Available online at www.sciencedirect.com

ScienceDirect

Nuclear Physics B 910 (2016) 496–527

NUCLEAR PHYSICS B

www.elsevier.com/locate/nuclphysb

Construction and classification of novel BPS Wilson loops in quiver Chern–Simons-matter theories

Hao Ouyang [a,b], Jun-Bao Wu [a,b], Jia-ju Zhang [a,b,*]

[a] *Theoretical Physics Division, Institute of High Energy Physics, Chinese Academy of Sciences, 19B Yuquan Rd, Beijing 100049, China*

[b] *Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, 19B Yuquan Rd, Beijing 100049, China*

Received 15 December 2015; accepted 14 July 2016

Available online 21 July 2016

Stephan Stieberger

Abstract

In this paper we construct and classify novel Drukker–Trancanelli (DT) type BPS Wilson loops along infinite straight lines and circles in $\mathcal{N} = 2, 3$ quiver superconformal Chern–Simons-matter theories, Aharony–Bergman–Jafferis–Maldacena (ABJM) theory, and $\mathcal{N} = 4$ orbifold ABJM theory. Generally we have four classes of Wilson loops, and all of them preserve the same supersymmetries as the BPS Gaiotto–Yin (GY) type Wilson loops. There are several free complex parameters in the DT type BPS Wilson loops, and for two classes of Wilson loops in ABJM theory and $\mathcal{N} = 4$ orbifold ABJM theory there are supersymmetry enhancements at special values of the parameters. We check that the differences of the DT type and GY type Wilson loops are Q-exact with Q being some supercharges preserved by both the DT type and GY type Wilson loops. The results would be useful to calculate vacuum expectation values of the DT type Wilson loops in matrix models if they are still BPS quantum mechanically.

* Corresponding author.

E-mail addresses: ouyangh@ihep.ac.cn (H. Ouyang), wujb@ihep.ac.cn (J.-B. Wu), jjzhang@ihep.ac.cn (J.-j. Zhang).

http://dx.doi.org/10.1016/j.nuclphysb.2016.07.018

1. Introduction

BPS (Bogomol'nyi–Prasad–Sommerfield) Wilson loops are important nonlocal objects in supersymmetric gauge theories, and in AdS/CFT correspondence [1–3] they are dual to probe F-strings/membranes in string/M theory [4–7], when they are in fundamental representation of the gauge group. BPS Wilson loops in three-dimensional quiver superconformal Chern–Simons-matter (CSM) theories are more complex than those in four-dimensional super Yang–Mills theories. One can use only bosonic fields and construct Gaiotto–Yin (GY) type Wilson loops [8], and also one can use both bosonic and fermionic fields and construct Drukker–Trancanelli (DT) type Wilson loops [9].

In $\mathcal{N}=2$ CSM theories there are 1/2 BPS GY type Wilson loops along infinite straight lines and circles, and in $\mathcal{N}=3$ CSM theories there are 1/3 BPS GY type Wilson loops [8]. The Aharony–Bergman–Jafferis–Maldacena (ABJM) theory is an $\mathcal{N}=6$ CSM theory with gauge group $U(N)\times U(N)$ and levels $(k,-k)$, and it is dual to M-theory in $\mathrm{AdS}_4\times \mathrm{S}^7/\mathrm{Z}_k$ spacetime or type IIA string theory in $\mathrm{AdS}_4\times \mathrm{CP}^3$ spacetime [10]. In ABJM theory, there are 1/6 BPS GY type Wilson loops [11–13] and 1/2 BPS DT type Wilson loops [9] along infinite straight lines and circles. The construction of the latter ones resolved the puzzle about the existence of the half BPS Wilson loop dual to half BPS F-string solution found in [11,13]. In ABJM theory there are more general BPS DT type Wilson loops along general curves that preserve fewer supersymmetries [14–18]. The $\mathcal{N}=4$ CSM theories were constructed in [19–21], and a special case is the $\mathcal{N}=4$ orbifold ABJM theory that has gauge group $U(N)^{2r}$ and alternating levels $(k,-k,\cdots,k,-k)$ and is dual to M-theory in $\mathrm{AdS}_4\times \mathrm{S}^7/(\mathrm{Z}_r\times \mathrm{Z}_{rk})$ spacetime [21–24]. Recently, 1/4 BPS GY type Wilson loops and 1/2 BPS DT type BPS Wilson loops in $\mathcal{N}=4$ orbifold ABJM theory were constructed in [25,26]. BPS Wilson loops in more general $\mathcal{N}=4$ CSM theories were also constructed in [26].

As announced in [27], we found novel DT type BPS Wilson loops in quiver superconformal CSM theories. In $\mathcal{N}=2$ quiver superconformal CSM theories, though the supersymmetries are relatively fewer, we found that there still exist DT type 1/2 BPS Wilson loops. This construction, when applied to theories with more supersymmetries, leads to DT type Wilson loops preserving two Poincaré supercharges (and also two superconformal charges), when the Wilson loops are along straight lines and the parameters in the Wilson loops are not under further constraints. In ABJM theory and $\mathcal{N}=4$ CSM theory, supersymmetry (SUSY) enhancements for Wilson loops can appear for special values of these parameters. We find no SUSY enhancement for Wilson loops in $\mathcal{N}=3$ CSM theories. This is consistent with the results in the dual M-theory side [28].

This paper is an extension of [27], with calculation details and more examples. We also pay more attention to classification of DT type Wilson loops. We construct novel DT type BPS Wilson loops along straight lines and circles in several quiver superconformal CSM theories. We investigate the case of a generic $\mathcal{N}=2$ quiver CSM theory with multiple bifundamental and anti-bifundamental matter in section 2, the case of an $\mathcal{N}=3$ quiver CSM theory in section 3, the case of ABJM theory in section 4, and the case of $\mathcal{N}=4$ orbifold ABJM theory in section 5. We conclude with discussion in section 6.

2. Generic $\mathcal{N}=2$ quiver CSM theory

We consider a generic $\mathcal{N}=2$ quiver superconformal CSM theory with bifundamental matter. We pick two adjacent nodes in the quiver diagram and the corresponding gauge groups are $U(N)$ and $U(M)$. The vector multiplet for gauge group $U(N)$ includes gauge field A_μ, and auxiliary fields σ, χ, D, with σ, D being bosonic and χ being fermionic. Similarly, for gauge group $U(M)$ we have the vector multiplet with fields $\hat{A}_\mu$, $\hat{\chi}$, $\hat{\sigma}$, $\hat{D}$. There are multiple matter fields in bifundamental and anti-bifundamental representations in the $\mathcal{N}=2$ theory. These multiplets include fields ϕ_i, ψ_i, F_i and $\phi_{\hat{\imath}}$, $\psi_{\hat{\imath}}$, $F_{\hat{\imath}}$, with $i = 1, 2, \cdots, N_f$, and $\hat{\imath} = \hat{1}, \hat{2}, \cdots, \hat{N}_{\hat{f}}$. Here F_i and $F_{\hat{\imath}}$ are bosonic auxiliary fields. Note that the number of chiral multiplets in bifundamental representation N_f does not necessarily equal to number of chiral multiplets in anti-bifundamental representation $\hat{N}_{\hat{f}}$. There could be other matter that couple to the two gauge fields, and they will change the on-shell values of σ and $\hat{\sigma}$ in the Wilson loops we will construct, but the structure of these Wilson loops will not change.

The SUSY transformations could be found in [29]. For the vector multiplet part, we only need the off-shell SUSY transformations of $A_\mu, \sigma, \hat{A}_\mu, \hat{\sigma}$

$$\begin{aligned}
\delta A_\mu &= \frac{1}{2}(\bar{\chi}\gamma_\mu\epsilon + \bar{\epsilon}\gamma_\mu\chi), \quad \delta\sigma = -\frac{\mathrm{i}}{2}(\bar{\chi}\epsilon + \bar{\epsilon}\chi), \\
\delta \hat{A}_\mu &= \frac{1}{2}(\bar{\hat{\chi}}\gamma_\mu\epsilon + \bar{\epsilon}\gamma_\mu\hat{\chi}), \quad \delta\hat{\sigma} = -\frac{\mathrm{i}}{2}(\bar{\hat{\chi}}\epsilon + \bar{\epsilon}\hat{\chi}),
\end{aligned} \tag{2.1}$$

and for the matter part we only need the off-shell transformations

$$\begin{aligned}
&\delta\phi_i = \mathrm{i}\bar{\epsilon}\psi_i, \quad \delta\bar{\phi}^i = \mathrm{i}\bar{\psi}^i\epsilon, \quad \delta\phi_{\hat{\imath}} = \mathrm{i}\bar{\epsilon}\psi_{\hat{\imath}}, \quad \delta\bar{\phi}^{\hat{\imath}} = \mathrm{i}\bar{\psi}^{\hat{\imath}}\epsilon \\
&\delta\psi_i = (-\gamma^\mu D_\mu\phi_i - \sigma\phi_i + \phi_i\hat{\sigma})\epsilon - \vartheta\phi_i + \mathrm{i}\bar{\epsilon}F_i, \\
&\delta\bar{\psi}^i = \bar{\epsilon}(\gamma^\mu D_\mu\bar{\phi}^i + \hat{\sigma}\bar{\phi}^i - \bar{\phi}^i\sigma) - \bar{\vartheta}\bar{\phi}^i - \mathrm{i}\epsilon\bar{F}^i, \\
&\delta\psi_{\hat{\imath}} = (-\gamma^\mu D_\mu\phi_{\hat{\imath}} - \hat{\sigma}\phi_{\hat{\imath}} + \phi_{\hat{\imath}}\sigma)\epsilon - \vartheta\phi_{\hat{\imath}} + \mathrm{i}\bar{\epsilon}F_{\hat{\imath}}, \\
&\delta\bar{\psi}^{\hat{\imath}} = \bar{\epsilon}(\gamma^\mu D_\mu\bar{\phi}^{\hat{\imath}} + \sigma\bar{\phi}^{\hat{\imath}} - \bar{\phi}^{\hat{\imath}}\hat{\sigma}) - \bar{\vartheta}\bar{\phi}^{\hat{\imath}} - \mathrm{i}\epsilon\bar{F}^{\hat{\imath}}.
\end{aligned} \tag{2.2}$$

The definitions of covariant derivatives are

$$\begin{aligned}
&D_\mu\phi_i = \partial_\mu\phi_i + \mathrm{i}A_\mu\phi_i - \mathrm{i}\phi_i\hat{A}_\mu, \quad D_\mu\bar{\phi}_i = \partial_\mu\bar{\phi}_i + \mathrm{i}\hat{A}_\mu\bar{\phi}_i - \mathrm{i}\bar{\phi}_i A_\mu, \\
&D_\mu\phi_{\hat{\imath}} = \partial_\mu\phi_{\hat{\imath}} + \mathrm{i}\hat{A}_\mu\phi_{\hat{\imath}} - \mathrm{i}\phi_{\hat{\imath}}A_\mu, \quad D_\mu\bar{\phi}_{\hat{\imath}} = \partial_\mu\bar{\phi}_{\hat{\imath}} + \mathrm{i}A_\mu\bar{\phi}_{\hat{\imath}} - \mathrm{i}\bar{\phi}_{\hat{\imath}}\hat{A}_\mu.
\end{aligned} \tag{2.3}$$

We have SUSY parameters $\epsilon = \theta + x^\mu\gamma_\mu\vartheta$, $\bar{\epsilon} = \bar{\theta} - \bar{\vartheta}x^\mu\gamma_\mu$, and terms with $\theta, \bar{\theta}$ denote Poincaré SUSY transformations and terms with ϑ, $\bar{\vartheta}$ denote superconformal transformations.

We adopt the spinor convention in [30]. The metric on the three-dimensional Minkowski spacetime is $\eta_{\mu\nu} = \mathrm{diag}(-++)$, the coordinates are $x^\mu = (x^0, x^1, x^2)$, and the gamma matrices are

$$\gamma^\mu{}_\alpha{}^\beta = (\mathrm{i}\sigma^2, \sigma^1, \sigma^3), \tag{2.4}$$

with $\sigma^{1,2,3}$ being Pauli matrices. Charge conjugate of a spinor is defined as complex conjugate

$$\bar{\theta}_\alpha = \theta^*_\alpha. \tag{2.5}$$

The Poincaré supercharges P, $\bar{P}$ and conformal supercharges S, $\bar{S}$ are related to the SUSY transformations as

$$\delta = \mathrm{i}(\bar{\theta}P + \bar{P}\theta + \bar{\vartheta}S + \bar{S}\vartheta). \tag{2.6}$$

There are constraints

$$\bar{\theta} = \theta^*, \quad \bar{\vartheta} = \vartheta^*, \quad \bar{P} = P^*, \quad \bar{S} = S^*. \tag{2.7}$$

After Wick rotation, we get a theory in Euclidean space. In three-dimensional Euclidean space, the metric is $\delta_{\mu\nu} = \mathrm{diag}(+++)$, the coordinates are $x^\mu = (x^1, x^2, x^3)$, and the gamma matrices are

$$\gamma^\mu{}_\alpha{}^\beta = (-\sigma^2, \sigma^1, \sigma^3). \tag{2.8}$$

Spinors θ and $\bar\theta$ are generally unrelated. Formally, the SUSY transformations (2.1) and (2.2) and the definitions of supercharges (2.6) still apply to Euclidean version of the $\mathcal{N} = 2$ CSM theory, but there are no longer the constraints (2.7).

2.1. Straight line in Minkowski spacetime

For BPS Wilson loops along infinite straight lines, the Poincaré and conformal supersymmetries are preserved separately, and the discussions of them are similar. So we will only consider Poincaré supercharges for Wilson loops along straight lines.

In Minkowski spacetime, one can construct the 1/2 BPS GY type Wilson loop along a timelike infinite straight line $x^\mu = \tau\delta_0^\mu$ as [8]

$$\begin{aligned}
&W_{\mathrm{GY}} = \mathcal{P}\exp\left(-\mathrm{i}\int \mathrm{d}\tau L_{\mathrm{GY}}(\tau)\right),\\
&L_{\mathrm{GY}} = \begin{pmatrix} A_\mu \dot x^\mu + \sigma|\dot x| & \\ & \hat A_\mu \dot x^\mu + \hat\sigma|\dot x| \end{pmatrix}.
\end{aligned} \tag{2.9}$$

The preserved Poincaré supersymmetries can be denoted as

$$\gamma_0\theta = \mathrm{i}\theta, \quad \bar\theta\gamma_0 = \mathrm{i}\bar\theta. \tag{2.10}$$

We construct the DT type Wilson loop along the line $x^\mu = \tau\delta_0^\mu$

$$\begin{aligned}
&W_{\mathrm{DT}} = \mathcal{P}\exp\left(-\mathrm{i}\int \mathrm{d}\tau L_{\mathrm{DT}}(\tau)\right), \quad L_{\mathrm{DT}} = \begin{pmatrix} \mathcal{A} & \bar f_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix},\\
&\mathcal{A} = A_\mu \dot x^\mu + \sigma|\dot x| + \mathcal{B}|\dot x|, \quad \hat{\mathcal{A}} = \hat A_\mu \dot x^\mu + \hat\sigma|\dot x| + \hat{\mathcal{B}}|\dot x|,\\
&\mathcal{B} = M^i{}_j\phi_i\bar\phi^j + M_{\hat\imath}{}^{\hat\jmath}\bar\phi^{\hat\imath}\phi_{\hat\jmath} + M^{i\hat\imath}\phi_i\phi_{\hat\imath} + M_{\hat\imath i}\bar\phi^{\hat\imath}\bar\phi^i,\\
&\hat{\mathcal{B}} = N_i{}^j\bar\phi^i\phi_j + N^{\hat\imath}{}_{\hat\jmath}\phi_{\hat\imath}\bar\phi^{\hat\jmath} + N_{i\hat\imath}\bar\phi^i\bar\phi^{\hat\imath} + N^{\hat\imath i}\phi_{\hat\imath}\phi_i,\\
&\bar f_1 = (\bar\zeta^i\psi_i + \bar\psi^{\hat\imath}\mu_{\hat\imath})|\dot x|, \quad f_2 = (\bar\psi^i\eta_i + \bar\nu^{\hat\imath}\psi_{\hat\imath})|\dot x|.
\end{aligned} \tag{2.11}$$

To make it preserve the supersymmetries (2.10), it is enough to require that [31]

$$\delta L_{\mathrm{DT}} = \partial_\tau G + \mathrm{i}[L_{\mathrm{DT}}, G], \tag{2.12}$$

for some Grassmann odd matrix

$$G = \begin{pmatrix} & \bar g_1 \\ g_2 & \end{pmatrix}. \tag{2.13}$$

Concretely, one needs

$$\begin{aligned}
&\delta\mathcal{A} = \mathrm{i}(\bar f_1 g_2 - \bar g_1 f_2),\\
&\delta\hat{\mathcal{A}} = \mathrm{i}(f_2\bar g_1 - g_2\bar f_1),\\
&\delta\bar f_1 = \partial_\tau\bar g_1 + \mathrm{i}\mathcal{A}\bar g_1 - \mathrm{i}\bar g_1\hat{\mathcal{A}},\\
&\delta f_2 = \partial_\tau g_2 + \mathrm{i}\hat{\mathcal{A}} g_2 - \mathrm{i} g_2\mathcal{A}.
\end{aligned} \tag{2.14}$$

From variations of $\mathcal{A}$ and $\hat{\mathcal{A}}$ we have

$$\begin{aligned}
&M^i{}_j = N_j{}^i, \quad M_{\hat{\imath}}{}^{\hat{\jmath}} = N^{\hat{\jmath}}{}_{\hat{\imath}}, \quad M^{i\hat{\imath}} = N^{\hat{\imath}i}, \quad M_{\hat{\imath}i} = N_{i\hat{\imath}}, \\
&(M^j{}_i\phi_j + M_{\hat{\imath}i}\bar{\phi}^{\hat{\imath}})\theta = \eta_i\bar{g}_1, \quad (M_{\hat{\jmath}}{}^{\hat{\imath}}\bar{\phi}^{\hat{\jmath}} + M^{i\hat{\imath}}\phi_i)\bar{\theta} = -\bar{g}_1\bar{\nu}^{\hat{\imath}}, \\
&(M^i{}_j\bar{\phi}^j + M^{i\hat{\imath}}\phi_{\hat{\imath}})\bar{\theta} = -g_2\bar{\zeta}^i, \quad (M_{\hat{\imath}}{}^{\hat{\jmath}}\phi_{\hat{\jmath}} + M_{\hat{\imath}i}\bar{\phi}^i)\theta = \mu_{\hat{\imath}}g_2.
\end{aligned} \tag{2.15}$$

From variations of $\bar{f}_1$ and f_2 we have

$$\begin{aligned}
&\bar{\zeta}^i\gamma_0 = \mathrm{i}\bar{\zeta}^i, \quad \gamma_0\mu_{\hat{\imath}} = \mathrm{i}\mu_{\hat{\imath}}, \quad \gamma_0\eta_i = \mathrm{i}\eta_i, \quad \bar{\nu}^{\hat{\imath}}\gamma_0 = \mathrm{i}\bar{\nu}^{\hat{\imath}}, \\
&\bar{g}_1 = \mathrm{i}\bar{\zeta}^i\theta\phi_i - \mathrm{i}\bar{\theta}\mu_{\hat{\imath}}\bar{\phi}^{\hat{\imath}}, \quad \mathcal{B}\bar{g}_1 = \bar{g}_1\hat{\mathcal{B}}, \\
&g_2 = -\mathrm{i}\bar{\theta}\eta_i\bar{\phi}^i + \mathrm{i}\bar{\nu}^{\hat{\imath}}\theta\phi_{\hat{\imath}}, \quad \hat{\mathcal{B}}g_2 = g_2\mathcal{B}.
\end{aligned} \tag{2.16}$$

We have the parameterizations

$$\begin{aligned}
&\bar{\zeta}^i = \bar{\alpha}^i\bar{\zeta}, \quad \mu_{\hat{\imath}} = \mu\gamma_{\hat{\imath}}, \quad \eta_i = \eta\beta_i, \quad \bar{\nu}^{\hat{\imath}} = \bar{\delta}^{\hat{\imath}}\bar{\nu}, \\
&\bar{\zeta}^\alpha = \bar{\nu}^\alpha = (1, \mathrm{i}), \quad \eta_\alpha = \mu_\alpha = (1, -\mathrm{i}),
\end{aligned} \tag{2.17}$$

with $\bar{\alpha}^i$, $\gamma_{\hat{\imath}}$, β_i and $\bar{\delta}^{\hat{\imath}}$ being complex constants. We have four classes of solutions, and all of them satisfy

$$\begin{aligned}
&M^{i\hat{\imath}} = M_{\hat{\imath}i} = \bar{\alpha}^i\bar{\delta}^{\hat{\imath}} = \gamma_{\hat{\imath}}\beta_i = 0, \\
&M^i{}_j = 2\mathrm{i}\bar{\alpha}^i\beta_j, \quad M_{\hat{\imath}}{}^{\hat{\jmath}} = 2\mathrm{i}\gamma_{\hat{\imath}}\bar{\delta}^{\hat{\jmath}}.
\end{aligned} \tag{2.18}$$

Class I

In the first solution we have

$$\gamma_{\hat{\imath}} = \bar{\delta}^{\hat{\imath}} = 0. \tag{2.19}$$

Here $\bar{\alpha}^i$ and β_j are free complex parameters.

We define that

$$L_{\mathrm{DT}} - L_{\mathrm{GY}} = L_B + L_F, \tag{2.20}$$

with L_B being the bosonic part and L_F being the fermionic part. We want to show that the difference of the DT and GY type BPS Wilson loop is Q-exact, i.e. that $W_{\mathrm{DT}} - W_{\mathrm{GY}} = QV$, with Q being some supercharge that is preserved by both the DT and GY type BPS Wilson loops. For the straight line case, it is enough to show [9,25,26]

$$\begin{aligned}
&\kappa\Lambda^2 = L_B, \quad Q\Lambda = L_F, \quad QL_{\mathrm{GY}} = 0, \\
&QL_F = \partial_\tau(\mathrm{i}\kappa\Lambda) + \mathrm{i}[L_{\mathrm{GY}}, \mathrm{i}\kappa\Lambda],
\end{aligned} \tag{2.21}$$

for some matrix Λ, factor κ and supercharge Q. Now we have

$$\begin{aligned}
&L_B = 2\mathrm{i}\bar{\alpha}^i\beta_j\begin{pmatrix}\phi_i\bar{\phi}^j & \\ & \bar{\phi}^j\phi_i\end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\alpha}^i\bar{\zeta}\psi_i \\ \bar{\psi}^i\eta\beta_i & \end{pmatrix}, \\
&\Lambda = \begin{pmatrix} & \bar{\alpha}^i\phi_i \\ \beta_i\bar{\phi}^i & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\zeta}P + \bar{P}\eta.
\end{aligned} \tag{2.22}$$

Class II

In the second solution we have

$$\bar{\alpha}^i = \beta_i = 0. \tag{2.23}$$

Now we have

$$L_B = 2\mathrm{i}\gamma_{\hat{\imath}}\bar{\delta}^{\hat{\jmath}} \begin{pmatrix} \bar{\phi}^{\hat{\imath}}\phi_{\hat{\jmath}} & \\ & \phi_{\hat{\jmath}}\bar{\phi}^{\hat{\imath}} \end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\psi}^{\hat{\imath}}\mu\gamma_{\hat{\imath}} \\ \bar{\delta}^{\hat{\imath}}\bar{\nu}\psi_{\hat{\imath}} & \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} & \gamma_{\hat{\imath}}\bar{\phi}^{\hat{\imath}} \\ \bar{\delta}^{\hat{\imath}}\phi_{\hat{\imath}} & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\nu}P + \bar{P}\mu. \tag{2.24}$$

Class III

In the third solution we have

$$\beta_i = \bar{\delta}^{\hat{\imath}} = 0. \tag{2.25}$$

Now we have

$$L_B = 0, \quad L_F = \begin{pmatrix} & \bar{\alpha}^i\bar{\zeta}\psi_i + \bar{\psi}^{\hat{\imath}}\mu\gamma_{\hat{\imath}} \\ 0 & \end{pmatrix}, \tag{2.26}$$

$$\Lambda = \begin{pmatrix} & \bar{\alpha}^i\phi_i + \gamma_{\hat{\imath}}\bar{\phi}^{\hat{\imath}} \\ 0 & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\zeta}P + \bar{P}\mu.$$

Class IV

In the fourth solution we have

$$\bar{\alpha}^i = \gamma_{\hat{\imath}} = 0. \tag{2.27}$$

Now we have

$$L_B = 0, \quad L_F = \begin{pmatrix} & 0 \\ \bar{\psi}^i\eta\beta_i + \bar{\delta}^{\hat{\imath}}\bar{\nu}\psi_{\hat{\imath}} & \end{pmatrix}, \tag{2.28}$$

$$\Lambda = \begin{pmatrix} & 0 \\ \beta_i\bar{\phi}^i + \bar{\delta}^{\hat{\imath}}\phi_{\hat{\imath}} & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\nu}P + \bar{P}\eta.$$

2.2. *Circle in Euclidean space*

People are more interested in circular BPS Wilson loops, since they usually have nontrivial vacuum expectation values. It was shown in [30] that there are no spacelike BPS Wilson loops in Minkowski spacetime, and so we have to consider circular BPS Wilson loops in Euclidean version of the $\mathcal{N} = 2$ superconformal CSM theory. For circular BPS Wilson loops there are no separately preserved Poincaré and conformal supercharges, and only special combinations of them are preserved. So we have to consider both the Poincaré and conformal supercharges for circular Wilson loops.

In Euclidean space we have the 1/2 BPS GY type Wilson loop along the circle $x^\mu = (\cos\tau, \sin\tau, 0)$

$$W_{\mathrm{GY}} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau L_{\mathrm{GY}}(\tau)\right), \tag{2.29}$$

$$L_{\mathrm{GY}} = \begin{pmatrix} A_\mu\dot{x}^\mu - \mathrm{i}\sigma|\dot{x}| & \\ & \hat{A}_\mu\dot{x}^\mu - \mathrm{i}\hat{\sigma}|\dot{x}| \end{pmatrix},$$

and the preserved supersymmetries are

$$\vartheta = \mathrm{i}\gamma_3\theta, \quad \bar{\vartheta} = \bar{\theta}\mathrm{i}\gamma_3. \tag{2.30}$$

We construct the DT type Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$

$$\begin{aligned}
&W_{\mathrm{DT}} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau L_{\mathrm{DT}}(\tau)\right), \quad L_{\mathrm{DT}} = \begin{pmatrix} \mathcal{A} & \bar{f}_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix}, \\
&\mathcal{A} = A_\mu \dot{x}^\mu - \mathrm{i}\sigma|\dot{x}| - 2(\bar{\alpha}^i\beta_j\phi_i\bar{\phi}^j + \gamma_{\hat{\imath}}\bar{\delta}^{\hat{\jmath}}\bar{\phi}^{\hat{\imath}}\phi_{\hat{\jmath}})|\dot{x}|, \\
&\hat{\mathcal{A}} = \hat{A}_\mu \dot{x}^\mu - \mathrm{i}\hat{\sigma}|\dot{x}| - 2(\bar{\alpha}^i\beta_j\bar{\phi}^j\phi_i + \gamma_{\hat{\imath}}\bar{\delta}^{\hat{\jmath}}\phi_{\hat{\jmath}}\bar{\phi}^{\hat{\imath}})|\dot{x}|, \\
&\bar{f}_1 = (\bar{\alpha}^i\bar{\zeta}\psi_i + \bar{\psi}^{\hat{\imath}}\mu\gamma_{\hat{\imath}})|\dot{x}|, \quad f_2 = (\bar{\psi}^i\eta\beta_i + \bar{\delta}^{\hat{\imath}}\bar{\nu}\psi_{\hat{\imath}})|\dot{x}|, \\
&\bar{\zeta}^\alpha = \bar{\nu}^\alpha = (\mathrm{e}^{\mathrm{i}\tau/2}, \mathrm{e}^{-\mathrm{i}\tau/2}), \quad \eta_\alpha = \mu_\alpha = (\mathrm{e}^{-\mathrm{i}\tau/2}, \mathrm{e}^{\mathrm{i}\tau/2}).
\end{aligned} \tag{2.31}$$

Similar to the case in Minkowski spacetime, we have four classes of solutions that make this circular DT type Wilson loop 1/2 BPS and preserve supersymmetries (2.30), and all of them satisfy

$$\bar{\alpha}^i\bar{\delta}^{\hat{\imath}} = \gamma_{\hat{\imath}}\beta_i = 0. \tag{2.32}$$

Class I

In the first solution we have

$$\gamma_{\hat{\imath}} = \bar{\delta}^{\hat{\imath}} = 0. \tag{2.33}$$

To show that the difference of the DT and GY type BPS Wilson loop is Q-exact, for the circle case we have to show [9,25,26]

$$\begin{aligned}
&\kappa\Lambda^2 = L_B, \quad \kappa(0) = \kappa(2\pi), \quad \Lambda(0) = -\Lambda(2\pi), \\
&Q\Lambda = L_F, \quad QL_{\mathrm{GY}} = 0, \quad QL_F = \partial_\tau(\mathrm{i}\kappa\Lambda) + \mathrm{i}[L_{\mathrm{GY}}, \mathrm{i}\kappa\Lambda],
\end{aligned} \tag{2.34}$$

for some matrix Λ, factor κ and supercharge Q. Now we have

$$\begin{aligned}
&L_B = -2\bar{\alpha}^i\beta_j\begin{pmatrix} \phi_i\bar{\phi}^j & \\ & \bar{\phi}^j\phi_i \end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\alpha}^i\bar{\zeta}\psi_i \\ \bar{\psi}^i\eta\beta_i & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2}\begin{pmatrix} & \bar{\alpha}^i\phi_i \\ \beta_i\bar{\phi}^i & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P + \mathrm{i}\gamma_3 S) + (\bar{P} + \bar{S}\mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1, 0), \quad b_\alpha = (0, 1).
\end{aligned} \tag{2.35}$$

Class II

In the second solution we have

$$\bar{\alpha}^i = \beta_i = 0. \tag{2.36}$$

Now we have

$$\begin{aligned}
&L_B = -2\gamma_{\hat{\imath}}\bar{\delta}^{\hat{\jmath}}\begin{pmatrix} \bar{\phi}^{\hat{\imath}}\phi_{\hat{\jmath}} & \\ & \phi_{\hat{\jmath}}\bar{\phi}^{\hat{\imath}} \end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\psi}^{\hat{\imath}}\mu\gamma_{\hat{\imath}} \\ \bar{\delta}^{\hat{\imath}}\bar{\nu}\psi_{\hat{\imath}} & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2}\begin{pmatrix} & \gamma_{\hat{\imath}}\bar{\phi}^{\hat{\imath}} \\ \bar{\delta}^{\hat{\imath}}\phi_{\hat{\imath}} & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P + \mathrm{i}\gamma_3 S) + (\bar{P} + \bar{S}\mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1, 0), \quad b_\alpha = (0, 1).
\end{aligned} \tag{2.37}$$

Class III

In the third solution we have

$$\beta_i = \bar{\delta}^{\hat{\imath}} = 0. \tag{2.38}$$

Now we have

$$\begin{aligned}
&L_B = 0, \quad L_F = \begin{pmatrix} & \bar{\alpha}^i \bar{\zeta} \psi_i + \bar{\psi}^{\hat{\imath}} \mu \gamma_{\hat{\imath}} \\ 0 & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & \bar{\alpha}^i \phi_i + \gamma_{\hat{\imath}} \bar{\phi}^{\hat{\imath}} \\ 0 & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P + \mathrm{i}\gamma_3 S) + (\bar{P} + \bar{S}\mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1, 0), \quad b_\alpha = (0, 1).
\end{aligned} \tag{2.39}$$

Class IV

In the fourth solution we have

$$\bar{\alpha}^i = \gamma_{\hat{\imath}} = 0. \tag{2.40}$$

Now we have

$$\begin{aligned}
&L_B = 0, \quad L_F = \begin{pmatrix} & 0 \\ \bar{\psi}^i \eta \beta_i + \bar{\delta}^{\hat{\imath}} \bar{\nu} \psi_{\hat{\imath}} & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & 0 \\ \beta_i \bar{\phi}^i + \bar{\delta}^{\hat{\imath}} \phi_{\hat{\imath}} & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P + \mathrm{i}\gamma_3 S) + (\bar{P} + \bar{S}\mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1, 0), \quad b_\alpha = (0, 1).
\end{aligned} \tag{2.41}$$

3. $\mathcal{N} = 3$ quiver CSM theory

We consider an $\mathcal{N} = 3$ quiver superconformal CSM theory and pick two adjacent nodes in the quiver diagram. The first vector multiplet includes gauge field A_μ and auxiliary fields $\sigma^a{}_b$, $\chi^a{}_b$, ξ, with $a, b, \cdots = 1, 2$ being indices of the $SU(2)$ R-symmetry. Here $\sigma^a{}_b$ is bosonic and $\chi^a{}_b$, ξ are fermionic. We have the constraints

$$\begin{aligned}
&\sigma^a{}_a = 0, \quad \sigma^a{}_b = (\sigma^b{}_a)^\dagger, \\
&\chi^a{}_a = 0, \quad \chi^a{}_b = (\sigma^b{}_a)^\dagger, \\
&\xi = \xi^\dagger,
\end{aligned} \tag{3.1}$$

with $\dagger$ being not only hermitian conjugate of color index but also complex conjugate of spinor index. Similarly, for the second vector multiplet we have fields $\hat{A}_\mu$, $\hat{\sigma}^a{}_b$, $\hat{\chi}^a{}_b$, $\hat{\xi}$. There are bifundamental matter fields ϕ^{ia}, ψ^{ia} with $i, j, \cdots = 1, 2, \cdots, N_f$ denoting indices of flavor. From the results in [8,32] we rewrite the off-shell SUSY transformations with manifest $SU(2)$ R-symmetry

$$\begin{aligned}
&\delta A_\mu = \frac{\mathrm{i}}{2} \chi^a{}_b \gamma_\mu \epsilon^b{}_a, \quad \delta \hat{A}_\mu = \frac{\mathrm{i}}{2} \hat{\chi}^a{}_b \gamma_\mu \epsilon^b{}_a, \\
&\delta \sigma^a{}_b = -\frac{1}{2}(\chi^a{}_c \epsilon^c{}_b - \epsilon^a{}_c \chi^c{}_b) + \mathrm{i}\xi \epsilon^a{}_b, \\
&\delta \hat{\sigma}^a{}_b = -\frac{1}{2}(\hat{\chi}^a{}_c \epsilon^c{}_b - \epsilon^a{}_c \hat{\chi}^c{}_b) + \mathrm{i}\hat{\xi} \epsilon^a{}_b,
\end{aligned} \tag{3.2}$$

$$\delta\phi^{ia} = \mathrm{i}\epsilon^a{}_b\psi^{ib}, \quad \delta\bar{\phi}_{ia} = \mathrm{i}\bar{\psi}_{ib}\epsilon^b{}_a,$$
$$\delta\psi^{ia} = -\gamma^\mu\epsilon^a{}_b D_\mu\phi^{ib} - \vartheta^a{}_b\phi^{ib} - \epsilon^b{}_c(\sigma^a{}_b\phi^{ic} - \phi^{ic}\hat{\sigma}^a{}_b),$$
$$\delta\bar{\psi}_{ia} = \epsilon^b{}_a\gamma^\mu D_\mu\bar{\phi}_{ib} - \vartheta^b{}_a\bar{\phi}_{ib} - \epsilon^c{}_b(\bar{\phi}_{ic}\sigma^b{}_a - \hat{\sigma}^b{}_a\bar{\phi}_{ic}).$$

We have covariant derivatives

$$D_\mu\phi^{ia} = \partial_\mu\phi^{ia} + \mathrm{i}A_\mu\phi^{ia} - \mathrm{i}\phi^{ia}\hat{A}_\mu,$$
$$D_\mu\bar{\phi}_{ia} = \partial_\mu\bar{\phi}_{ia} + \mathrm{i}\hat{A}_\mu\bar{\phi}_{ia} - \mathrm{i}\bar{\phi}_{ia}A_\mu. \tag{3.3}$$

We have the SUSY transformation parameter $\epsilon^a{}_b = \theta^a{}_b + x^\mu\gamma_\mu\vartheta^a{}_b$, and there are constraints

$$\theta^a{}_a = 0, \quad \theta^a{}_b = (\theta^b{}_a)^*, \quad \vartheta^a{}_a = 0, \quad \vartheta^a{}_b = (\vartheta^b{}_a)^*. \tag{3.4}$$

Note that each of $\theta^a{}_b$ and $\vartheta^a{}_b$ has the degrees of freedom of one Dirac spinor and one Majorana spinor. The Poincaré and conformal supercharges are defined as

$$\delta = \mathrm{i}(\theta^a{}_b P^b{}_a + \vartheta^a{}_b S^b{}_a) = \mathrm{i}(P^a{}_b\theta^b{}_a + S^a{}_b\vartheta^b{}_a), \tag{3.5}$$

with constraints

$$P^a{}_a = 0, \quad P^a{}_b = (P^b{}_a)^*, \quad S^a{}_a = 0, \quad S^a{}_b = (S^b{}_a)^*. \tag{3.6}$$

In Euclidean space, the SUSY transformations (3.2) and definitions of supercharges (3.5) still apply, but the constraints for the SUSY parameters and supercharges are

$$\theta^a{}_a = \vartheta^a{}_a = 0, \quad P^a{}_a = S^a{}_a = 0. \tag{3.7}$$

3.1. Straight line in Minkowski spacetime

We consider BPS GY and DT type Wilson loops along timelike infinite straight lines in Minkowski spacetime.

3.1.1. GY type Wilson loop

We construct a GY type Wilson loop along the infinite straight line $x^\mu = \tau\delta^\mu_0$ as

$$W_{\mathrm{GY}} = \mathcal{P}\exp\left(-\mathrm{i}\int \mathrm{d}\tau L_{\mathrm{GY}}(\tau)\right), \quad L_{\mathrm{GY}} = \begin{pmatrix} \mathcal{A}_{\mathrm{GY}} & \\ & \hat{\mathcal{A}}_{\mathrm{GY}} \end{pmatrix}, \tag{3.8}$$
$$\mathcal{A}_{GY} = A_\mu\dot{x}^\mu + R^a{}_b\sigma^b{}_a|\dot{x}|, \quad \hat{\mathcal{A}}_{GY} = \hat{A}_\mu\dot{x}^\mu + S^a{}_b\hat{\sigma}^b{}_a|\dot{x}|.$$

Without loss of generality we set $R^a{}_a = S^a{}_a = 0$. We take SUSY transformation $\delta\mathcal{A}_{\mathrm{GY}} = 0$ and get

$$\gamma_0\theta^a{}_b = \mathrm{i}(R^a{}_c\theta^c{}_b - \theta^a{}_c R^c{}_b), \quad R^a{}_b\theta^b{}_a = 0, \tag{3.9}$$

whose complex conjugates are

$$\gamma_0\theta^a{}_b = \mathrm{i}(R^{\dagger a}{}_c\theta^c{}_b - \theta^a{}_c R^{\dagger c}{}_b), \quad R^{\dagger a}{}_b\theta^b{}_a = 0. \tag{3.10}$$

Here the matrix $R^\dagger$ is the hermitian conjugate of R, i.e. $R^{\dagger a}{}_b = (R^b{}_a)^*$. We define

$$R = B + \mathrm{i}C, \quad B = \frac{R + R^\dagger}{2}, \quad C = -\mathrm{i}\frac{R - R^\dagger}{2}, \tag{3.11}$$

with B and C being traceless hermitian matrices, i.e. $B^a{}_a = 0$, $B = B^\dagger$, $C^a{}_a = 0$, $C = C^\dagger$. Then we get

$$\gamma_0 \theta^a{}_b = \mathrm{i}(B^a{}_c \theta^c{}_b - \theta^a{}_c B^c{}_b), \quad C^a{}_c \theta^c{}_b = \theta^a{}_c C^c{}_b, \quad B^a{}_b \theta^b{}_a = C^a{}_b \theta^b{}_a = 0. \tag{3.12}$$

We make an $SU(2)$ R-symmetry transformation and set $B^a{}_b = \mathrm{diag}(b, -b)$ with b being real. We have

$$\gamma_0 \theta^1{}_2 = 2\mathrm{i}b\theta^1{}_2, \quad \gamma_0 \theta^2{}_1 = -2\mathrm{i}b\theta^2{}_1, \quad \theta^1{}_1 = \theta^2{}_2 = 0. \tag{3.13}$$

Since the eigenvalues of γ_0 must be $\pm\mathrm{i}$, for $\theta^1{}_2 \neq 0$ we have $b = \pm\frac{1}{2}$, and without loss of generality we choose $b = \frac{1}{2}$. Then we get $C^a{}_b = 0$. From SUSY transformation $\delta\hat{\mathcal{A}}_{\mathrm{GY}} = 0$, we have

$$\gamma_0 \theta^a{}_b = \mathrm{i}(S^a{}_c \theta^c{}_b - \theta^a{}_c S^c{}_b), \quad R^a{}_b \theta^b{}_a = 0. \tag{3.14}$$

Then we get

$$(R^a{}_c - S^a{}_c)\theta^c{}_b = \theta^a{}_c (R^c{}_b - S^c{}_b), \tag{3.15}$$

and this leads to $R^a{}_b = S^a{}_b$.

In summary we have the GY type 1/3 BPS Wilson loop (3.8) with $R^a{}_b = S^a{}_b = \mathrm{diag}(\frac{1}{2}, -\frac{1}{2})$, and the preserved supersymmetries are

$$\gamma_0 \theta^1{}_2 = \mathrm{i}\theta^1{}_2, \quad \gamma_0 \theta^2{}_1 = -\mathrm{i}\theta^2{}_1, \quad \theta^1{}_1 = \theta^2{}_2 = 0. \tag{3.16}$$

This is just the Wilson loop that was constructed in [8]. Here we show that it is the only kind of BPS GY type Wilson loops along timelike infinite straight lines, up to some R-symmetry transformations.

3.1.2. DT type Wilson loop

Along $x^\mu = \tau\delta^\mu_0$ we construct the DT type Wilson loop

$$\begin{aligned}
&W_{\mathrm{DT}} = \mathcal{P}\exp\left(-\mathrm{i}\int \mathrm{d}\tau\, L_{\mathrm{DT}}(\tau)\right), \quad L_{\mathrm{DT}} = \begin{pmatrix} \mathcal{A} & \bar{f}_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix}, \\
&\mathcal{A} = \mathcal{A}_{\mathrm{GY}} + M_{ia}{}^{jb}\phi^{ia}\bar{\phi}_{jb}|\dot{x}|, \quad \bar{f}_1 = \bar{\zeta}_{ia}\psi^{ia}|\dot{x}|, \\
&\hat{\mathcal{A}} = \hat{\mathcal{A}}_{\mathrm{GY}} + N^{ia}{}_{jb}\bar{\phi}_{ia}\phi^{jb}|\dot{x}|, \quad f_2 = \bar{\psi}_{ia}\eta^{ia}|\dot{x}|.
\end{aligned} \tag{3.17}$$

We want the Wilson loop to preserve the supersymmetries (3.16). From variations of $\mathcal{A}$ and $\hat{\mathcal{A}}$ we get

$$M_{ia}{}^{jb} = N^{jb}{}_{ia}, \quad \eta^{ia}\bar{g}_1 = M_{jb}{}^{ic}\theta^a{}_c\phi^{jb}, \quad g_2\bar{\zeta}_{ia} = -M_{ic}{}^{jb}\theta^c{}_a\bar{\phi}_{jb}. \tag{3.18}$$

From variations of $\bar{f}_1$ and f_2 we have

$$\begin{aligned}
&\bar{\zeta}_{i1}\gamma_0 = \mathrm{i}\bar{\zeta}_{i1}, \quad \bar{\zeta}_{i2}\gamma_0 = -\mathrm{i}\bar{\zeta}_{i2}, \quad \gamma_0\eta^{i1} = \mathrm{i}\eta^{i1}, \quad \gamma_0\eta^{i2} = -\mathrm{i}\eta^{i2}, \\
&\bar{g}_1 = \bar{\zeta}_{ia}\gamma_0\theta^a{}_b\phi^{ib}, \quad M_{ia}{}^{jb}\phi^{ia}\bar{\phi}_{jb}\bar{g}_1 = \bar{g}_1 M_{jb}{}^{ia}\bar{\phi}_{ia}\phi^{jb}, \\
&g_2 = -\bar{\phi}_{ib}\theta^b{}_a\gamma_0\eta^{ia}, \quad M_{jb}{}^{ia}\bar{\phi}_{ia}\phi^{jb}g_2 = g_2 M_{ia}{}^{jb}\phi^{ia}\bar{\phi}_{jb}.
\end{aligned} \tag{3.19}$$

Then we get the parameterizations

$$\begin{aligned}
&\bar{\zeta}_{i1} = \bar{\alpha}_i\bar{\zeta}, \quad \bar{\zeta}^\alpha = (1, \mathrm{i}), \quad \bar{\zeta}_{i2} = \bar{\gamma}_i\bar{\mu}, \quad \bar{\mu}^\alpha = (-\mathrm{i}, -1), \\
&\eta^{i1} = \eta\beta^i, \quad \eta_\alpha = (1, -\mathrm{i}), \quad \eta^{i2} = \nu\delta^i, \quad \nu_\alpha = (-\mathrm{i}, 1),
\end{aligned} \tag{3.20}$$

as well as

$$\eta^{ia}\bar{\zeta}_{jc}\gamma_0\theta^c{}_b = M_{jb}{}^{ic}\theta^a{}_c, \quad M_{ia}{}^{jb}\bar{\zeta}_{kd}\gamma_0\theta^d{}_c = M_{kc}{}^{jb}\bar{\zeta}_{id}\gamma_0\theta^d{}_a,$$
$$\theta^b{}_c\gamma_0\eta^{jc}\bar{\zeta}_{ia} = M_{ic}{}^{jb}\theta^c{}_a, \quad M_{jb}{}^{ia}\theta^c{}_d\gamma_0\eta^{kd} = M_{jb}{}^{kc}\theta^a{}_d\gamma_0\eta^{id}. \tag{3.21}$$

We have four classes of solutions, and all of them must satisfy

$$M_{i1}{}^{j2} = M_{i2}{}^{j1} = \bar{\alpha}_i\delta^j = \bar{\gamma}_i\beta^j = 0,$$
$$M_{i1}{}^{j1} = 2\mathrm{i}\bar{\gamma}_i\delta^j, \quad M_{i2}{}^{j2} = 2\mathrm{i}\bar{\alpha}_i\beta^j. \tag{3.22}$$

Class I

In the first solution we have

$$\bar{\gamma}_i = \delta^i = 0. \tag{3.23}$$

Here $\bar{\alpha}_i$ and β^i are free complex parameters. Now we have

$$L_B = 2\mathrm{i}\bar{\alpha}_i\beta^j \begin{pmatrix} \phi^{i2}\bar{\phi}_{j2} & \\ & \bar{\phi}_{j2}\phi^{i2} \end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\alpha}_i\bar{\zeta}\psi^{i1} \\ \bar{\psi}_{i1}\eta\beta^i & \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} & \bar{\alpha}_i\phi^{i2} \\ \beta^i\bar{\phi}_{i2} & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\zeta}P^1{}_2 + P^2{}_1\eta, \tag{3.24}$$

and this makes that $W_{\mathrm{DT}} - W_{\mathrm{GY}} = QV$.

Class II

In the second solution we have

$$\bar{\alpha}_i = \beta^i = 0. \tag{3.25}$$

Now we have

$$L_B = 2\mathrm{i}\bar{\gamma}_i\delta^j \begin{pmatrix} \phi^{i1}\bar{\phi}_{j1} & \\ & \bar{\phi}_{j1}\phi^{i1} \end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\gamma}_i\bar{\mu}\psi^{i2} \\ \bar{\psi}_{i2}\nu\delta^i & \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} & \bar{\gamma}_i\phi^{i1} \\ \delta^i\bar{\phi}_{i1} & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\mu}P^2{}_1 + P^1{}_2\nu. \tag{3.26}$$

Class III

In the third solution we have

$$\beta^i = \delta^i = 0. \tag{3.27}$$

Now we have

$$L_B = 0, \quad L_F = \begin{pmatrix} & \bar{\alpha}_i\bar{\zeta}\psi^{i1} + \bar{\gamma}_i\bar{\mu}\psi^{i2} \\ 0 & \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} & \bar{\alpha}_i\phi^{i2} + \bar{\gamma}_i\phi^{i1} \\ 0 & \end{pmatrix}, \quad \kappa = -2\mathrm{i}, \quad Q = \bar{\zeta}P^1{}_2 + \bar{\mu}P^2{}_1. \tag{3.28}$$

Class IV

In the fourth solution we have

$$\bar{\alpha}_i = \bar{\gamma}_i = 0. \tag{3.29}$$

Now we have

$$L_B = 0, \quad L_F = \begin{pmatrix} & 0 \\ \bar{\psi}_{i1}\eta\beta^i + \bar{\psi}_{i2}\nu\delta^i & \end{pmatrix},$$
$$\Lambda = \begin{pmatrix} & 0 \\ \beta^i\bar{\phi}_{i2} + \delta^i\bar{\phi}_{i1} & \end{pmatrix}, \quad \kappa = -2\mathrm{i}, \quad Q = P^2{}_1\eta + P^1{}_2\nu. \tag{3.30}$$

3.2. *Circle in Euclidean space*

We consider circular BPS GY and DT type Wilson loops in Euclidean space.

3.2.1. *GY type Wilson loop*

We get the 1/3 BPS GY type Wilson loop along the circle $x^\mu = (\cos\tau, \sin\tau, 0)$

$$W_{\mathrm{GY}} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau\, L_{\mathrm{GY}}(\tau)\right), \quad L_{\mathrm{GY}} = \begin{pmatrix} \mathcal{A}_{\mathrm{GY}} & \\ & \hat{\mathcal{A}}_{\mathrm{GY}} \end{pmatrix}, \tag{3.31}$$
$$\mathcal{A}_{GY} = A_\mu\dot{x}^\mu + R^a{}_b\sigma^b{}_a|\dot{x}|, \quad \hat{\mathcal{A}}_{GY} = \hat{A}_\mu\dot{x}^\mu + R^a{}_b\hat{\sigma}^b{}_a|\dot{x}|,$$

with $R^a{}_b = \mathrm{diag}(-\mathrm{i}/2, \mathrm{i}/2)$, and the preserved supersymmetries are

$$\vartheta^1{}_2 = \mathrm{i}\gamma_3\theta^1{}_2, \quad \vartheta^2{}_1 = -\mathrm{i}\gamma_3\theta^2{}_1, \quad \theta^1{}_1 = \theta^2{}_2 = \vartheta^1{}_1 = \vartheta^2{}_2 = 0. \tag{3.32}$$

3.2.2. *DT type Wilson loop*

We construct the DT type Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$

$$\begin{aligned}
&W_{\mathrm{DT}} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau\, L_{\mathrm{DT}}(\tau)\right), \quad L_{\mathrm{DT}} = \begin{pmatrix} \mathcal{A} & \bar{f}_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix},\\
&\mathcal{A} = \mathcal{A}_{\mathrm{GY}} - 2(\bar{\gamma}_i\delta^j\phi^{i1}\bar{\phi}_{j1} + \bar{\alpha}_i\beta^j\phi^{i2}\bar{\phi}_{j2})|\dot{x}|,\\
&\hat{\mathcal{A}} = \hat{\mathcal{A}}_{\mathrm{GY}} - 2(\bar{\gamma}_i\delta^j\bar{\phi}_{j1}\phi^{i1} + \bar{\alpha}_i\beta^j\bar{\phi}_{j2}\phi^{i2})|\dot{x}|,\\
&\bar{f}_1 = (\bar{\alpha}_i\bar{\zeta}\psi^{1a} + \bar{\gamma}_i\bar{\mu}\psi^{2a})|\dot{x}|, \quad \bar{\zeta}^\alpha = (\mathrm{e}^{\mathrm{i}\tau/2}, \mathrm{e}^{-\mathrm{i}\tau/2}), \quad \bar{\mu}^\alpha = (\mathrm{e}^{\mathrm{i}\tau/2}, -\mathrm{e}^{-\mathrm{i}\tau/2}),\\
&f_2 = (\bar{\psi}_{i1}\eta\beta^i + \bar{\psi}_{i2}\nu\delta^i)|\dot{x}|, \quad \eta_\alpha = (\mathrm{e}^{-\mathrm{i}\tau/2}, \mathrm{e}^{\mathrm{i}\tau/2}), \quad \nu_\alpha = (-\mathrm{e}^{-\mathrm{i}\tau/2}, \mathrm{e}^{\mathrm{i}\tau/2}).
\end{aligned} \tag{3.33}$$

We want the Wilson loop to preserve the supersymmetries (3.32). We have four classes of solutions, and all of them must satisfy

$$\bar{\alpha}_i\delta^j = \bar{\gamma}_i\beta^j = 0. \tag{3.34}$$

Class I

In the first solution we have

$$\bar{\gamma}_i = \delta^i = 0. \tag{3.35}$$

Here $\bar{\alpha}_i$ and β^i are free complex parameters. Now we have

$$
\begin{aligned}
&L_B = -2\bar{\alpha}_i \beta^j \begin{pmatrix} \phi^{i2}\bar{\phi}_{j2} & \\ & \bar{\phi}_{j2}\phi^{i2} \end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\alpha}_i \bar{\zeta} \psi^{i1} \\ \bar{\psi}_{i1}\eta\beta^i & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & \bar{\alpha}_i \phi^{i2} \\ \beta^i \bar{\phi}_{i2} & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P^1{}_2 + \mathrm{i}\gamma_3 S^1{}_2) + (P^2{}_1 + S^2{}_1 \mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1,0), \quad b_\alpha = (0,1),
\end{aligned}
\tag{3.36}
$$

and this makes that $W_{\mathrm{DT}} - W_{\mathrm{GY}} = QV$.

Class II

In the second solution we have

$$\bar{\alpha}_i = \beta^i = 0. \tag{3.37}$$

Now we have

$$
\begin{aligned}
&L_B = -2\bar{\gamma}_i \delta^j \begin{pmatrix} \phi^{i1}\bar{\phi}_{j1} & \\ & \bar{\phi}_{j1}\phi^{i1} \end{pmatrix}, \quad L_F = \begin{pmatrix} & \bar{\gamma}_i \bar{\mu} \psi^{i2} \\ \bar{\psi}_{i2}\nu\delta^i & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & \bar{\gamma}_i \phi^{i1} \\ \delta^i \bar{\phi}_{i1} & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P^2{}_1 - \mathrm{i}\gamma_3 S^2{}_1) + (P^1{}_2 - S^1{}_2 \mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1,0), \quad b_\alpha = (0,1).
\end{aligned}
\tag{3.38}
$$

Class III

In the third solution we have

$$\beta^i = \delta^i = 0. \tag{3.39}$$

Now we have

$$
\begin{aligned}
&L_B = 0, \quad L_F = \begin{pmatrix} & \bar{\alpha}_i \bar{\zeta} \psi^{i1} + \bar{\gamma}_i \bar{\mu} \psi^{i2} \\ 0 & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & \bar{\alpha}_i \phi^{i2} + \bar{\gamma}_i \phi^{i1} \\ 0 & \end{pmatrix}, \quad \kappa = 2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P^1{}_2 + \mathrm{i}\gamma_3 S^1{}_2 + P^2{}_1 - \mathrm{i}\gamma_3 S^2{}_1), \quad \bar{a}^\alpha = (1,0).
\end{aligned}
\tag{3.40}
$$

Class IV

In the fourth solution we have

$$\bar{\alpha}_i = \bar{\gamma}_i = 0. \tag{3.41}$$

Now we have

$$
\begin{aligned}
&L_B = 0, \quad L_F = \begin{pmatrix} & 0 \\ \bar{\psi}_{i1}\eta\beta^i + \bar{\psi}_{i2}\nu\delta^i & \end{pmatrix}, \\
&\Lambda = \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & 0 \\ \beta^i \bar{\phi}_{i2} + \delta^i \bar{\phi}_{i1} & \end{pmatrix}, \quad \kappa = 2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = (P^1{}_2 - S^1{}_2 \mathrm{i}\gamma_3 + P^2{}_1 + S^2{}_1 \mathrm{i}\gamma_3)b, \quad b_\alpha = (0,1).
\end{aligned}
\tag{3.42}
$$

4. ABJM theory

The ABJM theory [10] is an $\mathcal{N}=6$ superconformal CSM theory, with gauge group $U(N)\times U(N)$ and levels $(k,-k)$. There are matter fields ϕ_I, ψ^I, with $I=1,2,3,4$ being the index of $SU(4)$ R-symmetry. The SUSY transformations of ABJM theory are [33–36]

$$\begin{aligned}
&\delta A_\mu=\frac{2\pi}{k}\left(\phi_I\bar\psi_J\gamma_\mu\epsilon^{IJ}+\bar\epsilon_{IJ}\gamma_\mu\psi^J\bar\phi^I\right),\\
&\delta\hat A_\mu=\frac{2\pi}{k}\left(\bar\psi_J\gamma_\mu\phi_I\epsilon^{IJ}+\bar\epsilon_{IJ}\bar\phi^I\gamma_\mu\psi^J\right),\\
&\delta\phi_I=\mathrm{i}\bar\epsilon_{IJ}\psi^J,\quad \delta\bar\phi^I=\mathrm{i}\bar\psi_J\epsilon^{IJ},\\
&\delta\psi^I=\gamma^\mu\epsilon^{IJ}D_\mu\phi_J+\vartheta^{IJ}\phi_J-\frac{2\pi}{k}\epsilon^{IJ}\left(\phi_J\bar\phi^K\phi_K-\phi_K\bar\phi^K\phi_J\right)-\frac{4\pi}{k}\epsilon^{KL}\phi_K\bar\phi^I\phi_L,\\
&\delta\bar\psi_I=-\bar\epsilon_{IJ}\gamma^\mu D_\mu\bar\phi^J+\bar\vartheta_{IJ}\bar\phi^J+\frac{2\pi}{k}\bar\epsilon_{IJ}\left(\bar\phi^J\phi_K\bar\phi^K-\bar\phi^K\phi_K\bar\phi^J\right)+\frac{4\pi}{k}\bar\epsilon_{KL}\bar\phi^K\phi_I\bar\phi^L.
\end{aligned}\tag{4.1}$$

The definitions of covariant derivatives are

$$\begin{aligned}
&D_\mu\phi_J=\partial_\mu\phi_J+\mathrm{i}A_\mu\phi_J-\mathrm{i}\phi_J\hat A_\mu,\\
&D_\mu\bar\phi^J=\partial_\mu\bar\phi^J+\mathrm{i}\hat A_\mu\bar\phi^J-\mathrm{i}\bar\phi^JA_\mu.
\end{aligned}\tag{4.2}$$

We have SUSY parameters $\epsilon^{IJ}=\theta^{IJ}+x^\mu\gamma_\mu\vartheta^{IJ}$ and $\bar\epsilon_{IJ}=\bar\theta_{IJ}-\bar\vartheta_{IJ}x^\mu\gamma_\mu$, with constraints

$$\begin{aligned}
&\theta^{IJ}=-\theta^{JI},\quad (\theta^{IJ})^*=\bar\theta_{IJ},\quad \bar\theta_{IJ}=\frac{1}{2}\epsilon_{IJKL}\theta^{KL},\\
&\vartheta^{IJ}=-\vartheta^{JI},\quad (\vartheta^{IJ})^*=\bar\vartheta_{IJ},\quad \bar\vartheta_{IJ}=\frac{1}{2}\epsilon_{IJKL}\vartheta^{KL}.
\end{aligned}\tag{4.3}$$

Symbol ϵ_{IJKL} is totally antisymmetric with $\epsilon_{1234}=1$. The supercharges are defined as

$$\delta=\frac{\mathrm{i}}{2}(\bar\theta_{IJ}P^{IJ}+\bar\vartheta_{IJ}S^{IJ})=\frac{\mathrm{i}}{2}(\bar P_{IJ}\theta^{IJ}+\bar S_{IJ}\vartheta^{IJ}),\tag{4.4}$$

with the constraints

$$\begin{aligned}
&P^{IJ}=-P^{JI},\quad (P^{IJ})^*=\bar P_{IJ},\quad \bar P_{IJ}=\frac{1}{2}\epsilon_{IJKL}P^{KL},\\
&S^{IJ}=-S^{JI},\quad (S^{IJ})^*=\bar S_{IJ},\quad \bar S_{IJ}=\frac{1}{2}\epsilon_{IJKL}S^{KL}.
\end{aligned}\tag{4.5}$$

In Euclidean space, the SUSY transformations (4.1) and definitions of supercharges (4.4) still apply, but the constraints for the SUSY parameters and supercharges are

$$\begin{aligned}
&\theta^{IJ}=-\theta^{JI},\quad \bar\theta_{IJ}=\frac{1}{2}\epsilon_{IJKL}\theta^{KL},\quad \vartheta^{IJ}=-\vartheta^{JI},\quad \bar\vartheta_{IJ}=\frac{1}{2}\epsilon_{IJKL}\vartheta^{KL},\\
&P^{IJ}=-P^{JI},\quad \bar P_{IJ}=\frac{1}{2}\epsilon_{IJKL}P^{KL},\quad S^{IJ}=-S^{JI},\quad \bar S_{IJ}=\frac{1}{2}\epsilon_{IJKL}S^{KL}.
\end{aligned}\tag{4.6}$$

4.1. Straight line in Minkowski spacetime

We consider BPS GY and DT type Wilson loops along timelike infinite straight lines in Minkowski spacetime.

4.1.1. GY type Wilson loop

A general GY type Wilson loop along the line $x^\mu = \tau\delta_0^\mu$ takes the form

$$W_{\rm GY} = \mathcal{P}\exp\left(-\mathrm{i}\int d\tau L_{\rm GY}(\tau)\right), \quad L_{\rm GY} = \begin{pmatrix} \mathcal{A}_{\rm GY} & \\ & \hat{\mathcal{A}}_{\rm GY} \end{pmatrix},$$
$$\mathcal{A}_{\rm GY} = A_\mu \dot{x}^\mu + \frac{2\pi}{k} R^I{}_J \phi_I \bar{\phi}^J |\dot{x}|, \quad \hat{\mathcal{A}}_{\rm GY} = \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} S_I{}^J \bar{\phi}^I \phi_J |\dot{x}|. \tag{4.7}$$

The SUSY transformation $\delta\mathcal{A}_{\rm GY} = 0$ leads to

$$\gamma_0 \theta^{IJ} = -\mathrm{i} R^I{}_K \theta^{KJ}, \quad \bar{\theta}_{IJ}\gamma_0 = -\mathrm{i} R^K{}_I \bar{\theta}_{KJ}. \tag{4.8}$$

Taking complex conjugate of the second equation we get

$$\gamma_0 \theta^{IJ} = -\mathrm{i} R^{\dagger I}{}_K \theta^{KJ}, \tag{4.9}$$

with the matrix $R^\dagger$ being the hermitian conjugate of R, i.e. $R^{\dagger I}{}_J = (R^J{}_I)^*$. We define

$$R = B + \mathrm{i}C, \quad B = \frac{R + R^\dagger}{2}, \quad C = -\mathrm{i}\frac{R - R^\dagger}{2}, \tag{4.10}$$

with B and C being hermitian matrices, i.e. $B = B^\dagger$ and $C = C^\dagger$. Then we have

$$\gamma_0 \theta^{IJ} = -\mathrm{i} B^I{}_K \theta^{KJ}, \quad C^I{}_K \theta^{KJ} = 0. \tag{4.11}$$

We use $SU(4)$ transformation of the R-symmetry to make the hermitian matrix B diagonal

$$B^I{}_J = \mathrm{diag}(b_1, b_2, b_3, b_4), \tag{4.12}$$

with b_1, b_2, b_3, and b_4 being real. Then we get

$$\gamma_0 \theta^{IJ} = -\mathrm{i} b_I \theta^{IJ}, \tag{4.13}$$

with no summation of index I in the right-hand side. Note that the eigenvalues of the matrix γ_0 can only be $\pm\mathrm{i}$. Without loss of generality, we suppose

$$\gamma_0 \theta^{12} = \mathrm{i}\theta^{12}, \quad \theta^{12} \neq 0, \tag{4.14}$$

and then using $\theta^{12*} = \theta^{34}$ we have

$$\gamma_0 \theta^{34} = -\mathrm{i}\theta^{34}, \quad \theta^{34} \neq 0. \tag{4.15}$$

We get $b_1 = b_2 = -1$, $b_3 = b_4 = 1$, and then we have

$$\theta^{13} = \theta^{14} = \theta^{23} = \theta^{24} = 0. \tag{4.16}$$

Then we get $C^I{}_J = 0$, i.e. that $R^I{}_J$ is a hermitian matrix. The SUSY transformation $\delta\hat{\mathcal{A}}_{\rm GY} = 0$ leads to

$$\gamma_0 \theta^{IJ} = -\mathrm{i} S_K{}^I \theta^{KJ}, \quad \bar{\theta}_{IJ}\gamma_0 = -\mathrm{i} S_I{}^K \bar{\theta}_{KJ}, \tag{4.17}$$

from which we get

$$(R^I{}_K - S_K{}^I)\theta^{KJ} = 0, \quad (R^K{}_I - S_I{}^K)\bar{\theta}_{KJ} = 0. \tag{4.18}$$

We get $S_I{}^J = R^J{}_I$.

In summary, we have the GY type 1/6 BPS Wilson loop (4.7) with $R^I{}_J = S_J{}^I = \mathrm{diag}(-1, -1, 1, 1)$, and the preserved supersymmetries are

$$\gamma_0\theta^{12} = \mathrm{i}\theta^{12}, \quad \gamma_0\theta^{34} = -\mathrm{i}\theta^{34},$$
$$\theta^{13} = \theta^{14} = \theta^{23} = \theta^{24} = 0. \tag{4.19}$$

This is just the Wilson loop that was constructed in [11–13]. Here we show that this is the only form of GY type BPS Wilson loop along a timelike straight line, up to some $SU(4)$ R-symmetry transformation. Especially, we do not require that $R^I{}_J$ or $S_I{}^J$ is a hermitian matrix *a priori*, and we show that it is the result of supersymmetric invariance.

4.1.2. DT type Wilson loop

We want a DT type Wilson loop that preserves the same supersymmetries (4.19). A general DT type Wilson loop is [9]

$$\begin{aligned}
&W_{\mathrm{DT}} = \mathcal{P}\exp\left(-\mathrm{i}\int \mathrm{d}\tau L_{\mathrm{DT}}(\tau)\right), \quad L_{\mathrm{DT}} = \begin{pmatrix} \mathcal{A} & \bar{f}_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix}, \\
&\mathcal{A} = \mathcal{A}_{\mathrm{GY}} + \frac{2\pi}{k} M^I{}_J \phi_I \bar{\phi}^J |\dot{x}|, \quad \bar{f}_1 = \sqrt{\frac{2\pi}{k}}\bar{\zeta}_I \psi^I |\dot{x}|, \\
&\hat{\mathcal{A}} = \hat{\mathcal{A}}_{\mathrm{GY}} + \frac{2\pi}{k} N_I{}^J \bar{\phi}^I \phi_J |\dot{x}|, \quad f_2 = \sqrt{\frac{2\pi}{k}}\bar{\psi}_I \eta^I |\dot{x}|.
\end{aligned} \tag{4.20}$$

From variations of $\mathcal{A}$ and $\hat{\mathcal{A}}$ we have

$$M^I{}_J = N_J{}^I, \quad \sqrt{\frac{2\pi}{k}} M^J{}_K \phi_J \theta^{KI} = \eta^I \bar{g}_1, \quad \sqrt{\frac{2\pi}{k}} M^J{}_K \bar{\phi}^K \bar{\theta}_{JI} = -g_2 \bar{\zeta}_I. \tag{4.21}$$

From variations of $\bar{f}_1$ and f_2 we have

$$\begin{aligned}
&\bar{\zeta}_{1,2}\gamma_0 = \mathrm{i}\bar{\zeta}_{1,2}, \quad \bar{\zeta}_{3,4}\gamma_0 = -\mathrm{i}\bar{\zeta}_{3,4}, \quad \gamma_0\eta_{1,2} = \mathrm{i}\eta_{1,2}, \quad \gamma_0\eta_{3,4} = -\mathrm{i}\eta_{3,4}, \\
&\bar{g}_1 = -\sqrt{\frac{2\pi}{k}}\bar{\zeta}_I \gamma_0 \theta^{IJ}\phi_J, \quad M^I{}_J \bar{\zeta}_L \gamma_0 \theta^{LK} = M^K{}_J \bar{\zeta}_L \gamma_0 \theta^{LI}, \\
&g_2 = \sqrt{\frac{2\pi}{k}}\bar{\theta}_{IJ}\gamma_0 \eta^I \bar{\phi}^J, \quad M^I{}_J \bar{\theta}_{KL}\gamma_0 \eta^L = M^I{}_K \bar{\theta}_{JL}\gamma_0\eta^L.
\end{aligned} \tag{4.22}$$

We make the parameterizations

$$\begin{aligned}
&\bar{\zeta}_{1,2} = \bar{\alpha}_{1,2}\bar{\zeta}, \quad \bar{\zeta}^\alpha = (1, \mathrm{i}), \quad \bar{\zeta}_{3,4} = \bar{\gamma}_{3,4}\bar{\mu}, \quad \bar{\mu}^\alpha = (-\mathrm{i}, -1), \\
&\eta^{1,2} = \eta\beta^{1,2}, \quad \eta_\alpha = (1, -\mathrm{i}), \quad \eta^{3,4} = \nu\delta^{3,4}, \quad \nu_\alpha = (-\mathrm{i}, 1).
\end{aligned} \tag{4.23}$$

Then equations (4.21) become

$$M^I{}_K \theta^{KJ} = -\eta^J \bar{\zeta}_K \gamma_0 \theta^{KI}, \quad M^K{}_I \bar{\theta}_{KJ} = -\bar{\theta}_{KI}\gamma_0 \eta^K \bar{\zeta}_J. \tag{4.24}$$

We find four classes of solutions, and all of them satisfy

$$\begin{aligned}
&M^I{}_J = 2\mathrm{i}\begin{pmatrix} \bar{\alpha}_2\beta^2 & -\bar{\alpha}_2\beta^1 & & \\ -\bar{\alpha}_1\beta^2 & \bar{\alpha}_1\beta^1 & & \\ & & \bar{\gamma}_4\delta^4 & -\bar{\gamma}_4\delta^3 \\ & & -\bar{\gamma}_3\delta^4 & \bar{\gamma}_3\delta^3 \end{pmatrix}, \\
&\bar{\alpha}_{1,2}\delta^{3,4} = \bar{\gamma}_{3,4}\beta^{1,2} = 0.
\end{aligned} \tag{4.25}$$

Class I

In the first solution we have

$$\bar{\gamma}_{3,4} = \delta^{3,4} = 0. \tag{4.26}$$

Note that there are four free complex parameters $\bar{\alpha}_{1,2}$ and $\beta^{1,2}$.

Now we have

$$L_B = \frac{2\pi}{k} M^I{}_J \begin{pmatrix} \phi_I \bar{\phi}^I & \\ & \bar{\phi}^J \phi_I \end{pmatrix}, \quad L_F = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & \bar{\zeta}_I \psi^I \\ \bar{\psi}_I \eta^I & \end{pmatrix},$$
$$\Lambda = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & \bar{\alpha}_2 \phi_1 - \bar{\alpha}_1 \phi_2 \\ \beta^2 \bar{\phi}^1 - \beta^1 \bar{\phi}^2 & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\zeta} P^{12} + \bar{P}_{12} \eta. \tag{4.27}$$

This shows that the difference between the DT type Wilson loop and GY type Wilson loop is Q-exact.

We wonder if there can be more preserved supersymmetries than (4.19) for the DT BPS Wilson loop, at least for some special values of the parameters $\bar{\alpha}_{1,2}$ and $\beta^{1,2}$. We solve the equations (2.14) of the Wilson loop with general SUSY transformation parameters θ^{IJ} and $\bar{\theta}_{IJ}$. From variations of $\bar{f}_1$ and f_2 we get

$$\gamma_0 \bar{\alpha}_I \theta^{IJ} = \mathrm{i} \bar{\alpha}_I \theta^{IJ}, \quad \bar{g}_1 = -\mathrm{i} \sqrt{\frac{2\pi}{k}} \bar{\zeta} \theta^{IJ} \bar{\alpha}_I \phi_J,$$
$$\beta^I \bar{\theta}_{IJ} \gamma_0 = \mathrm{i} \beta^I \bar{\theta}_{IJ}, \quad g_2 = \mathrm{i} \sqrt{\frac{2\pi}{k}} \bar{\theta}_{IJ} \eta \beta^I \bar{\phi}^J. \tag{4.28}$$

Then from variations of $\mathcal{A}$ and $\hat{\mathcal{A}}$ we get

$$\gamma_0 \theta^{IJ} + \mathrm{i} U^I{}_K \theta^{KJ} = 2 \bar{\alpha}_K \beta^J \theta^{KI}, \quad \bar{\theta}_{IJ} \gamma_0 + \mathrm{i} U^K{}_I \bar{\theta}_{KJ} = 2 \bar{\alpha}_J \beta^K \bar{\theta}_{KI}. \tag{4.29}$$

We define $\alpha^I = \bar{\alpha}_I^*$ and $\bar{\beta}_I = \beta^{I*}$. Using the fact that $\theta^{12} = \theta^{34*}$, $\theta^{13} = -\theta^{24*}$ and $\theta^{23} = \theta^{14*}$, we can show that equations (4.29) are equivalent to

$$\begin{aligned}
&\bar{\alpha}_1 \beta^1 + \bar{\alpha}_2 \beta^2 = -\mathrm{i}, \quad \gamma_0 \theta^{12} = \mathrm{i} \theta^{12}, \\
&\gamma_0 \theta^{13} = -(\mathrm{i} + 2\bar{\alpha}_1 \beta^1) \theta^{13} - 2\bar{\alpha}_2 \beta^1 \theta^{23}, \\
&\gamma_0 \theta^{23} = -2\bar{\alpha}_1 \beta^2 \theta^{13} + (\mathrm{i} + 2\bar{\alpha}_1 \beta^1) \theta^{23}, \\
&(\bar{\alpha}_1 \beta^1 + \bar{\beta}_1 \alpha^1) \theta^{13} + (\bar{\alpha}_2 \beta^1 + \bar{\beta}_2 \alpha^1) \theta^{23} = 0, \\
&(\bar{\alpha}_1 \beta^2 + \bar{\beta}_1 \alpha^2) \theta^{13} - (\bar{\alpha}_1 \beta^1 + \bar{\beta}_1 \alpha^1) \theta^{23} = 0.
\end{aligned} \tag{4.30}$$

Note that when an equation is satisfied, for consistency so is its complex conjugate. When there is SUSY enhancement for the Wilson loop, there must be nonvanishing solution for at least one of θ^{13} and θ^{23}, and this requires that

$$\begin{vmatrix} \bar{\alpha}_1 \beta^1 + \bar{\beta}_1 \alpha^1 & \bar{\alpha}_2 \beta^1 + \bar{\beta}_2 \alpha^1 \\ \bar{\alpha}_1 \beta^2 + \bar{\beta}_1 \alpha^2 & -(\bar{\alpha}_1 \beta^1 + \bar{\beta}_1 \alpha^1) \end{vmatrix} = -(\bar{\alpha}_1 \beta^1 + \bar{\beta}_1 \alpha^1)^2 - |\bar{\alpha}_1 \beta^2 + \bar{\beta}_1 \alpha^2|^2 = 0. \tag{4.31}$$

This leads to

$$\bar{\alpha}_1 \beta^1 + \bar{\beta}_1 \alpha^1 = \bar{\alpha}_1 \beta^2 + \bar{\beta}_1 \alpha^2 = 0. \tag{4.32}$$

Then we get the solution

$$\beta^I = -\frac{\mathrm{i}}{\bar{\alpha}_J \alpha^J}\alpha^I, \quad \bar{\alpha}_I \neq 0. \tag{4.33}$$

It can be checked that supersymmetries are enhanced to 1/2 BPS, and the preserved supersymmetries are

$$\gamma_0 \bar{\alpha}_I \theta^{IJ} = \mathrm{i}\bar{\alpha}_I \theta^{IJ}, \quad \gamma_0 \epsilon_{IJKL}\alpha^J \theta^{KL} = -\mathrm{i}\epsilon_{IJKL}\alpha^J \theta^{KL}. \tag{4.34}$$

Note that the two equations are consistent, since they are complex conjugates of each other. In this case we have

$$U^I{}_J = \delta^I_J - 2\mathrm{i}\bar{\alpha}_J \beta^I = \delta^I_J - \frac{2\bar{\alpha}_J \alpha^I}{\bar{\alpha}_K \alpha^K}. \tag{4.35}$$

The Wilson loop is global $SU(3)$ R-symmetry invariant. When $\bar{\alpha}_2 = 0$, it is just Wilson loop that was constructed in [9].

So for general parameters the Wilson loop with (4.26) is 1/6 BPS, and the preserved supersymmetries are (4.19). When there is also (4.33), the Wilson loop is enhanced to 1/2 BPS, and the preserved supersymmetries are (4.34).

Class II

In the second solution the calculation is parallel to that of the first one. We have

$$\bar{\alpha}_{1,2} = \beta^{1,2} = 0, \tag{4.36}$$

and the Wilson loop is 1/6 BPS and preserves the supersymmetries (4.19). Now we have

$$\begin{aligned} &L_B = \frac{2\pi}{k} M^I{}_J \begin{pmatrix} \phi_I \bar{\phi}^J & \\ & \bar{\phi}^J \phi_I \end{pmatrix}, \quad L_F = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & \bar{\zeta}_I \psi^I \\ \bar{\psi}_I \eta^I & \end{pmatrix}, \\ &\Lambda = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & \bar{\gamma}_4 \phi_3 - \bar{\gamma}_3 \phi_4 \\ \delta^4 \bar{\phi}^3 - \delta^3 \bar{\phi}^4 & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\mu} P^{34} + \bar{P}_{34} \nu. \end{aligned} \tag{4.37}$$

We want to find if there is SUSY enhancement for the Wilson loop with (4.36). From variations of $\bar{f}_1$ and f_2 we get

$$\begin{aligned} &\gamma_0 \bar{\gamma}_I \theta^{IJ} = -\mathrm{i}\bar{\gamma}_I \theta^{IJ}, \quad \bar{g}_1 = \mathrm{i}\sqrt{\frac{2\pi}{k}} \bar{\mu} \theta^{IJ} \bar{\gamma}_I \phi_J, \\ &\delta^I \bar{\theta}_{IJ} \gamma_0 = -\mathrm{i}\delta^I \bar{\theta}_{IJ}, \quad g_2 = -\mathrm{i}\sqrt{\frac{2\pi}{k}} \bar{\theta}_{IJ} \nu \delta^I \bar{\phi}^J. \end{aligned} \tag{4.38}$$

From variations of $\mathcal{A}$ and $\hat{\mathcal{A}}$ we get

$$\gamma_0 \theta^{IJ} + \mathrm{i}U^I{}_K \theta^{KJ} = 2\bar{\gamma}_K \delta^J \theta^{KI}, \quad \bar{\theta}_{IJ}\gamma_0 + \mathrm{i}U^K{}_I \bar{\theta}_{KJ} = 2\bar{\gamma}_J \delta^K \bar{\theta}_{KI}. \tag{4.39}$$

We define $\gamma^I = \bar{\gamma}_I^*$ and $\bar{\delta}_I = \delta^{I*}$. We can show that equations (4.39) are equivalent to

$$\begin{aligned} &\bar{\gamma}_3 \delta^3 + \bar{\gamma}_4 \delta^4 = \mathrm{i}, \quad \gamma_0 \theta^{12} = \mathrm{i}\theta^{12}, \\ &\gamma_0 \theta^{13} = (\mathrm{i} - 2\bar{\gamma}_3 \delta^3)\theta^{13} - 2\bar{\gamma}_4 \delta^3 \theta^{14}, \\ &\gamma_0 \theta^{14} = -2\bar{\gamma}_3 \delta^4 \theta^{13} - (\mathrm{i} - 2\bar{\gamma}_3 \delta^3)\theta^{14}, \\ &(\bar{\gamma}_3 \delta^3 + \bar{\delta}_3 \gamma^3)\theta^{13} + (\bar{\gamma}_4 \delta^3 + \bar{\delta}_4 \gamma^3)\theta^{14} = 0, \\ &(\bar{\gamma}_3 \delta^4 + \bar{\delta}_3 \gamma^4)\theta^{13} - (\bar{\gamma}_3 \delta^3 + \bar{\delta}_3 \gamma^3)\theta^{14} = 0. \end{aligned} \tag{4.40}$$

For the existence of nonvanishing solution for at least one of θ^{13} and θ^{14}, we have

$$\begin{vmatrix} \bar{\gamma}_3\delta^3+\bar{\delta}_3\gamma^3 & \bar{\gamma}_4\delta^3+\bar{\delta}_4\gamma^3 \\ \bar{\gamma}_3\delta^4+\bar{\delta}_3\gamma^4 & -(\bar{\gamma}_3\delta^3+\bar{\delta}_3\gamma^3) \end{vmatrix} = -(\bar{\gamma}_3\delta^3+\bar{\delta}_3\gamma^3)^2 - |\bar{\gamma}_3\delta^4+\bar{\delta}_3\gamma^4|^2 = 0. \tag{4.41}$$

This leads to

$$\bar{\gamma}_3\delta^3+\bar{\delta}_3\gamma^3 = \bar{\gamma}_3\delta^4+\bar{\delta}_3\gamma^4 = 0. \tag{4.42}$$

Then we get the solution

$$\delta^I = \frac{\mathrm{i}}{\bar{\gamma}_J\gamma^J}\gamma^I, \quad \bar{\gamma}_I \neq 0. \tag{4.43}$$

It can be checked that supersymmetries are enhanced to 1/2 BPS, and the preserved supersymmetries are[1]

$$\gamma_0\bar{\gamma}_I\theta^{IJ} = -\mathrm{i}\bar{\gamma}_I\theta^{IJ}, \quad \gamma_0\epsilon_{IJKL}\gamma^J\theta^{KL} = \mathrm{i}\epsilon_{IJKL}\gamma^J\theta^{KL}. \tag{4.44}$$

At present we have

$$U^I{}_J = -\delta^I_J - 2\mathrm{i}\delta^I\bar{\gamma}_J = -\delta^I_J + \frac{2\gamma^I\bar{\gamma}_J}{\bar{\gamma}_K\gamma^K}. \tag{4.45}$$

The Wilson loop is global $SU(3)$ R-symmetry invariant.

Class III

In the third solution we have

$$\beta^{1,2} = \delta^{3,4} = 0. \tag{4.46}$$

The Wilson loop is 1/6 BPS Wilson loop and preserves the supersymmetries (4.19). Now we have

$$\begin{aligned} & L_B = 0, \quad L_F = \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\zeta}_I\psi^I \\ 0 & \end{pmatrix}, \\ & \Lambda = \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\alpha}_2\phi_1 - \bar{\alpha}_1\phi_2 + \bar{\gamma}_4\phi_3 - \bar{\gamma}_3\phi_4 \\ 0 & \end{pmatrix}, \\ & \kappa = -2\mathrm{i}, \quad Q = \bar{\zeta}P^{12} + \bar{\mu}P^{34}. \end{aligned} \tag{4.47}$$

For the Wilson loop with (4.46), we have $f_2 = g_2 = 0$ and so $\delta\mathcal{A} = \delta\hat{\mathcal{A}} = 0$. This leads to

$$\gamma_0\theta^{IJ} = -\mathrm{i}R^I{}_K\phi^{KJ}, \quad \bar{\theta}_{IJ}\gamma_0 = -\mathrm{i}R^K{}_I\bar{\phi}_{KJ}. \tag{4.48}$$

The solution is (4.19). So there is no SUSY enhancement for this solution.

Class IV

The fourth solution is like the third one. We have

$$\bar{\alpha}_{1,2} = \bar{\gamma}_{3,4} = 0, \tag{4.49}$$

and the 1/6 BPS Wilson loop that preserves the supersymmetries (4.19). Now we have

[1] Please distinguish the gamma matrix γ_0 from the complex parameters $\bar{\gamma}_I$, γ^I.

$$
\begin{aligned}
&L_B = 0, \quad L_F = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & 0 \\ \bar{\psi}_I \eta^I & \end{pmatrix}, \\
&\Lambda = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & 0 \\ \beta^2\bar{\phi}^1 - \beta^1\bar{\phi}^2 + \delta^4\bar{\phi}^3 - \delta^3\bar{\phi}^4 & \end{pmatrix}, \\
&\kappa = -2\mathrm{i}, \quad Q = \bar{P}_{12}\eta + \bar{P}_{34}\nu.
\end{aligned} \tag{4.50}
$$

For the Wilson loop with (4.49), we have $\bar{f}_1 = \bar{g}_1 = 0$ and so $\delta\mathcal{A} = \delta\hat{\mathcal{A}} = 0$. This leads to the supersymmetries (4.19). There is no SUSY enhancement for this solution.

In [26], exactly $1/3$ BPS Wilson loops in ABJM theory were anticipated to exist. However we do not find $1/3$ BPS DT type Wilson loops within the Wilson loops at least preserving supersymmetries (4.19).

4.2. Circle in Euclidean space

We consider circular BPS GY and DT type Wilson loops in Euclidean space.

4.2.1. GY type Wilson loop

In Euclidean space, one can construct the circular 1/6 BPS GY type Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$

$$
\begin{aligned}
&W_{\mathrm{GY}} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau L_{\mathrm{GY}}(\tau)\right), \quad L_{\mathrm{GY}} = \begin{pmatrix} \mathcal{A}_{\mathrm{GY}} & \\ & \hat{\mathcal{A}}_{\mathrm{GY}} \end{pmatrix}, \\
&\mathcal{A}_{\mathrm{GY}} = A_\mu \dot{x}^\mu + \frac{2\pi}{k} R^I{}_J \phi_I \bar{\phi}^J |\dot{x}|, \quad \hat{\mathcal{A}}_{\mathrm{GY}} = \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} R^I{}_J \bar{\phi}^J \phi_I |\dot{x}|.
\end{aligned} \tag{4.51}
$$

with $R^I{}_J = \mathrm{diag}(\mathrm{i}, \mathrm{i}, -\mathrm{i}, -\mathrm{i})$. The preserved supersymmetries are

$$
\begin{aligned}
&\vartheta^{12} = \mathrm{i}\gamma_3\theta^{12}, \quad \vartheta^{34} = -\mathrm{i}\gamma_3\theta^{34}, \\
&\theta^{13} = \theta^{14} = \theta^{23} = \theta^{24} = 0, \\
&\vartheta^{13} = \vartheta^{14} = \vartheta^{23} = \vartheta^{24} = 0.
\end{aligned} \tag{4.52}
$$

4.2.2. DT type Wilson loop

We also construct the DT type Wilson loop along $x^\mu = (\cos\tau, \sin\tau, 0)$

$$
\begin{aligned}
&W_{\mathrm{DT}} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau L_{\mathrm{DT}}(\tau)\right), \quad L_{\mathrm{DT}} = \begin{pmatrix} \mathcal{A} & \bar{f}_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix}, \\
&\mathcal{A} = A_\mu \dot{x}^\mu + \frac{2\pi}{k} U^I{}_J \phi_I \bar{\phi}^J |\dot{x}|, \quad \bar{f}_1 = \sqrt{\frac{2\pi}{k}} (\bar{\alpha}_I \bar{\zeta} + \bar{\gamma}_I \bar{\mu}) \psi^I |\dot{x}|, \\
&\hat{\mathcal{A}} = \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} U^I{}_J \bar{\phi}^J \phi_I |\dot{x}|, \quad f_2 = \sqrt{\frac{2\pi}{k}} \bar{\psi}_I (\eta\beta^I + \nu\delta^I) |\dot{x}|, \\
&U^I{}_J = \begin{pmatrix} \mathrm{i} - 2\bar{\alpha}_2\beta^2 & 2\bar{\alpha}_2\beta^1 & & \\ 2\bar{\alpha}_1\beta^2 & \mathrm{i} - 2\bar{\alpha}_1\beta^1 & & \\ & & -\mathrm{i} - 2\bar{\gamma}_4\delta^4 & 2\bar{\gamma}_4\delta^3 \\ & & 2\bar{\gamma}_3\delta^4 & -\mathrm{i} - 2\bar{\gamma}_3\delta^3 \end{pmatrix}, \\
&\bar{\alpha}_I = (\bar{\alpha}_1, \bar{\alpha}_2, 0, 0), \quad \bar{\zeta}^\alpha = (\mathrm{e}^{\mathrm{i}\tau/2}, \mathrm{e}^{-\mathrm{i}\tau/2}), \quad \beta^I = (\beta^1, \beta^2, 0, 0), \quad \eta_\alpha = (\mathrm{e}^{-\mathrm{i}\tau/2}, \mathrm{e}^{\mathrm{i}\tau/2}), \\
&\bar{\gamma}_I = (0, 0, \bar{\gamma}_3, \bar{\gamma}_4), \quad \bar{\mu}^\alpha = (\mathrm{e}^{\mathrm{i}\tau/2}, -\mathrm{e}^{-\mathrm{i}\tau/2}), \quad \delta^I = (0, 0, \delta^3, \delta^4), \quad \nu_\alpha = (-\mathrm{e}^{-\mathrm{i}\tau/2}, \mathrm{e}^{\mathrm{i}\tau/2}).
\end{aligned} \tag{4.53}
$$

Similar to the case in Minkowski spacetime, we have four classes of solutions that make this DT type Wilson loop 1/6 BPS, and the preserved supersymmetries are (4.52). All of them must satisfy

$$\bar{\alpha}_I \delta^J = \bar{\gamma}_I \beta^J = 0. \tag{4.54}$$

Class I

In the first solution we have

$$\bar{\gamma}_I = \delta^I = 0. \tag{4.55}$$

Now we have

$$\begin{aligned}
&L_B = \frac{2\pi}{k} M^I{}_J \begin{pmatrix} \phi_I \bar{\phi}^I & \\ & \bar{\phi}^J \phi_I \end{pmatrix}, \quad L_F = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & \bar{\alpha}_I \bar{\zeta} \psi^I \\ \bar{\psi}_I \eta \beta^I & \end{pmatrix}, \\
&\Lambda = \sqrt{\frac{2\pi}{k}} \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & \bar{\alpha}_2 \phi_1 - \bar{\alpha}_1 \phi_2 \\ \beta^2 \bar{\phi}^1 - \beta^1 \bar{\phi}^2 & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P^{12} + \mathrm{i}\gamma_3 S^{12}) + (\bar{P}_{12} + \bar{S}_{12} \mathrm{i}\gamma_3) b, \quad \bar{a}^\alpha = (1, 0), \quad b_\alpha = (0, 1).
\end{aligned} \tag{4.56}$$

This shows that the difference between the DT type Wilson loop and GY type Wilson loop is Q-exact. When

$$\beta^I = \frac{\mathrm{i}}{\bar{\alpha}_J \alpha^J} \alpha^I, \quad \bar{\alpha}_I \neq 0, \tag{4.57}$$

the Wilson loop becomes 1/2 BPS, and the preserved supersymmetries are

$$\bar{\alpha}_I \vartheta^{IJ} = \mathrm{i}\gamma_3 \bar{\alpha}_I \theta^{IJ}, \quad \epsilon_{IJKL} \alpha^J \vartheta^{KL} = -\mathrm{i}\gamma_3 \epsilon_{IJKL} \alpha^J \theta^{KL}. \tag{4.58}$$

Class II

In the second solution we have

$$\bar{\alpha}_I = \beta^I = 0. \tag{4.59}$$

Now we have

$$\begin{aligned}
&L_B = \frac{2\pi}{k} M^I{}_J \begin{pmatrix} \phi_I \bar{\phi}^J & \\ & \bar{\phi}^J \phi_I \end{pmatrix}, \quad L_F = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & \bar{\gamma}_I \bar{\mu} \psi^I \\ \bar{\psi}_I \nu \delta^I & \end{pmatrix}, \\
&\Lambda = \sqrt{\frac{2\pi}{k}} \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & \bar{\gamma}_4 \phi_3 - \bar{\gamma}_3 \phi_4 \\ \delta^4 \bar{\phi}^3 - \delta^3 \bar{\phi}^4 & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau}, \\
&Q = \bar{a}(P^{34} - \mathrm{i}\gamma_3 S^{34}) + (\bar{P}_{34} - \bar{S}_{34} \mathrm{i}\gamma_3) b, \quad \bar{a}^\alpha = (1, 0), \quad b_\alpha = (0, 1).
\end{aligned} \tag{4.60}$$

When

$$\delta^I = -\frac{\mathrm{i}}{\bar{\gamma}_J \gamma^J} \gamma^I, \quad \bar{\gamma}_I \neq 0, \tag{4.61}$$

the Wilson loop is 1/2 BPS, and the preserved supersymmetries are

$$\bar{\gamma}_I \vartheta^{IJ} = -\mathrm{i}\gamma_3 \bar{\gamma}_I \theta^{IJ}, \quad \epsilon_{IJKL} \gamma^J \vartheta^{KL} = \mathrm{i}\gamma_3 \epsilon_{IJKL} \gamma^J \theta^{KL}. \tag{4.62}$$

Class III

In the third solution we have

$$\beta^I = \delta^I = 0. \tag{4.63}$$

Now we have

$$\begin{aligned}
&L_B = 0, \quad L_F = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & (\bar\alpha_I \bar\zeta + \bar\gamma_I \bar\mu)\psi^I \\ 0 & \end{pmatrix}, \quad \kappa = 2\mathrm{e}^{-\mathrm{i}\tau}, \\
&\Lambda = \sqrt{\frac{2\pi}{k}} \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & \bar\alpha_2\phi_1 - \bar\alpha_1\phi_2 + \bar\gamma_4\phi_3 - \bar\gamma_3\phi_4 \\ 0 & \end{pmatrix}, \\
&Q = \bar a(P^{12} + \mathrm{i}\gamma_3 S^{12} + P^{34} - \mathrm{i}\gamma_3 S^{34}), \quad \bar a^\alpha = (1, 0).
\end{aligned} \tag{4.64}$$

There is no SUSY enhancement for this solution.

Class IV

In the fourth solution we have

$$\bar\alpha_I = \bar\gamma_I = 0. \tag{4.65}$$

Now we have

$$\begin{aligned}
&L_B = 0, \quad L_F = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & 0 \\ \bar\psi_I(\eta\beta^I + \nu\delta^I) & \end{pmatrix}, \quad \kappa = 2\mathrm{e}^{-\mathrm{i}\tau}, \\
&\Lambda = \sqrt{\frac{2\pi}{k}} \mathrm{e}^{\mathrm{i}\tau/2} \begin{pmatrix} & 0 \\ \beta^2\bar\phi^1 - \beta^1\bar\phi^2 + \delta^4\bar\phi^3 - \delta^3\bar\phi^4 & \end{pmatrix}, \\
&Q = (\bar P_{12} + \bar S_{12}\mathrm{i}\gamma_3 + \bar P_{34} - \bar S_{34}\mathrm{i}\gamma_3)b, \quad b_\alpha = (0, 1).
\end{aligned} \tag{4.66}$$

There is no SUSY enhancement for this solution.

5. $\mathcal{N} = 4$ orbifold ABJM theory

The calculation for the Wilson loops in $\mathcal{N} = 4$ orbifold ABJM theory is very similar to the case of ABJM theory. It will be brief in this section.

Orbifolding the ABJM theory with gauge group $U(rN) \times U(rN)$ and levels $(k, -k)$ by Z_r, one gets the $\mathcal{N} = 4$ CSM theory with gauge group $U(N)^{2r}$ and Chern–Simons levels $(k, -k, \cdots, k, -k)$ [22]. We get the SUSY transformations of the $\mathcal{N} = 4$ orbifold ABJM theory from those of ABJM theory

$$\begin{aligned}
&\delta\phi_i^{(2\ell+1)} = \mathrm{i}\bar\epsilon_{i\hat\imath}\psi^{\hat\imath}_{(2\ell+1)}, \quad \delta\phi_{\hat\imath}^{(2\ell)} = -\mathrm{i}\bar\epsilon_{i\hat\imath}\psi^i_{(2\ell)}, \\
&\delta\bar\phi^i_{(2\ell+1)} = \mathrm{i}\bar\psi_{\hat\imath}^{(2\ell+1)}\epsilon^{i\hat\imath}, \quad \delta\bar\phi^{\hat\imath}_{(2\ell)} = -\mathrm{i}\bar\psi_i^{(2\ell)}\epsilon^{i\hat\imath}, \\
&\delta A_\mu^{(2\ell+1)} = \frac{2\pi}{k}\Big[\Big(\phi_i^{(2\ell+1)}\bar\psi_{\hat\imath}^{(2\ell+1)} - \phi_{\hat\imath}^{(2\ell)}\bar\psi_i^{(2\ell)}\Big)\gamma_\mu\epsilon^{i\hat\imath} \\
&\qquad + \bar\epsilon_{i\hat\imath}\gamma_\mu\Big(\psi^{\hat\imath}_{(2\ell+1)}\bar\phi^i_{(2\ell+1)} - \psi^i_{(2\ell)}\bar\phi^{\hat\imath}_{(2\ell)}\Big)\Big],
\end{aligned}$$

$$
\begin{aligned}
\delta\hat{A}_\mu^{(2\ell)} &= \frac{2\pi}{k}\bigg[\Big(\bar{\psi}_{\hat{\imath}}^{(2\ell-1)}\phi_i^{(2\ell-1)} - \bar{\psi}_i^{(2\ell)}\phi_{\hat{\imath}}^{(2\ell)}\Big)\gamma_\mu\epsilon^{i\hat{\imath}} \\
&\quad + \bar{\epsilon}_{i\hat{\imath}}\gamma_\mu\left(\bar{\phi}^i_{(2\ell-1)}\psi^{\hat{\imath}}_{(2\ell-1)} - \bar{\phi}^{\hat{\imath}}_{(2\ell)}\psi^i_{(2\ell)}\right)\bigg], \\
\delta\psi^i_{(2\ell)} &= \gamma^\mu\epsilon^{i\hat{\imath}}D_\mu\phi_{\hat{\imath}}^{(2\ell)} + \vartheta^{i\hat{\imath}}\phi_{\hat{\imath}}^{(2\ell)} - \frac{2\pi}{k}\epsilon^{i\hat{\imath}}\Big(\phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^j_{(2\ell-1)}\phi_j^{(2\ell-1)} \\
&\quad + \phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_{\hat{\jmath}}^{(2\ell)} - \phi_j^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)}\phi_{\hat{\imath}}^{(2\ell)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_{\hat{\imath}}^{(2\ell)}\Big) \\
&\quad - \frac{4\pi}{k}\epsilon^{j\hat{\jmath}}\left(\phi_j^{(2\ell+1)}\bar{\phi}^i_{(2\ell+1)}\phi_{\hat{\jmath}}^{(2\ell)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^i_{(2\ell-1)}\phi_j^{(2\ell-1)}\right), \\
\delta\psi^{\hat{\imath}}_{(2\ell+1)} &= -\gamma^\mu\epsilon^{i\hat{\imath}}D_\mu\phi_i^{(2\ell+1)} - \vartheta^{i\hat{\imath}}\phi_i^{(2\ell+1)} + \frac{2\pi}{k}\epsilon^{i\hat{\imath}}\Big(\phi_i^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)}\phi_j^{(2\ell+1)} \\
&\quad + \phi_i^{(2\ell+1)}\bar{\phi}^{\hat{\jmath}}_{(2\ell+2)}\phi_{\hat{\jmath}}^{(2\ell+2)} - \phi_j^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)}\phi_i^{(2\ell+1)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_i^{(2\ell+1)}\Big) \\
&\quad - \frac{4\pi}{k}\epsilon^{j\hat{\jmath}}\left(\phi_j^{(2\ell+1)}\bar{\phi}^{\hat{\imath}}_{(2\ell+2)}\phi_{\hat{\jmath}}^{(2\ell+2)} - \phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\phi_j^{(2\ell+1)}\right), \\
\delta\bar{\psi}_i^{(2\ell)} &= -\bar{\epsilon}_{i\hat{\imath}}\gamma^\mu D_\mu\bar{\phi}^{\hat{\imath}}_{(2\ell)} + \bar{\vartheta}_{i\hat{\imath}}\bar{\phi}^{\hat{\imath}}_{(2\ell)} + \frac{2\pi}{k}\bar{\epsilon}_{i\hat{\imath}}\Big(\bar{\phi}^{\hat{\imath}}_{(2\ell)}\phi_j^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)} \\
&\quad + \bar{\phi}^{\hat{\imath}}_{(2\ell)}\phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)} - \bar{\phi}^j_{(2\ell-1)}\phi_j^{(2\ell-1)}\bar{\phi}^{\hat{\imath}}_{(2\ell)} - \bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\Big) \\
&\quad + \frac{4\pi}{k}\bar{\epsilon}_{j\hat{\jmath}}\left(\bar{\phi}^j_{(2\ell-1)}\phi_i^{(2\ell-1)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)} - \bar{\phi}^{\hat{\jmath}}_{(2\ell)}\phi_i^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)}\right), \\
\delta\bar{\psi}_{\hat{\imath}}^{(2\ell+1)} &= \bar{\epsilon}_{i\hat{\imath}}\gamma^\mu D_\mu\bar{\phi}^i_{(2\ell+1)} - \bar{\vartheta}_{i\hat{\imath}}\bar{\phi}^i_{(2\ell+1)} - \frac{2\pi}{k}\bar{\epsilon}_{i\hat{\imath}}\Big(\bar{\phi}^i_{(2\ell+1)}\phi_j^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)} \\
&\quad + \bar{\phi}^i_{(2\ell+1)}\phi_{\hat{\jmath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)} - \bar{\phi}^j_{(2\ell+1)}\phi_j^{(2\ell+1)}\bar{\phi}^i_{(2\ell+1)} - \bar{\phi}^{\hat{\jmath}}_{(2\ell+2)}\phi_{\hat{\jmath}}^{(2\ell+2)}\bar{\phi}^i_{(2\ell+1)}\Big) \\
&\quad + \frac{4\pi}{k}\bar{\epsilon}_{j\hat{\jmath}}\left(\bar{\phi}^j_{(2\ell+1)}\phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)} - \bar{\phi}^{\hat{\jmath}}_{(2\ell+2)}\phi_{\hat{\imath}}^{(2\ell+2)}\bar{\phi}^j_{(2\ell+1)}\right).
\end{aligned} \tag{5.1}
$$

Here $\ell = 0, 1, \cdots, r-1$. There are no summations of ℓ here, and would not be summations of ℓ later unless it is pointed out explicitly. Here $i, j, \cdots = 1, 2$ and $\hat{\imath}, \hat{\jmath}, \cdots = \hat{1}, \hat{2}$ are indices of the $SU(2) \times SU(2)$ R-symmetry. We have definitions of covariant derivatives

$$
\begin{aligned}
D_\mu\phi_{\hat{\imath}}^{(2\ell)} &= \partial_\mu\phi_{\hat{\imath}}^{(2\ell)} + \mathrm{i}A_\mu^{(2\ell+1)}\phi_{\hat{\imath}}^{(2\ell)} - \mathrm{i}\phi_{\hat{\imath}}^{(2\ell)}\hat{A}_\mu^{(2\ell)}, \\
D_\mu\phi_i^{(2\ell+1)} &= \partial_\mu\phi_i^{(2\ell+1)} + \mathrm{i}A_\mu^{(2\ell+1)}\phi_i^{(2\ell+1)} - \mathrm{i}\phi_i^{(2\ell+1)}\hat{A}_\mu^{(2\ell+2)}, \\
D_\mu\bar{\phi}^{\hat{\imath}}_{(2\ell)} &= \partial_\mu\bar{\phi}^{\hat{\imath}}_{(2\ell)} + \mathrm{i}\hat{A}_\mu^{(2\ell)}\bar{\phi}^{\hat{\imath}}_{(2\ell)} - \mathrm{i}\bar{\phi}^{\hat{\imath}}_{(2\ell)}A_\mu^{(2\ell+1)}, \\
D_\mu\bar{\phi}^i_{(2\ell+1)} &= \partial_\mu\bar{\phi}^i_{(2\ell+1)} + \mathrm{i}\hat{A}_\mu^{(2\ell+2)}\bar{\phi}^i_{(2\ell+1)} - \mathrm{i}\bar{\phi}^i_{(2\ell+1)}A_\mu^{(2\ell+1)}.
\end{aligned} \tag{5.2}
$$

We have SUSY parameters $\epsilon^{i\hat{\imath}} = \theta^{i\hat{\imath}} + x^\mu\gamma_\mu\vartheta^{i\hat{\imath}}$ and $\bar{\epsilon}_{i\hat{\imath}} = \bar{\theta}_{i\hat{\imath}} - \bar{\vartheta}_{i\hat{\imath}}x^\mu\gamma_\mu$ with constraints

$$
(\theta^{i\hat{\imath}})^* = \bar{\theta}_{i\hat{\imath}}, \quad \bar{\theta}_{i\hat{\imath}} = \epsilon_{ij}\epsilon_{\hat{\imath}\hat{\jmath}}\theta^{j\hat{\jmath}}, \quad (\vartheta^{i\hat{\imath}})^* = \bar{\vartheta}_{i\hat{\imath}}, \quad \bar{\vartheta}_{i\hat{\imath}} = \epsilon_{ij}\epsilon_{\hat{\imath}\hat{\jmath}}\vartheta^{j\hat{\jmath}}. \tag{5.3}
$$

Symbols ϵ_{ij} and $\epsilon_{\hat{\imath}\hat{\jmath}}$ are antisymmetric with $\epsilon_{12} = \epsilon_{\hat{1}\hat{2}} = 1$. The supercharges are defined as

$$
\delta = \mathrm{i}(\bar{\theta}_{i\hat{\imath}}P^{i\hat{\imath}} + \bar{\vartheta}_{i\hat{\imath}}S^{i\hat{\imath}}) = \mathrm{i}(\bar{P}_{i\hat{\imath}}\theta^{i\hat{\imath}} + \bar{S}_{i\hat{\imath}}\vartheta^{i\hat{\imath}}), \tag{5.4}
$$

with the constraints

$$
(P^{i\hat{\imath}})^* = \bar{P}_{i\hat{\imath}}, \quad \bar{P}_{i\hat{\imath}} = \epsilon_{ij}\epsilon_{\hat{\imath}\hat{\jmath}}P^{j\hat{\jmath}}, \quad (S^{i\hat{\imath}})^* = \bar{S}_{i\hat{\imath}}, \quad \bar{S}_{i\hat{\imath}} = \epsilon_{ij}\epsilon_{\hat{\imath}\hat{\jmath}}S^{j\hat{\jmath}}. \tag{5.5}
$$

In Euclidean space, the SUSY transformations (5.1) and the definitions of supercharges (5.4) still apply, but the constraints become

$$\bar\theta_{i\hat\imath}=\epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}\theta^{j\hat\jmath},\quad \bar\vartheta_{i\hat\imath}=\epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}\vartheta^{j\hat\jmath},\quad \bar P_{i\hat\imath}=\epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}P^{j\hat\jmath},\quad \bar S_{i\hat\imath}=\epsilon_{ij}\epsilon_{\hat\imath\hat\jmath}S^{j\hat\jmath}. \tag{5.6}$$

5.1. Straight line in Minkowski spacetime

We consider BPS GY and DT type Wilson loops along timelike infinite straight lines in Minkowski spacetime.

5.1.1. GY type Wilson loop

There is GY type Wilson loop along the line $x^\mu=\tau\delta_0^\mu$

$$\begin{aligned}
&W^{(\ell)}_{\rm GY}=\mathcal{P}\exp\left(-\mathrm{i}\int d\tau L^{(\ell)}_{\rm GY}(\tau)\right),\quad L^{(\ell)}_{\rm GY}=\begin{pmatrix}\mathcal{A}^{(2\ell+1)}_{\rm GY} & \\ & \hat{\mathcal{A}}^{(2\ell)}_{\rm GY}\end{pmatrix},\\
&\mathcal{A}^{(2\ell+1)}_{\rm GY}=A^{(2\ell+1)}_\mu\dot x^\mu+\frac{2\pi}{k}(R_{(\ell)}{}^i{}_j\phi_i^{(2\ell+1)}\bar\phi^j_{(2\ell+1)}+R_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}\phi_{\hat\imath}^{(2\ell)}\bar\phi^{\hat\jmath}_{(2\ell)})|\dot x|,\\
&\hat{\mathcal{A}}^{(2\ell)}_{\rm GY}=\hat A^{(2\ell)}_\mu\dot x^\mu+\frac{2\pi}{k}(S^{(\ell)}{}_i{}^j\bar\phi^i_{(2\ell-1)}\phi_j^{(2\ell-1)}+S^{(\ell)}{}_{\hat\imath}{}^{\hat\jmath}\bar\phi^{\hat\imath}_{(2\ell)}\phi_{\hat\jmath}^{(2\ell)})|\dot x|.
\end{aligned} \tag{5.7}$$

The Poincaré SUSY transformation $\delta\mathcal{A}^{(2\ell+1)}_{\rm GY}=0$ leads to

$$\begin{aligned}
&\gamma_0\theta^{i\hat\imath}=-\mathrm{i}R_{(\ell)}{}^i{}_j\theta^{j\hat\imath},\quad \gamma_0\theta^{i\hat\imath}=-\mathrm{i}R_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}\theta^{i\hat\jmath},\\
&\bar\theta_{i\hat\imath}\gamma_0=-\mathrm{i}R_{(\ell)}{}^j{}_i\bar\theta_{j\hat\imath},\quad \bar\theta_{i\hat\imath}\gamma_0=-\mathrm{i}R_{(\ell)}{}^{\hat\jmath}{}_{\hat\imath}\bar\theta_{i\hat\jmath}.
\end{aligned} \tag{5.8}$$

Taking complex conjugates of the last two equations we have

$$\gamma_0\theta^{i\hat\imath}=-\mathrm{i}R^\dagger_{(\ell)}{}^i{}_j\theta^{j\hat\imath},\quad \gamma_0\theta^{i\hat\imath}=-\mathrm{i}R^\dagger_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}\theta^{i\hat\jmath}, \tag{5.9}$$

with $R^\dagger_{(\ell)}{}^i{}_j=(R_{(\ell)}{}^j{}_i)^*$ and $R^\dagger_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}=(R_{(\ell)}{}^{\hat\jmath}{}_{\hat\imath})^*$. We define

$$\begin{aligned}
&B_{(\ell)}{}^i{}_j=\frac{R_{(\ell)}{}^i{}_j+R^\dagger_{(\ell)}{}^i{}_j}{2},\quad C_{(\ell)}{}^i{}_j=-\mathrm{i}\frac{R_{(\ell)}{}^i{}_j-R^\dagger_{(\ell)}{}^i{}_j}{2},\\
&B_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}=\frac{R_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}+R^\dagger_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}}{2},\quad C_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}=-\mathrm{i}\frac{R_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}-R^\dagger_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}}{2},
\end{aligned} \tag{5.10}$$

and then we get

$$\begin{aligned}
&\gamma_0\theta^{i\hat\imath}=-\mathrm{i}B_{(\ell)}{}^i{}_j\theta^{j\hat\imath},\quad C_{(\ell)}{}^i{}_j\theta^{j\hat\imath}=0,\\
&\gamma_0\theta^{i\hat\imath}=-\mathrm{i}B_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}\theta^{i\hat\jmath},\quad C_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}\theta^{i\hat\jmath}=0.
\end{aligned} \tag{5.11}$$

We use $SU(2)\times SU(2)$ R-symmetry rotation and make

$$B_{(\ell)}{}^i{}_j=\mathrm{diag}(b_1^{(\ell)},b_2^{(\ell)}),\quad B_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}=\mathrm{diag}(b_{\hat 1}^{(\ell)},b_{\hat 2}^{(\ell)}). \tag{5.12}$$

Then we have

$$\gamma_0\theta^{i\hat\imath}=-\mathrm{i}b_i^{(\ell)}\theta^{i\hat\imath},\quad \gamma_0\theta^{i\hat\imath}=-\mathrm{i}b_{\hat\imath}^{(\ell)}\theta^{i\hat\imath}, \tag{5.13}$$

with no summation of indices on the right-hand sides of the two equations. Without loss of generality we choose

$$\gamma_0\theta^{1\hat{1}} = \mathrm{i}\theta^{1\hat{1}}, \quad \theta^{1\hat{1}} \neq 0. \tag{5.14}$$

Using $\theta^{1\hat{1}*} = \theta^{2\hat{2}}$, we get

$$\gamma_0\theta^{2\hat{2}} = -\mathrm{i}\theta^{2\hat{2}}, \quad \theta^{2\hat{2}} \neq 0. \tag{5.15}$$

This leads to that $b_1^{(\ell)} = -b_2^{(\ell)} = -1$ and $b_{\hat{1}}^{(\ell)} = -b_{\hat{2}}^{(\ell)} = -1$. Furthermore, we have $\theta^{1\hat{2}} = \theta^{2\hat{1}} = 0$. Then we get $C_{(\ell)}{}^i{}_j = C_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}} = 0$. The SUSY transformation $\delta\hat{\mathcal{A}}_{\rm GY}^{(2\ell)} = 0$ leads to

$$\begin{aligned}
&\gamma_0\theta^{i\hat{\imath}} = -\mathrm{i}S^{(\ell)}{}_j{}^i\theta^{j\hat{\imath}}, \quad \gamma_0\theta^{i\hat{\imath}} = -\mathrm{i}S^{(\ell)}{}_{\hat{\jmath}}{}^{\hat{\imath}}\theta^{i\hat{\jmath}},\\
&\bar{\theta}_{i\hat{\imath}}\gamma_0 = -\mathrm{i}S^{(\ell)}{}_i{}^j\bar{\theta}_{j\hat{\imath}}, \quad \bar{\theta}_{i\hat{\imath}}\gamma_0 = -\mathrm{i}S^{(\ell)}{}_{\hat{\imath}}{}^{\hat{\jmath}}\bar{\theta}_{i\hat{\jmath}}.
\end{aligned} \tag{5.16}$$

Then we get

$$\begin{aligned}
&(R_{(\ell)}{}^i{}_j - S^{(\ell)}{}_j{}^i)\theta^{j\hat{\imath}} = 0, \quad (R_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}} - S^{(\ell)}{}_{\hat{\jmath}}{}^{\hat{\imath}})\theta^{i\hat{\jmath}} = 0,\\
&(R_{(\ell)}{}^j{}_i - S^{(\ell)}{}_i{}^j)\bar{\theta}_{j\hat{\imath}} = 0, \quad (R_{(\ell)}{}^{\hat{\jmath}}{}_{\hat{\imath}} - S^{(\ell)}{}_{\hat{\imath}}{}^{\hat{\jmath}})\bar{\theta}_{i\hat{\jmath}} = 0,
\end{aligned} \tag{5.17}$$

and we get $S^{(\ell)}{}_j{}^i = R_{(\ell)}{}^i{}_j$ and $S^{(\ell)}{}_{\hat{\jmath}}{}^{\hat{\imath}} = R_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}}$.

In summary, we have the GY type 1/4 BPS Wilson loop (5.7) with $S^{(\ell)}{}_j{}^i = R_{(\ell)}{}^i{}_j = \mathrm{diag}(-1, 1)$ and $S^{(\ell)}{}_{\hat{\jmath}}{}^{\hat{\imath}} = R_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}} = \mathrm{diag}(-1, 1)$, and the preserved supersymmetries are

$$\gamma_0\theta^{1\hat{1}} = \mathrm{i}\theta^{1\hat{1}}, \quad \gamma_0\theta^{2\hat{2}} = -\mathrm{i}\theta^{2\hat{2}}, \quad \theta^{1\hat{2}} = \theta^{2\hat{1}} = 0. \tag{5.18}$$

This is just the 1/4 BPS Wilson loop that was constructed in [25,26].

5.1.2. *DT type Wilson loop*

We want a DT type Wilson loop that preserves the same supersymmetries as these of the GY type 1/4 BPS Wilson loop (5.18). A general DT type Wilson loop is

$$\begin{aligned}
&W_{\rm DT}^{(\ell)} = \mathcal{P}\exp\left(-\mathrm{i}\int \mathrm{d}\tau L_{\rm DT}^{(\ell)}(\tau)\right), \quad L_{\rm DT}^{(\ell)} = \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \bar{f}_1^{(\ell)} \\ f_2^{(\ell)} & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix},\\
&\mathcal{A}^{(2\ell+1)} = \mathcal{A}_{\rm GY}^{(2\ell+1)} + \frac{2\pi}{k}(M_{(\ell)}{}^i{}_j\phi_i^{(2\ell+1)}\bar{\phi}^j_{(2\ell+1)} + M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}}\phi_{\hat{\imath}}^{(2\ell)}\bar{\phi}^{\hat{\jmath}}_{(2\ell)})|\dot{x}|,\\
&\hat{\mathcal{A}}^{(2\ell)} = \hat{\mathcal{A}}_{\rm GY}^{(2\ell)} + \frac{2\pi}{k}(N^{(\ell)}{}_i{}^j\bar{\phi}^i_{(2\ell-1)}\phi_j^{(2\ell-1)} + N^{(\ell)}{}_{\hat{\imath}}{}^{\hat{\jmath}}\bar{\phi}^{\hat{\imath}}_{(2\ell)}\phi_{\hat{\jmath}}^{(2\ell)})|\dot{x}|,\\
&\bar{f}_1^{(\ell)} = \sqrt{\frac{2\pi}{k}}\bar{\zeta}_i^{(\ell)}\psi^i_{(2\ell)}|\dot{x}|, \quad f_2^{(\ell)} = \sqrt{\frac{2\pi}{k}}\bar{\psi}_i^{(2\ell)}\eta^i_{(\ell)}|\dot{x}|.
\end{aligned} \tag{5.19}$$

From SUSY variations of $\mathcal{A}^{(2\ell+1)}$ and $\hat{\mathcal{A}}^{(2\ell)}$, we get

$$\begin{aligned}
&M_{(\ell)}{}^i{}_j = N^{(\ell)}{}_j{}^i = 0, \quad M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}} = N^{(\ell)}{}_{\hat{\jmath}}{}^{\hat{\imath}},\\
&\sqrt{\frac{2\pi}{k}}M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}}\phi_{\hat{\imath}}^{(2\ell)}\theta^{i\hat{\jmath}} = -\eta^i_{(\ell)}\bar{g}_1^{(\ell)}, \quad \sqrt{\frac{2\pi}{k}}M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}}\bar{\phi}^{\hat{\jmath}}_{(2\ell)}\bar{\theta}_{i\hat{\imath}} = g_2^{(\ell)}\bar{\zeta}_i^{(\ell)}.
\end{aligned} \tag{5.20}$$

From variations of $\bar{f}_1^{(\ell)}$ and $f_2^{(\ell)}$ we get

$$\begin{aligned}
&\bar{\zeta}_1^{(\ell)}\gamma_0 = \mathrm{i}\bar{\zeta}_1^{(\ell)}, \quad \bar{\zeta}_2^{(\ell)}\gamma_0 = -\mathrm{i}\bar{\zeta}_2^{(\ell)}, \quad \gamma_0\eta^1_{(\ell)} = \mathrm{i}\eta^1_{(\ell)}, \quad \gamma_0\eta^2_{(\ell)} = -\mathrm{i}\eta^2_{(\ell)}, \\
&\bar{g}_1^{(\ell)} = -\sqrt{\frac{2\pi}{k}}\bar{\zeta}_i^{(\ell)}\gamma_0\theta^{i\hat{\imath}}\phi_{\hat{\imath}}^{(2\ell)}, \quad M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}}\bar{\zeta}_i^{(\ell)}\gamma_0\theta^{i\hat{k}} = M_{(\ell)}{}^{\hat{k}}{}_{\hat{\jmath}}\bar{\zeta}_i^{(\ell)}\gamma_0\theta^{i\hat{\imath}}, \\
&g_2^{(\ell)} = \sqrt{\frac{2\pi}{k}}\bar{\theta}_{i\hat{\imath}}\gamma_0\eta^i_{(\ell)}\bar{\phi}^{\hat{\imath}}_{(2\ell)}, \quad M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}}\bar{\theta}_{i\hat{k}}\gamma_0\eta^i_{(\ell)} = M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{k}}\bar{\theta}_{i\hat{\jmath}}\gamma_0\eta^i_{(\ell)}.
\end{aligned} \tag{5.21}$$

We make the parameterizations

$$\begin{aligned}
&\bar{\zeta}_1^{(\ell)} = \bar{\alpha}^{(\ell)}\bar{\zeta}, \quad \bar{\zeta}^\alpha = (1, \mathrm{i}), \quad \bar{\zeta}_2^{(\ell)} = \bar{\gamma}^{(\ell)}\bar{\mu}, \quad \bar{\mu}^\alpha = (-\mathrm{i}, -1), \\
&\eta^1_{(\ell)} = \eta\beta_{(\ell)}, \quad \eta_\alpha = (1, -\mathrm{i}), \quad \eta^2_{(\ell)} = \nu\delta_{(\ell)}, \quad \nu_\alpha = (-\mathrm{i}, 1).
\end{aligned} \tag{5.22}$$

Equations (5.20) become

$$M_{(\ell)}{}^{\hat{\imath}}{}_{\hat{\jmath}}\theta^{i\hat{\jmath}} = \eta^i_{(\ell)}\bar{\zeta}_j^{(\ell)}\gamma_0\theta^{j\hat{\imath}}, \quad M_{(\ell)}{}^{\hat{\jmath}}{}_{\hat{\imath}}\bar{\theta}_{i\hat{\jmath}} = \bar{\theta}_{j\hat{\imath}}\gamma_0\eta^j_{(\ell)}\bar{\zeta}_i^{(\ell)}, \tag{5.23}$$

from which we have

$$\begin{aligned}
&M_{(\ell)}{}^{\hat{1}}{}_{\hat{1}} = 2\mathrm{i}\bar{\alpha}^{(\ell)}\beta_{(\ell)}, \quad M_{(\ell)}{}^{\hat{2}}{}_{\hat{2}} = 2\mathrm{i}\bar{\gamma}^{(\ell)}\delta_{(\ell)}, \\
&M_{(\ell)}{}^{\hat{1}}{}_{\hat{2}} = M_{(\ell)}{}^{\hat{2}}{}_{\hat{1}} = \bar{\alpha}^{(\ell)}\delta_{(\ell)} = \bar{\gamma}^{(\ell)}\beta_{(\ell)} = 0.
\end{aligned} \tag{5.24}$$

We have four classes of solutions.

Class I

In the first solution we have

$$\bar{\gamma}^{(\ell)} = \delta_{(\ell)} = 0. \tag{5.25}$$

We have the DT type 1/4 BPS Wilson loop that preserves the supersymmetries (5.18), and there are two free complex parameters $\bar{\alpha}^{(\ell)}$ and $\beta_{(\ell)}$. Now we have

$$\begin{aligned}
&L_B^{(\ell)} = \frac{4\pi\mathrm{i}}{k}\bar{\alpha}^{(\ell)}\beta_{(\ell)}\begin{pmatrix} \phi_{\hat{1}}^{(2\ell)}\bar{\phi}^{\hat{1}}_{(2\ell)} & \\ & \bar{\phi}^{\hat{1}}_{(2\ell)}\phi_{\hat{1}}^{(2\ell)} \end{pmatrix}, \\
&L_F^{(\ell)} = \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\alpha}^{(\ell)}\bar{\zeta}\psi^1_{(2\ell)} \\ \bar{\psi}_1^{(2\ell)}\eta\beta_{(\ell)} & \end{pmatrix}, \\
&\Lambda^{(\ell)} = -\sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\alpha}^{(\ell)}\phi_{\hat{1}}^{(2\ell)} \\ \beta_{(\ell)}\bar{\phi}^{\hat{1}}_{(2\ell)} & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\zeta}P^{1\hat{1}} + \bar{P}_{1\hat{1}}\eta.
\end{aligned} \tag{5.26}$$

This shows that difference between the DT type Wilson loop and GY type Wilson loop is Q-exact.

We search for SUSY enhancement of the DT type BPS Wilson loop. One of the consequences of SUSY invariance of the Wilson loop is that

$$\gamma_0\theta^{i\hat{\imath}} = -\mathrm{i}R_{(\ell)}{}^i{}_j\theta^{j\hat{\imath}}, \quad \bar{\theta}_{i\hat{\imath}}\gamma_0 = -\mathrm{i}R_{(\ell)}{}^j{}_i\bar{\theta}_{j\hat{\imath}}, \tag{5.27}$$

from which we have

$$\gamma_0\theta^{1\hat{\imath}} = \mathrm{i}\theta^{1\hat{\imath}}, \quad \gamma_0\theta^{2\hat{\imath}} = -\mathrm{i}\theta^{2\hat{\imath}}, \quad \hat{\imath} = \hat{1}, \hat{2}. \tag{5.28}$$

From [37] we know that the only possibility of SUSY enhancement in the present case is that when

$$\bar{\alpha}^{(\ell)}\beta_{(\ell)} = -\mathrm{i}, \tag{5.29}$$

and supersymmetries are enhanced to 1/2 BPS. The preserved supersymmetries are (5.28). This is just the ψ_1-loop that was constructed in [25,26].

Class II

In the second solution we have

$$\bar{\alpha}^{(\ell)} = \beta_{(\ell)} = 0. \tag{5.30}$$

We have the DT type 1/4 BPS Wilson loop that preserves the supersymmetries (5.18). Now we have

$$\begin{aligned}
L_B^{(\ell)} &= \frac{4\pi \mathrm{i}}{k}\bar{\gamma}^{(\ell)}\delta_{(\ell)}\begin{pmatrix} \phi_{\hat{2}}^{(2\ell)}\bar{\phi}^{\hat{2}}_{(2\ell)} & \\ & \bar{\phi}^{\hat{2}}_{(2\ell)}\phi_{\hat{2}}^{(2\ell)} \end{pmatrix}, \\
L_F^{(\ell)} &= \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\gamma}^{(\ell)}\bar{\mu}\psi^2_{(2\ell)} \\ \bar{\psi}_2^{(2\ell)}\nu\delta_{(\ell)} & \end{pmatrix}, \\
\Lambda^{(\ell)} &= -\sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\gamma}^{(\ell)}\phi_{\hat{2}}^{(2\ell)} \\ \delta_{(\ell)}\bar{\phi}^{\hat{2}}_{(2\ell)} & \end{pmatrix}, \quad \kappa = 2\mathrm{i}, \quad Q = \bar{\mu}P^{2\hat{2}} + \bar{P}_{2\hat{2}}\nu.
\end{aligned} \tag{5.31}$$

When

$$\bar{\gamma}^{(\ell)}\delta_{(\ell)} = \mathrm{i}, \tag{5.32}$$

supersymmetries are enhanced to 1/2 BPS. The preserved supersymmetries are also (5.28). This is just the ψ_2-loop that was constructed in [26]. The ψ_1-loop and ψ_2-loop have the same preserved supersymmetries.

Class III

In the third solution we have

$$\beta_{(\ell)} = \delta_{(\ell)} = 0. \tag{5.33}$$

Now we have

$$\begin{aligned}
L_B^{(\ell)} &= 0, \quad L_F^{(\ell)} = \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\alpha}^{(\ell)}\bar{\zeta}\psi^1_{(2\ell)} + \bar{\gamma}^{(\ell)}\bar{\nu}\psi^2_{(2\ell)} \\ 0 & \end{pmatrix}, \\
\Lambda^{(\ell)} &= -\sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\alpha}^{(\ell)}\phi_{\hat{1}}^{(2\ell)} + \bar{\gamma}^{(\ell)}\phi_{\hat{2}}^{(2\ell)} \\ 0 & \end{pmatrix}, \\
\kappa &= -2\mathrm{i}, \quad Q = \bar{\zeta}P^{1\hat{1}} + \bar{\mu}P^{2\hat{2}}.
\end{aligned} \tag{5.34}$$

There is no SUSY enhancement in the present case.

Class IV

In the fourth solution we have

$$\bar{\alpha}^{(\ell)} = \bar{\gamma}^{(\ell)} = 0. \tag{5.35}$$

Now we have

$$
\begin{aligned}
& L_B^{(\ell)} = 0, \quad L_F^{(\ell)} = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & 0 \\ \bar\psi_1^{(2\ell)}\eta\beta_{(\ell)} + \bar\psi_2^{(2\ell)}\nu\delta_{(\ell)} & \end{pmatrix}, \\
& \Lambda^{(\ell)} = -\sqrt{\frac{2\pi}{k}} \begin{pmatrix} & 0 \\ \beta_{(\ell)}\bar\phi^{\hat 1}_{(2\ell)} + \delta_{(\ell)}\bar\phi^{\hat 2}_{(2\ell)} & \end{pmatrix}, \\
& \kappa = -2\mathrm{i}, \quad Q = \bar P_{1\hat 1}\eta + \bar P_{2\hat 2}\nu.
\end{aligned} \tag{5.36}
$$

There is no SUSY enhancement in the present case.

5.2. *Circle in Euclidean space*

We consider circular BPS GY and DT type Wilson loops in Euclidean space.

5.2.1. *GY type Wilson loop*

In Euclidean space we have the 1/4 BPS circular GY type Wilson loops along $x^\mu = (\cos\tau, \sin\tau, 0)$

$$
\begin{aligned}
& W_{\rm GY}^{(\ell)} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau L_{\rm GY}^{(\ell)}(\tau)\right), \quad L_{\rm GY}^{(\ell)} = \begin{pmatrix} \mathcal{A}_{\rm GY}^{(2\ell+1)} & \\ & \hat{\mathcal{A}}_{\rm GY}^{(2\ell)} \end{pmatrix}, \\
& \mathcal{A}_{\rm GY}^{(2\ell+1)} = A_\mu^{(2\ell+1)}\dot x^\mu + \frac{2\pi}{k}(R_{(\ell)}{}^i{}_j\phi_i^{(2\ell+1)}\bar\phi^j_{(2\ell+1)} + R_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}\phi_{\hat\imath}^{(2\ell)}\bar\phi^{\hat\jmath}_{(2\ell)})|\dot x|, \\
& \hat{\mathcal{A}}_{\rm GY}^{(2\ell)} = \hat A_\mu^{(2\ell)}\dot x^\mu + \frac{2\pi}{k}(R_{(\ell)}{}^i{}_j\bar\phi^j_{(2\ell-1)}\phi_i^{(2\ell-1)} + R_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath}\bar\phi^{\hat\jmath}_{(2\ell)}\phi_{\hat\imath}^{(2\ell)})|\dot x|,
\end{aligned} \tag{5.37}
$$

with $R_{(\ell)}{}^i{}_j = R_{(\ell)}{}^{\hat\imath}{}_{\hat\jmath} = \mathrm{diag}(\mathrm{i}, -\mathrm{i})$. The preserved supersymmetries are

$$
\vartheta^{1\hat 1} = \mathrm{i}\gamma_3\theta^{1\hat 1}, \quad \vartheta^{2\hat 2} = -\mathrm{i}\gamma_3\theta^{2\hat 2}, \quad \theta^{1\hat 2} = \theta^{2\hat 1} = \vartheta^{1\hat 2} = \vartheta^{2\hat 1} = 0. \tag{5.38}
$$

5.2.2. *DT type Wilson loop*

Along $x^\mu = (\cos\tau, \sin\tau, 0)$ we construct the circular DT type Wilson loop

$$
\begin{aligned}
& W_{\rm DT}^{(\ell)} = \mathrm{Tr}\mathcal{P}\exp\left(-\mathrm{i}\oint \mathrm{d}\tau L_{\rm DT}^{(\ell)}(\tau)\right), \quad L_{\rm DT}^{(\ell)} = \begin{pmatrix} \mathcal{A}^{(2\ell+1)} & \bar f_1^{(\ell)} \\ f_2^{(\ell)} & \hat{\mathcal{A}}^{(2\ell)} \end{pmatrix}, \\
& \mathcal{A}^{(2\ell+1)} = \mathcal{A}_{\rm GY}^{(2\ell+1)} - \frac{4\pi}{k}(\bar\alpha^{(\ell)}\beta_{(\ell)}\phi_{\hat 1}^{(2\ell)}\bar\phi^{\hat 1}_{(2\ell)} + \bar\gamma^{(\ell)}\delta_{(\ell)}\phi_{\hat 2}^{(2\ell)}\bar\phi^{\hat 2}_{(2\ell)})|\dot x|, \\
& \hat{\mathcal{A}}^{(2\ell)} = \hat{\mathcal{A}}_{\rm GY}^{(2\ell)} - \frac{4\pi}{k}(\bar\alpha^{(\ell)}\beta_{(\ell)}\bar\phi^{\hat 1}_{(2\ell)}\phi_{\hat 1}^{(2\ell)} + \bar\gamma^{(\ell)}\delta_{(\ell)}\bar\phi^{\hat 2}_{(2\ell)}\phi_{\hat 2}^{(2\ell)})|\dot x|, \\
& \bar f_1^{(\ell)} = \sqrt{\frac{2\pi}{k}}(\bar\alpha^{(\ell)}\bar\zeta\psi^1_{(2\ell)} + \bar\gamma^{(\ell)}\bar\mu\psi^2_{(2\ell)})|\dot x|, \quad \bar\zeta^\alpha = (\mathrm{e}^{\mathrm{i}\tau/2}, \mathrm{e}^{-\mathrm{i}\tau/2}), \quad \bar\mu^\alpha = (\mathrm{e}^{\mathrm{i}\tau/2}, -\mathrm{e}^{-\mathrm{i}\tau/2}), \\
& f_2^{(\ell)} = \sqrt{\frac{2\pi}{k}}(\bar\psi_1^{(2\ell)}\eta\beta_{(\ell)} + \bar\psi_2^{(2\ell)}\nu\delta_{(\ell)})|\dot x|, \quad \eta_\alpha = (\mathrm{e}^{-\mathrm{i}\tau/2}, \mathrm{e}^{\mathrm{i}\tau/2}), \quad \nu_\alpha = (-\mathrm{e}^{-\mathrm{i}\tau/2}, \mathrm{e}^{\mathrm{i}\tau/2}).
\end{aligned} \tag{5.39}
$$

To make it preserve the supersymmetries (5.38), we have four classes of solutions, and all of them satisfy

$$
\bar\alpha^{(\ell)}\delta_{(\ell)} = \bar\gamma^{(\ell)}\beta_{(\ell)} = 0. \tag{5.40}
$$

Class I

In the first solution we have

$$\bar{\gamma}^{(\ell)} = \delta_{(\ell)} = 0. \tag{5.41}$$

Now we have

$$\begin{aligned}
L_B^{(\ell)} &= -\frac{4\pi}{k}\bar{\alpha}^{(\ell)}\beta_{(\ell)}\begin{pmatrix}\phi_{\hat{1}}^{(2\ell)}\bar{\phi}^{\hat{1}}_{(2\ell)} & \\ & \bar{\phi}^{\hat{1}}_{(2\ell)}\phi_{\hat{1}}^{(2\ell)}\end{pmatrix},\\
L_F^{(\ell)} &= \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\alpha}^{(\ell)}\bar{\zeta}\psi^1_{(2\ell)}\\ \bar{\psi}_1^{(2\ell)}\eta\beta_{(\ell)} & \end{pmatrix},\\
\Lambda^{(\ell)} &= -\sqrt{\frac{2\pi}{k}}\mathrm{e}^{\mathrm{i}\tau/2}\begin{pmatrix} & \bar{\alpha}^{(\ell)}\phi_{\hat{1}}^{(2\ell)}\\ \beta_{(\ell)}\bar{\phi}^{\hat{1}}_{(2\ell)} & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau},\\
Q &= \bar{a}(P^{1\hat{1}} + \mathrm{i}\gamma_3 S^{1\hat{1}}) + (\bar{P}_{1\hat{1}} + \bar{S}_{1\hat{1}}\mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1,0), \quad b_\alpha = (0,1).
\end{aligned} \tag{5.42}$$

When

$$\bar{\alpha}^{(\ell)}\beta_{(\ell)} = \mathrm{i}, \tag{5.43}$$

supersymmetries are enhanced to 1/2 BPS. The preserved supersymmetries are

$$\vartheta^{1\hat{\imath}} = \mathrm{i}\gamma_3\theta^{1\hat{\imath}}, \quad \vartheta^{2\hat{\imath}} = -\mathrm{i}\gamma_3\theta^{2\hat{\imath}}, \quad \hat{\imath} = \hat{1},\hat{2}. \tag{5.44}$$

This is just the circular ψ_1-loop.

Class II

In the second solution we have

$$\bar{\alpha}^{(\ell)} = \beta_{(\ell)} = 0. \tag{5.45}$$

Now we have

$$\begin{aligned}
L_B^{(\ell)} &= -\frac{4\pi}{k}\bar{\gamma}^{(\ell)}\delta_{(\ell)}\begin{pmatrix}\phi_{\hat{2}}^{(2\ell)}\bar{\phi}^{\hat{2}}_{(2\ell)} & \\ & \bar{\phi}^{\hat{2}}_{(2\ell)}\phi_{\hat{2}}^{(2\ell)}\end{pmatrix},\\
L_F^{(\ell)} &= \sqrt{\frac{2\pi}{k}}\begin{pmatrix} & \bar{\gamma}^{(\ell)}\bar{\mu}\psi^2_{(2\ell)}\\ \bar{\psi}_2^{(2\ell)}\nu\delta_{(\ell)} & \end{pmatrix},\\
\Lambda^{(\ell)} &= -\sqrt{\frac{2\pi}{k}}\mathrm{e}^{\mathrm{i}\tau/2}\begin{pmatrix} & \bar{\gamma}^{(\ell)}\phi_{\hat{2}}^{(2\ell)}\\ \delta_{(\ell)}\bar{\phi}^{\hat{2}}_{(2\ell)} & \end{pmatrix}, \quad \kappa = -2\mathrm{e}^{-\mathrm{i}\tau},\\
Q &= \bar{a}(P^{2\hat{2}} - \mathrm{i}\gamma_3 S^{2\hat{2}}) + (\bar{P}_{2\hat{2}} - \bar{S}_{2\hat{2}}\mathrm{i}\gamma_3)b, \quad \bar{a}^\alpha = (1,0), \quad b_\alpha = (0,1).
\end{aligned} \tag{5.46}$$

When

$$\bar{\gamma}^{(\ell)}\delta_{(\ell)} = -\mathrm{i}, \tag{5.47}$$

supersymmetries are enhanced to 1/2 BPS. The preserved supersymmetries are also (5.44). This is just the circular ψ_2-loop.

Class III

In the third solution we have

$$\beta_{(\ell)} = \delta_{(\ell)} = 0. \tag{5.48}$$

Now we have

$$\begin{aligned} & L_B^{(\ell)} = 0, \quad L_F^{(\ell)} = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & \bar{\alpha}^{(\ell)} \bar{\zeta} \psi^1_{(2\ell)} + \bar{\gamma}^{(\ell)} \bar{\nu} \psi^2_{(2\ell)} \\ 0 & \end{pmatrix}, \\ & \Lambda^{(\ell)} = -\sqrt{\frac{2\pi}{k}} e^{i\tau/2} \begin{pmatrix} & \bar{\alpha}^{(\ell)} \phi_{\hat{1}}^{(2\ell)} + \bar{\gamma}^{(\ell)} \phi_{\hat{2}}^{(2\ell)} \\ 0 & \end{pmatrix}, \quad \kappa = 2e^{-i\tau}, \\ & Q = \bar{a}(P^{1\hat{1}} + i\gamma_3 S^{1\hat{1}} + P^{2\hat{2}} - i\gamma_3 S^{2\hat{2}}), \quad \bar{a}^\alpha = (1, 0). \end{aligned} \tag{5.49}$$

Class IV

In the fourth solution we have

$$\bar{\alpha}^{(\ell)} = \bar{\gamma}^{(\ell)} = 0. \tag{5.50}$$

Now we have

$$\begin{aligned} & L_B^{(\ell)} = 0, \quad L_F^{(\ell)} = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} & 0 \\ \bar{\psi}_1^{(2\ell)} \eta \beta_{(\ell)} + \bar{\psi}_2^{(2\ell)} \nu \delta_{(\ell)} & \end{pmatrix}, \\ & \Lambda^{(\ell)} = -\sqrt{\frac{2\pi}{k}} e^{i\tau/2} \begin{pmatrix} & 0 \\ \beta_{(\ell)} \bar{\phi}^{\hat{1}}_{(2\ell)} + \delta_{(\ell)} \bar{\phi}^{\hat{2}}_{(2\ell)} & \end{pmatrix}, \quad \kappa = 2e^{-i\tau}, \\ & Q = (\bar{P}_{1\hat{1}} + \bar{S}_{1\hat{1}} i\gamma_3 + \bar{P}_{2\hat{2}} - \bar{S}_{2\hat{2}} i\gamma_3) b, \quad b_\alpha = (0, 1). \end{aligned} \tag{5.51}$$

6. Conclusion and discussion

In this paper, we have constructed novel BPS Wilson loops along infinite straight lines and circles in several three-dimensional quiver superconformal CSM theories, especially the novel DT type BPS Wilson loops in ABJM theory and $\mathcal{N} = 4$ orbifold ABJM theory. There are several free complex parameters in these Wilson loops. There are SUSY enhancements at special values of the parameters for the DT type BPS Wilson loops in ABJM theory and $\mathcal{N} = 4$ orbifold ABJM theory. The construction here can be generalized for DT type BPS Wilson loops along circles. We also notice that our construction of DT type BPS Wilson loops in $\mathcal{N} = 4$ orbifold ABJM theory should be easily generalized to the one of similar Wilson loops in $\mathcal{N} = 4$ necklace quiver theory with gauge group $\prod_{i=1}^{2r} U(N_i)$ and Chern–Simons levels $(k, -k, \cdots, k, -k)$.

For the 1/6 BPS DT type Wilson loops in ABJM theory and 1/4 BPS DT type Wilson loops in $\mathcal{N} = 4$ orbifold ABJM theory, there are infinite degeneracies. We have an infinite number of Wilson loops along the same curve that preserve the same supersymmetries. In the spirit of [26], it is possible that not all of these Wilson loops are BPS quantum mechanically.[2] If it is the case one should find how the degeneracies are lifted. If it is not the case, one should identify their gravity duals.

[2] Recently it was found [38] that such degeneracies, at the level of vacuum expectation values, for the previously-constructed half BPS Wilson loops, the ψ_1- and ψ_2-loops, in $\mathcal{N} = 4$ orbifold ABJM theory are not lifted at two-loop level in the perturbation expansion. Possibility of lift at three-loop level was also discussed there.

Note added in proofreading. Recently in [39], it was found that degeneracy of vacuum expectation values of half-BPS ψ_1- and ψ_2-loops, mentioned in the footnote of this section, is lifted at three loops. And strong evidences were given that the average of these two Wilson loops are half-BPS quantum mechanically.

Acknowledgements

We would like to thank Nan Bai, Bin Chen, Nadav Drukker, Song He, Yu Jia, Wei Li, Jian-Xin Lu, Gary Shiu, Zhao-Long Wang and Piljin Yi for valuable discussions. JW would like to thank IMP, Northwest University for warm hospitality. The work was in part supported by National Natural Science Foundation of China Grants No. 11222549 and No. 11575202. JW also gratefully acknowledges the support of K. C. Wong Education Foundation. JW would also like to thank the participants of the advanced workshop "Dark Energy and Fundamental Theory" supported by the Special Fund for Theoretical Physics from the National Natural Science Foundation of China with Grant No. 11447613 for stimulating discussion.

References

[1] J.M. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231–252, arXiv:hep-th/9711200.
[2] S. Gubser, I.R. Klebanov, A.M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B 428 (1998) 105–114, arXiv:hep-th/9802109.
[3] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253–291, arXiv:hep-th/9802150.
[4] J.M. Maldacena, Wilson loops in large N field theories, Phys. Rev. Lett. 80 (1998) 4859–4862, arXiv:hep-th/9803002.
[5] S.-J. Rey, J.-T. Yee, Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity, Eur. Phys. J. C 22 (2001) 379–394, arXiv:hep-th/9803001.
[6] D.E. Berenstein, R. Corrado, W. Fischler, J.M. Maldacena, The operator product expansion for Wilson loops and surfaces in the large N limit, Phys. Rev. D 59 (1999) 105023, arXiv:hep-th/9809188.
[7] N. Drukker, D.J. Gross, H. Ooguri, Wilson loops and minimal surfaces, Phys. Rev. D 60 (1999) 125006, arXiv:hep-th/9904191.
[8] D. Gaiotto, X. Yin, Notes on superconformal Chern–Simons-matter theories, J. High Energy Phys. 0708 (2007) 056, arXiv:0704.3740 [hep-th].
[9] N. Drukker, D. Trancanelli, A supermatrix model for $\mathcal{N} = 6$ super Chern–Simons-matter theory, J. High Energy Phys. 1002 (2010) 058, arXiv:0912.3006 [hep-th].
[10] O. Aharony, O. Bergman, D.L. Jafferis, J. Maldacena, $\mathcal{N} = 6$ superconformal Chern–Simons-matter theories, M2-branes and their gravity duals, J. High Energy Phys. 0810 (2008) 091, arXiv:0806.1218 [hep-th].
[11] N. Drukker, J. Plefka, D. Young, Wilson loops in 3-dimensional $\mathcal{N} = 6$ supersymmetric Chern–Simons theory and their string theory duals, J. High Energy Phys. 0811 (2008) 019, arXiv:0809.2787 [hep-th].
[12] B. Chen, J.-B. Wu, Supersymmetric Wilson loops in $\mathcal{N} = 6$ super Chern–Simons-matter theory, Nucl. Phys. B 825 (2010) 38–51, arXiv:0809.2863 [hep-th].
[13] S.-J. Rey, T. Suyama, S. Yamaguchi, Wilson loops in superconformal Chern–Simons theory and fundamental strings in Anti-de Sitter supergravity dual, J. High Energy Phys. 0903 (2009) 127, arXiv:0809.3786 [hep-th].
[14] L. Griguolo, D. Marmiroli, G. Martelloni, D. Seminara, The generalized cusp in ABJ(M) $N = 6$ super Chern–Simons theories, J. High Energy Phys. 1305 (2013) 113, arXiv:1208.5766 [hep-th].
[15] V. Cardinali, L. Griguolo, G. Martelloni, D. Seminara, New supersymmetric Wilson loops in ABJ(M) theories, Phys. Lett. B 718 (2012) 615–619, arXiv:1209.4032 [hep-th].
[16] N. Kim, Supersymmetric Wilson loops with general contours in ABJM theory, Mod. Phys. Lett. A 28 (2013) 1350150, arXiv:1304.7660 [hep-th].
[17] M.S. Bianchi, L. Griguolo, M. Leoni, S. Penati, D. Seminara, BPS Wilson loops and Bremsstrahlung function in ABJ(M): a two loop analysis, J. High Energy Phys. 1406 (2014) 123, arXiv:1402.4128 [hep-th].
[18] D.H. Correa, J. Aguilera-Damia, G.A. Silva, Strings in $AdS_4 \times \mathbb{CP}^3$ Wilson loops in $\mathcal{N} = 6$ super Chern–Simons-matter and bremsstrahlung functions, J. High Energy Phys. 1406 (2014) 139, arXiv:1405.1396 [hep-th].

[19] D. Gaiotto, E. Witten, Janus configurations, Chern–Simons couplings, and the theta-angle in $\mathcal{N} = 4$ super Yang–Mills theory, J. High Energy Phys. 1006 (2010) 097, arXiv:0804.2907 [hep-th].

[20] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, J. Park, $\mathcal{N} = 4$ superconformal Chern–Simons theories with hyper and twisted hyper multiplets, J. High Energy Phys. 0807 (2008) 091, arXiv:0805.3662 [hep-th].

[21] Y. Imamura, K. Kimura, $\mathcal{N} = 4$ Chern–Simons theories with auxiliary vector multiplets, J. High Energy Phys. 0810 (2008) 040, arXiv:0807.2144 [hep-th].

[22] M. Benna, I. Klebanov, T. Klose, M. Smedback, Superconformal Chern–Simons theories and AdS(4)/CFT(3) correspondence, J. High Energy Phys. 0809 (2008) 072, arXiv:0806.1519 [hep-th].

[23] Y. Imamura, K. Kimura, On the moduli space of elliptic Maxwell–Chern–Simons theories, Prog. Theor. Phys. 120 (2008) 509–523, arXiv:0806.3727 [hep-th].

[24] S. Terashima, F. Yagi, Orbifolding the membrane action, J. High Energy Phys. 0812 (2008) 041, arXiv:0807.0368 [hep-th].

[25] H. Ouyang, J.-B. Wu, J.-j. Zhang, Supersymmetric Wilson loops in $\mathcal{N} = 4$ super Chern–Simons-matter theory, J. High Energy Phys. 1511 (2015) 213, arXiv:1506.06192 [hep-th].

[26] M. Cooke, N. Drukker, D. Trancanelli, A profusion of $1/2$ BPS Wilson loops in $\mathcal{N} = 4$ Chern–Simons-matter theories, J. High Energy Phys. 1510 (2015) 140, arXiv:1506.07614 [hep-th].

[27] H. Ouyang, J.-B. Wu, J.-j. Zhang, Novel BPS Wilson loops in three-dimensional quiver super Chern–Simons-matter theories, Phys. Lett. B 753 (2016) 215–220, arXiv:1510.05475 [hep-th].

[28] B. Chen, J.-B. Wu, M.-Q. Zhu, Holographical description of BPS Wilson loops in flavored ABJM theory, J. High Energy Phys. 1412 (2014), arXiv:1410.2311 [hep-th].

[29] J.H. Schwarz, Superconformal Chern–Simons theories, J. High Energy Phys. 0411 (2004) 078, arXiv:hep-th/0411077.

[30] H. Ouyang, J.-B. Wu, J.-j. Zhang, BPS Wilson loops in Minkowski spacetime and Euclidean space, Eur. Phys. J. C 75 (2015) 606, arXiv:1504.06929 [hep-th].

[31] K.-M. Lee, S. Lee, 1/2-BPS Wilson loops and vortices in ABJM model, J. High Energy Phys. 1009 (2010) 004, arXiv:1006.5589 [hep-th].

[32] H.-C. Kao, Selfdual Yang–Mills Chern–Simons Higgs systems with an $\mathcal{N} = 3$ extended supersymmetry, Phys. Rev. D 50 (1994) 2881–2892.

[33] D. Gaiotto, S. Giombi, X. Yin, Spin chains in $\mathcal{N} = 6$ superconformal Chern–Simons-matter theory, J. High Energy Phys. 0904 (2009) 066, arXiv:0806.4589 [hep-th].

[34] K. Hosomichi, K.-M. Lee, S. Lee, S. Lee, J. Park, $\mathcal{N} = 5, 6$ superconformal Chern–Simons theories and M2-branes on orbifolds, J. High Energy Phys. 0809 (2008) 002, arXiv:0806.4977 [hep-th].

[35] S. Terashima, On M5-branes in $\mathcal{N} = 6$ membrane action, J. High Energy Phys. 0808 (2008) 080, arXiv:0807.0197 [hep-th].

[36] M.A. Bandres, A.E. Lipstein, J.H. Schwarz, Studies of the ABJM theory in a formulation with manifest SU(4) R-symmetry, J. High Energy Phys. 0809 (2008) 027, arXiv:0807.0880 [hep-th].

[37] H. Ouyang, J.-B. Wu, J.-j. Zhang, Exact results for Wilson loops in orbifold ABJM theory, Chin. Phys. C 40 (2016) 083101, arXiv:1507.00442 [hep-th].

[38] L. Griguolo, M. Leoni, A. Mauri, S. Penati, D. Seminara, Probing Wilson loops in $\mathcal{N} = 4$ Chern–Simons-matter theories at weak coupling, Phys. Lett. B 753 (2016) 500–505, arXiv:1510.08438 [hep-th].

[39] M.S. Bianchi, L. Griguolo, M. Leoni, A. Mauri, S. Penati, D. Seminara, The quantum 1/2 BPS Wilson loop in $\mathcal{N} = 4$ Chern–Simons-matter theories, arXiv:1606.07058 [hep-th].