lambda p\colon\ \lambda \geqslant 0$ and $p \in c_j\}$. It is clear that $\mathbb{R}^3 = \bigcup_{j\in\{1,\dots,6\}} C_j$.

Let g be the homeomorphism of the first example. Let $\pi^+ = \{(x, y, z) \in \mathbb{R}^3\colon\ z \geqslant 0\}$. The restriction $g|_{\pi^+}$ is conjugated to a homeomorphism $g_j : C_j \to C_j$. Consider the orientation preserving homeomorphism $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $\phi|_{C_j} = g_j$ for $j \in \{1, \dots, 6\}$.

We have that $\mathrm{Per}(\phi) = \mathrm{Fix}(\phi) = \{0\}$ and, again, $\{0\}$ is the unique compact ϕ-invariant set. Now, the stable "manifold" is the cone of the one-dimensional skeleton of ∂C and the unstable manifold decomposes into six one-dimensional branches (the union of the half lines joining 0 with the center of each face of ∂C).

It is not difficult to check that there exists a filtration pair (N, L) such that N is an isolating block 3-ball and L is disjoint union of six 3-balls.

If we identify each component of L to a different point we have the space N_L and the induced map $\bar{\phi} : N_L \to N_L$. Now, $\mathrm{Per}(\bar{\phi}) = \mathrm{Fix}(\bar{\phi}) = \{0, q_1, q_2, \dots, q_6\}$. All q_j's are attractors and then they have index $= 1$.

Therefore, $i(\phi^k, 0) = 1 - 6 = -5$ for every $k \in \mathbb{N}$. As a consequence, $i(\phi^{-k}, 0) = -i(\phi^k, 0) = 5$ for every $k \in \mathbb{N}$.

We can compute the sequence of indices of ϕ^{-1} directly by considering a filtration pair for ϕ^{-1}. Indeed, there is a pair (N, E) where N is a closed ball and E is an adequate tubular neighborhood of $\partial N \cap (\bigcup_{j\in\{1,\dots,6\}} \partial C_j)$. Now, the quotient space N_E is an ANR having the homotopy type of the wedge of five 2-spheres. Each of these five 2-spheres corresponds to one of the faces of ∂C. The remaining one represents the sum of the others.

Now, it is easy to check, by choosing obvious generators, that the matrix of $\overline{\phi^{-1}}{}^*_2 : H_2(N_E) \to H_2(N_E)$ can be assumed to be the identity.

Then, the Lefschetz number $\Lambda((\overline{\phi^{-1}})^k) = 6$ for all $k \in \mathbb{N}$ and $i(\phi^{-k}, 0) = 6 - 1 = 5$, $k \in \mathbb{N}$.

Example 3. Let $\psi : \mathbb{R}^3 \to \mathbb{R}^3$ be the homeomorphism given by the composition of $r_{\pi/2} \circ \phi^{-1}$ where ϕ is the homeomorphism of Example 2 and $r_{\pi/2}$ is the $\pi/2$-rotation with respect to the axis $\{(x, y, z) \in \mathbb{R}^3\colon\ x = y = 0\}$.

Here, the sequence is periodic of period 4 and it is not difficult to show that

$$i(\psi^{-k}, p) = \begin{cases} -5 & \text{if } k \in 4\mathbb{N}, \\ -1 & \text{if } k \notin 4\mathbb{N}, \end{cases}$$

and

$$i\big(\psi^k, p\big) = \begin{cases} 5 & \text{if } k \in 4\mathbb{N}, \\ 1 & \text{if } k \notin 4\mathbb{N}. \end{cases}$$

3. The construction of the homeomorphisms. Proof of the theorems

Proof of Theorem 1. Since we will be interested just in the elements of the sequence $\{c_m\}_{m\in\mathbb{N}}$ with m prime, we shall rename some of the terms of that sequence in the following way. If $q \in \mathbb{N}$ is the kth prime number, we will write $c_q = c'_k$.

Our aim is to construct a $\mathbb{R}^3$-homeomorphism f such that $\mathrm{Fix}(f) = \mathrm{Per}(f) = \{0\}$ and $\limsup \frac{|i(f^m,0)|}{c_m} = \infty$. For this end it is enough that for each k, if p denotes the kth prime, we get the index of f^p to be

$$i\big(f^p, 0\big) = -p^{c_p} = -p^{c'_k}.$$

To simplify the notation we will write again the new sequence $\{c'_m\}_{m\in\mathbb{N}}$ as $\{c_m\}_{m\in\mathbb{N}}$.

Let $N = B(0,1)$ be the unit closed ball. Our first step is to make a partition of N in solid regions. In each of these regions we will have different dynamics.

Let $A_i \subset N$ be the solid region limited by a cone C_i with vertex 0 and axis the line joining the poles n and s of N in such a way that

$$A_i \subsetneq A_{i+1} \subsetneq \cdots \subset N^+ \quad \text{and} \quad \mathrm{cl}\Big(\bigcup A_i\Big) = N^+$$

with $N^+ = \{\bar{x} = (x,y,z) \in N\colon z \geqslant 0\}$.

We define the different solid regions on which we will have the characteristic dynamics of f in the next way:

Let $S_0 = A_0$, $S_i = \mathrm{cl}(A_i \setminus A_{i-1})$ for $i \in \mathbb{N}$ and let $S_\infty = \{\bar{x} = (x,y,z) \in N\colon z \leqslant 0\} = N^-$.

We have a decomposition of N,

$$N = \bigcup_{m=0}^{\infty} S_m,$$

with $S_i \cap S_{i+1} = C_i \cap N^+$ and $S_i \cap S_j = \{0\}$ if $j \notin \{i-1, i, i+1\}$.

We construct the sets S_i, $i \notin \{0,\infty\}$, in such a way that the length of each of the two arcs $c_i \cup c'_i = \partial(N) \cap S_i \cap \{x = 0\}$ is $l_i = \frac{\pi}{2^{i+2}}$. The length of the arc $c_0 = \partial(N) \cap S_0 \cap \{x = 0\}$ is $\frac{\pi}{2}$. Let us observe that if we work with spherical coordinates (ρ, θ, ϕ), with

$$x = \rho \sin\phi \cos\theta, \qquad y = \rho \sin\phi \sin\theta, \qquad z = \rho \cos\theta,$$
$$\rho \geqslant 0,\ 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \phi \leqslant \pi,$$

the angle ϕ of the points in $C_n^+ = C_n \cap \pi^+$ is

$$\phi = \sum_{i=0}^{n} \frac{\pi}{2^{i+2}} = \frac{\pi}{2} - \frac{\pi}{2^{n+2}}.$$

We will define the homeomorphism f as the composition of two homeomorphisms $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ and $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ with $f = f_0 \circ g_0 : \mathbb{R}^3 \to \mathbb{R}^3$.

Let

$$E(A_i) = \{\lambda \bar{x} \colon \lambda \in \mathbb{R}^+,\ \bar{x} \in A_i\}.$$

In the same way we define the sets $E(S_i)$ with $i = 0, \ldots, \infty$.

Let $\{r_m\}_{m\in\mathbb{N}} = \{p_m/q_m\}_{m\in\mathbb{N}}$ be a sequence of rational numbers converging to an irrational number r with $\{q_m\}_{m\in\mathbb{N}}$ the sequence of prime numbers and such that $\gcd(p_m, q_m) = 1$. We can construct the sequence $\{r_m\}_{m\in\mathbb{N}}$ with $0 < r < 1$ in the following way:

For each q_m we consider a partition of the unit interval $[0, 1]$ in q_m intervals of length $1/q_m$ and select $p_m < q_m$ as the natural number such that $d(p_m/q_m, r) = \min\{d(n/q_m, r)\}$ with $n \in \mathbb{N}$. Then, the sequence $\{p_m/q_m\}_{m\in\mathbb{N}} \to r$ when $m \to \infty$ and $\gcd(p_m, q_m) = 1$.

In S_n with $n = 2m - 1$ odd we consider a family of $q_m^{c_m}$ isometric solid regions $\{T_{j,m}\}$, linearly isomorphic to the sets A_i, and such that $T_{j,m} \subset S_n$, $T_{j,m} \cap \partial(S_n) = D_{j,m} \cup \{0\}$ with $D_{j,m}$ a closed disc. We put these solid regions in S_n with constant angle $\frac{2\pi}{q_m^{c_m}}$ around the vertical axis (which joins the poles of N) and define $E(T_{j,m}) = \{\lambda \bar{x} \colon \lambda \in \mathbb{R}^+,\ \bar{x} \in T_{j,m}\}$. See Fig. 1.

We define f_0 in $E(S_\infty)$ as

$$f_0(\bar{x}) = \left(1 - \frac{1}{\pi}\lambda(\bar{x})\right)\bar{x}$$

with $\lambda(\bar{x})$ defined as the length of the obvious arc of $\partial(N)$ joining $\frac{\bar{x}}{|\bar{x}|}$ with the plane $\{z = 0\}$. See Fig. 2. Working with spherical coordinates we have

$$f_0(\rho, \theta, \phi) = \left(\left(\frac{1}{2} + \frac{-\phi}{\pi}\right)\rho, \theta, \phi\right).$$

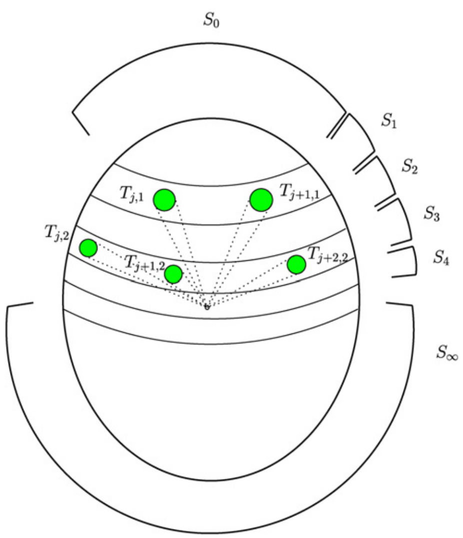

Fig. 1.

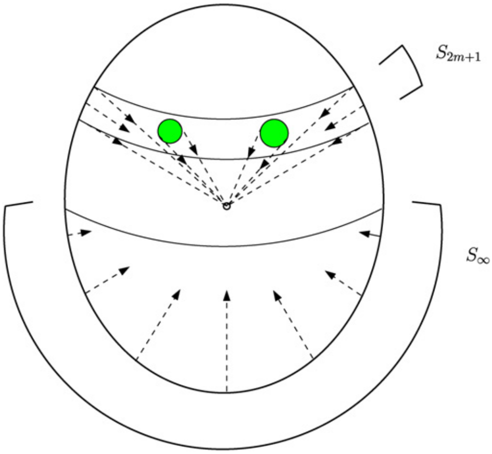

Fig. 2.

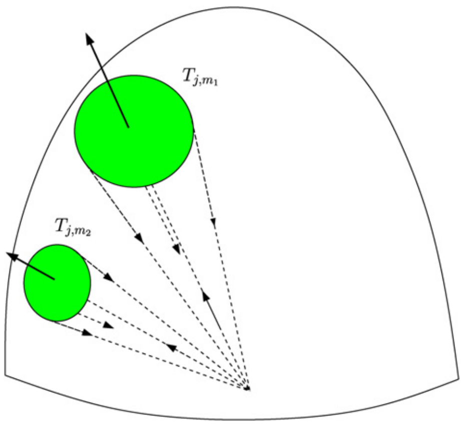

Fig. 3.

The definition of f_0 in two consecutive conical regions $C_{2m}^+ \cup C_{2m+1}^+$ is

$$f_0(\rho,\theta,\phi) = \left(\left(1 - \frac{1}{2^{2m}}\right)\rho,\theta,\phi\right).$$

In the same way, we define f_0 in each region $\mathrm{cl}(E(S_{2m+1}) \setminus \bigcup E(T_{j,m+1}))$ (see Fig. 2) as

$$f_0(\rho,\theta,\phi) = \left(\left(1 - \frac{1}{2^{2m}}\right)\rho,\theta,\phi\right).$$

On the other hand, the dynamical behavior of f_0 in the sets $E(T_{j,m}) \subset S_n$ for $n = 2m - 1$ is conjugated with the one given in Example 1 for the map $g|_{\pi^+} : \pi^+ \to \pi^+$ and commutes with a rotation of angle $\frac{2\pi}{q_m^{c_m}}$. Moreover, we construct f_0 in $E(T_{j,m}) \subset S_n$ in such a way that $d(f_0(\bar{x}), \bar{x}) \leqslant k_n \|\bar{x}\|$ for all $\bar{x} \in E(T_{j,m})$ and with $k_n \to 0$ when $n \to \infty$. See Fig. 3.

Let us suppose that $n = 2m$ even. For every point $\bar{x} \in E(S_{2m})$, the coordinate ϕ is in the interval $[\frac{\pi}{2} - \frac{\pi}{2^{2m+1}}, \frac{\pi}{2} - \frac{\pi}{2^{2m+2}}]$. The behavior of f_0 in the regions $E(S_n)$ with $n = 2m$, written in spherical coordinates, is

$$f_0(\rho, \theta, \phi) = \big(k_n(\phi)\rho, \theta, \phi\big)$$

with

$$k_n : \left[\frac{\pi}{2} - \frac{\pi}{2^{2m+1}}, \frac{\pi}{2} - \frac{\pi}{2^{2m+2}}\right] \to \left[1 - \frac{1}{2^{2(m-1)}}, 1 - \frac{1}{2^{2m}}\right]$$

an increasing, bijective linear map. See Fig. 4.

For each solid region $T_{j,m}$, the exit set of $f_0|_{T_{j,m}}$ is a topological closed ball $L_{j,m}$ such that $L_{j,m} \cap \partial(N)$ is a closed disc. These closed balls are the exit regions of N for $f_0|_{S_n}$ and have constant angle $\frac{2\pi}{q_m^{c_m}}$ around the vertical axis (which joins the poles of N). See Fig. 5.

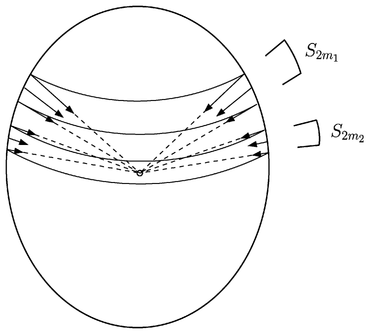

Fig. 4.

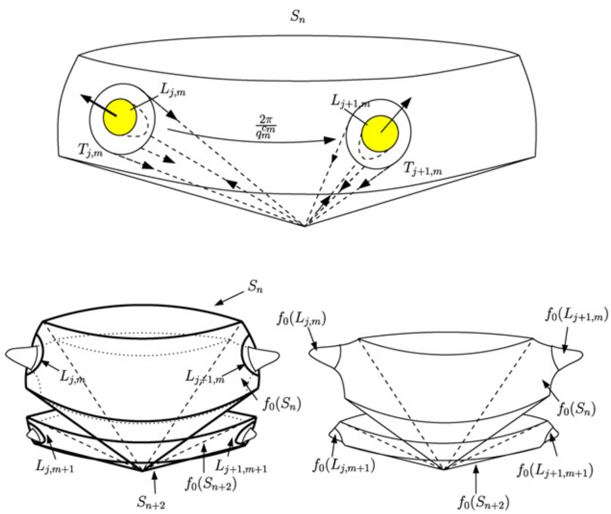

Fig. 5.

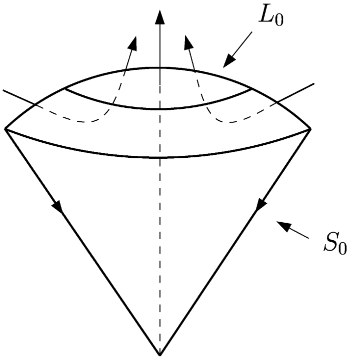

Fig. 6.

It only remains to construct $f_0|_{E(S_0)}$. It is topologically conjugated with dynamics given in Example 1 for $g|_{\pi^+}$. We obtain an exit region for $f_0|_{S_0}$ which is a closed ball $L_0 \subset S_0$ such that $L_0 \cap \partial(N)$ is a closed disc. The dynamical behavior is equivalent to the dynamics obtained in the sets $T_{j,m}$. See Fig. 6.

It is easy to check that the map $f_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism, limit of homeomorphisms $\{f_{0,n}\}_n$, with $f_{0,n} : \mathbb{R}^3 \to \mathbb{R}^3$ defined as

$$f_{0,n}(\bar{x}) = \begin{cases} f_0(\bar{x}) & \text{if } \bar{x} \in E(A_n) \cup E(A_n^-), \\ k(n)\bar{x} & \text{if } \bar{x} \in N \setminus (E(A_n) \cup E(A_n^-)), \end{cases}$$

where $A_n^- = \{\bar{x} \in N \text{ such that } -\bar{x} \in A_n\}$ and $k(n) = 1 - \frac{1}{2^{n-1}}$ if n is odd and $k(n) = 1 - \frac{1}{2^n}$ if n is even.

Let us observe that $\mathrm{Fix}(f_{0,n}) = \mathrm{Per}(f_{0,n}) = \mathrm{Inv}(N, f_{0,n}) = \{0\}$ and $\mathrm{Fix}(f_0|_N) = \mathrm{Per}(f_0|_N) = \mathrm{Inv}(N, f_0) = N \cap \{z = 0\}$ with $N^-(f_0) = \{\bar{x} \in N\colon\ f_0(x) \notin \mathrm{int}(N)\} = \bigcup L_{j,m} \cup (\{z = 0\} \cap \partial(N)) \cup L_0$.

The homeomorphism $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is defined in the following way:

The map $g_0|_{E(S_n)}$ with $n = 2m - 1$ odd is a rotation around the vertical axis with angle $2\pi \frac{p_m}{q_m}$, that is,

$$g_0|_{E(S_{2m-1})}(\rho, \theta\, \phi) = \left(\rho, \theta + 2\pi \frac{p_m}{q_m}, \phi\right).$$

The restrictions $g_0|_{E(S_\infty)}$ and $g_0|_{E(S_0)}$ are rotations around the vertical axis with angles $2\pi r$ and $2\pi \frac{p_1}{q_1}$, respectively.

The dynamical behavior of $g_0|_{E(S_n)}$ with $n = 2m$ even is as follows:

Since $g_0|_{C_{n-1} \cap \pi^+}$ and $g_0|_{C_n \cap \pi^+}$ are rotations with angles $2\pi \frac{p_m}{q_m}$ and $2\pi \frac{p_{m+1}}{q_{m+1}}$, given a cone C with vertex 0 and axis the line joining the poles of N such that $C \cap \pi^+ \subset E(S_n)$, we construct the dynamics in $C \cap \pi^+$ as a rotation with angle $c \in [2\pi \frac{p_m}{q_m}, 2\pi \frac{p_{m+1}}{q_{m+1}}]$ in such a way that c tends to $2\pi \frac{p_m}{q_m}$ $(2\pi \frac{p_{m+1}}{q_{m+1}})$ if C tends to C_{n-1} (C_n), see Fig. 7. Working with spherical coordinates

$$g_0|_{S_{2m}}(\rho, \theta\, \phi) = \big(\rho, \theta + k_n(\phi), \phi\big)$$

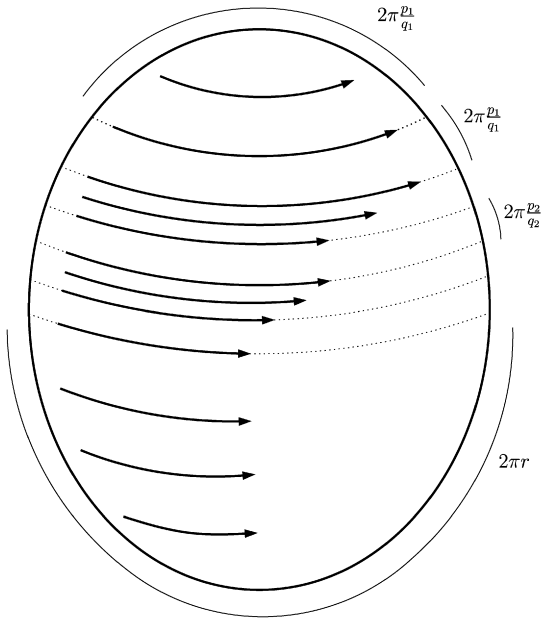

Fig. 7.

with

$$k_n : \left[\frac{\pi}{2} - \frac{\pi}{2^{2m+1}}, \frac{\pi}{2} - \frac{\pi}{2^{2m+2}}\right] \to \left[2\pi \frac{p_m}{q_m}, 2\pi \frac{p_{m+1}}{q_{m+1}}\right]$$

an increasing, bijective linear map.

The constructed map $g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism and limit of homeomorphisms $\{g_{0,n}\}_n$ with $g_{0,n}$ defined as follows:

Let us define $g_{0,n}$ for $n = 2m - 1$ odd (if n is even, the construction is analogous). Given $\bar{x} \in E(A_n) \cup E(A_n^-)$ we define $g_{0,n}(\bar{x}) = g_0(\bar{x})$. On the other hand, let us observe that for every $\bar{x} \in C_n \cap \pi^+$ the spherical coordinates are (ρ, θ, ϕ_n) with $\phi_n = \frac{\pi}{2} - \frac{\pi}{2^{n+1}}$ fixed. Since $g_0|_{C_n \cap \pi^+}$ and $g_0|_{C_n \cap \pi^-}$ are rotations with angles $2\pi \frac{p_m}{q_m}$ and $2\pi r$, for each $\bar{x} \in \mathrm{cl}(\mathbb{R}^3 \setminus (E(A_n) \cup E(A_n^-)))$ with spherical coordinates (ρ, θ, ϕ) we construct $g_{0,n}$ as a rotation with angle $c(\phi) \in [2\pi \frac{p_m}{q_m}, 2\pi r]$ in such a way that c tends to $2\pi \frac{p_m}{q_m}$ $(2\pi r)$ if ϕ tends to ϕ_n $(\frac{\pi}{2} - \phi_n)$. Then, given $n = 2m - 1$,

$$g_{0,n}|_{\mathrm{cl}(\mathbb{R}^3 \setminus (E(A_n) \cup E(A_n^-)))}(\rho, \theta, \phi) = \big(\rho, \theta + k_n(\phi), \phi\big)$$

with

$$k_n : \left[\frac{\pi}{2} - \frac{\pi}{2^{2m}}, \frac{\pi}{2} + \frac{\pi}{2^{2m}}\right] \to \left[2\pi \frac{p_m}{q_m}, 2\pi r\right]$$

an increasing, bijective linear map.

The map $f = f_0 \circ g_0 : \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism with $\mathrm{Fix}(f) = \mathrm{Per}(f) = \{0\}$ and $\mathrm{Inv}(N, f) = N \cap \{z = 0\}$. If we consider the sequence of homeomorphisms $\{f_n\}_n$ with $f_n = f_{0,n} \circ g_{0,n} : \mathbb{R}^3 \to \mathbb{R}^3$ we have

$$\mathrm{Fix}(f_n) = \mathrm{Per}(f_n) = \mathrm{Inv}(N, f_n) = \{0\}$$

and it is obvious that f is limit of the homeomorphisms $\{f_n\}$.

Let us compute the fixed point index $i(f^n, 0)$ for $n \in \mathbb{N}$. For this purpose we will use the next two results of existence of homotopies between close maps and the homotopy invariance of the fixed point index.

Proposition 1. *Let $f : X \subset \mathbb{R}^m \to \mathbb{R}^n \setminus \{0\}$ be a continuous map. Then if $g : X \to \mathbb{R}^n \setminus \{0\}$ is a continuous map close enough to f, then f and g are homotopic.*

Proposition 2. *Let X be a metric ANR, W an open subset of X and $F : \mathrm{cl}(W) \times [0, 1] \to X$ a continuous and compact map such that $F(x, t) \neq x$ for $(x, t) \in \partial(W) \times [0, 1]$. Then $i_X(F_t, W)$ is constant for $0 \leqslant t \leqslant 1$.*

Let us fix $d \in \mathbb{N}$. Since the map $f^d|_N : N \to \mathbb{R}^3$ can be approximated by maps of the type $f_n^d|_N : N \to \mathbb{R}^3$, from the first of the two propositions there exists $n_0 \in \mathbb{N}$ such that for each $n \geqslant n_0$ there exists a homotopy $H : N \times I \to \mathbb{R}^3$ with $H_0 = f^d$, $H_1 = f_n^d$ and $H(x, t) \neq x$ for all $x \in \partial(N)$ and $t \in [0, 1]$. From the second we obtain that $i(f^d, 0) = i(f_n^d, 0)$.

Let us compute $i(f_n^d, 0)$. There exists a finite family of closed balls $\{L_{j,m}\}$ contained in N which are the exit regions of N for $f_n|_N$. Identifying the sets $\{L_{j,m}\}$ to points $\{l_{j,m}\}$ we obtain a quotient space N_L, which is a closed ball, and an induced map $\bar{f}_n : N_L \to N_L$. It is obvious that $i_{N_L}(\bar{f}_n^d, 0) = i(f_n^d, 0)$. Given m fixed, the action of the map $\bar{f}_n$ on the family of points $\{l_{j,m}\}_j$, with $j = 1, \ldots, q_m^{c_m}$, give us a union of $q_m^{c_m - 1}$ cycles of length q_m,

$$\{l_{j,m}\}_j = \bigcup_k \bigl\{l(k, 1), \ldots, l(k, q_m)\bigr\},$$

with $k = 1, \ldots, q_m^{c_{m-1}}$, such that

$$\bar{f}_n\bigl(l(k, r)\bigr) = l(k, r + 1)$$

for $r = 1, \ldots, q_m$. See Fig. 8.

It is obvious that

$$i_{N_L}\bigl(\bar{f}_n^d, l_{j,m}\bigr) = \begin{cases} 1 & \text{if } d \in q_m\mathbb{N}, \\ 0 & \text{if } d \notin q_m\mathbb{N}. \end{cases}$$

We obtain the equality

$$1 = i_{N_L}\bigl(\bar{f}_n^d, N\bigr) = i\bigl(f^d, 0\bigr) + \sum_{\substack{j=1,\ldots,q_m^{c_m} \\ q_m | d}} i_{N_L}\bigl(\bar{f}_n^d, l_{j,m}\bigr) + 1,$$

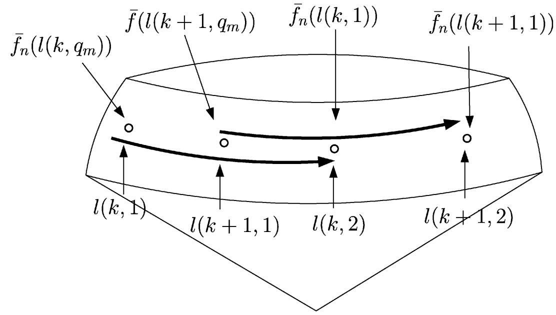

Fig. 8.

where the last 1 corresponds to the closed ball L_0. Then,

$$i\left(f^d,0\right) = -\sum_{\substack{j=1,\dots,q_m^{c_m}\\ q_m|d}} i_{N_L}\left(\bar{f}_n^d, l_{j,m}\right) = -\sum_{q_m|d} q_m^{c_m}.$$

If we consider $d = q_m$ prime,

$$i\left(f^{q_m},0\right) = -q_m^{c_m}$$

and the result is proved.

On the other hand, let us observe that if we consider the sequence of natural numbers $\{q_m^k\}_k$ with q_m the mth prime number and $k \in \mathbb{N}$, then

$$i\left(f^{q_m^k},0\right) = -q_m^{c_m} \quad \text{for all } k \in \mathbb{N}.$$

Let us observe also that for each $m = p_1^{r_1} \cdots p_k^{r_k} \in \mathbb{N}$, with $p_1, \dots, p_k$ different prime numbers, $i(f^m,0) = i(f_n^m,0) = -\sum_{j=1,\dots,k} p_j^{c_j}$ for every $n \geqslant \max\{p_1,\dots,p_k\}$. $\square$

Remark 2. One can consider the dual construction of Theorem 1, i.e. the map f at ∞ and the inverse homeomorphism f^{-1}. In the first case, for every closed ball, B, centered in ∞, $\mathrm{Inv}(B,f) \cap \partial B \neq \emptyset$ and in the latter the exit sets for each of the analogous approaching homeomorphisms, h_m, are solid $(1 + \sum q_{k_m}^{c_{k_m}})$-tori. Following similar arguments (see also the examples in Section 2), one has that

$$i\left(f^{-n},0\right) = i\left(f^n,\infty\right) = \sum_{q_m|n} q_m^{c_m}.$$

Consequently, if we see the homeomorphism f as a S^3-homeomorphism such that $\mathrm{Fix}(f) = \mathrm{Per}(f) = \{0,\infty\}$, it follows that $\limsup \frac{|i(f^m,0)|}{c_m} = \limsup \frac{|i(f^m,\infty)|}{c_m} = \infty$.

Proof of Theorem 2. The ingredients of the proof of Theorem 2 are the homeomorphisms given in Theorem 1 and the plug construction developed by Wilson in [23] (see also [3]). We shall maintain the notation of Theorem 1.

Consider the solid cylinder $B = \{(x, y, z) \in \mathbb{R}^3\colon\ x^2 + y^2 \leqslant 1,\ z \in [a, b]\}$ and the flow induced by the constant vector field $Y = (0, 0, 1)$. Denote respectively by $\sigma(B)$, $\tau(B)$ and $\beta(B)$ the lateral, top and bottom boundaries of B.

A *flow box* (U, g) for a vector field X at a point p consists of a neighborhood U of p and a diffeomorphism $g : B \to U$ such that

(i) X is transverse to $g(\beta(B))$;
(ii) there is a positive constant c such that $\phi(ct, g(x)) = g(\psi(t, x))$ where $\phi(t,\cdot)$ and $\psi(t,\cdot)$ denote the flows induced by X and Y on B, respectively. When it is clear from the context, we shall omit the diffeomorphism g.

Let U and V be two flow boxes with $V \subset U$. Then V is called *a shrinkage* of U if $\sigma(V) \subset \operatorname{int}(U)$, $\tau(V) \subset \tau(U)$ and $\beta(V) \subset \beta(U)$.

Let us recall the following version of Wilson's theorem [23] that we will need.

Theorem 3. *Let X be a C^∞ $\mathbb{R}^3$-vector field. Let U be a flow box of X and let V be a shrinkage of U. Then, there exists a C^∞ vector field X^1 on U such that*

(a) *X^1 coincides with X on a neighborhood of ∂U*;
(b) *the limit sets of X^1 are a finite collection of invariant circles on which the restricted flow is minimal*;
(c) *every trajectory of X^1 which intersects $\beta(V)$ remains in positive time inside U*;
(d) *each trajectory of X^1 which leaves U in positive and negative time coincides as a point set with some trajectory of X in a neighborhood of ∂U.*

Consider now the semi-space $\pi^+ = \{(x, y, z) \in \mathbb{R}^3\colon\ z \geqslant 0\}$ and let $X : \pi^+ \to \mathbb{R}^3$ be the vector field $X(x, y, z) = (-x, -y, z)$. Let ϕ be the flow in π^+ induced by X and let $D_{m,l} = \{(x, y, z) \in \mathbb{R}^3\colon\ x^2 + y^2 \leqslant 1/m,\ z = 1/l\}$.

For every natural number $n \geqslant 2$, take the cylinder $B_n = \{(x, y, z) \in \mathbb{R}^3\colon\ x^2 + y^2 \leqslant 1/n,\ z \in [1/n, 1/(n-1)]\}$. Now for every positive even integer k, we define the flow boxes $U_k = \{\phi(\bar{x}, t)\colon\ \bar{x} \in D_{k,k},\ t \geqslant 0\} \cap B_k$ and $V_k = \{\phi(\bar{x}, t)\colon\ \bar{x} \in D_{k/2,k},\ t \geqslant 0\} \cap B_k$. It is clear that V_k is a shrinkage of U_k. On the other hand, $U_k \cap U_{k'} = \emptyset$ if $k \neq k'$.

For each $k \in 2\mathbb{N}$, let X^1_k be the vector field obtained by applying Wilson's theorem to X and the pair (U_k, V_k).

Now let $G : \pi^+ \to \mathbb{R}^3$ be the vector field defined as $G(p) = X(p)$ if $p \notin \bigcup_{k\in 2\mathbb{N}} U_k$ and $G(p) = X^1_k(p)$ if $p \in U_k$. Finally consider a flat enough (in 0) smooth non-negative real map γ, depending of $\|p\|^2$, such that $\gamma^{-1}(0) = \{0\}$ to obtain $X_1 = \gamma G$ to be smooth.

Let ψ be the flow in π^+ associated to X_1. The set of periodic orbits of ψ is countable. Then we can choose a positive and decreasing sequence $t_n \to 0$ such that $\operatorname{Fix}(\psi(t_n,\cdot)) = \operatorname{Per}(\psi(t_n,\cdot)) = \{0\}$. Since each $D_{k/2,k}$ is a section that captures every orbit in $\operatorname{int}(\pi^+)$ near 0, it is clear that 0 is Lyapunov stable.

Now, we shall apply the same construction of Theorem 1 but we will paste adequately, in every cone, copies of homeomorphims conjugated to $\psi(t_n,\cdot) : \pi^+ \to \pi^+$ instead of homeomorphisms conjugated to the map $g|_{\pi^+}$ of Fig. 3 and Example 1.

As in Theorem 1, for every $n = 2m - 1$ odd we have in each sector $E(S_n)$ a finite family of identical cones $E(T_{j,m})$, $j \in \{1, 2, \ldots, q_m^{c_m}\}$. For every m there is a canonical cone $E(T_m) \subset$

$\mathrm{int}(\pi^+)$ which is isometric to every $E(T_{j,m})$. Let $h_m : \pi^+ \to E(T_m)$ be a homeomorphisms such that for every $\bar{x} \in \partial(E(T_m))$, $\|h_m^{-1}(\bar{x})\| = \|\bar{x}\|$.

Now define the homeomorphisms $\psi'_m = h_m \circ \psi(t_m, \cdot) \circ h_m^{-1} : E(T_m) \to E(T_m)$.

Begin with an $\mathbb{R}^3$-homeomorphism (dynamically equivalent to the homeomorphism f_0 of Theorem 1 in $\mathbb{R}^3 \setminus \bigcup \mathrm{int}(E(T_{j,m}))$), $h_1 : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathrm{Fix}(h_1) = \{z = 0\}$, h_1 is decreasing in each ray $\{\lambda\bar{x}\colon \lambda \geqslant 0, \bar{x} \in \mathbb{R}^3 \setminus \{z = 0\}\}$ and h_1 behaves in each ray in $\partial(E(T_{j,m}))$ as ψ'_m.

Replacing, in each cone $E(T_{j,m})$, h_1 by copies of ψ'_m we obtain an $\mathbb{R}^3$-homeomorphism h_0.

Let $h = g_0 \circ h_0$. We obtain in this way an $\mathbb{R}^3$-homeomorphism such that $\mathrm{Fix}(h) = \mathrm{Per}(h) = \{0\}$ and 0 is Lyapunov stable. It is easy to see that also h is limit of a sequence of homeomorphisms for which every closed ball centered in 0 and large enough radius is still an isolating block with the same exit sets and the same behavior as in Theorem 1. Then, the sequence of fixed point indices of the iterates of h and f coincide. □

Final remarks

(i) In Theorem 1, if $B \subset \mathbb{R}^3$ is any closed ball centered in the origin, $\mathrm{Inv}(B, f)$ is the closed 2-disc $B \cap \{z = 0\}$. For this kind of nice compacta there is a 3-dimensional Carathéodory's compactification (see [2]) and one could try to apply the ideas of Le Calvez to reduce the problem of the computation of the indices to the case where the fixed point is an isolated invariant set. Unfortunately this method is not longer valid because, in this case, the two associated fixed prime ends are not isolated invariant sets.

In Theorem 2, if $B \subset \mathbb{R}^3$ is any closed ball centered in the origin, $\mathrm{Inv}(B, f)$ contains the union of the closed 2-disc $B \cap \{z = 0\}$ and a countable family of circles.

(ii) Consider the restriction to π^+ of the homeomorphisms f and h of Theorems 1 and 2. We can define, by symmetry, global $\mathbb{R}^3$-homeomorphisms, F and H. Now let S be the symmetry with respect to the plane $\pi = \{z = 0\}$. Now, $S \circ F$ and $S \circ H$ are orientation reversing homeomorphisms such that $\mathrm{Fix}(S \circ F) = \mathrm{Per}(S \circ F) = \mathrm{Fix}(S \circ H) = \mathrm{Per}(S \circ H) = \{0\}$. Of course 0 is again Lyapunov stable for $S \circ H$ and, in this case, $i((S \circ F)^{2k+1}, 0) = i((S \circ H)^{2k+1}, 0) = 1$ for every $k \in \mathbb{N}$. For even iterates we have that $i((S \circ F)^{2k}, 0) = i((S \circ H)^{2k}, 0) = -1 + 2i(f^{2k}, 0)$.

References

[1] I.K. Babenko, S.A. Bogatyi, The behavior of the index of periodic points under iterations of a mapping, Math. USSR Izvestiya 38 (1992) 1–26.

[2] B.L. Brechner, J.S. Lee, A three dimensional prime end theory, Topology Proc. 20 (1995) 15–47.

[3] C. Bonatti, J. Villadelprat, The index of stable critical points, Topology Appl. 126 (1–2) (2002) 263–271.

[4] R.F. Brown, The Lefschetz Fixed Point Theorem, Scott Foresman Co., Glenview, IL, 1971.

[5] S.N. Chow, J. Mallet-Paret, J.A. Yorke, A periodic orbit index which is a bifurcation invariant, in: Geometric Dynamics, Rio de Janeiro, 1981, in: Lect. Notes in Math., vol. 1007, Springer, Berlin, 1983, pp. 109–131.

[6] E.N. Dancer, R. Ortega, The index or Lyapunov stable fixed points, J. Dynam. Differential Equations 6 (1994) 631–637.

[7] A. Dold, Fixed point indices of iterated maps, Invent. Math. 74 (1983) 419–435.

[8] E. Erle, Stable equilibria and vector field index, Topology Appl. 49 (1993) 231–235.

[9] J. Franks, D. Richeson, Shift equivalence and the Conley index, Trans. Amer. Math. Soc. 352 (7) (2000) 3305–3322.

[10] G. Graff, P. Nowak-Przygodzki, Fixed point indices of iterations of C^1-maps in $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. 4 (2006) 843–856.

[11] S.T. Hu, Theory of Retracts, Wayne State University Press, 1965.

[12] J. Jezierski, W. Marzantowicz, Homotopy Methods in Topological Fixed and Periodic Points Theory, Springer, 2005.

[13] P. Le Calvez, J.C. Yoccoz, Un théoréme d'indice pour les homéomorphismes du plan au voisinage d'un poin fixe, Ann. of Math. 146 (1997) 241–293.
[14] P. Le Calvez, J.C. Yoccoz, Suite des indices de Lefschetz des itérés pour un domaine de Jordan qui est un bloc isolant, unpublished.
[15] P. Le Calvez, Dynamique des homéomorphismes du plan au voisinage d'un point fixe, Ann. Sci. Ecole Norm. Sup. (4) 36 (1) (2003) 139–171.
[16] R.D. Nussbaum, The fixed point index and some applications, in: Séminaire de Mathématiques supérieures, Les Presses de L'Université de Montréal, 1985.
[17] F.R. Ruiz del Portal, Planar isolated and stable fixed points have index = 1, J. Differential Equations 199 (2004) 179–188.
[18] F.R. Ruiz del Portal, J.M. Salazar, Fixed point index of iterations of local homeomorphisms of the plane: A Conley-index approach, Topology 41 (2002) 1199–1212.
[19] F.R. Ruiz del Portal, J.M. Salazar, A Poincaré formula for the fixed point indices of the iterations of arbitrary planar homeomorphisms, preprint.
[20] F.R. Ruiz del Portal, J.M. Salazar, Fixed point indices of the iterations of $\mathbb{R}^3$-homeomorphisms, preprint.
[21] M. Shub, D. Sullivan, A remark on the Lefschetz fixed point formula for differentiable maps, Topology 13 (1974) 189–191.
[22] J. Vaughan (Ed.), Open problems collected from the Spring Topology and Dynamical Systems Conference, 2006, http://www.uncg.edu/mat/stdc.
[23] F.W. Wilson, On the minimal sets of non-singular vector fields, Ann. of Math. 84 (1966) 529–536.