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ON THE ELEMENTARY THEORY OF BANACH ALGEBRAS

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1. Introduction

1.1. Origin of the problem. At the end of the survey article [4] there is a list of open questions concerning the decidability of certain important elementary theories.

One of these questions asks if the elementary theory of complete normed rings (i.e. Banach algebras) is decidable. By classical work of Tarski the rings **C** and **R** are decidable [15]. It is easy to extend Tarski's results and prove that **C** and **R** are decidable normed algebras. From this we see that the elementary theory of Banach algebras is not essentially undecidable. Moreover, most known decidable rings are elementarily equivalent to complete topological rings [1]. Thus, it is reasonable to hope to obtain positive decidability results for Banach algebras.

However, in this paper we prove rather general undecidability results for Banach algebras.

1.2. Different formulations. There are various formulations of our problem, according to the way we construe Banach algebras as structures for a first-order logic.

The natural way to construe Banach algebras is as algebras (over **C** or **R**) endowed with a norm map to **R**. In this formulation, to be called *the first formulation*, Banach algebras are many-sorted structures for a certain many-sorted logic. For many-sorted model-theory, consult [15].

Other formulations are obtained by forgetting some of the preceding structure. We get three other formulations in this way.

Second formulation. If we forget the algebra structure, we construe Banach algebras as rings with norms.

Third formulation. If we forget the norm, we construe Banach algebras as algebras.

Fourth formulation. If we forget the algebra structure and the norm, we construe Banach algebras simply as rings.

1.3. Main results. Clearly a negative answer to the original problem in the fourth formulation (i.e. a proof that the theory of the underlying rings of Banach algebras is undecidable) implies a negative answer to the problem in the other formulations. Similarly, a negative answer in either of the second or third formulations implies a negative answer in the first formulation.

Our central result is that the theory of Banach algebras, construed simply as rings, is hereditarily undecidable.

We present three proofs of this result, each giving significantly different information.

The first proof uses spectral theory, and establishes that if A is a Banach algebra over **C**, with unit and with trivial centre, and if A is infinite-dimensional over its radical, then the ring-theory of A is hereditarily undecidable. An example of such an A is $\mathcal{L}(H, H)$, the ring of continuous linear operators on an infinite-dimensional Hilbert space H.

The second proof depends on Ersov's [3], and establishes the strong result that the theory of the class of groups of invertible elements of finite-dimensional semi-simple Banach algebras is hereditarily undecidable.

The third proof uses Grzegorczyk's theorem [6] on the undecidability of the algebra of closed subsets of the Euclidean plane, and establishes that the theory of commutative semi-simple Banach algebras, construed as rings, is hereditarily undecidable.

En route to the first proof, we prove that if A is a Banach algebra infinite-dimensional over its radical, and we construe A simply as an algebra, then Th(A) is hereditarily undecidable. Later we prove the analogous result for normed rings.

2. Preliminaries

2.1. For definitions and basic facts about Banach algebras, see [2, 13]. From now until Section 9, we restrict ourselves to Banach algebras over **C**, with unit. In Sections 9 and 10, we combine our main results with the techniques of complexification and adjoining a unit, to get results about algebras over **R** and algebras without unit.

2.2. We list the basic ingredients of a Banach algebra A over **C**, with unit. These ingredients are as follows.

2.2.1. Three sets M_0, M_1, M_2. M_0 is the set of elements of the algebra, M_1 is the set **C** of scalars, and M_2 is the set **R**.

2.2.2. Individuals $a_0, ..., a_5$. a_0 and a_1 are respectively the zero and unit elements of M_0. a_2 and a_3 are respectively the zero and unit elements of M_1. a_4 and a_5 are respectively the zero and unit of M_2.

2.2.3. Operations $F_0, ..., F_8$.
$F_0 : M_0^2 \to M_0$ is addition in M_0.
$F_1 : M_1^2 \to M_1$ is addition in M_1.
$F_2 : M_2^2 \to M_2$ is addition in M_2.
$F_3 : M_0^2 \to M_0$ is multiplication in M_0.
$F_4 : M_1^2 \to M_1$ is multiplication in M_1.
$F_5 : M_2^2 \to M_2$ is multiplication in M_2.
$F_6 : M_1 \times M_0 \to M_0$ is scalar multiplication.
$F_7 : M_0 \to M_2$ is the norm map from M_0 to M_2.
$F_8 : M_1 \to M_2$ is the standard norm map from **C** to **R**.

2.2.4. A binary relation R_0. R_0 is the natural order $<$ on M_2.

Thus a Banach algebra over **C**, with unit, is a 3-sorted structure

$$(\langle M_i \rangle_{i<3}, \langle R_i \rangle_{i<3}, \langle a_i \rangle_{i<6}, \langle F_i \rangle_{i<9}),$$

and so has a signature σ as in [5]. Now we introduce a first-order finitary language appropriate to structures of signature σ.

2.3. Let $\mathcal{L}_1$ be a first-order finitary language of signature σ. The ingredients of $\mathcal{L}_1$ are as follows.

2.3.1. Infinitely many variables of each of three sorts, corresponding to M_0, M_1, M_2.

2.3.2. Individual constants corresponding to $a_0, ..., a_5$.

2.3.3. Operation-symbols of appropriate arity, corresponding to $F_0, ..., F_8$.

2.3.4. A binary relation-symbol corresponding to R_0.

2.3.5. An equality symbol =.

2.3.6. The usual quantifiers and connectives.

2.4. Thus, a Banach algebra over **C**, with unit, is an $\mathcal{L}_1$-structure. The language $\mathcal{L}_1$ corresponds to the first formulation of 1.2.

Next, we describe sublanguages of $\mathcal{L}_1$ corresponding to the other formulations.

$\mathcal{L}_2$ is got from $\mathcal{L}_1$ by forgetting the symbols of $\mathcal{L}_1$ corresponding to $a_2, a_3, F_1, F_4, F_6, F_8$, and the variables of $\mathcal{L}_1$ of the sort corresponding to M_1. Then $\mathcal{L}_2$ corresponds to the second formulation.

$\mathcal{L}_3$ is got from $\mathcal{L}_1$ by forgetting the symbols of $\mathcal{L}_1$ corresponding to $a_4, a_5, F_2, F_5, F_7, F_8, R_0$ and the variables of the sort corresponding to M_2. Then $\mathcal{L}_3$ corresponds to the third formulation.

$\mathcal{L}_4$ is got from $\mathcal{L}_1$ by forgetting the symbols of $\mathcal{L}_1$ corresponding to $a_2, a_3, a_4, a_5, F_1, F_2, F_4, F_5, F_6, F_7, F_8, R_0$ and the variables of sorts corresponding to M_1 and M_2. Then $\mathcal{L}_4$ corresponds to the fourth formulation. $\mathcal{L}_4$ is just the language of ring-theory.

2.5. We define Ban_1 as the class of $\mathcal{L}_1$-structures which are Banach algebra over **C** with unit.

For $2 \leq i \leq 4$, we define Ban_i as the class of $\mathcal{L}_i$-structures which are reducts of members of Ban_1.

Thus, Ban_2 is the class of underlying normed rings of Banach algebras over **C** with unit. Similarly Ban_3 is the class of underlying algebras, and Ban_4 the class of underlying rings, of Banach algebras over **C** with unit.

2.6. Theories. We assume the basic notions and results of model-theory and recursion-theory. See [4].

If $\mathcal{L}$ is a first-order language, an $\mathcal{L}$-theory is a consistent set of $\mathcal{L}$-sentences closed under deduction. If $\mathfrak{M}$ is an $\mathcal{L}$-structure than Th($\mathfrak{M}$), the theory of $\mathfrak{M}$, is the set of all $\mathcal{L}$-sentences Φ such that $\mathfrak{M} \models \Phi$. If k is a class of $\mathcal{L}$-structures then Th(k), the theory of k, is the set of all $\mathcal{L}$-sentences Φ such that $\mathfrak{M} \models \Phi$ for all $\mathfrak{M} \in k$.

2.7. Undecidability. We assume a fixed Gödel numbering of the languages $\mathcal{L}_i$ $(1 \leq i \leq 4)$, so we can talk freely about recursive sets of $\mathcal{L}_i$-sentences. An $\mathcal{L}_i$-theory T is decidable if it is a recursive set. An $\mathcal{L}_i$-theory T is hereditarily undecidable if all subtheories of T are undecidable.

All the languages $\mathcal{L}_i$ $(1 \leq i \leq 4)$ contain the standard language for ring theory, and the models in which we are interested are rings, possibly with extra structure. If $A \in \mathrm{Ban}_i$ then A contains a subring isomorphic to $\mathbf{Z}$, namely the subring of all elements $n \cdot e$ where $n \in \mathbf{Z} \subseteq \mathbf{C}$, and e is the unit of A. For convenience we identify this subring with $\mathbf{Z}$. Let N be the subsemiring of non-negative integers.

Lemma 1. *Suppose $A \in \mathrm{Ban}_i$ and N is definable in A by a formula of $\mathcal{L}_i$. Then* Th(A) *is hereditarily undecidable.*

Proof. Standard. See [4].

We refer to [4] for the notion of the interpretability of one theory in another.

An $\mathcal{L}$-theory T is essentially undecidable if T_1 is undecidable for each $\mathcal{L}$-theory T_1 with $T \subseteq T_1$.

Lemma 2. *If a finitely axiomatizable essentially undecidable theory can be interpreted in T then T is hereditarily undecidable.*

Proof. See [4].

2.8. Informal notation. Suppose $A \in \mathrm{Ban}_1$. Then A is an $\mathcal{L}_1$-structure.

$$(\langle M_i \rangle_{i<3}, \langle R_i \rangle_{i<1}, \langle a_i \rangle_{i<6}, \langle F_i \rangle_{i<9}),$$

where M_1 is $\mathbf{C}$ and M_2 is $\mathbf{R}$.

$\mathcal{L}_1$ has formal symbols corresponding to the ingredients of A. Now we provide familiar notation, in terms of which we will give our definitions.

Suppose x and y are members of M_0, λ and μ are members of $\mathbf{C}$ and r and s are members of $\mathbf{R}$. We put:

$$
\begin{aligned}
0 &= a_0,\\
e &= a_1,\\
0_{\mathbf{C}} &= a_2,\\
1_{\mathbf{C}} &= a_3,\\
0_{\mathbf{R}} &= a_4,\\
1_{\mathbf{R}} &= a_5,\\
x+y &= F_0(x, y),\\
\lambda+\mu &= F_1(\lambda, \mu),\\
r+s &= F_2(r, s),\\
x \cdot y &= F_3(x, y),\\
\lambda \cdot \mu &= F_4(\lambda, \mu),\\
r \cdot s &= F_5(r, s),\\
\lambda \cdot x &= F_6(\lambda, x),\\
\|x\| &= F_7(x),\\
|\lambda| &= F_8(\lambda),
\end{aligned}
$$

and finally

$$r < s \Leftrightarrow R_0(r, s).$$

Obviously the graphs of the operations of subtraction in M_0, M_1 and M_2 are $\mathcal{L}_1$-definable. In fact, subtraction in M_0 is $\mathcal{L}_4$-definable, subtraction in M_1 is $\mathcal{L}_3$-definable, and subtraction in M_2 is $\mathcal{L}_2$-definable.

We will sometimes write

$$x \in A$$

when we mean

$$x \in M_0.$$

3. Spectra

Suppose $A \in \mathrm{Ban}_1$, and $x \in A$. $\mathrm{Sp}_A(x)$, the spectrum of x in A, is the set

$$\{\lambda \in \mathbf{C} : x - \lambda \cdot e \text{ is not invertible in } A\}.$$

3.1. For reasons that will emerge in the next section, we want to know when A satisfies the following condition: For each non-negative integer n there exists $x_n \in A$ such that each of the integers 0, 1, ..., n is a member of $\mathrm{Sp}_A(x_n)$. (*)

Lemma 3. *Suppose* $A \in \mathrm{Ban}_1$. *Then* A *satisfies* (*) *if and only if for each non-negative* n *there exists* $y_n \in A$ *such that* $\mathrm{Sp}_A(y_n)$ *has a least* $n + 1$ *members.*

Proof. Necessity is trivial.

Sufficiency. Suppose A satisfies the condition in the lemma. Let n be a non-negative integer. Select $y_n \in A$ such that $\mathrm{Sp}_A(y_n)$ contains distinct elements $\lambda_0, ..., \lambda_n$. Using the Lagrange Interpolation Theorem, select $f \in \mathbf{C}[x]$ such that $f(\lambda_m) = m$ for $0 \leq m \leq n$. Let $x_n = f(y_n)$. Then, by 1.6.10 of [13], $\mathrm{Sp}_A(x_n) = \{f(\lambda) : \lambda \in \mathrm{Sp}_A(y_n)\}$, so $0, 1, ..., n \in \mathrm{Sp}_A(x_n)$. Since n was arbitrary, A satisfies (*).

3.2. **The radical.** For the definition and basic properties of the Jacobson radical of an algebra, see [7, 13].

If $A \in \mathrm{Ban}_1$, let $J(A)$ be the radical of A. $J(A)$ is a closed ideal of A, and $A/J(A)$ has a natural structure of Banach algebra. A is called semi-simple if $J(A) = 0$.

Some useful facts about $J(A)$ are *3.2.1. – 3.2.3.* below.

3.2.1. $A/_{J(A)}$ is semi-simple.

3.2.2. $J(A) = \{y \in A : 1 + ty$ is invertible for all $t \in A\}$.

3.2.3. If $\eta : A \to A/J(A)$ is the natural projection then $\mathrm{Sp}_{A/_{J(A)}}(\eta(x)) = \mathrm{Sp}_A(x)$ for all $x \in A$.

For 3.2.1 see [13, page 56]. For 3.2.2 see [13, page 55]. In 3.2.3 the inclusion $\mathrm{Sp}_A(x) \subseteq \mathrm{Sp}_{A/_{J(A)}}(\eta(x))$ is clear. The reverse inclusion follows from the observation that an element invertible modulo the radical is invertible, and this is almost trivial from 3.2.3.

3.3. **Lemma 4.** *Suppose* $A \in \mathrm{Ban}_1$, *and* $\dim_{\mathbf{C}}(A/J(A)) = n < \infty$. *Then, if* $y \in A$, $\mathrm{Sp}_A(y)$ *has at most* n *elements,*

Proof. By 3.2.1 and 3.2.3 we may assume A is semi-simple and $\dim_{\mathbf{C}}(A) = n$.

Let $y \in A$. Then there exists $f \in \mathbf{C}[x]$ such that $f(y) = 0$ and $1 \leq \deg(f) \leq n-1$. Now by 1.6.10 of [13]

$$\mathrm{Sp}_A(f(y)) = \{f(\lambda) : \lambda \in \mathrm{Sp}_A(y)\}.$$

But

$$\mathrm{Sp}_A(f(y)) = \mathrm{Sp}_A(0) = \{0\}.$$

$$\therefore \lambda \in \mathrm{Sp}_A(y) \to f(\lambda) = 0.$$

$$\therefore \quad \mathrm{Sp}_A(y) \text{ has at most } n \text{ members.}$$

3.5. Lemma 5. *Suppose* $A \in \mathrm{Ban}_1$ *and* $\dim_{\mathbf{C}}(A/J(A))$ *is infinite. Then there exists* $x \in A$ *such that* $\mathrm{Sp}_A(x)$ *is infinite.*

Proof. By 3.2.1 and 3.2.3 we may assume A is semi-simple. But then the lemma is a result of Kaplansky [8, Lemma 7].

Lemma 6. *Suppose* $A \in \mathrm{Ban}_1$. *Then* A *satisfies* (*) *if and only if* $\dim_{\mathbf{C}}(A/J(A))$ *is infinite.*

Sufficiency. Suppose $\dim_{\mathbf{C}}(A/J(A))$ is infinite. Then by Lemma 3 and 5 A satisfies (*).

4. Constants and the Definition of N

Suppose $A \in \mathrm{Ban}_1$. We define Con as $\{\lambda \cdot e : \lambda \in \mathbf{C}\}$. Then Con is a subring of A, isomorphic to $\mathbf{C}$. With the convention of 2.7, $N \subseteq$ Con. The elements of Con are called constants.

We will show that if A satisfies (*) then N is definable from Con using only the notions of elementary ring theory.

4.1. Some definitions. Suppose $x, y, z, t, u \in A$.

4.1.1. $x \| y \leftrightarrow (\exists z)(y = xz \wedge xz = zx)$.

4.1.2. $D(x, y, z) \leftrightarrow (y-x) \| z \wedge (y-x)$ is not invertible.

4.1.3. $\Psi(x) \longleftrightarrow x \in \text{Con} \;\wedge.$

$$(\exists t)(\exists u)[D(0, t, u) \wedge (\forall y)((y \in \text{Con} \wedge y \neq x \wedge D(y, t, u)) \rightarrow D(y + e, t, u))].$$

4.2. Lemma 7. *Suppose* $A \in \text{Ban}_1$ *and* $x \in A$. *Then* $\Psi(x) \rightarrow x \in N$.

Proof. Suppose $x \notin N$ and $\Psi(x)$. Select t, u such that $D(0, t, u)$ and

$$(\forall y)((y \in \text{Con} \wedge y \neq x \wedge D(y, t, u)) \rightarrow D(y + e, t, u)).$$

By induction we get

$$D(n \cdot e, t, u)$$

for all $n \in N$.

But then

$$n \in N \rightarrow t - n \cdot e \text{ not invertible}$$

$$\therefore N \subseteq \text{Sp}_A(t).$$

But $\text{Sp}_A(t)$ is bounded [13, 1.6.4].

This gives a contradiction.

$$\therefore \Psi(x) \rightarrow x \in N.$$

Lemma 8. *Suppose* $A \in \text{Ban}_1$, *and* $n \in N$. *Suppose* $t \in A$ *and* $\{0, 1, ..., n\} \subseteq \text{Sp}_A(t)$. *Let* $u = t \cdot (t-e) \ldots (t-n \cdot e)$. *Then for* $\lambda \in \mathbf{C}$,

$$D(\lambda \cdot e, t, u) \longleftrightarrow \lambda \in \{0, 1, ..., n\}.$$

Proof. *Case 1. A commutative and semi-simple.*

By Gelfand's Theorem [13, 3.1.20], A is algebraically isomorphic to an algebra of continuous functions on a compact Hausdorff space X. The isomorphism preserves spectra, and the property D of 4.1.2, so we may assume without loss of generality that A is an algebra of continuous functions on a space X.

Clearly, if $m \in \{0, 1, ..., n\}$, $t-m \cdot e \| u$ and $t-m \cdot e$ is not invertible, so $D(m \cdot e, t, u)$.

Conversely, suppose $\lambda \in \mathbf{C}$ and $D(\lambda \cdot e, t, u)$. Then $t-\lambda \cdot e$ is not invertible, so $\lambda \in \mathrm{Range}(t)$. Select $x \in X$ such that $t(x) = \lambda$. Since $D(\lambda \cdot e, t, u)$, we have $t-\lambda \cdot e \| u$, so there exists z such that $u = (t-\lambda \cdot e) \cdot z$. Now, $(t-\lambda \cdot e)(x) = 0$, so $u(x) = 0$. $\therefore t(x)(t(x)-1) \dots (t(x)-n) = 0$, so $t(x) = m$ for some $m \in \{0, 1, ..., n\}$. But $t(x) = \lambda$, so $\lambda \in \{0, 1, ..., n\}$.

This proves the result.

Case 2. A commutative. Let $\eta : A \to A/J(A)$ be the natural projection. Then $\{0, 1, ..., n\} \subseteq \mathrm{Sp}_{A/J(A)}(t)$ by 3.2.3. Also $\eta(u) = \eta(t) \cdot (\eta(t)-e) \dots,$

$(\eta(t)-n \cdot e)$. Since $A/J(A)$ is semi-simple, we conclude from the first case that for $\lambda \in \mathbf{C}$, $D(\lambda \cdot e, \eta(t), \eta(u)) \leftrightarrow \lambda \in \{0, 1, ..., n\}$.

As in Case 1, it is clear that if $m \in \{0, 1, ..., n\}$ then $D(m \cdot e, t, u)$.

Suppose $\lambda \in \mathbf{C}$ and $D(\lambda \cdot e, t, u)$. Then $t-\lambda \cdot e \| u$. By definition of $\|$, it follows that $\eta(t) - \lambda \cdot e \| \eta(u)$. Similarly, since $t-\lambda \cdot e$ is not invertible, $\eta(t) - \lambda \cdot e$ is not invertible, by 3.2.3.

$$\therefore D(\lambda \cdot e, \eta(t), \eta(u)).$$

$$\therefore \lambda \in \{0, 1, ..., n\}.$$

This concludes the proof.

Case 3. A arbitrary. As before, it is clear that if $m \in \{0, 1, ..., n\}$ then $D(m \cdot e, t, u)$.

Conversely, suppose $\lambda \in \mathbf{C}$ and $D(\lambda \cdot e, t, u)$. Then $\lambda \in \mathrm{Sp}_A(t)$, and $t-\lambda \cdot e \| u$. Thus there exists z such that $u = (t-\lambda \cdot e) \cdot z$, and $(t-\lambda \cdot e) \cdot z = z \cdot (t-\lambda \cdot e)$. Then clearly $tz = zt$. By [13, 1.6.14] there exists a maximal commutative subalgebra B of A such that $t \in B$, $z \in B$, and $e \in B$. By [13, 1.6.14], $\mathrm{Sp}_B(x) = \mathrm{Sp}_A(x)$ for $x \in B$. Thus $\{0, 1, ..., n\} \subseteq \mathrm{Sp}_B(t)$. Also, $\lambda \in \mathrm{Sp}_B(t)$. Since t and z are in B, $u \in B$ and $t-\lambda \cdot e \| u$ in B. Thus $D(\lambda \cdot e, t, u)$ in B. Since B is commutative, Case 2 implies that $\lambda \in \{0, 1, ..., n\}$, proving the lemma.

Lemma 9. *Suppose $A \in \mathrm{Ban}_1$ and A satisfies (*). If $x \in A$, then*

$$x \in N \to \Psi(x).$$

Proof. Suppose $x \in N$. Then $x = n \cdot e$ for some non-negative integer n. Since A satisfies (*), we may select $t \in A$ such that $\{0, 1, ..., n\} \subseteq \mathrm{Sp}_A(t)$. Let $u = t \cdot (t-e) \dots (t-n \cdot e)$. By Lemma 8, if $\lambda \in \mathbf{C}$ then

$$D(\lambda \cdot e, t, u) \leftrightarrow \lambda \in \{0, 1, ..., n\}.$$

Thus $D(0, t, u)$ and

$$(\forall y)((y \in \mathrm{Con} \wedge y \neq x \wedge D(y, t, u)) \to D(y + e, t, u).$$

$\therefore \Psi(x)$.

4.3. Undecidability of algebras satisfying (*).

Suppose $A \in \mathrm{Ban}_1$ and $A/J(A)$ is infinite-dimensional. Then A satisfies (*), by Lemma 6. By Lemmas 7 and 9, Ψ defines N in A. So, by Lemma 1, if Ψ is definable in $\mathcal{L}_i$ then the $\mathcal{L}_i$-theory of A is hereditarily undecidable.

Clearly $D(x, y, z)$ is definable in $\mathcal{L}_4$. Thus Ψ will be $\mathcal{L}_i$-definable provided Con is $\mathcal{L}_i$-definable.

Trivially, Con is $\mathcal{L}_3$-definable, by:

$$x \in \mathrm{Con} \leftrightarrow (\exists \lambda \in M_1)(x = \lambda \cdot e).$$

The preceding observations immediately give

Theorem 1. *Suppose* $A \in \mathrm{Ban}_3$ *and* $A/J(A)$ *is infinite-dimensional. Then* $\mathrm{Th}(A)$ *is hereditarily undecidable.*

Remark. This implies that all infinite-dimensional semi-simple Banach algebras, construed simply as algebras, are undecidable. Examples are:

a) $C(X)$, the algebra of continuous complex functions on an infinite compact Hausdorff space X. See [13, Chapter III].

b) $\mathcal{L}(H, H)$, the algebra of continuous linear operators on an infinite-dimensional Hilbert space H. See [13, Appendix A.1.1].

We will prove in [10] that all finite-dimensional algebras over $\mathbf{C}$ (not just Banach algebras) are decidable.

Corollary 1 to Theorem 1. Th(Ban_3) *is hereditarily undecidable.*

Proof. Let X be the unit interval [0, 1]. Then $C(X) \in Ban_3$. By Theorem 1, Th($C(X)$) is hereditarily undecidable. Since Th(Ban_3) $\subseteq$ Th($C(X)$), Th(Ban_3) is hereditarily undecidable.

Corollary 2. *The theory of commutative Banach algebras, construed simply as algebras, is hereditarily undecidable.*

Proof. Immediate from the proof of Corollary 1, since $C(X)$ is commutative.

5. Defining N in $\mathcal{L}_4$

Since we want undecidability results for Banach algebras as rings, we want definitions of N in $\mathcal{L}_4$, and so we want definitions of Con in $\mathcal{L}_4$.

5.1. Suppose we can define in $\mathcal{L}_i$ a subset Con_1 of Con, such that $N \subseteq Con_1$. Replace Con by Con_1 in the definition of Ψ, to get a new notion Ψ_1. The following lamme is proved in the same way as Lemmas 7 and 9.

Lemma 10. *Suppose $A \in Ban_1$, and A satisfies (*). If $x \in A$, then*

$$x \in N \leftrightarrow \Psi_1(x).$$

Thus if $A \in Ban_i$, and A satisfies (*), and an appropriate set Con_1 is $\mathcal{L}_i$-definable, then Th(A) is hereditarily undecidable. In Section 6 we will apply a variant of this observations when $i = 2$, to get the analogue of Theorem 1 for Banach algebras as normed rings.

5.2. The centre. Suppose $A \in Ban_1$. We define Cen, the centre of A, by:

$$x \in \text{Cen} \leftrightarrow (\forall y)(xy = yx).$$

Obviously Cen is definable in $\mathcal{L}_4$. Clearly Con $\subseteq$ Cen.

Now let A be $\mathcal{L}(H, H)$, the ring of continuous linear operators on an infinite-dimensional Hilbert space H. As observed before, A satisfies (*). But also [13, 2.4.5] the centre of A is Con. For this A, Con is $\mathcal{L}_4$-definable. More generally we have:

Theorem 2. *Suppose* $A \in \mathrm{Ban}_4$, $A/J(A)$ *is infinite-dimensional, and the centre of* A *is* Con. *Then* Th(A) *is hereditarily undecidable.*

Proof. Con = Cen, so Con is $\mathcal{L}_4$-definable. Since $A/J(A)$ is infinite-dimensional, Ψ defines N. Thus N is $\mathcal{L}_4$-definable. The result follows by Lemma 1.

Corollary. Th(Ban_4) *is hereditarily undecidable.*

Proof. Let A be $\mathcal{L}(H, H)$ as above. Then Th(Ban_4) $\subseteq$ Th(A). By Theorem 2, Th(A) is hereditarily undecidable, whence the result.

Remark. We do not know any commutative A satisfying (*) for which Con, or an appropriate Con_1, is $\mathcal{L}_4$-definable. At the same time we do not know any such A for which no Con_1 is definable, and in the light of Kaplansky's [8] it will be difficult to prove undefinability results.

6. Interpreting Number Theory in $\mathcal{L}_2$

The main technical result of this section is that if $A \in \mathrm{Ban}_1$ then we can define in $\mathcal{L}_2$, the language for normed rings, a set Kon $\subseteq A$ such that

$$N \subseteq \mathrm{Kon} \subseteq \{\lambda \cdot e + j : \lambda \in \mathbf{C} \wedge j \in J(\mathrm{Cen})\}.$$

Then, if A satisfies (*), we can extend the techniques of Section 4 to get an interpretation of number theory in the $\mathcal{L}_2$-theory of A, whence the $\mathcal{L}_2$-theory of A is hereditarily undecidable.

6.1. Spectral Radius. Suppose $A \in \mathrm{Ban}_1$ and $x \in A$. Then $\lim_{n\to\infty} \|x^n\|^{1/n}$ exists, and $\nu(x)$, the spectral radius of x is defined as this limit. In fact,

$$\nu(x) = \sup_{\lambda \in \mathrm{Sp}_A(x)} |\lambda|. \qquad \text{See [13, 1.4.1].}$$

Neither of the above characterizations of spectral radius is in the form of an $\mathcal{L}_2$-definition.

Definition 6.1.1. $\mathrm{Cen}_1 = \{y \in \mathrm{Cen}: (y = 0) \vee (y \text{ is invertible} \wedge \|y^{-1}\| = \|y\|^{-1}\}$.

Definition 6.1.2. $\mathrm{Sp}_A^*(x) = \{y \in \mathrm{Cen}_1 : x - y \text{ is not invertible}\}$.

Lemma 10. *If* $y \in \mathrm{Sp}_A^*(x)$ *then* $\|y\| \leq \nu(x)$.

Proof. Suppose $y \in \mathrm{Sp}_A^*(x)$. Then $y \in \mathrm{Cen}_1$. If $y = 0$ then $\|y\| = 0 \leq \nu(x)$.

Suppose $y \neq 0$. Then y is invertible, and $\|y^{-1}\| = \|y\|^{-1}$. Suppose $\|y\| > \nu(x)$. Then, by [13, page 10],

$$\begin{aligned} \nu(xy^{-1}) &\leq \nu(x)\,\nu(y^{-1}) \\ &\leq \nu(x)\,\|y^{-1}\| \\ &= \nu(x)\,\|y\|^{-1} \\ &< 1. \end{aligned}$$

Then by [13, page 18], $e - xy^{-1}$ is invertible, and since $x - y = y(xy^{-1} - e)$ it follows that $x - y$ is invertible.

$\therefore\ y \notin \mathrm{Sp}_A^*(x)$.

We conclude that $\|y\| \leq \nu(x)$.

Corollary 1. $\nu(x) = \sup_{y \in \mathrm{Sp}_A^*(x)} \|y\|$.

Proof. By the lemma,

$$\sup_{y \in \mathrm{Sp}_A^*(x)} \|y\| \leq \nu(x).$$

But there exists $\lambda \in \mathbf{C}$ such that $\lambda \in \mathrm{Sp}_A(x)$ and $\nu(x) = |\lambda|$, since $\mathrm{Sp}_A(x)$ is compact. Let $y = \lambda \cdot e$. Then $y \in \mathrm{Sp}_A^*(x)$, and $\|y\| = |\lambda| = \nu(x)$.

$$\therefore\ \nu(x) \leq \sup_{y \in \mathrm{Sp}_A^*(x)} \|y\|.$$

$$\therefore\ \nu(x) = \sup_{y \in \mathrm{Sp}_A^*(x)} \|y\|.$$

Corollary 2. *ν is $\mathcal{L}_2$-definable.*

Proof. Clearly Cen_1 and Sp_A^* are $\mathcal{L}_2$-definable. The result follows.

6.2. Definition 6.2.1.

$$M(x) \longleftrightarrow x \in \text{Cen} \wedge (\forall y \in \text{Cen})(\nu(xy) = \nu(x)\nu(y) \wedge \nu((x-e)y) = \nu(x-e)\nu(y)).$$

Clearly the predicate M is $\mathcal{L}_2$-definable.

Lemma 11. *Suppose* $A \in \text{Ban}_1$, *and* $x \in A$. *Suppose* $M(x)$. *Then there exists* $\lambda \in \mathbf{C}$ *such that* $\text{Sp}_A(x) \subseteq \{\lambda, \bar{\lambda}\}$.

Proof. Firstly, we show that the proof reduces to the case when A is commutative and semi-simple. Cen is a closed subalgebra of A, and claerly if $M(x)$ then $x \in$ Cen and $M(x)$ in Cen. Further, $\text{Sp}_A(x) = \text{Sp}_{\text{Cen}}(x)$. Thus the proof certainly reduces to the case when A is commutative. The general commutative case reduces to the semi-simple case, by the usual device of factoring out the radical. We now assume A is commutative and semi-simple.

Consider the Gelfand isomorphism $\hat{}: A \to \hat{A}$. See [13, Chapter III] for details. For each y in A, $\nu(\hat{y}) = \nu(y)$, so $M(\hat{x})$ in $\hat{A}$. If we have the lemma for $\hat{A}$, we can conclude that there exists $\lambda \in \mathbf{C}$ such that $\text{Sp}_{\hat{A}}(\hat{x}) \subseteq \{\lambda, \bar{\lambda}\}$. But $\text{Sp}_{\hat{A}}(\hat{x}) = \text{Sp}_A(x)$, whence the result.

Thus our proof is finally reduced to the case where A is a closed subalgebra of $C(X)$ with the sup norm, where X is a compact Hausdorff space. Let ∂A be the Šilov Boundary of A. See [13, 3,3].

Suppose $M(x)$. For $y \in A$, $\nu(y) = \|y\|$, by [13, 1.4.2]. Thus $\|xy\| = \|x\| \cdot \|y\|$, and $\|(x-e)\cdot y\| = \|x-e\| \cdot \|y\|$.

We claim that if $\alpha \in \partial A$ then $|x(\alpha)| = \|x\|$, and $|x(\alpha) - 1| = \|x-e\|$.

By [13, Theorem 3.3.6],

$$\inf_{\alpha \in \partial A} |x(\alpha)| = \inf_{y \in A} \frac{\|xy\|}{\|y\|}$$

$$= \|x\|, \text{ since } M(x).$$

But $|x(\alpha)| \leq \|x\|$ for all $\alpha \in X$.

$\therefore$ $|x(\alpha)| = \|x\|$ for $\alpha \in \partial A$.

Similarly, $|x(\alpha) - 1| = \|x-e\|$ for $\alpha \in \partial A$.

Thus, for $\alpha \in \partial A$,

$x(\alpha) \in \{\lambda \in \mathbf{C} : |\lambda| = \|x\|\} \cap \{\lambda \in \mathbf{C} : |\lambda - 1| = \|x-e\|\}$,

and this intersection is clearly of the form $\{\lambda, \bar{\lambda}\}$ for some $\lambda \in \mathbf{C}$. (The intersection is not empty, since $x(\alpha)$ is a member, for $\alpha \in \partial A$).

Thus $x(\alpha)$ is λ or $\bar{\lambda}$ for each $\alpha \in \partial A$. Let $y = (x-\lambda \cdot e)(x-\bar{\lambda}\cdot e))$. Then if $\alpha \in \partial A$, $y(\alpha) = 0$. $\therefore$ $y = 0$. $\therefore$ Range $(x) \subseteq \{\lambda, \bar{\lambda}\}$, i.e.

$$\mathrm{Sp}_A\ (x) \subseteq \{\lambda, \bar{\lambda}\},$$

Lemma 12. *Suppose $A \in \mathrm{Ban}_1$, and $x \in A$. Suppose $x \in$ Cen and there exists $\lambda \in \mathbf{C}$ such that $\mathrm{Sp}_A(x) \subseteq \{\lambda, \bar{\lambda}\}$. Then $M(x)$.*

Proof. Just as in Lemma 11, we need only consider the case where A is a closed subalgebra of $C(X)$ with the sup norm, where X is a compact Hausdorff space.

Since $\nu(y) = \|y\|$ for $y \in A$, we want to establish that $\|xy\| = \|x\| \cdot \|y\|$ and $\|(x-e)\cdot y\| = \|(x-e)\cdot y\|$, for all $y \in A$.

Since $\mathrm{Sp}_A(x) \subseteq \{\lambda, \bar{\lambda}\}$,
Range $(x) \subseteq \{\lambda, \bar{\lambda}\}$, so
$|x(\alpha)| = \|x\|$ for all $\alpha \in X$.

$$\text{Thus } \|xy\| = \sup_{\alpha \in X} |(xy)(\alpha)| = \sup_{\alpha \in X} |x(\alpha)\cdot y(\alpha)|$$

$$= \sup_{\alpha \in X} \|x\| \cdot |y(\alpha)| = \|x\| \cdot \sup_{\alpha \in X} |y(\alpha)|$$

$$= \|x\| \cdot \|y\|.$$

Similarly, $\mathrm{Sp}_A(x-e) \subseteq \{\lambda-1, \overline{\lambda-1}\}$, so $\|(x-e)\cdot y\| = \|x-e\| \cdot \|y\|$.
$\therefore M(x)$.

Corollary. *$M(x)$ if and only if $x \in$ Cen and there exists $\lambda \in$ C such that $\mathrm{Sp}_A(x) \subseteq \{\lambda, \lambda\}$.*

Proof. Immediate from Lemmas 11 and 12.

6.3. Definition 6.3.1. $x \in \mathrm{Kon} \leftrightarrow (\forall y)(M(y) \to M(xy))$. Clearly Kon is $\mathcal{L}_2$-definable.

Lemma 13. *Suppose $A \in \mathrm{Ban}_1$. Then*
i) Kon $\subseteq \{\lambda \cdot e + j : \lambda \in \mathbf{C} \wedge j \in J(\mathrm{Cen})\}$;
ii) *if $\lambda \in$ C and λ is real then* $\lambda \cdot e \in$ Kon.

Proof. i) First we observe that we need only consider the case when A is commutative. For if $x \in$ Kon then $M(x)$, since $M(e)$. Thus $x \in$ Cen.

Moreover, $x \in$ Kon in Cen. So if we have the result for Cen then we get it for A.

So, assume A is commutative and $x \in$ Kon. Then $M(x)$. Let i be a square root of -1 in **C**. Then $\mathrm{Sp}_A(i\cdot e) = \{i\}$, so $M(i\cdot e)$ by corollary to Lemma 12. Since $x \in$ Kon, $M(x\cdot(i\cdot e))$, so $M(i\cdot x)$. Now, since $M(x)$, there exists $\lambda \in \mathbf{C}$ such that $\mathrm{Sp}_A(x) \subseteq \{\lambda, \bar{\lambda}\}$, by the corollary to Lemma 12. Similarly, there exists $\mu \in \mathbf{C}$ such that $\mathrm{Sp}_A(ix) \subseteq \{\mu, \bar{\mu}\}$. But $\mathrm{Sp}_A(i\cdot x) \subseteq \{i\lambda, i\bar{\lambda}\}$. If $\lambda \neq 0$, $\overline{i\lambda} \neq i\bar{\lambda}$. We conclude that $\mathrm{Sp}_A(x)$ is a singleton, and without loss of generality $\mathrm{Sp}_A(x) = \{\lambda\}$. Therefore $\mathrm{Sp}_A(x-\lambda\cdot e) = \{0\}$. Since A is commutative, [13, 2.4.6] implies that $x-\lambda\cdot e \in J(A) = J(\mathrm{Cen})$. Thus $x = \lambda\cdot e + j$, where $j \in J(\mathrm{Cen})$, as required.

ii) Suppose $\lambda \in \mathbf{C}$ and λ is real. Suppose $M(y)$. Then by the corollary to Lemma 12, there exists $\mu \in \mathbf{C}$ such that $\mathrm{Sp}_A(y) \subseteq \{\mu, \bar{\mu}\}$. Thus $\mathrm{Sp}_A((\lambda\cdot e) \subseteq \{\lambda\mu, \lambda\bar{\mu}\}$, since $\lambda = \bar{\lambda}$. By the corollary to Lemma 12, $M((\lambda e)\cdot y)$. We conclude that $\lambda\cdot e \in$ Kon.

6.4. Definition 6.4.1. $\mathrm{Kon}_N = \mathrm{Kon} \cap \{\lambda\cdot e + j : \lambda \in N \wedge j \in J(\mathrm{Cen})\}$.

Our objective is to show that Kon_N is $\mathcal{L}_2$-definable, provided A satisfies (*). This involves modifying the technique of Section 4.

Definition 6.4.2. $\Psi_K(x) \leftrightarrow x \in \mathrm{Kon} \wedge (\exists t)(\exists u)[D(0, t, u) \wedge$
$(\forall y)((y \in \mathrm{Kon} \wedge y-x \notin J(\mathrm{Cen}) \wedge D(y, t, u))$
$\to (\exists z)(D(z, t, u) \wedge z - (y + e) \in J(\mathrm{Cen})))]$.

Lemma 14. *Suppose $A \in \mathrm{Ban}_1$ and $x \in A$. Then $\Psi_K(x) \to x \in \mathrm{Kon}_N$.*

Proof. Suppose $x \notin \mathrm{Kon}_N$ and $\Psi_K(x)$. Select t, u such that $D(0, t, u)$ and $(\forall y)[(y \in \mathrm{Kon} \wedge y - x \notin J(\mathrm{Cen}) \wedge D(y, t, u))$
$\to (\exists z)(D(z, t, u) \wedge z-(y + e) \in J(\mathrm{Cen}))]$.
Since $\Psi_K(x)$, $x \in$ Kon, so $x = \lambda\cdot e + j$, where $\lambda \in \mathbf{C}$ and $j \in J(\mathrm{Cen})$. If $y = n\cdot e + l$, where $n \in N$ and $l \in J(\mathrm{Cen})$, then $y-x \in J(\mathrm{Cen})$, since $\lambda \notin N$. It follows by induction that for each $n \in N$ there exists $l_n \in J(\mathrm{Cen})$ such that $D(n\cdot e + l_n, t, u)$. In particular, $t-(n\cdot e + l_n)$ is not invertible. Now, $l_n \in$ Cen. Let B be a maximal commutative subalgebra of A, containing t. Since $l_n \in J(\mathrm{Cen})$, $\mathrm{Sp}_{\mathrm{Cen}}(l_n) = \{0\}$, so by [13, 1.6.14]

$Sp_B(l_n) = \{0\}$, so $l_n \in J(B)$, by [13, 2.4.6]. Therefore, by 3.2.3
$Sp_B(t) = Sp_B\ (t - l_n)$
$\quad = Sp_A\ (t - l_n)$.
Since $n \in Sp_A\ (t - l_n)$, $n \in Sp_B(t)$.
$\therefore N \subseteq Sp_B(t)$, a contradiction, since $Sp_B(t)$ is bounded.
$\therefore \Psi_K(x) \to x \in \mathrm{Kon}_N$.

Lemma 15. *Suppose $A \in \mathrm{Ban}_1$ and A satisfies* (*). *Let $x \in A$. Then*
$$x \in \mathrm{Kon}_N \to \Psi_K(x).$$

Proof. Suppose $x \in \mathrm{Kon}_N$. Then there exists $n \in N$ and $j \in J(\mathrm{Cen})$ such that $x = n \cdot e + j$, and $x \in \mathrm{Kon}$. Since A satisfies (*) we can pick $t \in A$ such that $\{0, ..., n\} \subseteq Sp_A(t)$. Let $u = t(t-e) \ldots (t-n \cdot e)$. Then, by Lemma 8, if $\lambda \in \mathbf{C}$, $D(\lambda \cdot e, t, u) \leftrightarrow \lambda \in \{0, 1, ..., n\}$.
In particular, $D(0, t, u)$.

Suppose $y \in \mathrm{Kon}$, and $y - x \notin J(\mathrm{Cen})$, and $D(y, t, u)$. By Lemma 13, there exists $\lambda \in \mathbf{C}$ and $l \in J(\mathrm{Cen})$ such that $y = \lambda \cdot e + l$. Since $D(y, t, u)$, $(t-y) \| u$, so there exists v such that $u = (t-y) \cdot v = v \cdot (t-y)$. Let B be a maximal commutative subalgebra of A containing e, t, and v. Then $\mathrm{Cen} \subseteq B$. Thus $y \in B$ and $(t-y) \| u$ in B. Since $D(y, t, u)$, $t-y$ is not invertible, so it follows that $D(y, t, u)$ in B. Also, $l \in J(\mathrm{Cen})$, so $Sp_{\mathrm{Cen}}(l) = \{0\}$, so $Sp_B(l) = \{0\}$, so $l \in J(B)$ since B is commutative. Let $\eta: B \to B/J(B)$ be the natural projection. Just as in the proof of Lemma 8, $D(\eta(y), \eta(t), \eta(u))$, so $D(\lambda \cdot e + \eta(l), \eta(t), \eta(u))$, so $D(\lambda \cdot e, \eta(t), \eta(u))$. Therefore, by Lemma 8, $\lambda \in \{0, 1, ..., n\}$. But $y-x \notin J(\mathrm{Cen})$, so $\lambda \in \{0, 1, ..., n-1\}$. Let $z = (\lambda + 1) \cdot e$. Then $z - (y + e) = l \in J(\mathrm{Cen})$, and $D(z, t, u)$ by Lemma 8.

We conclude that $\Psi_K(x)$.

Corollary. Kon_N *is $\mathcal{L}_2$-definable, if A satisfies* (*).

Proof. Ψ_K is clearly $\mathcal{L}_2$-definable.

6.5. Suppose $A \in \mathrm{Ban}_1$ and A satisfies (*). Then Kon_N is $\mathcal{L}_2$-definable.

Definition 6.5.1.
$x \in P \leftrightarrow (\exists y)(y \in \mathrm{Kon}_N \wedge y-x \in J(\mathrm{Cen}))$.

Then P is $\mathcal{L}_2$-definable, because J is $\mathcal{L}_2$-definable via 3.2.2.

Note. $P = \mathrm{Kon}_N$, but there is no need to prove this. Trivially, $\mathrm{Kon}_N \subseteq P$.

Lemma 16. i) *P is closed under $+$ and $\cdot$.*
ii) *Let $\equiv$ be congruence modulo J(Cen). Then* $\langle P, +, \cdot\rangle/\equiv\ \cong \langle N, +, \cdot\rangle$.

Proof. Trivial.

Thus we have proved that if $A \in \mathrm{Ban}_2$ and A satisfies (*) then number theory is interpretable in Th(A). Thus Th(A) is hereditarily undecidable.

Theorem 3. *Suppose $A \in \mathrm{Ban}_2$, and $A/J(A)$ is infinite-dimensional. Then* Th(A) *is hereditarily undecidable.*

Proof. Done.

Corollary 1. Th(Ban_2) *is undecidable.*

Proof. Let $A = C(I)$. Then Th(A) is hereditarily undecidable, and Th(Ban_2) $\subseteq$ Th(A).

Corollary 2. *The $\mathcal{L}_2$-theory of commutative Banach algebras is undecidable.*

Proof. See Corollary 1.

Remark. Con is not in general definable in $\mathcal{L}_2$. Take $A = \mathbf{C}^2$, with $\|(x, y)\| = \max(|x|, |y|)$. Define σ: A $\to$ A by $\sigma((x, y)) = (x, \bar{y})$.

Then σ is a ring isomorphism of A, and σ is an isometry. But Con is not closed under σ, for $(i, i) \in$ Con but $\sigma((i, i)) = (i, -i) \notin$ Con. By Padoa's test, Con is not $\mathcal{L}_2$-definable.

However, in this example, Kon $= \{\lambda \cdot e : \lambda \in \mathbf{R}\}$, so we can define in $\mathcal{L}_2$ a subset Con_1 of Con with $\mathrm{N} \subseteq \mathrm{Con}_1$.

We know of no A for which no such subset Con_1 is $\mathcal{L}_2$-definable.

7. Finite-dimensional semi-simple Banach algebras

Suppose $i = 2$ or 3, and $A \in \mathrm{Ban}_i$. We known that if A is infinite-dimensional and semi-simple then Th(A) is hereditarily undecidable.

Now we look at finite-dimensional semi-simple algebras. By [10] each such algebra has decidable $\mathcal{L}_3$-theory. We do not know any such algebra with undecidable $\mathcal{L}_2$-theory, but we conjecture that such algebras exist.

7.1. The group of invertible elements. Let $A \in \mathrm{Ban}_1$. The set of invertible elements of A forms a group Gp(A) under $\cdot$. The elementary theory of Gp(A) is formalized in a sublanguage $\mathcal{L}_5$ of $\mathcal{L}_4$. The only non-logical symbols of $\mathcal{L}_5$ are the constant corresponding to e, and the operation symbol corresponding to $\cdot$. $\mathcal{L}_5$ is just the usual language for group theory.

Let n be an integer ≥ 1. $\mathbf{C}^n$ has a natural structure of n-dimensional Hilbert space over $\mathbf{C}$. Consider $\mathcal{L}(\mathbf{C}^n, \mathbf{C}^n)$ the Banach algebra of continuous linear operators on $\mathbf{C}^n$. The underlying ring of $\mathcal{L}(\mathbf{C}^n, \mathbf{C}^n)$ is just $M_n(\mathbf{C})$, the ring of $n \times n$ matrices over $\mathbf{C}$. Let $A = \mathcal{L}(\mathbf{C}^n, \mathbf{C}^n)$. Then Gp($A$) = $\mathrm{GL}_n(\mathbf{C})$, the group of $n \times n$ invertible matrices over $\mathbf{C}$.

For any field K, define $M(K) = \{\mathrm{GL}_n(K) : 1 \leq n < \omega\}$. Ersov [3] proved that Th($M(K)$) is hereditarily undecidable.

From this we deduce:

Theorem 4. *Let* Γ *be the class of groups of invertible elements of finite-dimensional semi-simple Banach algebras. Then* Th(Γ) *is hereditarily undecidable.*

Proof. $M(\mathbf{C}) \subseteq \Gamma$, since $M_n(\mathbf{C})$ is semi-simple for each n. The result follows immediately from Ersov's result.

Corollary 1. *The* $\mathcal{L}_5$*-theory of the class of finite-dimensional semi-simple Banach algebras is hereditarily undecidable.*

Proof. Immediate from Theorem 4, and the observation that Th(Γ) is interpretable in the $\mathcal{L}_5$-theory of the class of finite-dimensional semi-simple Banach algebras.

Corollary 2. *Let* $1 \leq i \leq 4$. *The* $\mathcal{L}_i$*-theory of the class of finite-dimensional semi-simple Banach algebras is hereditarily undecidable.*

Proof. Immediate from Corollary 1, since $\mathcal{L}_5$ is a sublanguage of $\mathcal{L}_i$.

Remarks. 1. Ersov's result, with $K = \mathbf{R}$, establishes the hereditary undecidability of the class of finite-dimensional semi-simple real Banach algebras.

With $K = \mathbf{Q}_p$, the field of p-adic numbers, we get a corresponding result for normed algebras over $\mathbf{Q}_p$.

2. We do not know any $A \in \mathrm{Ban}_1$ for which $\mathrm{Gp}(A)$ is undecidable.

3. The additive group of a Banach algebra is a torsion-free divisible abelian group, and so is decidable [4]. Also by completeness [4] of the theory of non-trivial torsion-free divisible abelian groups, the theory of the class of additive groups of Banach algebras is decidable.

7.2. The commutative case. It is known [13, 2.4] that if $A \in \mathrm{Ban}_1$ and A is finite-dimensional, commutative and semi-simple then A is isomorphic as an algebra to one of the algebras $\mathbf{C}^n$ $(1 \leq n < \omega)$.

Theorem 5. *The* $\mathcal{L}_4$*-theory of the class of finite-dimensional, commutative, semi-simple Banach algebras over* $\mathbf{C}$*, with unit, is decidable.*

Proof. By the remark above, it suffices to prove that the theory of the class of rings $\{\mathbf{C}^n : 1 \leq n < \omega\}$ is decidable. But $\mathbf{C}$ is a decidable ring, by Tarski's theorem [15], so by a result of Feferman-Vaught [4] the class $\{\mathbf{C}^n : 1 \leq n < \omega\}$ is decidable.

One might hope to extend the above theorem from $\mathcal{L}_4$ to $\mathcal{L}_3$. However, this cannot be done.

Theorem 6. *For* $i = 1, 2$ *or* 3 *the* $\mathcal{L}_i$*-theory of the class of finite-dimensional, commutative, semi-simple Banach algebras over* $\mathbf{C}$*, with unit, is hereditarily undecidable.*

Proof. (In outline). It suffices to consider the cases $i = 2$ or 3.

$i = 3$. In [10] it is proved that the theory of the class of *algebras* $\{\mathbf{C}^n : n < \omega\}$ is hereditarily undecidable. (The proof is a variant of those in Section 4). The result follows.

$i = 2$. Suppose $A = \mathbf{C}^n$, $n > 1$. A minor extension of the proof of Lemma 13 shows that Kon is the set of real constants. For $n = 1$, Kon = Con. Now, our proof in [10] shows that $\{\mathbf{C}^n : 2 \leq n < \omega\}$ is a hereditarily undecidable class of algebras over $\mathbf{R}$. But $\mathbf{R}$ is uniformly definable in $\mathcal{L}_2$, for these algebras. It follows that the $\mathcal{L}_2$-theory of the class $\{\mathbf{C}^n : 2 \leq n < \omega\}$ is hereditarily undecidable, whence the result.

8. Commutative Banach algebras with undecidable $\mathcal{L}_4$-theory

So far we have not found any undecidable commutative ring which is the underlying ring of a Banach algebra. There are in fact many such rings, as we now prove by a new technique. However, our results do not have the generality of those in Sections 5 and 6.

8.1. We will consider only the case where $A = C(X)$, and X is a compact Hausdorff space.

Lemma 17. *The maximal ideals of $C(X)$ are precisely the sets of the form $\{f \in C(X): f(\alpha) = 0\}$, where $\alpha \in X$.*

Proof. See [13, 3.1].

Definition 8.1.1. Suppose $f \in C(X)$. Then $Z(f) = \{\alpha \in X: f(\alpha) = 0\}$.

Clearly $Z(f)$ is closed if $f \in C(X)$.

Definition 8.1.2. Zer(X) is the partially ordered set consisting of the sets $Z(f)$, where $f \in C(X)$, under $\subseteq$.

Lemma 18. *Suppose $f, g \in C(X)$. Then $Z(f) \cap Z(g) \neq \emptyset$.*
$\leftrightarrow (\forall r, s \in C(X))(rf + sg \neq e)$.

Proof. *Necessity.* Suppose $Z(f) \cap Z(g) \neq \emptyset$. Then $f(\alpha) = g(\alpha) = 0$, for some $\alpha \in X$. Then clearly $rf + sg \neq e$, for all $r, s \in C(X)$.
Sufficiency. Suppose $rf + sg \neq e$, for all $r, s \in C(X)$. Then f and g generate a proper ideal, which extends to a maximal ideal. By Lemma 17, it follows that there exists $\alpha \in X$ such that $f(\alpha) = g(\alpha) = 0$. Thus $Z(f) \cap Z(g) \neq \emptyset$.

Lemma 19. *Suppose* $f, g \in C(X)$. *Then*
$Z(f) \subseteq Z(g) \leftrightarrow (\forall h \in C(X))\, [Z(f) \cap Z(h) \neq \emptyset \rightarrow Z(g) \cap Z(h) \neq \emptyset]$.

Proof. Necessity is trivial.
Sufficiency. Suppose $Z(f) \not\subseteq Z(g)$. Select $\alpha \in X$ such that $f(\alpha) = 0$ and $g(\alpha) \neq 0$. Consider the disjoint closed sets $\{\alpha\}$ and $Z(g)$. Since X is compact, X is normal [9], so there exists $h \in C(X)$ such that $h(\alpha) = 0$ and $h(\beta) = 1$ if $\beta \in Z(g)$.
Then $Z(f) \cap Z(h) \neq \emptyset$, but

$$Z(g) \cap Z(h) = \emptyset.$$

This proves the result.

Lemma 20. *The theory of* Zer(X) *can be interpreted in the* $\mathcal{L}_4$*-theory of* $C(X)$.

Proof. By Lemma 18, the relation $Z(f) \cap Z(g) \neq \emptyset$ is $\mathcal{L}_4$-definable. Then by Lemma 19 the relation $Z(f) \subseteq Z(g)$ is $\mathcal{L}_4$-definable. Then clearly $Z(f) = Z(g)$ is $\mathcal{L}_4$-definable, since
$Z(f) = Z(g) \leftrightarrow Z(f) \subseteq Z(g) \cap Z(g) \subseteq Z(f)$.
Define $f \equiv g$ by $Z(f) = Z(g)$.
On $C(X)_{/\equiv}$ define $\leq$ by:
$f_{/\equiv} \leq g_{/\equiv} \leftrightarrow Z(f) \subseteq Z(g)$.
Then $(C(X)_{/\equiv}, \leq) \cong \mathrm{Zer}(X)$, and so $\mathrm{Zer}(X)$ is interpretable in the $\mathcal{L}_4$-theory of $C(X)$.

8.2. **Lemma 21.** *Suppose X is compact and metrizable. Then every closed subset of X is of the form $Z(f)$, for some $f \in C(X)$.*

Proof. Let d be the metric on X. Let M be a closed subset of X. Define $f \in C(X)$ by $f(\alpha) = \inf_{\beta \in M} d(\alpha, \beta)$.
Then $M = Z(f)$.

Corollary. *Suppose X is compact and metrizable. Then the theory of the lattice of closed subsets of X is interpretable in the $\mathcal{L}_4$-theory of $C(X)$.*

Proof. Immediate from Lemmas 20 and 21.

8.3. **Grzegorczyk's Conditions.** We say a topological space X satisfies Grzegorczyk's conditions if it satisfies (a)-(e) below.

a) X is metrizable.
b) X has at least two points.
c) X is connected.
d) If A and B are two closed isolated disjoint sets, and $A \cup B \subseteq E$, and E is a connected open set, then there exist two connected open sets C and D such that $A \subseteq C$, $B \subseteq D$, $C \cup D \subseteq E$, and the closures of C and D are disjoint.
e) If A and B are two closed isolated disjoint sets, and there exists a 1-1 mapping of A into B, then there exists a closed set C such that $A \cup B \subseteq C$, and every component D of C contains exactly one point of the set A and one point of the set B.

Grzegorczyk [6] proved:
If X satisfies Grzegorczyk's conditions then the theory of the lattice of closed subsets of X has a finitely axiomatizable essentially undecidable subtheory.

Remarks. i) Condition (a) is equivalent to conditions A1-A4 of Grzegorczyk's paper, as he remarks in footnote 7. (d) is his A′6, which implies his A6. Although he actually considers the Brouwer algebra of closed sets, his results hold for the lattice of closed sets, since his operation $\dot{-}$ on page 143 of his paper is obviously definable in terms of the lattice operations.
ii) If X is compact, closed isolated sets are finite, and condition (e) follows from the other condition (cf. the argument on page 140 of [6]).
iii) Examples of spaces satisfying Grzegorczyk's conditions are E_n (Euclidean n-space) for $n \geq 2$, and the sphere S_2. These facts are used in [6], without proof. A fact not used in [6], but relevant here, is that S_n satisfies Grzegorczyk's conditions for $n \geq 2$. For all these spaces X, the

crucial observation in verifying Grzegorczyk's conditions is: if A is a closed isolated subset of X, and E is an open connected subset of X, and $A \subseteq E$, then there is a closed subspace X_0 of X, such that $A \subseteq X_0 \subseteq E$, and X_0 is homeomorphic to either the real line E_1 or the unit interval I.

This observation applies also when $X = I^n$, for $n \geq 3$.

Thus we have the following list of compact Hausdorff spaces satisfying Grzegorczyk's conditions:
S_n $(n \geq 2)$; I^n $(n \geq 3)$.

The spaces S_1, I and I^2 do not satisfy Grzegorczyk's conditions. For S_1 and I this is obvious. For I^2, take $A = \{(0, 0), (1, 1)\}$, $B = \{(0, 1), (1, 0)\}$, $E = I^2$, and then clearly condition (d) fails.

8.4. **Theorem** 7. *Suppose X is a compact Hausdorff space satisfying Grzegorczyk's conditions. Then the $\mathcal{L}_4$-theory of $C(X)$ is hereditarily undecidable.*

Proof. Since X is compact metrizable, the corollary to Lemma 12 implies that the theory of the lattice of closed subsets of X is interpretable in the $\mathcal{L}_4$-theory of $C(X)$. By Grzegorczyk's theorem and Lemma 2, the $\mathcal{L}_4$-theory of $C(X)$ is hereditarily undecidable.

Corollary 1. *$C(S_n)$, for $n \geq 2$, and $C(I^n)$, for $n \geq 3$, are hereditarily undecidable.*

Proof. Immediate.

Corollary 2. *The $\mathcal{L}_4$-theory of commutative semi-simple Banach algebras over* C, *with unit, is hereditarily undecidable.*

Proof. Immediate from Corollary 1, since $C(S_n)$ is commutative and semi-simple.

8.5. Although I^2 does not satisfy Grzegorczyk's conditions, we can show that its lattice of closed subsets has a hereditarily undecidable theory. We sketch a proof, which is simply a variant of Grzegorczyk's proof of Theorem 5 in [6].

I^2 is a subspace of E_2, and has a boundary B in E_2. B is a closed subset of I^2. It turns out that B is a definable element of the lattice of closed subsets of I^2. Accept this for the moment. Then we can define the class of those closed subsets of I^2 which are subsets of the interior of I^2. The finite subsets of the interior of I^2 are the closed isolated subsets of I^2 not intersecting B, Now by using just the definitions given by Grzegorczyk, one can interpret, in the theory of the lattice of closed subsets of I^2, the arithmetic of the finite subsets of the interior of I^2, and this of course has a finitely axiomatizable, essentially undecidable subtheory, whence the required result.

Definition of B. Suppose $p \in I^2$. Then $p \in B$ if and only if there exist $q, r, s \in I^2$ such that whenever A and B are closed connected subsets of I^2 with $\{p, q\} \subseteq A$ and $\{r, s\} \subseteq B$, then $A \cap B \neq \emptyset$. From this we get a definition of B by replacing points by atoms as in [6].

A corollary of this is that $C(I^2)$ is undecidable.

8.6. Open Problems. 1. Is $C(I)$ a decidable ring?

Rabin [12] proved that the lattice of closed subsets of I is decidable, so our method breaks down. By Theorem 1, $C(I)$ is an undecidable algebra. Note that there are undecidable algebras over **C**, whose underlying ring is decidable. An example is $\mathbf{C}^\omega$ See [11].

2. Are there Banach algebras of analytic functions with undecidable $\mathcal{L}_4$-theory?

9. Banach algebras over R

The reason that we have until now restricted ourselves to algebras over **C** is that the spectral theory is smoother. The reader may have noticed that the results obtained via the theorems of Ersov and Grzegorczyk remain valid when **C** is replaced by **R**.

We work in the same languages $\mathcal{L}_i$ as before. We consider Banach algebras over **R**. These are construed as in 2.2, except that now the set M_1 of scalars is **R**. As before, we consider only algebras with unit. For $1 \leq i \leq 4$ let $\mathrm{Ban}_i^{(\mathbf{R})}$ be the real analogue of Ban_i.

9.1. Analogue of Section 7. 9.1.1. Replace **C** by **R** in 7.1, and the next theorem is proved.

Theorem 8. *Let* $\Gamma_{\mathbf{R}}$ *be the class of groups of invertible elements of finite-dimensional semi-simple Banach algebras over* **R**, *with unit. Then* $\mathrm{Th}(\Gamma_{\mathbf{R}})$ *is hereditarily undecidable.*

Corollary 1. *The* $\mathcal{L}_5$*-theory of the class of finite-dimensional semi-simple Banach algebras over* **R**, *with unit, is hereditarily undecidable.*

Corollary 2. *Let* $1 \leq i \leq 4$. *The* $\mathcal{L}_i$*-theory of the class of finite-dimensional semi-simple Banach algebras over* **R**, *with unit, is hereditarily undecidable.*

9.1.2. Now we get analogues of Theorems 5 and 6.

It is known [13, 2.4.4; 7] that if $A \in \mathrm{Ban}_i^{(\mathbf{R})}$ and A is finite-dimensional, commutative and semi-simple than A is isomorphic as an algebra to one of the algebras $\mathbf{C}^m \times \mathbf{R}^n$ $(0 \leq m < \omega, 0 \leq n < \omega, m + n > 0)$. The next theorem is proved just as Theorem 5.

Theorem 9. *The* $\mathcal{L}_4$*-theory of the class of finite-dimensional commutative, semi-simple Banach algebras over* **R**, *with unit, is decidable.*

Replace **C** by **R** in the proof of Theorem 6 to get

Theorem 10. *For* $i = 1, 2,$ *or,* 3 *the* $\mathcal{L}_i$*-theory of the class of finite-dimensional, commutative, semi-simple Banach algebras over* **R**, *with unit, is hereditarily undecidable.*

9.2. Analogue of Section 8. For a compact Hausdorff space X, let $C_{\mathbf{R}}(X)$ be the real Banach algebra of real-valued continuous functions on X. Then one may easily check that all the results of Section 8 remain valid. Thus we get:

Theorem 11. *Suppose* X *is a compact Hausdorff space satisfying Grzegorczyk's conditions. Then the* $\mathcal{L}_4$*-theory of* $C_{\mathbf{R}}(X)$ *is hereditarily undecidable. In particular* $C_{\mathbf{R}}(X)$ *is an undecidable ring.*

Corollary 1. *$C_{\mathbf{R}}(I^n)$, for $n > 2$, and $C_{\mathbf{R}}(S_n)$, for $n > 1$, are hereditarily undecidable.*

Corollary 2. *The $\mathcal{L}_4$-theory of commutative semi-simple Banach algebras over* **R**, *with unit, is hereditarily undecidable.*

Note. As in Section 8, we can show that $C_{\mathbf{R}}(I^2)$ is hereditarily undecidable.

9.3. Analogue of Section 4. For the notion of complexification of a real algebra, see [13, 1.3]. Let $A \in \mathrm{Ban}_1^{(\mathbf{R})}$, let $A_{\mathbf{C}}$ be the complexification of A. Then $A_{\mathbf{C}}$ is $A \times A$, where (x, y) is to mimic $x + iy$. Precisely,
$(x, y) + (u, v) = (x + u, y + v)$
$(\lambda + i\mu)(x, y) = (\lambda x - \mu y, \lambda y + \mu x)$, and
$(x, y)\cdot(u, v) = (xu - yv, xv + yu)$.

Then clearly the $\mathcal{L}_3$-theory of $A_{\mathbf{C}}$ is interpretable in the $\mathcal{L}_3$-theory of A. We know by Section 4 that if $A_{\mathbf{C}}$ satisfies (*) then number theory, and in particular the finitely axiomatizable essentially undecidable system Q of R.M. Robinson [4, 2.2], is interpretable in the $\mathcal{L}_3$-theory of $A_{\mathbf{C}}$. Thus by [4, 3.4] if $A_{\mathbf{C}}$ satisfies (*) then the $\mathcal{L}_3$-theory of A is hereditarily undecidable. Thus:

Theorem 12. *Suppose $A \in \mathrm{Ban}_3^{(\mathbf{R})}$, and $\dim_{\mathbf{C}}(A_{\mathbf{C}}/J(A_{\mathbf{C}}))$ is infinite. Then* Th(A) *is hereditarily undecidable.*

Examples. i) $C_{\mathbf{R}}(X)$, where X is an infinite compact Hausdorff space.

ii) The algebra of continuous linear operators on an infinite-dimensional real Hilbert space.

Corollary 1. $\mathrm{Th}(\mathrm{Ban}_3^{(\mathbf{R})})$ *is hereditarily undecidable.*

Corollary 2. *The theory of commutative Banach algebras over* **R**, *construed simply as algebras, is undecidable.*

9.4. Analogue of Section 6. Suppose $A \in \mathrm{Ban}_1$. In [13, 1.3] it is shown that $A_{\mathbf{C}}$ has a norm $\|\cdot\|$ under which it is a Banach algebra over **C**, and

$x \to (x, 0)$ is an isometry. Actually [13, 1.3] produces many norms, corresponding to normed real representations of A. For definiteness we take the left regular representation in Theorem (1.3.2) of [13, 1.3]. Unfortunately, we do not see how to interpret within the $\mathcal{L}_2$-theory of A the $\mathcal{L}_2$-theory of $A_{\mathbf{C}}$. The snag occurs in Theorem (1.3.1) of [13, 1.3]. We define first

$|(x, y)| = \|x\| + \|y\|$, and then

$\|(x, y)\| = {}^1/_{\sqrt{2}} \sup_\theta |e^{i\theta} \cdot (x, y)|$.

If we have a definition of $\mathrm{Con}_{\mathbf{R}}$, the constants of A, then we may define

$$\|(x, y)\| = {}^1/_{\sqrt{2}} \sup_{\substack{u, v \in \mathrm{Con}_{\mathbf{R}} \\ u^2 + v^2 = 1}} |(u, v) \cdot (x, y)| .$$

Then it turns out (we omit the details) that if $\mathrm{Con}_{\mathbf{R}}$ is $\mathcal{L}_2$-definable then we can interpret within the $\mathcal{L}_2$-theory of A the $\mathcal{L}_2$-theory of $A_{\mathbf{C}}$. But without a definition of $\mathrm{Con}_{\mathbf{R}}$ we do not see what to do.

When is $\mathrm{Con}_{\mathbf{R}}$ $\mathcal{L}_2$-definable? We content ourselves with a sufficient condition. Suppose the centre of $A_{\mathbf{C}}$ is semi-simple. Then by Lemma 13 we can define the real constants of $A_{\mathbf{C}}$ in $\mathcal{L}_2$, whence we can define $\mathrm{Con}_{\mathbf{R}}$ in $\mathcal{L}_2$.

These remarks and Theorem give:

Theorem 13. *Suppose* $A \in \mathrm{Ban}_2^{(\mathbf{R})}$, $\dim_{\mathbf{C}} ({}^{A_{\mathbf{C}}}/_{J(A_{\mathbf{C}})})$ *is infinite, and the centre of* $A_{\mathbf{C}}$ *is semi-simple. Then* $\mathrm{Th}(A)$ *is hereditarily undecidable.*

10. Algebras without unit

Since many important Banach algebras do not have a unit, it seems worthwhile to consider briefly such algebras. Most of our techniques break down. Certainly our approaches via the theorems of Eršov and Grzegorczyk yield nothing. There is, however, an analogue of Theorem 1.

10.1. We formulate the elementary theories of algebras without unit in languages $\mathcal{L}_i^1$ $(1 \leq i \leq 4)$ obtained from the languages $\mathcal{L}_i$ by dropping the constant corresponding to the unit.

We will consider only algebras over **C**. To get results about algebras over **R**, combine the methods of Sections 9 and 10.

We will use the standard device of adjoining a unit. If A has no unit, we define A_e as $A \times \mathbf{C}$ with operations as follows:

$$(x, \alpha) + (y, \beta) = (x + y, \alpha + \beta)$$
$$(x, \alpha)\cdot(y, \beta) = (\alpha y + \beta x + xy, \alpha\beta)$$
$$\|(x, \alpha)\| = \|x\| + |\alpha|.$$

Then A_e is a Banach algebra with unit (0,1), and the map $x \to (x, 0)$ is an isometric isomorphism of A onto a maximal ideal of A_e. It follows that $J(A_e) = J(A)$.

In $\mathcal{L}^1$ and $\mathcal{L}_3^1$ we have a separate sort of variable for the scalars $\mathbf{C}$. It then follows from the definition of A_e that for $i = 1$ or 3 the $\mathcal{L}_i$-theory of A_e is interpretable in the $\mathcal{L}_i^1$-theory of A. This gives:

Theorem 14. *Suppose A is a Banach algebra over* $\mathbf{C}$ *without unit, such that* $\dim_{\mathbf{C}}(A/J(A))$ *is infinite. Then the $\mathcal{L}_3^1$-theory of A is hereditarily undecidable.*

10.2. We do not see how to interpret within the $\mathcal{L}_i^1$-theory of A, for $i = 2$ or 4, the $\mathcal{L}_i$-theory of A_e.

However, we do have:

Theorem 15. *Let $1 \leq i \leq 4$. Let A be a Banach algebra over* $\mathbf{C}$ *without unit. If the $\mathcal{L}_i^1$-theory of A is decidable then the $\mathcal{L}_i$-theory of A_e is decidable.*

We will prove this for $i = 4$ in much greater generality in [10]. The other cases are proved similarly.

10.3. Having given considerable attention to semi-simple algebras, we now raise the question of the decidability of radical algebras, i.e. algebras A for which $J(A) = A$. Such algebras have of course no unit.

We have found no undecidability results for such algebras.

Problem. Is the theory of radical Banach algebras decidable?

We conjecture not. A candidate for a hereditarily undecidable radical Banach algebra is $L(0, 1)$ under addition and convolution [13, Appendix A.2.11].

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